BACHELOR

On the convergence of infinite products of 2x2 matrices

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On the convergence of infinite products of 2x2 matrices

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Abstract

For years, humanity has found ever-improving ways to evaluate irrational numbers with arbitrary numerical precision. In this report, we will discuss a method that can evaluate an infinite, countable but unknown subset of the irrational numbers: those that can be expressed as the limit of the ratio of the two leftmost numbers of an infinite matrix product of the following form:

\[
\begin{pmatrix}
u_1 & u_0 \\
v_1 & v_0
\end{pmatrix}
\prod_{n=2}^{\infty}
\begin{pmatrix}
an + b & en + f \\
cn + d & 0
\end{pmatrix}
\]

where all variables \(u_0, u_1, v_0, v_1, a, b, c, d, e, f\) are integers. We will not focus on the convergence speed of this method, but we will give a few specific solutions for some ‘special’ numbers, a few general solutions for different sub-problems and give possible methods to find more general solutions. Furthermore, we will also provide a general answer in the form of a continued fraction by showing a relation between the infinite matrix products and the continued fractions.
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Chapter 1

Introduction

1.1 Introduction to the subject

This report is inspired by a thesis by P.J.Potts [1]. In this PhD thesis, Potts makes a lot of interesting remarks about infinite 2x2 matrix products, but some part might be unclear for some readers. The aim of this report, therefore, is to clarify some of his work. In his thesis, Potts discusses, among many other things, infinite matrix products of the following form:

\[
\begin{pmatrix}
u_1 & u_0 \\
v_1 & v_0 \\
\end{pmatrix} \prod_{n=2}^{\infty} \begin{pmatrix} an + b & en + f \\
fn + d & gn + h \\
\end{pmatrix}
\]

where \(\prod\) is used in the following way:

\[
\prod_{n=k}^{\infty} \begin{pmatrix} an + b & en + f \\
fn + d & gn + h \\
\end{pmatrix} = \prod_{n=k+1}^{\infty} \begin{pmatrix} an + b & en + f \\
fn + d & gn + h \\
\end{pmatrix}
\]

So, for example:

\[
\prod_{n=2}^{\infty} \begin{pmatrix} an + b & en + f \\
fn + d & gn + h \\
\end{pmatrix} = \prod_{n=3}^{\infty} \begin{pmatrix} 2a + b & 2e + f \\
2c + d & 2g + h \\
\end{pmatrix} \prod_{n=4}^{\infty} \begin{pmatrix} 3a + b & 3e + f \\
3c + d & 3g + h \\
\end{pmatrix} \prod_{n=5}^{\infty} \begin{pmatrix} 4a + b & 4e + f \\
4c + d & 4g + h \\
\end{pmatrix} \ldots
\]

In this report, however, we will only focus on infinite matrix products of the following form:

\[
\begin{pmatrix}
u_1 & u_0 \\
v_1 & v_0 \\
\end{pmatrix} \prod_{n=2}^{\infty} \begin{pmatrix} an + b & en + f \\
fn + d & 0 \\
\end{pmatrix}
\]

We do so to reduce the complexity of the problem; even after making this simplification, the subject is still sufficiently complex. This simplification has been chosen to define \(u_n\) in an easy way; we will do this in the next chapter.

1.2 Outline of the report

In chapter 2, we will create a ‘starting point’. We introduce the series \(u_n\), give a definition of the limit using this series, and introduce the formal power series \(U(x)\), a recurrence relation for
CHAPTER 1. INTRODUCTION

$u_n$ and a differential equation for $U(x)$, all of which are used regularly in the other chapters.

In chapter 3, the first main chapter, we will focus on the convergence of the infinite matrix product. We will start with some special cases, including trivial cases and the work of Cats ([2]). Then, we move on to a formula for the limit for one subproblem by directly solving $u_n$. Finally, we find a general formula for the limit for some subproblems by using $U(x)$ and its differential equation.

In chapter 4, the second main chapter, we will provide formulas for specific numbers, like $\sqrt{2}$, $e$, $\pi$, etc. Most of these can be found by using the general formulas for the limit introduced in chapter 3. One of these, $\ln(2)$, is a special case, and we will explain how to get this limit separately.

In chapter 5, the third and final main chapter, we will explain the relation between our problem and the continued fractions, in order to know something about the convergence speed of the method, and to express the specific numbers used in chapter 4 as continued fractions.

In chapter 6, we will draw our conclusions and give some ideas for further research.
Chapter 2

Definitions and starting point

Let $a, b, c, d, e, f, u_0, v_0, u_1, v_1 \in \mathbb{Z}$. Consider the matrix product

$$L = \begin{pmatrix} u_1 & u_0 \\ v_1 & v_0 \end{pmatrix} \prod_{n=2}^{\infty} \begin{pmatrix} an+b & en+f \\ cn+d & 0 \end{pmatrix}$$

(2.1)

This equation should be read as follows: if we write

$$\begin{pmatrix} p_M & q_M \\ r_M & s_M \end{pmatrix} = \begin{pmatrix} u_1 & u_0 \\ v_1 & v_0 \end{pmatrix} \prod_{n=2}^{M} \begin{pmatrix} an+b & en+f \\ cn+d & 0 \end{pmatrix}$$

then (2.1) is equivalent to

$$L = \lim_{M \to \infty} \frac{p_M}{r_M} = \lim_{M \to \infty} \frac{q_M}{s_M}$$

provided this limit exists. One clear example where the limit does not exist is $u_0 = u_1 = v_0 = v_1 = 0$, in which case $\forall M \in \mathbb{N}^+ : p_M = q_M = r_M = s_M = 0$. Most of these cases where the limit does not exist will be discussed in section 3.1, but we will have to make some assumptions along the way sometimes to avoid non-well-defined limits. We can rewrite this definition to another one, which is easier to work with, namely:

$$\begin{pmatrix} u_M & u_{M-1} \\ v_M & v_{M-1} \end{pmatrix} = \begin{pmatrix} u_1 & u_0 \\ v_1 & v_0 \end{pmatrix} \prod_{n=2}^{M} \begin{pmatrix} \frac{an+b}{en+f} & 1 \\ \frac{cn+d}{en+f} & 0 \end{pmatrix}$$

thus reducing the four recurrence relations we had to only two recurrence relations,

$$u_n = \frac{an+b}{en+f} u_{n-1} + \frac{cn+d}{en+f} u_{n-2}$$

(2.2)

and

$$v_n = \frac{an+b}{en+f} v_{n-1} + \frac{cn+d}{en+f} v_{n-2}$$

(2.3)

We are now interested in $L := \lim_{n \to \infty} \frac{u_n}{v_n}$. We will focus on this limit for the rest of this report, so whenever we talk about ‘the limit’, the limit meant is this limit. Again, this limit does not always exist, for example when $u_0 = u_1 = v_0 = v_1 = 0$. Note that scaling $u_0, v_0, u_1$ and $v_1$ (all at once) with some $\lambda \neq 0$ does not change the limit, as does scaling $a, b, c, d, e$ and $f$ with a $\lambda \neq 0$. On the convergence of infinite products of 2x2 matrices 3
CHAPTER 2. DEFINITIONS AND STARTING POINT

Define the formal power series
\[ U(x) = \sum_{n=0}^{\infty} u_n x^n \]  \hspace{1cm} (2.4)
and its derivative
\[ U'(x) = \sum_{n=1}^{\infty} n u_n x^{n-1} \]

Analogously for \( v_n, V(x) \) and \( V'(x) \). Multiply (2.2) with \((en + f)x^n\), and sum for \( n = 2 \) to \( \infty \):
\[
\sum_{n=2}^{\infty} (en + f)u_n x^n = \sum_{n=2}^{\infty} (a(n-1) + b + a)u_{n-1}x^n + (c(n-2) + 2c + d)u_{n-2}x^n
\]
to obtain the following differential equation for \( U \):
\[
ex(U' - u_1) + f(U - u_0 - u_1 x) = ax^2U' + (b + a)x(U - u_0) + cx^3U' + (2c + d)x^2U
\]
which can be rewritten as
\[
(ex^3 + ax^2 - ex)U' + ((2c + d)x^2 + (b + a)x - f)U = ((b + a)u_0 - (e + f)u_1)x - fu_0 \]  \hspace{1cm} (2.5)
Chapter 3

Convergence of the infinite matrix product

In this chapter, we will explain some of the general formulas for the limit. We will start with some special cases, then find general formulas for the limit in two different ways: using a direct solution for $u_n$, and using the differential equation for $U(x)$.

