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Geometry of tracer trajectories in turbulent rotating convection

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Abstract

Convection appears in multiple situations in nature and technology. Rotation changes the flow structure, causing different behavior. Researching this concept can give a better understanding of natural processes or may help making machinery more efficient. Previous studies to rotating turbulent convection already showed that rotation can change the flow structure, causing vertically aligned plumes to appear, and that rotation can enhance the heat flow. For this research, rotating turbulent convection has been studied by looking at the geometry of tracer trajectories. This has been done by looking at the curvature and torsion of the trajectories. The results from situations with rotation have been compared to a situation without rotation, showing how the geometry changes. The results have also been compared to results from studies to homogeneous isotropic turbulence, to see if it gives different results.

For this research a rotating cylindrical Rayleigh-Bénard cell has been used. Rayleigh-Bénard convection occurs in a fluid between two horizontal plates, where the bottom plate is heated and the top plate is cooled. The cylinder can rotate around its vertical axis and contains one million tracer particles. The tracer particles are studied by using a direct numerical simulation (DNS) in a Lagrangian framework. The DNS solves the Navier-Stokes equations in cylindrical coordinates and makes use of a finite difference scheme. The properties of the particles are calculated using a trilinear interpolation and Adams-Bashforth time integration.

The simulations have been executed for four situations: $Ro = \infty$, $Ro = 2.5$, $Ro = 1.0$, and $Ro = 0.1$, where the Rossby number $Ro$ is inversely proportional to the rotation rate. The curvature and torsion have been calculated on the trajectory of all particles. This data has been used to calculate probability density functions (PDFs) of the curvature and torsion, and joint PDFs of the curvature and torsion with the velocity. The differences in the PDFs have been used to show the changes as a result of the rotation.

The geometry of the tracer particles behaves the same in turbulent Rayleigh-Bénard convection as it does in homogeneous isotropic turbulence. The PDFs of the curvature $\kappa$ scale as $\kappa^1$ for low values of the curvature and as $\kappa^{−2.5}$ for high values of the curvature, and the PDFs of the torsion $\tau$ scale as $\tau^0$ for low values of the torsion and as $\tau^{−3}$ for high values of the torsion. The joint PDFs showed that high values of the curvature and torsion are related to low velocities. This suggests that the high values appear when a particle reverses direction. It also showed that high values of torsion are related to high and low values of curvature, indicating that high values of torsion appear when a particle changes direction fast or barely changes direction.

The rotation of the Rayleigh-Bénard cell does show some changes. As the rotation speed increases, the values of the curvature and torsion also increase, while the scaling of the PDFs remain the same. This is probably because of the vortices that appear as a result of the rotation.

An extra method for finding information about the geometry that makes use of the angle between subsequent particle increments verified the results from the curvature. Rotation caused a higher probability for finding large angles for small time-increments. This suggests that rotation gives a higher probability for finding larger curvature values.
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1 Introduction

Convection plays a big part in a lot of processes in nature and technology. A natural system to investigate is Rayleigh-Bénard convection: a type of convection that occurs in a fluid between two horizontal plates, where the bottom plate is heated and the top plate is cooled, causing a buoyancy driven flow. This type of convection behaves different when the fluid is rotated. Some examples of rotational convection are geophysical flows like sea currents, astrophysical flows like in the outer layer of the sun, or flows in industrial machines [1].

Previous studies to rotating Rayleigh-Bénard convection already showed some interesting results. In Figure 1.1 (a) it is shown that in a rotating cylinder, the flow structure in the cylinder changes. Because of the rotation, vortices appear that enhance the heat flow due to Ekman pumping. This is further elaborated in Figure 1.1 (b). As the rotation speed increases, the convective heat flow increases, but becomes destructive as the rotation gets too high and suppresses the turbulence.

![Figure 1.1: (a) 3D visualization of the temperature isosurfaces in a cylinder for Rayleigh number $Ra = 10^8$ and Prandtl number $Pr = 0.7$ for the upper pictures and $Pr = 6.4$ for the lower pictures. The left pictures have $Ro = \infty$, the right pictures have $Ro = 0.3$. (b) The Nusselt number ratio $Nu(\Omega)/Nu(\Omega = 0)$ for heat transfer as a function of the Rossby number $Ro$ for multiple situations. [2]](image)

