Dissipation of coherent structures in confined two-dimensional turbulence

Citation for published version (APA):

Clercx, H. J. H., & van Heijst, G. J. F. (2017). Dissipation of coherent structures in confined two-dimensional turbulence. *Physics of Fluids*, 29(11), 1-15. [1111103]. <https://doi.org/10.1063/1.4993488>

DOI:

[10.1063/1.4993488](https://doi.org/10.1063/1.4993488)

Document status and date:

Published: 18/07/2017

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
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Citation: *Physics of Fluids* **29**, 111103 (2017); doi: 10.1063/1.4993488

View online: <http://dx.doi.org/10.1063/1.4993488>

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Dissipation of coherent structures in confined two-dimensional turbulence

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(Received 29 March 2017; accepted 27 June 2017; published online 18 July 2017)

Since the seminal article “Inertial ranges in two-dimensional turbulence” by Kraichnan in 1967, our understanding of the dynamics and transport properties of two-dimensional turbulence is largely built on the assumption of homogeneity and isotropy of statistically steady or decaying turbulence. In the last two decades, more attention has been paid to the presence of lateral walls, either with stress-free or no-slip boundary conditions, and also considering a variety of geometries such as square, rectangular, or circular domains. The impact of confining boundaries on the dynamics of two-dimensional turbulence is important. This is in sharp contrast with three-dimensional turbulence, where homogeneity and isotropy are locally restored due to the cascade process in the inertial range. The impact of confining boundaries is therefore limited in three-dimensional turbulence. The presence of an inverse energy cascade in two-dimensional turbulence, however, will continuously generate large-scale energy-containing eddies, and their vigorous interaction with, in particular, no-slip walls generates large amounts of vorticity and contributes significantly to the dissipation of kinetic energy of the flow. The dissipation is even strongly enhanced compared with the unbounded case. In this review, we will focus on one of the elementary structures observed in two-dimensional turbulent flows: the dipolar vortex. With its self-induced velocity, it propagates easily through the domain and is hence likely to interact with domain boundaries. Standard vortex generation mechanisms allow to create well-defined dipoles and to investigate the collision of such structures with rigid domain boundaries in detail. Relevant aspects of the collision process concern the dynamics and stability of the generated boundary layers, the vorticity, and vorticity gradients contained in these boundary layers, and the dissipation of kinetic energy when dipoles collide with walls. Some of these aspects will be discussed in this review. Moreover, we are interested in the Reynolds-number dependence of these processes, including exploration of the vanishing viscosity limit. *Published by AIP Publishing.* [<http://dx.doi.org/10.1063/1.4993488>]

I. INTRODUCTION

Two-dimensional (2D) flows are characterized by self-organization. For example, in freely decaying 2D turbulence starting from a random initial condition (without coherent vorticity patches), large and approximately axisymmetric vorticity patches will emerge during the decay with lifetimes long compared with the characteristic advection time scale of the flow.¹ This process is basically the result of an interplay between the transport of vorticity to smaller scales by elongation of thin vorticity filaments and the transfer of kinetic energy to larger scales by the so-called merging events of like-sign vortical structures. The situation is even more clear in the present day numerical simulations of forced 2D turbulence with flow forcing occurring at a certain intermediate length scale, see, for example, the very-high resolution simulation by Boffetta.² In that situation, the kinetic energy, largely supplied at the injection scale, is transported from this injection scale to large and even domain-size scales, as a result of the inverse energy cascade, as predicted by Kraichnan.^{3,4} The ultimate consequence is that the kinetic energy may condense in the largest scale available.^{5,6} Transport of enstrophy occurs downscale towards the viscous dissipation range, which is usually referred to as the direct enstrophy cascade. The inverse energy cascade is in strong contrast with cascade processes in three-dimensional

(3D) turbulence, where injected kinetic energy is transported from the macroscale, containing the energy-rich eddies, to the small-scale Kolmogorov or dissipative scale. The theoretical studies by Kraichnan,³ Leith,⁷ and Batchelor,⁸ and also most of the subsequent investigations, have as starting point statistically isotropic and homogeneous 2D turbulence, assuming that the flow is sufficiently well separated from the domain boundaries. In practice, this implies that numerical studies of homogeneous isotropic 2D turbulence can be conducted in computational domains with periodic boundary conditions in both directions.