3.1 Special cases

3.1.1 Trivial cases

- $a = b = 0$
  In this case, the sequence does not converge, unless $v_0, v_1 \neq 0, \frac{u_n}{v_n} = \frac{u_n}{v_n} = L, \forall n \geq 2 : cn + d \neq 0, cn + f \neq 0$, in which case the limit is $L$.

- $c = d = 0$
  In this case, the limit is $\frac{u_n}{v_n}$, assuming $\forall n \geq 2 : cn + f \neq 0$ and $\forall n \geq 2 : an + b \neq 0$. If the first assumption does not hold, the limit is not well-defined. Else, if the second assumption does not hold, it does not converge.

- $\exists n \geq 2 : en + f = 0$
  In this case the limit is not well-defined, since we try to divide by zero.

From now on, we will assume these trivial cases do not hold.

3.1.2 Fully known

The case $a = c = e = 0$ has been fully explored in [2]. In this case, the limit exists if and only if $(d \neq 0) \land (b \neq 0) \land (f \neq 0) \land (b^2 + 4fd \geq 0)$, in which case the limit is the solution for $z$ of $dz^2 - bz - f = 0$ that has the greatest absolute value.

3.1.3 Independence of $u_0, u_1, v_0$ or $v_1$

Independence of $u_0$

Suppose we want to make the limit independent of $u_0$. Logically, if $u_2$ is independent of $u_0$, then $\forall N \geq 2 \in \mathbb{N} : u_n$ independent of $u_0$, since $u_n$ is only dependent of $u_{n-1}$ and $u_{n-2}$. By definition,
\[ u_2 = \frac{2a+b}{2c+f}u_1 + \frac{2c+d}{2c+f}u_0, \] so when \( 2c + d = 0 \), the solution is independent of \( u_0 \). To find the limit, look at:

\[
\begin{pmatrix}
    u_M \\
    u_{M-1}
\end{pmatrix} = \begin{pmatrix}
    u_1 \\
    u_0
\end{pmatrix} \prod_{n=2}^{M} \begin{pmatrix}
    \frac{an+b}{cn+f} & 1 \\
    \frac{cn-f}{cn-f} & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
    u_1 \\
    u_0
\end{pmatrix} \begin{pmatrix}
    \frac{2a+b}{2c+f} & 1 \\
    0 & 0
\end{pmatrix} \prod_{n=3}^{M} \begin{pmatrix}
    \frac{an+b}{cn+f} & 1 \\
    \frac{cn-f}{cn-f} & 0
\end{pmatrix}
\]

\[
= u_1 \begin{pmatrix}
    \frac{2a+b}{2c+f} & 1 \\
    0 & 0
\end{pmatrix} \prod_{n=3}^{M} \begin{pmatrix}
    \frac{an+b}{cn+f} & 1 \\
    \frac{cn-f}{cn-f} & 0
\end{pmatrix}
\]

Now, it should not be hard to see that \( \frac{u_n}{v_n} = \frac{u_1}{v_1} \) for all \( n \geq 1 \), so \( \lim_{n \to \infty} \frac{u_n}{v_n} = \frac{u_1}{v_1} \).

And indeed, when we take \( 2c + d = 0 \), using (5.2) we find:

\[
L = \frac{u_0}{v_0} + \frac{u_1 v_0}{v_1} = \frac{u_0}{v_0} + \frac{u_1}{v_1} = \frac{u_1}{v_1}
\]

However, we have only proven existence. Now, let us look at uniqueness. Remember (2.5). Suppose we have found a solution \( U \). If \( u_0 \) has no effect on \( u_2, u_3, ... \), then after increasing \( u_0 \) - and with it \( U(x) \) - by a constant \( C \) the equation should still hold. We get:

\[
(cx^3 + ax^2 - cx)(U+C)' + ((2c+d)x^2 + (a+c)x - f)(U+C) = ((b+a)(u_0+C) - (e+f)u_1)x - f(u_0+C)
\]

Because \( U(x) \) is a solution, we can subtract the original differential equation from this equation:

\[
(2c + d)C x^2 = 0
\]

Since this should hold for all \( C \) and \( x \), we deduce \( 2c + d = 0 \). So, the solution is independent of \( u_0 \) iff \( 2c + d = 0 \).

**Independence of \( u_1 \)**

Suppose we want the limit to be independent of \( u_1 \). If \( u_2 \) is independent of \( u_1 \), and \( u_3 \) is too, \( \forall N \geq 2 \in \mathbb{N} : u_n \) independent of \( u_0 \), since \( u_n \) is only dependent of \( u_{n-1} \) and \( u_{n-2} \). For \( u_2 \) to be independent of \( u_1 \), \( 2a + b = 0 \). For \( u_3 \) to be independent of \( u_1 \), \( 3c + d = 0 \). We obtain

\[
\begin{pmatrix}
    u_M \\
    u_{M-1}
\end{pmatrix} = \begin{pmatrix}
    u_1 \\
    u_0
\end{pmatrix} \prod_{n=2}^{M} \begin{pmatrix}
    \frac{an+b}{cn+f} & 1 \\
    \frac{cn-f}{cn-f} & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
    u_1 \\
    u_0
\end{pmatrix} \begin{pmatrix}
    0 & 1 \\
    \frac{a}{3c} & 0
\end{pmatrix} \prod_{n=4}^{M} \begin{pmatrix}
    \frac{an-2a}{cn-3c} & 1 \\
    \frac{cn}{cn-3c} & 0
\end{pmatrix}
\]

\[
= u_0 \begin{pmatrix}
    0 & 1 \\
    \frac{a}{3c} & 0
\end{pmatrix} \prod_{n=4}^{M} \begin{pmatrix}
    \frac{an-2a}{cn-3c} & 1 \\
    \frac{cn}{cn-3c} & 0
\end{pmatrix}
\]

from which we deduce that \( \frac{u_n}{v_n} = \frac{u_0}{v_0} \) for all \( n \geq 2 \). Therefore, \( L = \lim_{n \to \infty} \frac{u_n}{v_n} = \frac{u_0}{v_0} \).
When we take \(2a + b = 0 = 3c + d\), using (5.2) we find
\[
L = \frac{u_0}{v_0} + \frac{u_1}{v_0} - \frac{u_0v_1}{v_0^2} + \frac{2c + d}{0 + 0}
\]
which is of course not well-defined, but if we take \(3c + d = 0\) and then take the limit of \(2a + b \to 0\), we indeed find a limit of \(\frac{u_0}{v_0}\), as expected.

Again, we have not yet proven uniqueness, so let us do so now. If we want \(u_1\) to have no effect, increasing \(u_1\) by \(C\) and increasing \(U(x)\) by \(Cx\) should have no effect. We get:
\[
(cx^3 + ax^2 - cx)(U + Cx)' + ((2c + d)x^2 + (b + a)x - f)(U + Cx) = ((b + a)u_0 - (e + f)(u_1 + C))x - fu_0
\]
Again, we subtract the original equation:
\[
C(cx^3 + ax^2 - cx) + Cx((2c + d)x^2 + (b + a)x - f) = -(e + f)Cx
\]
\[
Cx^2((3c + d)x + b + 2a) = 0
\]
In the same way as before, we deduce \(3c + d = 0\) and \(2a + b = 0\). In conclusion, the solution is independent of \(u_1\) iff \(3c + d = 0\) and \(2a + b = 0\).
3.2 Finding the limit using a direct solution for \( u_n \)

In this section, we will find a formula for \( L \) by finding a direct solution for \( u_n \) and \( v_n \). Unfortunately, but not completely unexpected, there is only one case where we can do so.