To investigate the effect of rotation on Rayleigh-Bénard convection further, the geometrical characteristics of the particle trajectories will be studied. These characteristics give information about the coherent structures in the flow, but the effect of rotation has not yet been explained. The trajectory geometry will be studied by looking at the curvature and the torsion. The curvature indicates how sharply the trajectory turns. This means that a high value of curvature corresponds with a sharp turn and a small radius of curvature, and a low value of curvature corresponds with a slight turn and a big radius of curvature. The torsion indicates how sharply the trajectory twists out of the plane of curvature. A high value of the torsion corresponds with a sharp twist and a small radius of torsion, and a small value of the torsion corresponds with a slight twist and a big radius of torsion.
The geometrical characteristics of particle trajectories in turbulent flows have already been studied extensively by Braun et al [3], Xu et al [4], and Scagliarini [5]. These studies have all given similar results for the Probability Density Function (PDF) of curvature $\kappa$. The PDF exhibits two power-law behaviors, one for large and one for small values of curvature. For small values, the PDF has a linear scaling $\kappa^1$, and for large values, the PDF scales as $\kappa^{-2.5}$ (Figure 1.2). The PDF of torsion $\tau$ has been studied by Scagliarini [5]. This study shows that for small values of the torsion, the PDF scales as $\tau^0$, and for large values, the PDF scales as $\tau^{-3}$.

![Figure 1.2: PDF of (a) the curvature and (b) the torsion in a homogeneous and isotropic turbulent flow, calculated in a direct numerical simulation and normalized by the Kolmogorov scale $\eta$. [5]](image)

Xu et al [4] also studied the relation between the curvature of a particle and its velocity by studying the joint PDFs and the PDF quotients, which gives information about the correlation between two variables. Figure 1.3 shows that high values of curvature are correlated with small values of velocity. This suggests that the highest values for curvature appear when a particle reverses direction.

![Figure 1.3: (a) Joint PDF of the curvature and the velocity magnitude. (b) PDF quotient of the curvature and the velocity magnitude [4]](image)

In this report, the effects of rotation on Rayleigh-Bénard convection will be studied by investigating the geometrical characteristics of the particle trajectories. This will be done using numerical simulations, but the results will also be compared to results from experiments. Furthermore, the
results are compared to the results of previous studies, to see if turbulent Rayleigh-Bénard convection gives the same results as homogeneous isotropic turbulence.

In the second chapter of this report, the theory that has been used will be described, followed by the methods that have been used to obtain the results. In chapter 4 the results will be displayed with a discussion. The report will close with a conclusion of the findings and recommendations for further research.
2 Theory

2.1 Navier-Stokes equations

The conservation of momentum, energy and mass for a Newtonian fluid can be expressed with the following set of equations:

\[
\rho \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = - \frac{\partial P}{\partial x_i} - \rho g \delta_{ij} + \frac{\partial \sigma'_{ij}}{\partial x_j}, \tag{2.1}
\]

\[
\rho T \left( \frac{\partial s}{\partial t} + u_i \frac{\partial s}{\partial x_i} \right) = \frac{\partial}{\partial x_i} \left( k \frac{\partial T}{\partial x_i} \right) + \sigma_{ij} \frac{\partial u_i}{\partial x_j}, \tag{2.2}
\]

\[
\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x_i} (\varrho u_i) = 0, \tag{2.3}
\]

with

\[
\sigma_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \mu \frac{\partial u_k}{\partial x_k} \delta_{ij} + \zeta \frac{\partial u_k}{\partial x_k} \delta_{ij}, \tag{2.4}
\]

the viscous stress tensor. These equations give the change of velocity \( \mathbf{u} = (u_1, u_2, u_3) \), entropy \( s \) and density \( \varrho \) for time, with \( P \) the pressure, \( g \) the gravitational acceleration, \( T \) the temperature, \( k \) the thermal conductivity of the fluid, \( \mu \) the dynamic viscosity of the fluid, \( \zeta \) the bulk viscosity of the fluid, and \( \delta_{ij} \) the Kronecker delta function.

These equations can be reduced with the Boussinesq approximation. According to this approximation, the density variations and compressibility effects can be neglected, apart from the gravitational term. In vector notation, the resulting Boussinesq equations are given by

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + 2 \Omega \times \mathbf{u} = -\nabla P + g \alpha T \hat{k} + \nu \nabla^2 \mathbf{u}, \tag{2.5}
\]

\[
\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \chi \nabla^2 T, \tag{2.6}
\]

\[\nabla \cdot \mathbf{u} = 0, \tag{2.7}\]

with \( \Omega \) the rotation vector, \( \alpha \) the thermal expansion coefficient, \( \hat{k} \) the unit vector in the z-direction, \( \nu \) the kinematic viscosity, and \( \chi \) the thermal diffusivity.