Since the seminal contribution of Kraichnan³ on the inertial ranges in unbounded two-dimensional turbulence, within a few years complemented with the analyses of Leith⁷ and Batchelor,⁸ many studies have been conducted on the dynamics and evolution of two-dimensional turbulence, its statistical properties, the emergence of coherent structures and condensation phenomena, the transport of material and other properties by the flow, etc. A comprehensive overview of investigations up to about 1980 is provided in the review paper of Kraichnan and Montgomery.⁹ They focussed on both hydrodynamic and plasma applications. Developments up to the beginning of this century, including a review of some recent experiments, are provided by Tabeling¹⁰ and by Kellay and Goldburg.¹¹ An overview of the most recent results and developments on

fluid turbulence confined to two spatial dimensions is given by Boffetta and Ecke.¹²

Despite the reasonable assumption that sufficiently far from domain boundaries, the dynamics and statistical properties of 2D turbulence should hardly be affected by the presence of these boundaries; it was found that the presence of solid (no-slip) lateral boundaries (and also the geometry of the domain) on the behaviour of decaying 2D turbulence is of crucial importance, see the recent overviews by Clercx and van Heijst.^{13,14} In Sec. II, we will summarize some of the most striking observations from (mostly numerical) studies of 2D turbulence on bounded domains, and special emphasis is given to enstrophy production near solid walls in Sec. III. The dipole-wall collision as an exemplary case to investigate enstrophy production and other phenomena is reviewed and discussed in Sec. IV. Subsequently, we focus on the scaling of vorticity production during the dipole-wall collision in different Reynolds number regimes, see Sec. V. In Sec. VI, we summarize and discuss some recent results in the context of attempts to address dipole-wall collisions in the vanishing viscosity limit, and in Sec. VII, we briefly summarize our main findings on the consequences of vorticity production on 2D bounded turbulence.

II. 2D TURBULENCE ON BOUNDED DOMAINS

One of the most striking differences between 2D turbulence in confined domains with no-slip boundaries and unbounded 2D turbulence (with periodic boundary conditions) concerns the quasi-stationary final state of decaying 2D turbulence: on a periodic domain, it is usually a large-scale dipolar structure, see for example Refs. 5 and 15–17, while on a square bounded domain at moderate integral-scale Reynolds numbers usually a quasi-steady monopolar or tripolar vortex is observed in both laboratory experiments (in linearly density-stratified or in two-layer stratified fluids¹⁸) and numerical studies.^{19–24} However, decaying 2D turbulence in circular domains shows a different quasi-steady final state.^{22,25–28} It is either a monopolar structure or a dipolar/quadrupolar structure, depending on the amount of angular momentum (or swirl) added to the

initial flow field. In absence of initial swirl, a dipolar/quadrupolar state is found.

Clercx and co-workers also have shown that the decaying turbulent flow in square containers acquires spontaneously angular momentum due to flow-wall interaction.^{19,21,23} This phenomenon is known as “spontaneous spin-up” and signifies a clear symmetry breaking: for higher integral-scale Reynolds numbers, about 50% of the runs show quasi-stationary states with positive angular momentum and the remaining 50% with negative angular momentum. The size and signature of angular momentum are already obtained at the early stage of the decay when no quasi-stationary final state yet exists.²³ An illustrative example is shown in Fig. 1 where vorticity snapshots of a simulation with an initial integral-scale Reynolds number $Re = 5 \times 10^4$ (based on the domain size and initial root-mean-square velocity; for computational details, see Ref. 29). This phenomenon of symmetry breaking and spontaneous spin-up due to interaction of the flow with no-slip sidewalls has also been reported and discussed recently in the framework of plasma turbulence.^{30–32} The interaction of the (turbulent) flow with the rigid lateral walls in square domains turns out to be crucial for the production of angular momentum and the mentioned process of symmetry breaking. For 2D decaying turbulence in domains with circular no-slip walls, however, production of angular momentum is negligible and not observed in numerical studies^{25,28} and laboratory experiments²⁷ (see the work of Keetels *et al.*³³ for an explanation).