3.2.1 \( a \neq 0, c = 0, e = 0 \)

If we take \( c = e = 0 \), we can solve the recurrence directly:

\[
 u_n = \left( \frac{a}{f} + \frac{b}{f} \right) u_{n-1} + \frac{d}{f} u_{n-2}
\]

Take \( w_n = \left( \frac{d}{f} \right) \frac{2}{3} u_n \) to obtain: \( w_n = \sqrt{\frac{2}{3}} \left( n \frac{2}{3} + \frac{d}{3} \right) u_{n-1} + w_{n-2} \). Mathematica tells us this has the form

\[
 w_n = w_0 \frac{K_{1+\frac{d}{3}}(k)}{I_{1+\frac{d}{3}}(k)} - I_{1+\frac{d}{3}}(k) I_{n+1+\frac{d}{3}}(k) - I_{1+\frac{d}{3}}(k) K_{n+1+\frac{d}{3}}(k) + \ldots
\]

where \( f \) and \( K \) are the modified Bessel functions of the first and second kind, respectively. If we write \( \frac{2d}{3} = k \), we get:

\[
 w_n = \frac{K_{2+\frac{d}{3}+\frac{d}{3}}(k)}{I_{2+\frac{d}{3}+\frac{d}{3}}(k)} - I_{2+\frac{d}{3}+\frac{d}{3}}(k) K_{n+1+\frac{d}{3}}(k) + \frac{K_{2+\frac{d}{3}+\frac{d}{3}}(k)}{I_{n+1+\frac{d}{3}}(k)} - I_{2+\frac{d}{3}+\frac{d}{3}}(k) K_{n+1+\frac{d}{3}}(k) + \ldots
\]

So,

\[
 L = \lim_{n \to \infty} u_n = \lim_{n \to \infty} x_n
\]

If \( k > 0 \), then \( I_{\frac{d}{3}+\frac{d}{3}+\frac{d}{3}}(k) \to 0 \) when \( n \to \infty \), else if \( k < 0 \), then \( K_{\frac{d}{3}+\frac{d}{3}+\frac{d}{3}}(k) \to 0 \). For now, assume \( k > 0 \). We get:

\[
 L = \frac{u_1 I_{\frac{d}{3}+\frac{d}{3}+\frac{d}{3}}(k) - u_0 I_{\frac{d}{3}+\frac{d}{3}+\frac{d}{3}}(k) - u_1 I_{\frac{d}{3}+\frac{d}{3}+\frac{d}{3}}(k)}{v_1 I_{\frac{d}{3}+\frac{d}{3}+\frac{d}{3}}(k) - v_0 I_{\frac{d}{3}+\frac{d}{3}+\frac{d}{3}}(k)}
\]
3.3 Finding the limit using the differential equation

In this section we will find general formulas for $L$ under different assumptions, by combining knowledge of both $u_n$ and $U(x)$. If we assume that $u_n = D_u x^n + o(x)^n$, we can get:

$$\frac{U(x)}{V(x)} = \frac{\sum_{n=0}^{\infty} u_n x^n}{\sum_{n=0}^{\infty} v_n x^n} = \frac{\sum_{n=0}^{\infty} D_u (ax)^n + o((ax)^n)}{\sum_{n=0}^{\infty} D_v (ax)^n + o((ax)^n)}$$

Now, if we take $x \to \frac{1}{a}$, the leftmost parts of the infinite sums $(D_u (ax)^n)$ and $(D_v (ax)^n)$ will grow with an ever-increasing speed, while the rightmost parts $(o((ax)^n))$ will remain relatively small. Therefore,

$$\lim_{x \to \frac{1}{a}} \frac{U(x)}{V(x)} = \frac{D_u}{D_v} = \lim_{n \to \infty} \frac{u_n}{v_n} = L$$

(3.2)

So, we need to have a formula for $U(x)$ and to confirm $u_n$ is of the form $D_u x^n + o(x)^n$. This last one can be done by checking that $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \alpha$.

3.3.1 A simple general solution ($a, c, e \neq 0, f = 0$)

Let us start with the case $a \neq 0, c \neq 0, e \neq 0$. We will make some assumptions along the way, however, to reduce the complexity of the deduction.

A simple general solution for $U(x)$

Remember (2.5). We would like to simplify this equation by dividing it by $x$, so we take $f = 0$:

$$(e - ax - cx^2)U' + (-(2c + d)x - (b + a))U = -(b + a)u_0 + cu_1 =: u$$

We want to integrate this equation directly, using the following:

$$\frac{((e - ax - cx^2)^{1+\alpha} U')}{(e - ax - cx^2)^{\alpha}} = (e - ax - cx^2) U' + (-2(1 + \alpha)cx - a(1 + \alpha)) U$$

To do so, we need $-2(1 + \alpha)cx - a(1 + \alpha) = -(2c + d)x - (b + a)$. From this we can deduce $\alpha = \frac{b}{a}, 2bc = ad$. So, from now on, we assume $2bc = ad$ and set $\alpha = \frac{b}{a}$ to obtain:

$$\frac{((e - ax - cx^2)^{1+\frac{b}{a}} U')}{(e - ax - cx^2)^{\frac{b}{a}}} = (e - ax - cx^2) U' + (-2e + d)x - (b + a))U = u$$

$$(e - ax - cx^2)^{1+\frac{b}{a}} U' = u(e - ax - cx^2)^{\frac{b}{a}}$$

$$U = C(e - ax - cx^2)^{-1 - \frac{b}{a}} + u(e - ax - cx^2)^{-1 - \frac{b}{a}} \int_0^x (e - at - ct^2)^{\frac{b}{a}} dt$$

Furthermore, $U(0) = u_0$ (remember (2.4)) and $U(0) = \frac{C}{e^{1+\frac{b}{a}} u_0}$ (using the equation above), so $C = e^{1+\frac{b}{a}} u_0$. In conclusion, we have:

$$U(x) = e^{1+\frac{b}{a}} u_0 (e - ax - cx^2)^{-1 - \frac{b}{a}} + u(e - ax - cx^2)^{-1 - \frac{b}{a}} \int_0^x (e - at - ct^2)^{\frac{b}{a}} dt$$

(3.3)

On the convergence of infinite products of 2x2 matrices
CHAPTER 3. CONVERGENCE OF THE INFINITE MATRIX PRODUCT

Approximation of \( u_n \)

From (2.2), we find:

\[
z_n := \frac{u_n}{u_{n-1}} = \frac{an + b}{en + f} \cdot \frac{u_{n-1}}{u_{n-2}} + \frac{cn + d}{en + f} \cdot \frac{1}{z_{n-1}}.
\]

So,

\[
z_n = \frac{an + b}{en + f} + \frac{cn + d}{en + f} \cdot \frac{1}{z_{n-1}} + \frac{a(n-1) + b}{e(n-1) + f} \cdot \frac{1}{z_{n-2}} + \frac{c(n-1) + d}{e(n-1) + f} \cdot \frac{1}{z_{n-3}} + \ldots
\]

When \( n \to \infty \), we get, assuming \( a > 0, (\frac{e}{c})^2 + 4 \frac{f}{e} > 0 \):

\[
\lim_{n \to \infty} z_n = z = \frac{a}{e} + \frac{c}{e} \cdot \frac{1}{\frac{a}{e} + \frac{c}{e}} = \frac{\frac{a}{e} + \sqrt{\left(\frac{a}{e}\right)^2 + 4 \frac{f}{e}}}{2}.
\]