These equations can be nondimensionalised by introducing the dimensionless position \( \tilde{x} = x/H \), velocity \( \tilde{\mathbf{u}} = \mathbf{u}/U \), time \( \tilde{t} = t U/H \), pressure \( \tilde{P} = P/U^2 \), and temperature fluctuation \( \tilde{T} = T/\Delta T \). This transforms the equations into

\[
\frac{\partial \tilde{\mathbf{u}}}{\partial \tilde{t}} + (\tilde{\mathbf{u}} \cdot \nabla) \tilde{\mathbf{u}} + \frac{2 \Omega H}{U} \hat{k} \times \tilde{\mathbf{u}} = -\nabla \tilde{P} + \frac{g \alpha \Delta T H}{U^2} \tilde{T} \hat{k} + \frac{\nu}{UH} \nabla^2 \tilde{\mathbf{u}}, \tag{2.8}
\]
The used parameters are the height of the cylinder $H$, and the free-fall velocity $U = \sqrt{\frac{g \alpha \Delta T H}{\nu \chi}}$. Two dimensionless numbers related to Rayleigh-Bénard convection that are relevant for this situation, are the Rayleigh number $Ra$ and the Prandtl number $Pr$:

\begin{equation}
Ra = \frac{g \alpha \Delta T H^3}{\nu \chi},
\end{equation}

\begin{equation}
Pr = \frac{\nu}{\chi}.
\end{equation}

The Rayleigh number gives a ratio of buoyancy to dissipation in a fluid. For small Rayleigh numbers below a critical value, the heat flow is stable and dominated by conduction. As the Rayleigh number becomes higher, the heat flow will become dominated by convection, and convection cells will appear, as shown in Figure 2.1. This is known as Rayleigh-Bénard convection. As the Rayleigh number becomes even higher, the heat flow will become turbulent.

The Prandtl number gives the ratio of momentum diffusivity to thermal diffusivity. For values much smaller than one, $Pr \ll 1$, thermal diffusivity dominates, while for $Pr \gg 1$, momentum diffusivity dominates.

A dimensionless number that is relevant for rotation is the Rossby number $Ro$, defined by

\begin{equation}
Ro = \frac{U}{2\Omega H}.
\end{equation}

The Rossby number gives the ratio of inertial forces to Coriolis forces. For $Ro \ll 1$, Coriolis forces dominate, while for $Ro \gg 1$, inertial forces dominate. [1]

![Convection Cells](image)

**Figure 2.1:** The convection cells in a Rayleigh-Bénard convection

### 2.2 Lagrangian and Eulerian approach

In fluid mechanics, there are two major methods for describing a flow field: using the Lagrangian approach or using the Eulerian approach. In the Eulerian approach, a fixed grid is used. The
properties of the fluid (velocity, density, temperature, etc) are measured at fixed positions as a function of time. The fluid particles that are measured may differ each measurement step. It can be seen as putting a measuring device in a fixed location in the fluid.

In the Lagrangian approach, individual fluid particles are followed. The measuring grid basically moves along with the particles. The properties of the particles are measured as a function of time. It can be seen as putting a measuring device in the fluid and letting it go with the flow.

2.3 Curvature and torsion

Since the Lagrangian approach follows a particle around, it can be used to describe its trajectory. Using this approach, the position of a particle as a function of time \( \mathbf{r}(t) \), moving along the trajectory, can be described by

\[
\mathbf{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}, \tag{2.14}
\]

where \( \hat{i}, \hat{j}, \) and \( \hat{k} \) are the unit vectors in the x-, y-, and z-direction respectively.

Its corresponding velocity \( \mathbf{u}(t) \) and acceleration \( \mathbf{a}(t) \) can be found by using

\[
\mathbf{u}(t) = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}(t) \tag{2.15}
\]

and

\[
\mathbf{a}(t) = \frac{d\mathbf{u}}{dt} = \dot{\mathbf{u}}(t). \tag{2.16}
\]

For the trajectory, a set of tangential, normal, and binormal unit vectors can be described as shown in Figure 2.2.