The evolution of the vorticity field and, in particular, the dynamics of the vorticity patches, such as shown in Fig. 1 (but also in similar runs with $Re = 10^4$ and $Re = 10^5$, see Ref. 23), reveal features we are already familiar with from standard high-resolution simulations of 2D decaying turbulence on periodic domains, like in the pioneering studies reported in Refs. 1, 16, and 36–41. Now the following question emerges: How does decaying 2D turbulence in a square domain with no-slip boundaries compare with the case of a double-periodic domain? At a first glance, the evolution of the flow and the vortex distributions look similar for both cases, irrespective of the kind of boundary conditions. After a closer inspection of

FIG. 1. A series of vorticity snapshots showing the process of spontaneous spin-up from a simulation with $Re = 5 \times 10^4$. The simulation is based on solving the penalized Navier-Stokes equations^{34,35} on a square periodic domain and is performed with a resolution of 2048^2 Fourier modes (see Ref. 29 for details of the computational method). The initial flow field consisted of an array of 10×10 vortices with positions slightly distorted to enhance the rapid evolution towards an irregular turbulent flow field. The dimensionless time (based on the initial eddy turnover time of the initial vortices) is given by τ .

the images, however, one observes important differences that can be illustrated with a comparison between bounded and unbounded 2D turbulence. In Fig. 2, we show a comparison of vorticity snapshots from two numerical simulations: one with no-slip lateral walls (left panel) and another with periodic boundary conditions (right panel).²³ These simulations are initialized with exactly the same initial conditions as the run shown in Fig. 1: a regular array of 10×10 slightly disturbed Gaussian vortices with an alternating sign and typical sizes of one-twentieth of the container size each. The dimensionless root-mean-square velocity of the initial flow field is normalized to unity; thus for the Gaussian vortices, we then obtain $\omega_{max} \approx 100$. The integral-scale Reynolds number is $Re = 10^5$, and the snapshots are taken both at the dimensionless time $\tau = 15$ (where time has been made dimensionless with the eddy turnover time of the Gaussian vortices from the initial conditions). The vortices are initially well away from the boundaries. It appears, in particular, in the simulations at higher integral-scale Reynolds number ($Re \gtrsim 5 \times 10^4$), that a much larger number of smaller and stronger vortices is observed for the case of confined 2D decaying turbulence. Moreover, the typical diameter of these small-scale newly generated vortices turns out to be substantially smaller than the diameter of the Gaussian vortices in the 10×10 vortex array of the initial vorticity field. The vorticity snapshots shown in Fig. 2 indeed suggest that the smallest vortices must result from flow-wall interactions, or more precisely, from vortex-wall collisions. Moreover, these small-scale vortices are distributed over the entire domain (left panel), thus not restricted to a thin layer near the solid walls. Such small vortices are almost absent in the simulation with periodic boundary conditions. Although vortex population statistics have not been explored in detail for this particular set of simulations, a numerical comparison between runs of 2D decaying turbulence with periodic boundary conditions and simulations in confined domains with no-slip walls, all for smaller Reynolds number cases, has been

FIG. 2. Vorticity snapshots, taken at dimensionless time $\tau = 15$, from a simulation with no-slip boundary conditions (a) and with periodic boundary conditions (b). The Reynolds number for both simulations is $Re = 10^5$, and the initial conditions were exactly the same for both simulations. The simulations of the penalized Navier-Stokes equations are carried out with 4096^2 Fourier modes in each coordinate direction (see Ref. 29 for details of the computational method and validation of the proper resolution of the boundary layers and vorticity production at the no-slip walls). The grey levels indicate vorticity values ($-80 \leq \omega \leq +80$) with dark grey indicating positive vorticity values and light grey, negative values. Reproduced with permission from G. H. Keetels, H. J. H. Clercx, and G. J. F. van Heijst, “On the origin of spin-up processes in decaying two-dimensional turbulence,” *Eur. J. Mech. B Fluids* **29**, 1–8 (2010). Copyright 2010 Elsevier.

reported by Clercx and Nielsen.⁴² This comparison was based on three series of simulations with the integral-scale Reynolds number $Re = 5 \times 10^3$ (12 runs), 1×10^4 (8 runs), and 2×10^4 (2 runs), and they compared the results with available scaling theories, see Refs. 43–45. A clear difference has been found between the time evolution of, for example, the kinetic energy, the enstrophy, the vortex density and the vortex separation (all ensemble averaged for their respective Reynolds numbers) for the runs with periodic boundary conditions and those with no-slip boundaries (all runs started with similar initial conditions). The comparison displayed in Fig. 2 is in qualitative agreement with the results reported by Clercx and Nielsen.⁴² Additionally, data from laboratory experiments of decaying turbulence in shallow fluid layers have been reported, see Refs. 46–48. For a more detailed overview of these studies and contributions on this topic by other investigators, see Ref. 14.