Since \( \frac{u_n}{u_{n-1}} \) has the limit \( z \), we conclude:

\[
u_n = D_u z^n + o(z^n)
\]

\[
u_n = D_u \left(\frac{a}{e} + \sqrt{\left(\frac{a}{e}\right)^2 + 4 \frac{f}{e}} \right)^n + o\left(\frac{a}{e} + \sqrt{\left(\frac{a}{e}\right)^2 + 4 \frac{f}{e}} \right)^n
\]

A formula for the limit

Since we have now confirmed \( u_n \) is of the right form and we have a formula for \( U(x) \), we can use (3.2) to find:

\[
L = \lim_{x \to \frac{a}{e} + \sqrt{\left(\frac{a}{e}\right)^2 + 4 \frac{f}{e}}} \frac{U(x)}{V(x)}
\]

(3.4)

If we combine this with (3.3), we get:

\[
L = \lim_{x \to \frac{a}{e} + \sqrt{\left(\frac{a}{e}\right)^2 + 4 \frac{f}{e}}} \frac{\int I^n(x) \cdot u_0(e - ax - cx^2)^{-1 - \frac{1}{e}} + v(e - ax - cx^2)^{-1 - \frac{1}{e}} \int_I (e - at - ct^2)^{\frac{1}{e}} dt}{\int I^n(x) \cdot v_0(e - ax - cx^2)^{-1 - \frac{1}{e}} + v(e - ax - cx^2)^{-1 - \frac{1}{e}} \int_I (e - at - ct^2)^{\frac{1}{e}} dt}
\]
If we assume $e^{1/2}v_0(x - ax - cx^2)^{-1/2} + v(e - ax - cx^2)^{-1/2} \int_0^x (e - at - ct^2)^{1/2} dt \neq 0$, we get

$$L = \lim_{x \to \frac{1}{2} + \sqrt{\frac{1}{2}^2 + 4\frac{2}{c}}} \frac{e^{1/2}u_0 + u \int_0^x (e - at - ct^2)^{1/2} dt}{e^{1/2}v_0 + v \int_0^x (e - at - ct^2)^{1/2} dt} \frac{e^{1/2}v_0 + v \int_0^x (e - at - ct^2)^{1/2} dt}{e^{1/2}v_0 + v \int_0^x (e - at - ct^2)^{1/2} dt}$$

(3.5)

which means we now have a formula for the limit. The assumptions made for this formula are:

\[
\begin{align*}
a &\neq 0 \\
c &\neq 0 \\
e &\neq 0 \\
f &\neq 0 \\
2bc &= ad
\end{align*}
\]

3.3.2 A more general solution ($a, c, e \neq 0$)

Start again with (2.5). Write $P_0 = ((b+a)u_0 - (e+f)u_1)x - f u_0$, $P_2 = (2c+d)x^2 + (b+a)x - f$, $P_3 = cx^3 + ax^2 - ex$ (so $P_3U' + P_2U = P_1$). Then, we can get

$$U(x) = e^{-\frac{F(x)}{P_3(z)}} \int_{z=0}^{x} \frac{P_0(z)}{P_3(z)} e^{\int_{z}^{x} P_2(z)} dz$$

where $F$ is a primitive of $\frac{P_2}{P_3}$. Note the difference between the variable $e$ and the Euler constant $e$. Write $P_3$ as $cx(x - s)(x - t)$, with $0 \neq s \neq t \neq 0$. To do so, $e$ must be unequal to zero and $a^2 + 4ac$ should be strictly bigger than 0. Then, we obtain $e^F = e^x A(x - s)^B(x - t)^C$, where

\[
\begin{align*}
A &= \frac{cf}{e} \\
B &= \frac{1}{2}(2c + d - \frac{cf}{e} + \frac{2b - a(f + \frac{d}{e})}{\sqrt{(\frac{d}{e})^2 + 4\frac{f}{e}}}) \\
C &= \frac{1}{2}(2c + d - \frac{cf}{e} - \frac{2b - a(f + \frac{d}{e})}{\sqrt{(\frac{d}{e})^2 + 4\frac{f}{e}}})
\end{align*}
\]

Of course we need $a \neq 0 \neq e$ for the above to be well-defined, so let us make that assumption. We now have

$$\begin{align*}
U(x) &= e^{\frac{-F(x)}{P_3(z)}} \int_{z=0}^{x} \frac{P_0(z)}{P_3(z)} e^{\int_{z}^{x} P_2(z)} dz \\
V(x) &= e^{\frac{-F(x)}{P_3(z)}} \int_{z=0}^{x} \frac{P_0(z)}{P_3(z)} e^{\int_{z}^{x} P_2(z)} dz
\end{align*}$$

On the convergence of infinite products of 2x2 matrices
CHAPTER 3. CONVERGENCE OF THE INFINITE MATRIX PRODUCT

On the convergence of infinite products of 2x2 matrices

\[ U(x) = \frac{\int_{z=0}^{x} ((b+\alpha)u_0 - (c+f)v_0)z-fu_0 \, dz}{\int_{z=0}^{x} ((b+\alpha)u_0 - (c+f)v_0)z-fu_0 \, dz} \]

If we combine this with (3.2), we get:

\[ L = \lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{u_n}{v_n} \cdots \]

The following assumptions were made to get this monstrous equation:

\[
\begin{align*}
& a \neq 0 \\
& c \neq 0 \\
& e \neq 0 \\
& a > 0 \\
& c > 0 \\
& a^2 + 4ce > 0
\end{align*}
\]

3.3.3 \( a \neq 0, c = 0, e \neq 0 \)

Suppose we take \( c = 0, a \neq 0, e \neq 0 \). We will try to use the same tactic as used in the previous section, but because \( c = 0 \), the complexity of the equations and formulas should be reduced.

The recursion

From (2.2), we find:

\[ z_n := \frac{u_n}{u_{n-1}} = \frac{an+b}{en+f} u_{n-1} + \frac{d}{en+f} u_{n-2} \]

So,

\[ z_n = \frac{an+b}{en+f} + \frac{d}{en+f} \frac{1}{z_{n-1}} \]

\[ z_n = \frac{an+b}{en+f} + \frac{d}{en+f} \frac{1}{z_{n-2}} \]

\[ z_n = \frac{an+b}{en+f} + \frac{d}{en+f} \frac{1}{z_{n-3}} \]

\[ \vdots \]

\( a \neq 0 \)

On the convergence of infinite products of 2x2 matrices
When \( n \to \infty \), we get:

\[
\lim_{n \to \infty} z_n = z = \frac{a}{e} + 0 \frac{1}{\frac{a}{e} + 0} = \frac{a}{e}.
\]

Since \( \frac{u_n}{u_{n-1}} \) has the limit \( z \), we conclude:

\[
u_n = D_u z^n + o(z^n)
\]

\[
u_n = D_u \left( \frac{a}{e} \right)^n + o\left( \left( \frac{a}{e} \right)^n \right)
\]

The differential equation

Start at (2.5). Fill in \( c = 0 \):

\[
(ax^2 - ex)U' + (dx^2 + (b + a)x - f)U = -u_1 ex - f u_0 - f u_1 x + (b + a)u_0 x
\]

Write \( P_u = ((b+a)u_0 - (e+f)u_1)x - f u_0, P_2 = dx^2 + (b+a)x - f, P_3 = ax^2 - ex \) (so \( P_3 U' + P_2 U = P_u \)) Then, we can get:

\[
U(x) = e^{-F(x)} \int_{z=0}^x P_u(z) e^{F(z)} dz
\]

where \( F \) is a primitive of \( \frac{P_u}{P_3} \).