The tangential vector \( \mathbf{t} \) points in the direction of motion and is given by

\[
\mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{u}}{|\mathbf{u}|} \tag{2.17}
\]

The normal vector \( \mathbf{n} \) points in the direction of the normal acceleration and is perpendicular to the tangential vector, which is given by

\[
\mathbf{n} = \frac{1}{\kappa} \frac{d^2\mathbf{r}}{ds^2} = \frac{1}{\kappa} \frac{dt}{ds}, \tag{2.18}
\]

where \( \kappa \) is the curvature.

The binormal vector \( \mathbf{b} \) is perpendicular to both the tangential vector and normal vector and is defined as

\[
\mathbf{b} = \mathbf{t} \times \mathbf{n}. \tag{2.19}
\]
These unit vectors are used in the Frenet–Serret formulas:

\[
\frac{dt}{ds} = \kappa n, \tag{2.20}
\]

\[
\frac{dn}{ds} = \tau b - \kappa t, \tag{2.21}
\]

\[
\frac{db}{ds} = -\tau n, \tag{2.22}
\]

where \(\tau\) is the torsion.

The curvature and the torsion give information about the geometry of the curve. The curvature can be calculated as

\[
\kappa = \frac{|u \times a|}{|u|^3}, \tag{2.23}
\]

and the torsion can be calculated as

\[
\tau = \frac{u \cdot (a \times \dot{a})}{(u \cdot u)^3 \kappa^2}. \tag{2.24}
\]

Figure 2.2: A curve with the tangential vector \(t\), normal vector \(n\) and binormal vector \(b\) for several positions on the curve. [3]
3 Numerical method and setup

3.1 Setup

The situation that has been simulated using Direct Numerical Simulation (DNS) is a cylinder filled with water, as shown in Figure 3.1. The cylinder can rotate around its vertical axis, and is heated at the bottom and cooled at the top. The cylinder makes it possible to compare the results with the results from experiments, which have also been done in a cylindrical setup. The cylinder also gives the opportunity to study the role of boundary layers. In this report, however, only the center of the cylinder will be studied. In this region, the flow will be close to homogeneous isotropic turbulence. The water contains one million tracer particles, whose position, velocity and acceleration are measured for different Rossby numbers.

The simulations in this study have been carried out for $Ra = 1.28 \times 10^9$ and $Pr = 6.7$, giving a turbulent flow. To describe the cylinder, the aspect ratio

$$\Gamma = \frac{D}{H}$$  (3.1)

is introduced, where $D$ is the diameter. For this study, an aspect ratio $\Gamma = 1$ has been used. The volume that has been investigated is a cube centered around the center of the cylinder. The edges of the cube have a length $L = H/4$, giving a volume that is approximately one fiftieth of the volume of the cylinder. The tracer particles will be distributed evenly over the cylinder because of the turbulent flow, so the control volume will contain approximately 20000 particles at all times.

Figure 3.1: A sketch of the cylinder used in the calculations. The dashed box indicates the area of interest for data processing.
3.2 Numerical simulations

The Navier-Stokes equations have been solved in cylindrical coordinates using a DNS. The differential equations have been discretized using a second order finite difference method and are solved by using a fractional step procedure. Using cylindrical coordinates usually gives complications since the Navier-Stokes equations in cylindrical coordinates have a singularity at $r = 0$. This is solved by using the radial flux $q_r = r \cdot \nu_r$ and by using a staggered grid, as described by Verzicco and Orlandi [6]. In a staggered grid, scalar variables are stored in the center of the computational cells and vectors are stored in the cell faces, as shown in Figure 3.2. This way, the only variable that is calculated at $r = 0$ is $q_r$, which is zero by definition. The grid is non-uniform, as more grid points are used at the walls of the cell to get better results for the boundary layers.

The particle temperature and velocity are calculated using tri-linear interpolation. The position of the particle in time is calculated using the second order Adams-Bashforth method. The particle acceleration is calculated using numerical differentiation:

$$a(t) = \frac{u(t) - u(t - \Delta t)}{\Delta t}.$$  \hspace{1cm} (3.2)

Figure 3.2: The computational cells used in the DNS. The scalar variables like the pressure are stored in the center, while the flux is stored in the cell faces for every direction. [6]

3.3 Experiments

The results from the DNS will be compared to the results from experiments. These experiments have been carried out as a separate study, and only the results will be used. This setup also makes use of a cylindrical convection cell. The cell consists of a cylindrical Plexiglass vessel with a heating part at the bottom and a cooling part at the top. The heating part consists of an electrical resistance heater attached to a copper plate. A sapphire plate is positioned under the cooling chamber for better thermal contact with the fluid.