These observations are an indication that such small-scale vortices are indeed formed due to flow-wall interaction events. It is expected that the larger and stronger vortices will contribute significantly to this process. The collision of these vortices with the walls is most vigorous, thus leading to the formation of many thin boundary layers. After subsequent detachment, they will roll-up into tiny vortices.

III. ENSTROPY PRODUCTION AT RIGID NO-SLIP WALLS

The collision of vortices with rigid no-slip walls will result in thin boundary layers with large velocity gradients (equivalent to high-amplitude induced vorticity) and thus strong dissipation. Moreover, when these boundary layers detach from the wall in the form of extremely thin filaments, they tend to roll up, thus forming tiny but strong vortices. These structures, both the thin vorticity filaments and the small vortices, contain large-amplitude vorticity and are susceptible to viscous dissipation. Enhanced dissipation of the kinetic energy of the flow and the vorticity production during vortex-wall collisions must be reflected in the time evolution of the kinetic energy, defined as

$$E = \frac{1}{2} \int_{\mathcal{D}} |\mathbf{u}|^2 dA, \quad (1)$$

and the enstrophy, which is defined as

$$Z = \frac{1}{2} \int_{\mathcal{D}} \omega^2 dA, \quad (2)$$

with $\mathbf{u} = (u, v)$ the fluid velocity and its Cartesian components u and v , $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ the vorticity of the flow field, $\mathcal{D}$ the (computational) domain, and dA an infinitesimal area element. The kinetic energy and the enstrophy have been computed for four decaying turbulence simulations with $Re = 10^5$: three with no-slip walls and one with periodic boundary conditions. The results are shown in Fig. 3, where these quantities are plotted as a function of the dimensionless time τ ($\tau_{max} = 400$ for the no-slip runs and $\tau_{max} = 100$ for the case with periodic boundary conditions). In the case of periodic boundaries, the enstrophy of the flow (dotted line) shows a rapid and monotonous decay, which can be understood by the relation between the rate of change of enstrophy Z and the palinstrophy P , which is defined

FIG. 3. (a) The time evolution of the kinetic energy as a function of the dimensionless time τ of a simulation of 2D decaying turbulence on a domain with no-slip boundaries (three runs are shown, solid lines) and on a periodic domain (shorter run, up to $\tau = 100$, indicated by the dashed line). (b) Similar as in (a) but now the enstrophy evolution is shown. The Reynolds number is $Re = 10^5$, and the resolution of the simulations was 4096^2 Fourier modes for all four cases. Reproduced with permission from G. H. Keetels, H. J. H. Clercx, and G. J. F. van Heijst, “On the origin of spin-up processes in decaying two-dimensional turbulence,” *Eur. J. Mech. B Fluids* **29**, 1–8 (2010). Copyright 2010 Elsevier.

as

$$P = \frac{1}{2} \int_{\mathcal{D}} |\nabla \omega|^2 dA, \quad (3)$$

and in dimensionless form, this relation can be expressed for unbounded flows as

$$\frac{dZ(t)}{dt} = -\frac{2}{Re} P(t). \quad (4)$$

The palinstrophy is always positive, thus the enstrophy decays monotonously. It does so even rapidly as a result of the vorticity gradient amplification in 2D turbulence. This is in sharp contrast with the enstrophy evolution for 2D decaying turbulence in containers with no-slip walls. We observe an extremely vigorous first stage of the decay process with strong production of small-scale vorticity in the form of earlier mentioned thin boundary layers and the resulting small but high-amplitude vortices. For a substantial time during the decay of the turbulent flow, the vorticity production partly balances the dissipation

mentioned above. Note that for bounded domains, an additional contribution, the boundary integral $\frac{1}{Re} \int_{\partial \mathcal{D}} \omega \frac{\partial \omega}{\partial n} ds$, with $\frac{\partial}{\partial n}$ representing the wall-normal derivative and ds , an infinitesimal element of the boundary $\partial \mathcal{D}$, needs to be added to the expression for the time rate of change of the enstrophy, see, e.g., Refs. 49 and 50, or

$$\frac{dZ(t)}{dt} = -\frac{2}{Re} P(t) + \frac{1}{Re} \int_{\partial \mathcal{D}} \omega \frac{\partial \omega}{\partial n} ds. \quad (5)$$