If we write \( P_3 = ax(x - \frac{e}{a}) \) we obtain \( P_3 \left( \frac{a}{e} \right) = A + \frac{B}{x} + \frac{C}{x - \frac{e}{a}} \), \( F(x) = Ax + B \ln(x) + C \ln(x - \frac{e}{a}) \) and \( e^{F(x)} = e^{A x} x^B (x - \frac{e}{a})^C \). Here,

\[
\begin{align*}
A & = \frac{d}{a} \\
C & = 1 + \frac{b}{a} + \frac{de}{a^2} - \frac{f}{e} \\
B & = \frac{f}{e}
\end{align*}
\]

\[
\frac{U(x)}{V(x)} = \frac{e^{-F} \int_{t=0}^x P_u e^F dt}{e^{-F} \int_{t=0}^x P_3 e^F dt} = \frac{\int_{t=0}^x P_u e^F dt}{\int_{t=0}^x P_3 e^F dt}
\]

Furthermore:

\[
\frac{P_u}{P_3} = \frac{D}{x} + \frac{E}{x - \frac{e}{a}}
\]

where

\[
\begin{align*}
D & = \frac{f u_0}{e} \\
E & = \frac{(b + a)u_0 - (e + f)u_1}{a} - \frac{f u_0}{e}
\end{align*}
\]

\[
\frac{U(x)}{V(x)} = \frac{\int_{t=0}^x \left( \frac{f u_0}{e} (t - \frac{e}{a})^{1 + \frac{b}{a} + \frac{de}{a^2} - \frac{f}{e}} + \frac{(b+a)u_0 - (e+f)u_1}{a} \right) t^{\frac{b}{a}} e^{\frac{f}{e} t} dt}{\int_{t=0}^x \left( \frac{f u_0}{e} (t - \frac{e}{a})^{1 + \frac{b}{a} + \frac{de}{a^2} - \frac{f}{e}} + \frac{(b+a)u_0 - (e+f)u_1}{a} \right) t^{\frac{b}{a}} e^{\frac{f}{e} t} dt}
\]

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If we write \( f = k \cdot e \):

\[
U(x) = \frac{\int_{x_0}^{x} k u_0(t - \frac{e}{a})^{1 + \frac{k}{a} + \frac{d}{a}} - k t e \frac{e}{a} dt}{\int_{x_0}^{x} k v_0(t - \frac{e}{a})^{1 + \frac{k}{a} + \frac{d}{a}} - k t e \frac{e}{a} dt}
\]

So our limit is:

\[
L = \frac{U(x)}{V(x)} = \frac{\int_{x_0}^{x} k u_0(t - \frac{e}{a})^{1 + \frac{k}{a} + \frac{d}{a}} - k t e \frac{e}{a} dt}{\int_{x_0}^{x} k v_0(t - \frac{e}{a})^{1 + \frac{k}{a} + \frac{d}{a}} - k t e \frac{e}{a} dt}
\]
Chapter 4

Some examples for specific numbers

4.1 Using found formulas

In this chapter, we will give some of the possible limits and how to obtain these limits, in order to show the diversity of the infinite matrix products.

4.1.1 A formula for $\sqrt{k}$

Let us try to obtain a simple result from (3.5). The integral is relatively easy when $\frac{b}{a} \in \mathbb{N}$. Therefore, choose $a = b$:

$$L = e^{\frac{2}{u_0}} + u \int_{\frac{1}{u_0}}^{1+\sqrt{k}} (e - at - ct^2) \, dt$$

We now make the observation that the limit is of the form $A + B\sqrt{k}$, where $k = (\frac{a}{e})^2 + 4c$. Note that this is true whenever $a|b$, not only when $a = b$. Now, take $k \geq 2 \in \mathbb{N}$ fixed where $k$ does not contain a square, i.e. $m^2|k \Rightarrow m = \pm 1$. Let us try to obtain the limit $\sqrt{k}$. First, we need $z\sqrt{k} = \sqrt{(\frac{z}{e})^2 + 4\frac{z}{e}}$ for some $z \in \mathbb{N}^+$. We can obtain this by choosing $a = 2$, $e = 1$, $c = k - 1$. We then get $\sqrt{(\frac{z}{e})^2 + 4\frac{z}{e}} = 2\sqrt{k}$. Using this, we get:

$$L = \frac{u_0 + u \int_{\frac{1}{u_0}}^{1+\sqrt{k}} (1 - 2t - (k - 1)t^2) \, dt}{v_0 + v \int_{\frac{1}{v_0}}^{1+\sqrt{k}} (1 - 2t - (k - 1)t^2) \, dt}$$

$$L = \frac{(k - 1)^2u_0 - u(k - \frac{1}{3}) + \frac{2}{\sqrt{k}}}{(k - 1)^2v_0 - v(k - \frac{1}{3}) + \frac{2}{\sqrt{k}}} = \sqrt{k}$$

Since $u_0, u, v_0, v \in \mathbb{Q}$ and $\sqrt{k} \notin \mathbb{Q}$ ($k$ did not contain a square and was bigger than 1), we can split this equation into two equations:

$$\left\{ \begin{array}{l}
\frac{2}{3} = (k - 1)^2u_0 - u(k - \frac{1}{3}) \\
((k - 1)^2v_0 - v(k - \frac{1}{3}))\sqrt{k} = \frac{2}{3}\sqrt{k} \end{array} \right.$$
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\[
\begin{align*}
2k^2v & = 3(k-1)^2u_0 - u(3k-1) \\
3(k-1)^2v_0 - v(3k-1) & = 2ku
\end{align*}
\]

Choose \( v = -(k-1)^2\alpha, u = -(k-1)^2\beta \), so we can divide by \((k-1)^2\).

\[
\begin{align*}
-2k^2(k-1)^2\alpha - 3(k-1)^2u_0 - (k-1)^2\beta(3k-1) & = 0 \\
3(k-1)^2v_0 + (k-1)^2\alpha(3k-1) + 2k(k-1)^2\beta & = 0
\end{align*}
\]

Choose \( v_0 = \alpha \).

\[
\begin{align*}
-2k^2\alpha - 3u_0 - \beta(3k-1) & = 0 \\
3v_0 + \alpha(3k-1) + 2k\beta & = 0
\end{align*}
\]

Choose \( \alpha = 2k\gamma \), to make sure the bottom equation has a solution:

\[
\begin{align*}
-4k^3\gamma - 3u_0 - \beta(3k-1) & = 0 \\
-(3k+2)\gamma & = \beta
\end{align*}
\]

\[-4k^3\gamma + (3k+2)(3k-1)\gamma = 3u_0
\]

\[-(4k^3 + 9k^2 + 3k - 2)\gamma = 3u_0
\]

Choose \( \gamma = 3 \):

\[u_0 = -4k^3 + 9k^2 + 3k - 2\]

Which brings us to the following statements:

\[
\begin{align*}
a & = 2, b = 2, c = k - 1, 2bc = ad, e = 1, f = 0 \\
v & = -(k-1)^2\alpha \\
u & = -(k-1)^2\beta \\
v_0 & = \alpha \\
\alpha & = 2k\gamma \\
\beta & = -(3k+2)\gamma \\
\gamma & = 3 \\
eu_1 & = (b+a)u_0 + u \\
ev_1 & = (b+a)v_0 + v \\
u_0 & = -4k^3 + 9k^2 + 3k - 2
\end{align*}
\]

which can be simplified to:

\[
\begin{align*}
a & = 2, b = 2, c = k - 1, d = 2(k-1), e = 1, f = 0 \\
u_0 & = -4k^3 + 9k^2 + 3k - 2 \\
u_1 & = -7k^3 + 24k^2 + 9k - 2 \\
v_0 & = 6k \\
v_1 & = 6k(-k^2 + 2k + 3)
\end{align*}
\]

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4.1.2 A formula for \( \pi \)

Back to (3.5). Choose \( a = 2, b = 1, c = 1, d = 1, e = 1, f = 0, u_0 = 2, u_1 = 10, v_0 = 1, v_1 = 3 \).