The particles in the fluid are recorded with four charge-coupled device cameras with 50 mm lenses, located above the cylinder. The camera frequency is set to 30 Hz, which is good enough to capture
small scale structures. A particle tracking velocimetry method has been used that has been developed at ETH Zurich and has been used successfully before. Up to 2000 particles are tracked per time-step.

### 3.4 Data processing

The DNS will be used on the cylinder described in paragraph 3.1 for four different rotations. The Rossby number will be used to specify the situations for easier comparison to other results. The four situations are $Ro = \infty$, $Ro = 2.5$, $Ro = 1.0$, and $Ro = 0.1$. For $Ro = 0.1$, the rotation speed is the highest, while for $Ro = \infty$, the cylinder does not rotate at all. The situation with no rotation will be for comparison with results from previous studies, where rotation was not researched. The data will be calculated using the Lagrangian approach, calculating the properties of every particle in time. For every time step, the data for all particles will be stored in one file.

The data from the DNS will be used to calculate the curvature and torsion using equation (2.23) and (2.24). The curvature can be calculated directly with the data from the DNS, but to calculate the torsion, the derivative of the acceleration is required. This is not given by the DNS, so it is calculated by using numerical differentiation. The numerical differentiation that has been used is a symmetric derivative:

$$\ddot{a}(t) = \frac{a(t + \Delta t) - a(t - \Delta t)}{2\Delta t}.$$ (3.3)

The values of the curvature and torsion will be used to calculate PDFs. Regular PDFs will be calculated, but also joint PDFs of the curvature and torsion with the velocity, with accompanying PDF quotients are determined. The PDF quotient is defined as

$$Q(x, y) = \log_{10}\left(\frac{P(x, y)}{P(x)P(y)}\right).$$ (3.4)

With this definition, $Q(x, y) < 0$ means anti-correlation between variables $x$ and $y$, $Q(x, y) > 0$ means positive correlation, and $Q(x, y) = 0$ means $x$ and $y$ are uncorrelated.

The PDFs of different Rossby numbers will be compared to each other to see how the rotation influences the flow. The PDFs of the numerical simulation will also be compared to experimental results to see if the results agree.
4 Results

4.1 Curvature

To get an indication of the differences in particle trajectory for a situation with or without rotation of the cylinder, some trajectories of random particles have been plotted. A trajectory for $Ro = \infty$ and for $Ro = 0.1$ have been visualized in Figure 4.1. The color indicates the logarithm of the curvature of the particle at that location. The particle in the cylinder without rotation moves around a lot without many sudden changes of direction. The particle in the cylinder with rotation only stays in a small part of the cylinder and changes direction more often.

![Figure 4.1](image)

Figure 4.1: A random particle trajectory for $Ro = \infty$ (left) and for $Ro = 0.1$ (right). Both trajectories have the same time span. The color indicates the logarithm of the curvature of the particle at that location.

This difference is also visible in the PDFs of curvature shown in Figure 4.2. As the rotation speed increases, the curvature values become higher, causing the PDF to shift to the right. The shape of the PDFs is very similar to the PDFs of previous studies in Figure 1.2. For all rotation speeds, the PDFs scale as $\kappa^{1}$ for small values of curvature and as $\kappa^{-2.5}$ for large values, just like in the studies to isotropic homogeneous turbulence. This is better visible in Figure 4.3. The straight parts of the PDFs have been fitted and the resulting scalings have been plotted for $Ro$. The scalings are close to the theoretical values and do not change significantly for different rotations.
Figure 4.2: PDFs of the curvature, nondimensionalized by the cylinder height $H$, for all rotations calculated using DNS. The straight lines represent the $\kappa^1$ and $\kappa^{-2.5}$ scaling.

Figure 4.3: The scaling of the PDFs of curvature for (a) small values of curvature and (b) large values of curvature.

The results from experiments only agree with the numerical results partially. For small values of curvature, the PDF scales as $\kappa^1$, just like in the numerical results. But for large values of curvature, the PDF does not scale as $\kappa^{-2.5}$, as can be seen in Figure 4.4. Instead, the scaling of the right side of the PDF decreases as the curvature rises. This is probably because of the limitations of the experimental measuring devices. For high values of curvature, a particle makes a sudden turn, causing the measuring device not to recognize the particle anymore. Figure 4.4 (b) does show that the peak positions of the PDFs agree for the numerical and experimental results. This is better visualized in Figure 4.5. This figure also shows something about the shift of the PDFs. The values saturate for high values of $Ro$, but the behavior for smaller values is not clear.
Figure 4.4: (a) PDFs of the curvature, nondimensionalized by the cylinder height $H$, calculated in experiments for $Ro = \infty$, $Ro = 2.5$, $Ro = 0.5$, and $Ro = 0.1$. The straight lines represent the $\kappa^1$ and $\kappa^{-2.5}$ scaling. (b) Direct comparison of the PDFs of curvature for the numerical and experimental method for $Ro = \infty$. The figures have been shifted vertically for clarity.