It may even substantially increase the enstrophy at some instants of time. Persistence of enstrophy has immediate consequences for the dissipation of kinetic energy. The time rate of change of the energy, expressed by

$$\frac{dE(t)}{dt} = -\frac{2}{Re} Z(t) \quad (6)$$

(regardless of the type of boundary conditions for this square domain) and persistence of enstrophy automatically means stronger decay of the kinetic energy of the flow. This is illustrated in Fig. 3(a) where flows on bounded domains decay much more rapidly than in domains with periodic boundary conditions.

From the overview on the evolution of 2D turbulence on confined domains in this and Sec. II, we can conjecture that the interaction of the flow (in general) and that of vortices (in particular) with domain walls may have important consequences. Symmetry breaking of the flow reflected by the spontaneous spin-up of the fluid in the clockwise or anticlockwise direction, the production of many small-scale vortices with high-amplitude vorticity due to vortex-wall interactions, and strong indications of enhanced dissipation of 2D turbulence in confined domains immediately put forward the fundamental question: what will happen when $Re \rightarrow \infty$? In particular, will these dissipation processes and the relevant energy-dissipating structures persist in the limit of vanishing viscosity, and will there be finite dissipation in the inviscid limit for bounded flows? Note that for 2D decaying flows with periodic boundary conditions, energy dissipation will tend to zero in the inviscid limit as the enstrophy is always bounded

FIG. 4. Sequence of vorticity contour plots showing the flow evolution of a dipole-like vortex colliding with a rigid no-slip wall. The integral-scale Reynolds number is $Re = 2500$ (see Sec. IV B for further details on the initial conditions and definition of the Reynolds number). Reproduced with permission from W. Kramer, H. J. H. Clercx, and G. J. F. van Heijst, “Vorticity dynamics of a dipole colliding with a no-slip wall,” *Phys. Fluids* **19**, 126603 (2007). Copyright 2007 AIP Publishing LLC.

by its initial value, resulting in $\frac{dE(t)}{dt} \propto \text{Re}^{-1} \rightarrow 0$ for $\text{Re} \rightarrow \infty$. We will address this issue by reviewing the results of a specific flow problem with direct relevance for 2D turbulence: the collision of a vortex dipole (i.e., a couple of vortices with opposite sign and a self-induced velocity) with a rigid (no-slip) boundary. A typical example of a collision of a dipole with a rigid no-slip wall is illustrated by the snapshots in Fig. 4 from Kramer *et al.*,⁵⁰ showing features already discussed by Orlandi.⁵¹ This is an example of a clean numerical experiment, which has the advantage to investigate the main dissipation processes related to the formation of thin boundary layers at the wall during a vortex-wall collision and the detachment and formation of filamentary structures in a well-defined problem accessible with high-resolution numerical simulations.

IV. THE DIPOLE-WALL COLLISION

The first detailed numerical investigation of the collision of a dipolar vortex with a flat no-slip wall was conducted by Orlandi.⁵¹ One of the prime motivations of this numerical investigation was at that time not directly related to applications in 2D turbulence, but with the problem of trailing vortices from an aircraft, which interact with the ground, resulting in possible vortex rebounds. Better knowledge of the time evolution of such vortices after interaction with the ground might provide clues how to optimize landing and takeoff processes and procedures at airports. Several experimental studies have been reported at that time, for example, on the interaction of a single trailing vortex interacting with a moving bottom plate.⁵² They observed the vortex rebound from the wall and explained this phenomenon, including the fact that the vortex travels away from the bottom wall, by the creation of a secondary vortex (with opposite vorticity) at the rigid bottom wall. Similar vortex rebounds were observed in experiments where a pair of (two-dimensional) line vortices was generated, which subsequently interacted with the ground plane and also with a free surface.⁵³ Later on, several numerical and observational studies have been reported on the interaction of vortex pairs with the ground during takeoff and landing phases.^{54–56} In some of these studies, also the effect of the cross-wind on the temporal evolution of a dipole-runway collision and eventual vortex rebound has been explored, which are examples of asymmetric dipole-wall collisions.^{54,56}