Note that all assumptions hold, so we can do this without any problems. Then we get \( a = -(b+a)u_0 + ev_1 = -3u_0 + u_1 = 4, v = -(b+a)v_0 + ev_1 = -3v_0 + v_1 = 0 \). We obtain:

\[
L = 2 + 4 \int_0^1 \frac{(1 - 2t - t^2)^{\frac{1}{2}}}{1} \, dt \\
= 2 + 4\int_0^1 \frac{(t + 1)\sqrt{1 - 2t - t^2 + \arcsin\left(\frac{t + 1}{\sqrt{2}}\right) \big|_{t=0}^{t=2-1}} = \pi
\]

4.1.3 \( c = 0 \)

Starting from (3.7), take \( a = b = d = e = f = 1 \):

\[
L = \frac{U(1)}{V(1)} = \frac{\int_0^1 (\frac{m_0}{n_0} + \frac{m_0 - 2m_n}{n_1}) e^t \, dt}{\int_0^1 (\frac{m_0}{n_0} + \frac{m_0 - 2m_n}{n_1}) e^t \, dt} = \frac{(3u_0 - 2u_1)e - 2(4u_0 - 3u_1)}{(3v_0 - 2v_1)e - 2(4v_0 - 3v_1)}
\]

where \( e \) is Euler’s number (2.71828...). Suppose we want to get \( e \), we could take \( u_0 = 6, u_1 = 8, v_0 = 2, v_1 = 3 \).

Let us go one step back to (3.7). It can be easily seen that as long as \( \frac{b}{a} + \frac{c}{d} - k \geq 0 \) and \( k - 1 \geq 0 \), we get ‘something with \( e \) to the power something’. So, let us be bold and take \( a = b = d = a \): We get:

\[
\lim_{n \to \infty} \frac{u_n}{v_n} = \frac{\int_0^e k u_0 (t - e)^{2+e-k} \, dt}{\int_0^e k v_0 (t - e)^{2+e-k} \, dt} = \left( \frac{3u_0 - 2u_1}{3v_0 - 2v_1} \right) \left( \frac{e}{1} \right)
\]

which converges when \( 1 + e \geq k \geq 1 \). If \( k = 2 \), we get:

\[
\lim_{n \to \infty} \frac{u_n}{v_n} = \frac{\int_0^e 2u_0 (t - e)^3 \, dt}{\int_0^e 2v_0 (t - e)^3 \, dt} = \frac{e}{1} \left( \frac{e}{1} \right)
\]

Now, as a general rule, when \( p(t) \) is a polynomial:

\[
\int_0^e p(t) e^t \, dt = \sum_{k=0}^{\infty} (-1)^k \frac{d^k}{dt^k} p(t) e^t \bigg|_{t=0}
\]

from which we after some struggling obtain:

\[
\int_0^e (2u_0 (t - e)^3 t - 3eu_1 (t - e)^3 t^3) e^t \, dt = (-1)^e e! \ldots
\]

\[
\left(2u_0 (1 + e + e^{e+1} \sum_{k=1}^{e} \frac{k}{e^{k(e-k+1)!}}) - 3u_1 (e^3 + e^{e+1} \sum_{k=2}^{e} \frac{k(k-1)}{e^{k(e-k+1)!}}) \right) + e^e (-2u_0 + 3eu_1)
\]

which in turn gives the limit

\[
L = \frac{2u_0 (1 + e + e^{e+1} \sum_{k=1}^{e} \frac{k}{e^{k(e-k+1)!}}) - 3u_1 (e^2 + 1) + e^{e+1} \sum_{k=2}^{e} \frac{k(k-1)}{e^{k(e-k+1)!}}) + e^e (-2u_0 + 3eu_1)}{2v_0 (1 + e + e^{e+1} \sum_{k=1}^{e} \frac{k}{e^{k(e-k+1)!}}) - 3v_1 (e^2 + 1) + e^{e+1} \sum_{k=2}^{e} \frac{k(k-1)}{e^{k(e-k+1)!}}) + e^e (-2v_0 + 3ev_1)}
\]

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So, if we take

\[
\begin{align*}
  u_0 &= 3M(e^2 + 1) + e^{\epsilon + 1} \sum_{k=2}^{c-1} \frac{k(k-1)}{e^k(e-k+1)!} \\
  u_1 &= 2M(1 + e + e^{\epsilon + 1} \sum_{k=1}^{\epsilon} \frac{k}{e^k(e-k+1)!}) \\
  v_0 &= 3\epsilon M \\
  v_1 &= 2M \\
  M &= \prod_{k=2}^{c-1} e^k(e-k+1)!
\end{align*}
\]

we get a limit of $e^\epsilon$ ($M$ is a multiplier to make sure $u_0, u_1 \in \mathbb{Z}$).

4.1.4 A formula for $e$ ($a \neq 0, c = 0, e = 0$)

To avoid confusion: in this report Euler’s number will be written as $e$, whereas the variable will be written as $\epsilon$.

Back to (2.2). Pick $c = e = 0, d = f = 1, a = 4, b = 2$.

\[ u_n = (4n + 2)u_{n-1} + u_{n-2} \]

Mathematica can give us the direct solution for this equation:

\[ u_n = \frac{(u_1 K_{\frac{1}{2}}(\frac{1}{2}) - u_0 K_{\frac{1}{2}}(\frac{3}{2}))I_{n+\frac{1}{2}}(-\frac{1}{2}) + (u_0 I_{\frac{1}{2}}(-\frac{1}{2}) - u_1 I_{\frac{1}{2}}(-\frac{1}{2}))K_{n+\frac{1}{2}}(\frac{3}{2})}{I_{\frac{1}{2}}(-\frac{1}{2})K_{\frac{1}{2}}(\frac{3}{2}) - I_{\frac{1}{2}}(-\frac{3}{2})K_{\frac{1}{2}}(\frac{5}{2})} \]

Here, $I$ and $K$ are the modified Bessel functions of the first and second kind, respectively.

\[ L = \lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{(u_1 K_{\frac{1}{2}}(\frac{1}{2}) - u_0 K_{\frac{1}{2}}(\frac{3}{2}))I_{n+\frac{1}{2}}(-\frac{1}{2}) + (u_0 I_{\frac{1}{2}}(-\frac{1}{2}) - u_1 I_{\frac{1}{2}}(-\frac{1}{2}))K_{n+\frac{1}{2}}(\frac{3}{2})}{(v_1 K_{\frac{1}{2}}(\frac{1}{2}) - v_0 K_{\frac{1}{2}}(\frac{3}{2}))I_{n+\frac{1}{2}}(-\frac{1}{2}) + (v_0 I_{\frac{1}{2}}(-\frac{1}{2}) - v_1 I_{\frac{1}{2}}(-\frac{1}{2}))K_{n+\frac{1}{2}}(\frac{3}{2})} \]

\[ \frac{I_{\frac{1}{2}}(-\frac{1}{2})}{K_{\frac{1}{2}}(\frac{3}{2})} \to 0 \text{ when } n \to \infty, \text{ so} \]

\[ L = \frac{-u_1 I_{\frac{1}{2}}(-\frac{1}{2}) + u_0 I_{\frac{1}{2}}(-\frac{1}{2})}{-v_1 I_{\frac{1}{2}}(-\frac{1}{2}) + v_0 I_{\frac{1}{2}}(-\frac{1}{2})} \]

which already looks a lot better. Now, choose $u_0 = 3, u_1 = 19, v_0 = 1, v_1 = 7$ for a reason that will become obvious in just a moment:

\[ L = \frac{-19I_{\frac{1}{2}}(-\frac{1}{2}) + 3I_{\frac{1}{2}}(-\frac{1}{2})}{-7I_{\frac{1}{2}}(-\frac{1}{2}) + I_{\frac{1}{2}}(-\frac{1}{2})} \]

This may look like some weird number, but

\[ \frac{I_{\frac{1}{2}}(-\frac{1}{2})}{I_{\frac{1}{2}}(-\frac{1}{2})} = \frac{19 - 7e}{3 - e} \]

which we can use to obtain

\[ L = \frac{-19I_{\frac{1}{2}}(-\frac{1}{2}) + 3\frac{19-7e}{3-e}I_{\frac{1}{2}}(-\frac{1}{2})}{-7I_{\frac{1}{2}}(-\frac{1}{2}) + \frac{19-7e}{3-e}I_{\frac{1}{2}}(-\frac{1}{2})} = \frac{-19 + 3\frac{19-7e}{3-e}}{-7 + \frac{19-7e}{3-e}} = e \]
4.1.5 \(c = e = 0\)

which results in a whole range of interesting numbers:

- If we take \(b = 0, k = 2\), we get:

\[
L = \frac{u_1 C_{\text{CF}} - u_0}{v_1 C_{\text{CF}} - v_0}
\]

where \(C_{\text{CF}}\) is the continued fraction constant \(0 + \frac{1}{1 + \frac{1}{2 + \frac{1}{3 + \ldots}}}\). This can easily verified by using (5.2)

- If we take \(b/a = 1/2\), we get:

\[
L = \frac{k(\sinh(k) - k \cosh(k))}{v_1 (k^2 + 3)|\sinh(k) - 3k \cosh(k)| - u_0}
\]

\[
= \frac{u_1}{v_1 (k^2 + 3)(|e^{-k} - e^{-k} - ke^{-k}|)} - u_0
\]

\[
= \frac{u_1}{v_1 (k^2 + 3)(|e^{-k} - e^{-k} - ke^{-k}|)} - u_0
\]

\[
= \frac{u_1}{v_1 (k^2 + 3)(|e^{-k} - e^{-k} - ke^{-k}|)} - u_0
\]

\[
= \frac{u_1}{v_1 (k^2 + 3)(|e^{-k} - e^{-k} - ke^{-k}|)} - u_0
\]

\[
= \frac{u_1}{v_1 (k^2 + 3)(|e^{-k} - e^{-k} - ke^{-k}|)} - u_0
\]

So, suppose we take \(k = 1/3\):

\[
L = \frac{u_1 5 - 3e^{1/3}}{v_1 5 - 3e^{1/3}} - u_0
\]

In this way, we can construct \(e^A\) for all \(A, B \in \mathbb{Z}\). However, we can also take \(k\) a square root. For example, take \(b = 2, a = 4, d = 1, f = 2\) (so \(k = \sqrt{2}\)) to get:

\[
L = \frac{u_1 (1 + \sqrt{2}) + (1 - \sqrt{2})e^{\sqrt{2}}}{v_1 (1 + \sqrt{2}) + (1 - \sqrt{2})e^{\sqrt{2}} - u_0}
\]

\[
= \frac{u_1 (1 + \sqrt{2}) + (1 - \sqrt{2})e^{\sqrt{2}} - u_0((1 + \sqrt{2}) + (1 - \sqrt{2})e^{\sqrt{2}})}{v_1 (1 + \sqrt{2}) + (1 - \sqrt{2})e^{\sqrt{2}} - u_0((1 + \sqrt{2}) + (1 - \sqrt{2})e^{\sqrt{2}})}
\]

\[
= \frac{u_1 - 7u_0 + (u_1 - 3u_0)\sqrt{2} + (u_1 + 7u_0 - (u_1 + 3u_0)\sqrt{2})e^{\sqrt{2}}}{v_1 - 7v_0 + (v_1 - 3v_0)\sqrt{2} + (v_1 + 7v_0 - (v_1 + 3v_0)\sqrt{2})e^{\sqrt{2}}}
\]

- If we take \(u_1 = 2, u_0, v_0, v_1 = 1\), we get:

\[
L = \frac{-5 - \sqrt{2} + (9 - 5\sqrt{2})e^{\sqrt{2}}}{-6 - 2\sqrt{2} + (8 - 4\sqrt{2})e^{\sqrt{2}}}
\]
CHAPTER 4. SOME EXAMPLES FOR SPECIFIC NUMBERS

4.2 Special cases

4.2.1 A formula for $\ln(2)$ ($a = 0, c \neq 0, e \neq 0$)

There is another case where Mathematica can solve (2.2) directly:

$$u_n = \frac{1}{n} u_{n-1} + \frac{n-1}{n} u_{n-2}$$

In this case, the direct solution is:

$$u_n = (-1)^n u_0 \Phi(-1, 1, n + 1) - (-1)^n u_1 \Phi(-1, 1, n + 1) + u_0 - u_0 \ln(2) + u_1 \ln(2)$$

where $\Phi$ is the Lerch transcendent. Now

$$L = \lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{(-1)^n u_0 \Phi(-1, 1, n + 1) - (-1)^n u_1 \Phi(-1, 1, n + 1) + u_0 - u_0 \ln(2) + u_1 \ln(2)}{(-1)^n v_0 \Phi(-1, 1, n + 1) - (-1)^n v_1 \Phi(-1, 1, n + 1) + v_0 - v_0 \ln(2) + v_1 \ln(2)}$$

$$\lim_{n \to \infty} \Phi(-1, 1, n + 1) = 0,$$ so:

$$L = \frac{u_0 - u_0 \ln(2) + u_1 \ln(2)}{v_0 - v_0 \ln(2) + v_1 \ln(2)}$$

If we take $u_0 = 0, u_1 = v_1 = v_0 = 1$, we get:

$$L = \frac{\ln(2)}{1 - \ln(2) + \ln(2)} = \ln(2)$$
Chapter 5

The relation with continued fractions

In this chapter, we will explain the relation between the infinite matrix products and the continued fractions. This will lead to an expression for $L$ in the form of a continued fraction. We can then use this expression for the special numbers found in the previous chapter.

5.1 Making the connection to continued fractions

Let us take a look at the continued fraction $a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \cdots}}}$.

Let $p_{-1} = 1, q_{-1} = 0$. Define

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \prod_{k=0}^{n} \begin{pmatrix} a_k & 1 \\ b_k & 0 \end{pmatrix}$$

Now, note that

$$\frac{p_n}{q_n} = \frac{1}{b_0} \begin{pmatrix} a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \cdots}}}{b_0} \end{pmatrix}$$

(5.1)

If we assume $v_0 \neq 0$ and write

$$\begin{pmatrix} u_1 & u_0 \\ v_1 & v_0 \end{pmatrix} = \begin{pmatrix} u_0 & 1 \\ v_0 & 0 \end{pmatrix} \begin{pmatrix} \frac{v_1}{v_0} & 0 \\ -\frac{u_0}{v_0} & 0 \end{pmatrix}$$

we get:

$$\begin{pmatrix} u_M & u_{M-1} \\ v_M & v_{M-1} \end{pmatrix} = \begin{pmatrix} u_1 & u_0 \\ v_1 & v_0 \end{pmatrix} \prod_{n=2}^{M} \begin{pmatrix} \frac{a_n+b}{c_n+d} & 1 \\ \frac{c_n+d}{c_n+d} & 0 \end{pmatrix}$$

On the convergence of infinite products of 2x2 matrices
\[
\begin{align*}
\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 - \frac{u_0 v_1}{v_0} \\ v_0 \end{pmatrix} \prod_{n=2}^M \begin{pmatrix} \frac{a_n + b}{en + f} & 0 \\ \frac{en + f}{en + f} & 0 \end{pmatrix} \\
\begin{pmatrix} a_0 & 1 \\ b_0 & 0 \end{pmatrix} &= \begin{pmatrix} a_1 & 1 \\ b_1 & 0 \end{pmatrix} \prod_{n=2}^M \begin{pmatrix} a_n & 1 \\ b_n & 0 \end{pmatrix}
\end{align*}
\]