Figure 4.5: The peaks of the PDFs of the curvature from the numerical simulations and the experiments.

The results of the joint PDFs and PDF Quotients are displayed in Figure 4.6. These figures are also similar to the results from previous studies shown in Figure 1.3. There is a highest probability of finding a particle with low curvature and a high velocity and the smallest probability of finding a particle with high curvature and a low velocity. It also shows that high curvature is positively correlated with low velocity. This means that high values of curvature appear when a particle slows down and reverses direction. Rotation slightly changes the shape of the PDFs, but the same conclusions can be taken.
Figure 4.6: Joint PDFs of the curvature, nondimensionalized by the cylinder height $H$, and velocity, nondimensionalized by its standard deviation $\sigma_u$, for (a) $Ro = \infty$ and (b) $Ro = 0.1$. (c) and (d) are the PDF quotients for $Ro = \infty$ and $Ro = 0.1$ respectively.

4.2 Torsion

Rotation does the same to torsion as it does to curvature, as can be seen in Figure 4.7. For lower $Ro$, the torsion values become higher and the PDFs shift to the right. The shape of the PDFs looks like the PDF from previous studies in Figure 1.2 (b). For all $Ro$, the PDFs scale as $\tau^0$ for small values of torsion, and scale as $\tau^{-3}$ for large values of torsion. This is better visible in Figure 4.8. The scalings are close to the theoretical values, especially for small torsion values, and do not seem to change as $Ro$ changes.
Figure 4.7: PDF of torsion, nondimensionalized by the cylinder height $H$, for all rotations calculated using the DNS. The straight lines represent the $\tau_0^0$ and $\tau_{-3}^3$ scaling.

Figure 4.8: The scaling of the PDFs of torsion for (a) small values of torsion and (b) large values of torsion.

The results from the experiments again agree partially. The PDFs scale correctly for small values of the torsion, but for large values there is no constant scaling, as can be seen in Figure 4.9. This is again the result of the limitations of the measuring devices, but also because of the third derivative of the position $\hat{a}$ in the equation for torsion, equation (2.24). This makes the results for torsion less accurate than the results for curvature.
Figure 4.9: (a) PDFs of the torsion, nondimensionalized by the cylinder height $H$, calculated in experiments for $Ro = \infty$, $Ro = 2.5$, $Ro = 0.5$, and $Ro = 0.1$. The straight lines represent the $\tau^0$ and $\tau^{-3}$ scaling. (b) Direct comparison of the PDFs of torsion for the numerical and experimental method for $Ro = \infty$. The figures have been shifted vertically for clarity.

The PDFs of torsion in Figure 4.10 show that there is a high probability for finding a particle with low torsion and high velocity and a low probability for finding a particle with high torsion for any velocity. The PDF quotients show that there is a positive correlation between high values of the torsion and low velocities. This suggests that, just like for the curvature, the highest values of the torsion are found when a particle reverses direction.

Figure 4.10: Joint PDFs of the torsion, nondimensionalized by the cylinder height $H$, and velocity, nondimensionalized by its standard deviation $\sigma_u$, for (a) $Ro = \infty$ and (b) $Ro = 0.1$. (c) and (d) are the PDF quotients for $Ro = \infty$ and $Ro = 0.1$ respectively.
4.3 Curvature and torsion

Figure 4.11 shows a high probability for finding a particle with a low value of torsion and a medium value of curvature, and a low probability for finding a particle with a high value of torsion and a high value of curvature. This agrees with the PDFs found in Figure 4.2 and Figure 4.7. The previous paragraphs already suggested that high values of curvature and high values of torsion are positively correlated, since both are correlated with low velocities. This is also visible in Figure 4.11 (c), but it also shows other relations. High values of torsion are also correlated with low values of curvature, and low values of torsion are correlated with medium values of curvature.