Saffman analyzed the approach of a symmetrical vortex pair (consisting of equal-sized vortices with opposite circulation) perpendicular to a plane surface in an inviscid fluid.⁵⁷ He showed that under inviscid conditions, the vortex pair must approach the wall monotonically. This implies the absence of any vortex rebound, and he concluded that the observed rebounding of finite vortices, such as in earlier mentioned studies,^{51–53} must have another explanation than the inviscid dynamics of finite vortices near a plane surface. Although viscous vorticity production at rigid no-slip walls will be the main reason for the formation of secondary vortices, needed for a vortex rebound and further discussed in the remaining of this overview, another mechanism is worthwhile to mention here. It is the mechanism for an inviscid dipole-vortex rebound applicable in geophysical flows: vorticity production by vortex tube stretching in the presence of a sloping bottom

near the coast.⁵⁸ The inviscid vortex rebound originates from parameterization of some three-dimensional vortex dynamics into a two-dimensional description of the flow dynamics (in pure 2D flows, no vortex stretching can occur).

These experimental examples of trailing vortices mentioned above concern essentially three-dimensional flows, but the simulations of the dipole-wall collision by Orlandi⁵¹ were also motivated by a different set of experiments at that time closely related to the topic of interest in this review, exploring quasi-2D flow dynamics. Dipolar vortices have been generated in linearly stratified fluids (to generate approximate 2D flows), and by adding dye to the fluid, it has been possible to visualize the collision of such dipoles with a rigid sidewall.^{59,60}

The first numerical results hinting at the role of viscous effects were presented by Peace and Riley,⁶¹ who calculated the flow induced by a pair of vortices with opposite circulation in a viscous fluid in the presence of a no-slip or stress-free flat boundary. It was concluded that the vortex rebound from the wall occurs for both no-slip and stress-free boundary conditions, and an explanation for the vortex rebound should thus be offered in terms of viscous effects. In those years, and also more recently, several model studies have been reported that focussed on the behavior of the boundary layer at a rigid no-slip wall due to the presence of a nearby (point) vortex.^{62–69} The advantage of that approach in several of these investigations is the possibility of using the Prandtl boundary-layer equations instead of the full Navier-Stokes equations allowing studies at much large Reynolds numbers. A more detailed explanation of the vortex rebounding process by vorticity production at the (no-slip) wall was provided by Orlandi,⁵¹ and this work can be considered as the starting point for 2D flow simulations of dipole-wall collisions. During the last 25 years, this study by Orlandi was complemented with more detailed numerical studies of this particular problem (higher resolutions and vortex-wall interactions at larger Reynolds number).^{50,70,71} Some aspects of the investigations by Orlandi and other more recent studies on the perpendicular dipole-wall collision will be summarized below.

A. The normal collision of a dipole with a flat rigid wall

The numerical experiments by Orlandi⁵¹ on the normal dipole-wall collision were based on a Lamb dipole, with its axis perpendicular to and released at a certain distance from a rigid stress-free or no-slip flat wall, as an initial condition. The Lamb dipole is an exact solution of the 2D Euler equations in an unbounded fluid.⁷² After release at $t = 0$, it will propagate towards the rigid wall, while the vortex cores constituting the dipole will slowly increase in size due to viscous diffusion. The computations by Orlandi⁵¹ of the two-dimensional flow of an incompressible fluid in the x, y -plane were based on the dimensionless 2D Navier-Stokes equations in the vorticity-stream function formulation, which has the form

$$\frac{\partial \omega}{\partial t} + J(\omega, \psi) = \frac{1}{\text{Re}} \nabla^2 \omega, \quad (7)$$

with ω and $\mathbf{u}$ defined as before, and the relation between the fluid velocity and the stream function according to $u = \frac{\partial \psi}{\partial y}$ and

$v = -\frac{\partial\psi}{\partial x}$. The vorticity $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ can then be written as

$$\omega = -\nabla^2\psi. \quad (8)$$

Apparently, the stream function ψ is governed by a Poisson-type equation, with the vorticity ω appearing as a source term. In Eq. (7), the Jacobian $J(\omega, \psi)$ is a shorthand notation for the nonlinear term: $J(\omega, \psi) = u \frac{\partial\omega}{\partial x} + v \frac{\partial\omega}{\partial y} = \frac{\partial\omega}{\partial x} \frac{\partial\psi}{\partial y} - \frac{\partial\omega}{\partial y} \frac{\partial\psi}{\partial x}$. Two types of boundary conditions are assumed. The no-slip condition, which, for a non-moving wall, means $\mathbf{u} = \mathbf{0}$ at the wall, and the stress-free condition which, for a flat plate, reads $\omega|_{\text{wall}} = 0$. The vorticity boundary condition at a no-slip wall cannot be prescribed *a priori*, and a variety of approaches exist to deal with this issue. In the numerical simulations of 2D turbulence in confined domains or dipole-wall collisions by Clercx and co-workers,^{19,20,42,49,50,73} the influence matrix technique has been applied to account for the correct boundary values of the vorticity at the boundary at each time step during the integration.^{74–76}