Using (5.1), we find the limit:

\[
\begin{align*}
L &= \frac{1}{b_0} \begin{pmatrix} a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \frac{b_4}{\vdots}}}} \\ \frac{v_1}{v_0} \end{pmatrix} \\
L &= \frac{1}{v_0} \begin{pmatrix} u_0 + \frac{u_1 - \frac{u_0 v_1}{v_0}}{2c + d} \frac{2c + d}{2e + f} \frac{2a + b}{2e + f} \frac{3a + b}{3e + f} \frac{4a + b}{4e + f} \frac{5a + b}{5e + f} \vdots \\ \frac{v_1}{v_0} \end{pmatrix} \\
L &= \frac{u_0}{v_0} + \frac{u_1 - \frac{u_0 v_1}{v_0}}{2c + d} \frac{2c + d}{2e + f} \frac{2a + b}{2e + f} \frac{3a + b}{3e + f} \frac{4a + b}{4e + f} \frac{5a + b}{5e + f} \vdots \\
L &= \frac{u_0}{v_0} + \frac{u_1 - \frac{u_0 v_1}{v_0}}{2c + d} \frac{2c + d}{2e + f} \frac{2a + b}{2e + f} \frac{(3c + d)(2e + f)}{3a + b + (4c + d)(3e + f)} \frac{4a + b + \cdots}{4a + b + \cdots}
\end{align*}
\]

\[
L = \frac{u_0}{v_0} + \frac{u_1 - \frac{u_0 v_1}{v_0}}{1} \frac{1}{(2a + b) \left( \frac{1}{2c + d} + \frac{1}{3a + b} \frac{1}{4a + b} \frac{1}{5a + b} \cdots \right)}
\]
We will not use this way to find solutions to the problem due to time restrictions, but we will refer to this now and then. There are a few interesting consequences we will note, however. First of all, from (5.2) we find that if \( nc + d = 0 \) for some \( n \in \mathbb{N}^+ \), \( n \geq 2 \), the limit is a rational number.

Secondly, by using (5.3) and (5.1), we can say something about the speed \( \frac{p_n}{q_n} \) converges towards the limit. The convergents converge faster when \( |a| \) is bigger and when \( |c| \) or \( |e| \) is smaller. The best case therefore is \( a \neq 0, c = e = 0 \). In this case, we get

\[
L \approx \frac{u_0}{v_0} + \frac{u_1}{v_1} \frac{v_0}{v_1} \frac{v_2}{v_3} \frac{1}{1} \frac{2a}{d} + \frac{1}{1} \frac{3c}{f} + \frac{1}{1} \frac{4a}{d} + \cdots
\]

which converges faster and faster. In the worst case scenario, \( a = 0, c \neq 0 \neq e \). Then, we obtain

\[
L \approx \frac{u_0}{v_0} + \frac{u_1}{v_1} \frac{v_0}{v_1} \frac{v_2}{v_3} \frac{1}{1} \frac{b}{c} + \frac{1}{1} \frac{h}{d} + \frac{1}{1} \frac{b}{e} + \cdots
\]

which converges slower and slower. Thirdly and finally, if we write

\[
D = \frac{2c + d}{2a + b + \frac{(3c + d)(2e + f)}{3a + b + \frac{(4c + d)(3e + f)}{4a + b + \cdots}}}
\]

we find:

\[
L = \frac{u_0}{v_0} + \frac{u_1}{v_1} \frac{v_0}{v_1} \frac{v_2}{v_3} \frac{1}{1} \frac{u_0 (v_1 + D) + u_1}{v_0} \frac{v_0}{v_1} \frac{v_2}{v_3} \frac{u_0 D + u_1}{v_0 D + v_1}
\]

Note that \( D \) is independent of \( u_0, v_0, u_1 \) and \( v_1 \). So, if we have found some limit \( L \), we can easily influence the limit by changing \( u_0, u_1 \) and \( v_0, v_1 \). Some possibilities are multiplying \( L \) by any rational number, adding any rational number to \( L \) and getting a limit of \( \frac{1}{L} \); this can be done by swapping \( u_i \) with \( v_i \) for \( i = 0, 1 \).
CHAPTER 5. THE RELATION WITH CONTINUED FRACTIONS

5.2 Continued fractions found using the known limits

5.2.1 \( \sqrt{2} \)

If we take \( k = 2 \) and combine this with (5.2), we get:

\[
\sqrt{2} = \frac{2}{3} + \frac{2}{\frac{7}{4} + \frac{1}{\frac{5}{2} + \frac{1}{\frac{6}{3} + \frac{1}{\frac{7}{4} + \ldots}}}}
\]

5.2.2 \( \pi \)

Which also means that, using (5.2), we find:

\[
\pi = 2 + \frac{4}{\frac{2^2 - 1}{3} + \frac{4^2 - 1}{5} + \frac{6^2 - 1}{7} + \ldots}
\]

5.2.3 \( e \)

Combining this with (5.2) gives

\[
e = 3 - \frac{1}{\frac{2}{3} + \frac{3}{3 + \frac{4}{5 + \frac{5}{6 + \ldots}}}}
\]

and, using (5.2), we find:

\[
e = 3 - \frac{2}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{\ldots}}}}}}
\]
which can be rewritten to

\[
e = 1 + \frac{2}{1 + \frac{1}{6 + \frac{1}{10 + \frac{1}{14 + \frac{1}{18 + \frac{1}{22 + \cdots}}}}}}
\]

which can be found in [3], p114.

5.2.4  \( e^2 \)

So, for example, we can take \( e = 2 \) and combine it with (5.2) to get:

\[
\begin{aligned}
u_0 &= 15 \\
u_1 &= 9 \\
v_0 &= 3 \\
v_1 &= 1
\end{aligned}
\]

\[
e^2 = 5 + \frac{4}{1 + \frac{3}{1 + \frac{8}{4 + \frac{10}{5 + \frac{12}{6 + \cdots}}}}}
\]

5.2.5  \( \ln(2) \)

(5.2) gives:

\[
\ln(2) = \frac{1}{1 + \frac{1^2}{1 + \frac{2^2}{1 + \frac{3^2}{1 + \frac{4^2}{1 + \cdots}}}}}
\]

which is a well-known continued fraction for \( \ln(2) \); for example, it can be found in [3], p111.
Chapter 6

Conclusions and ideas for further research

In this report, we have explained a few interesting steps. The most promising step is probably the link with continued fractions. Not only can these be used to find solutions for the original problem; we also have a direct formula for a set of continued fractions. It should be clear that many of the most elemental irrational numbers (\(e, \pi, \sqrt{2}, \ln(2)\)) can be formed this way, which makes it likely many other ‘special’ numbers can be formed this way.

Even though we have not focused on the convergence speed, we can conclude this method will probably never be used to evaluate these irrational numbers to an arbitrary amount of digits, as the convergence speed is only slightly better than linear in the best case scenario, and even worse than linear in the worst case. However, as long as matrix multiplication can be done relatively quickly, this method might still be the best in cases where a small to moderate amount of precision is needed for a lot of different numbers.

For further research, there are quite some possibilities. First, the link with the continued fractions can be used to find more formulas for the original problem. Second, as can be seen in 6.1 there are some cases completely left unexplored: \(c \neq 0, e = 0\) and \(a = c = 0, e \neq 0\), and some others were only used in special cases. Many times, \(u_n \neq \alpha^n\), in which case the trick used in (3.2) cannot be used. However, the formal power series does not have to be defined as in (2.4).

For example, if \(u_n \approx n! \cdot \alpha^n\), \(U(x) = \sum_{n=0}^{\infty} \frac{u_n}{n!} x^n\) can be used.

Furthermore, the original problem can be seen as a simplified version of a bigger problem. For example, the 0 in the lower right hand corner of (2.1) can be replaced by \(gn + h\). Another possibility is expanding the matrices to 3x3 or even nxn matrices, in which case the limit becomes two- or (n - 1)-dimensional and the possibilities may be practically endless.

In many cases, I have used Mathematica (version 9.0) to solve subproblems. Many times, it failed to give me an answer, even though one existed. For example, as can be seen in section 3.2, Mathematica only managed to solve the equation for \(w_n\), even though it was a relatively simple substitution of \(u_n\). We therefore might conclude that many - or even all - of the missing general solutions are (almost) within our grasp, with or without help from computers.
Figure 6.1: A Venn-diagram of the different subproblems. For the black areas, a general solution has been presented. In the grey area, only one specific solution has been found. The white areas are not discussed in this report.
Bibliography