Figure 4.11: Joint PDFs of the curvature and torsion, both nondimensionalized by the cylinder height $H$, for (a) $Ro = \infty$ and (b) $Ro = 0.1$. (c) and (d) are the PDF quotients for $Ro = \infty$ and $Ro = 0.1$ respectively.
5 Time-resolved angular statistics (extension)

5.1 Introduction

In the first part of this report, Rayleigh-Bénard convection in a rotating cylinder has been investigated using a Lagrangian approach. This may give a better understanding of convection in other situations, like the atmosphere where rotation also plays a big part. Tracer particles have been tracked in the rotating cylindrical Rayleigh-Bénard setup using a DNS model and the results are compared with experiments. The main focus has been on geometrical characteristics of particle trajectories measured in the form of curvature and torsion. Curvature has been calculated using the Frenet-Serret formulas and a given formula for the curvature \( \kappa \):

\[
\kappa = \frac{|\mathbf{u} \times \mathbf{a}|}{|\mathbf{u}|^3},
\]

where \( \mathbf{u} \) is the velocity vector and \( \mathbf{a} \) is the acceleration vector at the particle’s position [3].

The probability density function (PDF) of the curvature has been calculated for approximately 20000 particles in the center of the cylinder for multiple Rossby numbers. Using these PDFs, the effects of the rotation of the cylinder on the particle trajectories have been studied. The results have been compared to experimental results and results from previous studies on particles in turbulent flows.

The shape of the PDFs for curvature has been similar to results from previous studies: for small curvatures the PDFs scale as \( \kappa^1 \), while for large curvatures the PDFs scale as \( \kappa^{-2.5} \). Comparing the PDFs for different rotations revealed that faster rotations give higher curvature, as can be seen in Figure 5.1.

![Figure 5.1: PDFs of the curvature, nondimensionalized by the cylinder height \( H \), for all rotations calculated using DNS. The straight lines represent the \( \kappa^1 \) and \( \kappa^{-2.5} \) scaling.](image)

Unfortunately, curvature mainly contains information about small scale structures and only little information about multi-scale structures. In turbulence, many length and timescales are present and
therefore it is interesting to also focus on multi-scale dynamics of the particle trajectories. A method to measure geometrics properties, including multiple time scales, is described by Bos et al [7]. This method uses the angle $\theta$ between subsequent particle increments, as shown in Figure 5.2. The distance between subsequent particle increments is defined as

$$\delta X(x_0, t, \Delta t) = X(x_0, t) - X(x_0, t - \Delta t),$$  

(5.2)

where $X(x_0, t)$ is the position of a particle at time $t$, starting at position $x_0$ at $t = t_0$ and $\tau$ is the time step between measurements.

![Figure 5.2: Visualization of the angle $\theta$ between subsequent particle increments [7]](image)

With this, the angle $\theta$ is defined as

$$\cos(\theta(t, \Delta t)) = \frac{\delta X(x_0, t, \Delta t) \cdot \delta X(x_0, t + \Delta t, \Delta t)}{|\delta X(x_0, t, \Delta t)||\delta X(x_0, t + \Delta t, \Delta t)|}. \quad (5.3)$$

For small time steps, $\theta$ can be used to calculate the curvature:

$$\kappa(t) = \lim_{\Delta t \to 0} \frac{|\theta(t, \Delta t)|}{2\Delta t \|u(t)\|}. \quad (5.4)$$

For more information about the trajectories, the angle $\theta$ and the time step $\Delta t$ can be researched. The ensemble average of $\theta$ is given by

$$\theta(\Delta t) = \langle |\theta(t, \Delta t)| \rangle. \quad (5.5)$$

This ensemble average should be close to zero for small time steps and should tend to $\pi/2$ for large time steps.

In the extension of the report, the methods described above will be applied to the particle data described in the first part of the report. PDFs and ensemble averages of the angle $\theta$ between subsequent particle increments will be measured for multiple time steps and for multiple rates. In
the limit of very small time increments the curvature can be calculated from the angle $\theta$ and this result can be compared to the results of the original method. Again, the main goal will be to study the effects of the rotation of the cylinder on the movement of the particles.

5.2 Results

Figure 5.3 (a) shows that for the smallest time-step, there is a high probability for finding a small angle and a small chance for finding a large angle. This corresponds with the results from the first method. For such a small time-step, a large angle corresponds with a high value of the curvature and a small angle corresponds with a low value of the curvature. As the time-step increases, the particle will have more time to move around, causing higher probabilities for the larger angles. The time-step will eventually become so large that a subsequent particle position will become random. Because of this, the PDF will become symmetrical around $\pi/2$.