To describe the translating Lamb dipole, we use a co-moving polar coordinate system with the origin in the center of the dipole. The initial propagation speed of the dipole is U , and the dipole moves in the negative y -direction. The vorticity distribution of the Lamb dipole in this co-moving frame is given by the relation $\omega = k^2\psi$ in a region $r < a$ with a the width of the dipole half (also known as the radius of the circular separatrix distinguishing the interior part of the dipole from the exterior part), with ψ from hereon the stream function in the co-moving frame of reference. The associated vorticity distribution and stream function (both in the interior part of the dipole, with $r \leq a$) are given by

$$\omega = 2Uk \frac{J_1(kr)}{J_0(ka)} \cos\theta \quad \text{and} \quad \psi = 2U \frac{J_1(kr)}{kJ_0(ka)} \cos\theta \\ = U_d a J_1(kr) \cos\theta, \quad (9)$$

and $\omega = 0$ for $r \geq a$ (potential flow). Here, J_0 and J_1 are Bessel functions of the first kind, and k is chosen such that $J_1(ka) = 0$ (the first non-trivial zero of J_1 , thus $ka \approx 3.8317$). The Reynolds number in the simulations by Orlandi was defined as $\text{Re}_d = \frac{U_d a}{\nu}$, with ν the kinematic viscosity of the fluid and U_d defined in Eq. (9). Three values of the Reynolds number were explored: $\text{Re} = 800$, 1600, and 3200.

First, a comparison has been made with the analytical solution by Saffman for the inviscid motion of two elliptical vortices towards a rigid free-slip wall.⁵⁷ According to Saffman, the finite size of the vorticity patches does not cause a serious departure from the trajectories of a set of point vortices with opposite circulation travelling to a free-slip wall.⁷⁷ When the vortex pair approaches the free-slip wall, the patches start to separate slowly, and when coming close to the wall, the separation speeds up dramatically due to the interaction with mirror vorticity patches. From these observations, it can be conjectured that also the trajectories of the two halves of the Lamb dipole for a normal collision with a free-slip wall will show similar dynamics. The numerical experiment by Orlandi was slightly different. He did not consider a pure inviscid flow but a viscous simulation, and instead of a free-slip wall, a stress-free wall was used. No vorticity was created at the walls, and the strength of the vortex dipole slowly decreases due to viscous diffusion of vorticity. For sufficiently high Reynolds number

(here $\text{Re} = 800$), a good agreement with both the inviscid Saffman solution and the point-vortex solution is found. This comparison provided indirect evidence that viscous boundary layers are required for the vortex rebound.

In a few follow-up studies,^{50,78} comparisons of the normal collision of a Lamb-like dipole with a stress-free flat wall with a modified point-vortex model were made. In these models, effects of viscosity are included by accounting for the change of the size of the dipole halves due to viscous diffusion, thus including a slowly increasing distance between the vortex centers. The numerical experiments by Kramer *et al.* (conducted for a dipole with $\text{Re} = 1250$ colliding with a stress-free wall) showed vortex trajectories in excellent agreement with the predictions of the collision of a point-vortex couple including a viscosity correction (and without rebounds).⁵⁰

The collision of a dipole (or any other vortex) with a flat rigid no-slip wall will introduce another phenomenon: the generation of intense thin vorticity layers at the wall. Due to the curvature of the primary vortex flows, a pressure gradient in the direction parallel to the wall arises, and the boundary layer may detach and subsequently roll up to form the earlier mentioned relatively small vortices containing a substantial amount of vorticity. At intermediate Reynolds numbers, like in Orlandi's simulations,⁵¹ such secondary vortices pair with the primary vortices, and the formed couples will follow a strongly curved trajectory (due to asymmetry of the vortex couple) and interact with the rigid wall once again, see also Fig. 4.