Figure 5.3 (b) shows that the addition of rotation gives some notable differences. For the smallest time-step, there is still a highest probability for finding a small angle, but the probability for finding a higher angle is significantly higher than in the situation without rotation. This means that there is a higher probability for finding higher curvature values, which was also shown in section 4.1. The PDF will also go towards the symmetrical limit, but not as fast as in the situation without rotation. This is better visible in Figure 5.4. The average angle $\theta$ is higher in the situation with rotation for small time-steps $\Delta t$, but the situation without rotation reaches the asymptote of $\pi/2$ faster.

![Figure 5.3: PDF of the angle $\theta$ for (a) $Ro = \infty$ and (b) $Ro = 0.1$.](image)
Figure 5.4: The average angle $\theta$ as a function of the time-step $\Delta t$. The horizontal black line is the value $\pi/2$.

The PDFs of the curvature for different time-steps $\Delta t$ are shown in Figure 5.5. For the smallest time-step, the PDF is similar to the PDF found in chapter 4.1 for the scaling and peak position. As the time-step increases, the PDF will stay the same except for the large values. Since the highest values of the curvature take place in a small time window, a bigger time-step will cause these high values to be filtered out. As $\Delta t$ becomes even bigger, the PDFs will become less accurate and not relevant anymore.

Figure 5.5: PDFs of the curvature for different time-steps $\Delta t$ for $Ro = \infty$. The straight lines represent the $\kappa^1$ and $\kappa^{-2.5}$ scaling.
6 Conclusion

In this report, the geometry of tracer trajectories in turbulent rotating convection has been studied. Previous studies have already shown that rotation changes the flow structure, causing vertically aligned plumes to appear, and that rotation can enhance heat flow. Studying the geometry of particle trajectories will give more information about coherent structures in the flow. This research has been done by calculating the curvature and torsion of one million tracer particles in a rotating cylindrical Rayleigh-Bénard cell. One of the main goals of this research is to see whether turbulent convection behaves the same as homogeneous isotropic turbulence and to see how rotation influences the geometrical characteristics of the particle trajectories.

The geometry of tracer trajectories in turbulent Rayleigh-Bénard convection appears to behave the same as it does in homogeneous isotropic turbulence. The PDF of the curvature scales as $\kappa^4$ for low values of the curvature and as $\kappa^{-2.5}$ for high values of the curvature, and the PDF of the torsion scales as $\tau^0$ for low values of the torsion and as $\tau^{-3}$ for high values of the torsion. The joint PDF and PDF quotient of the curvature and velocity also gave the same results. High values of the curvature mainly appear when a particle has a low velocity and reverses direction. It was discovered that high values of the torsion also appear when a particle reverses direction, but also when a particle barely changes direction.

Rotation of the Rayleigh-Bénard cell caused some changes in the geometrical characteristics of the particle trajectories. Both the values of the curvature and the torsion increase as the Rossby number decreases, but the scaling of the PDFs remains the same. This is probably the result of the vortices that appear when the cell is rotated. The joint PDFs and PDF quotients slightly change shape and position because of the rotation, but they give the same conclusion.

The method where the angle $\theta$ between subsequent particle increments is used verifies the curvature results. For a small time-step $\Delta t$, the addition of rotation gives a higher probability for finding larger angles $\theta$. This suggests that the addition of rotation gives a higher probability for finding higher values of the curvature, which agrees with the results from the curvature method. Looking at small time-steps, the equation for the curvature that makes use of $\theta$ and $\Delta t$ also gives similar results to the curvature equation that makes use of the particle velocity and acceleration. Large scale structures give some other information about the addition of rotation. For larger $\Delta t$, the average angle $\theta$ reaches the asymptote $\pi/2$. However, this asymptote is reached faster when no rotation is applied.

There are a few things that can be done for further research to complement this investigation. The first is to study more rotation speeds. By doing this, a relation might be found between the Rossby number and the shift of the PDFs. Research to lower Rossby numbers will show if the values of the curvature and the torsion will keep rising or saturate. A second possibility for further research is to focus more on torsion. Torsion is harder to determine than curvature and has not yet been studied as extensively. Further research will give more clarification about its behavior.

The final point of interest are the boundary layers of the cylinder. One of the reasons the cylinder has been chosen for this research is because of the possibility to study the different boundary layers at the walls. Because of time restraints, this report has only covered the behavior of the flow in the center of the cylinder. Studying the boundary layers will give important information for understanding the flow behavior.
7 Bibliography