Orlandi⁵¹ compared a numerical experiment of a normal dipole-wall collision at $\text{Re} = 800$ with a laboratory experiment carried out in a stratified fluid. The dipole was generated in the experiment by turbulent horizontal injection of a small volume of dyed fluid, with dye added for visualization purposes.⁵⁹ After the collapse of the turbulent jet in the experiment due to gravity, a well-defined dipole results (for the mechanisms and scaling analysis, one is referred to Refs. 60, 79, and 80). The orientation of the jet (and the resulting dipole) was such that a normal dipole-wall collision would occur. The Reynolds number of the dipolar flow in this experiment was $\text{Re} \approx 900$. The agreement between simulations and experiments is striking, with all main phenomena such as boundary-layer formation and detachment, secondary vortex formation, and the resulting rebound process clearly present.

At much higher Reynolds number, the secondary vortices are much smaller (with high-amplitude vorticity) and even much more abundant⁵⁰ and contribute significantly to additional dissipation. This is our main case of interest and will be discussed in more detail below.

B. Vorticity production at (flat) rigid no-slip walls

During the last 15 years, a variety of numerical dipole-wall collision experiments have been conducted to shed light on the vorticity production process, the rebound process of the primary vortex, and the dissipation of kinetic energy during the collision.^{49,50,71,73,81} As the numerical experiments by Orlandi⁵¹ considered a range of Reynolds numbers up to about $\text{Re} = 3200$, which turns out to be relatively small in the context of 2D turbulence simulations, these more recent studies explored dipole-wall collisions with the Reynolds number as

FIG. 5. The formation and early-time evolution of the vortex dipole. (a) The initial configuration consisting of a couple of isolated Gaussian monopoles; (b) the shedding process of the surrounding vorticity shields; and (c) the formation of a downward-travelling Lamb-like dipole. The contour levels are drawn for $\dots, -150, -90, -30, 30, 90, 150, \dots$. Reproduced with permission from W. Kramer, H. J. H. Clercx, and G. J. F. van Heijst, “Vorticity dynamics of a dipole colliding with a no-slip wall,” *Phys. Fluids* **19**, 126603 (2007). Copyright 2007 AIP Publishing LLC.

large as $\text{Re} = 1.6 \times 10^5$, see Ref. 49, and $\text{Re} = 10^5$, see Ref. 71.

In the numerical studies by Clercx and co-workers,^{49,50,73} an initial dipole is constructed by combining two isolated Gaussian monopoles with opposite vorticity (of same amplitude). We consider a square domain $\mathcal{D} = [-1, 1] \times [-1, 1]$ (dimensionless units; here and later on, the domain half-width is used to non-dimensionalize length scales) with no-slip walls at all four sides in Ref. 49 or a 2D periodic channel domain $\mathcal{D} = [0, 2] \times [-1, 1]$ in Ref. 50. The periodic direction of the channel is aligned along the x -axis. The boundary conditions at the lateral sidewalls of the channel, at $y = \pm 1$, are no-slip. By using the couple of isolated Gaussian vortices, we avoid the formation of artificial boundary layers due to enforcement of the no-slip boundary condition at $t = 0$. The centers of these two vortices are placed at a distance $d = 0.2$ apart. The initial vorticity distribution of one of the isolated Gaussian monopoles is

$$\omega(\mathbf{r}, t = 0) = \omega_0 [1 - (r/r_0)^2] \exp[-(r/r_0)^2], \quad (10)$$

with r_0 the (dimensionless) core radius (this is actually the radius at which the vorticity changes sign), which is set to $r_0 = 0.1$, and $\mathbf{r} = \mathbf{x} - \mathbf{x}_0$ with $\mathbf{x}_0$ being the position of the center of the vortex. The two isolated vortices are placed at the positions $\mathbf{x}_0 = (0, 0.1)$ and $\mathbf{x}_0 = (0, -0.1)$ for the square container experiments and $\mathbf{x}_0 = (1.1, 0)$ and $\mathbf{x}_0 = (0.9, 0)$ for the cases with channel geometry. Finally, ω_0 is the dimensionless extremum vorticity value (in $r = 0$), and its value was taken $\omega_0 \approx 320$, see Refs. 49, 50, and 73.

The initial stage of the evolution from a couple of isolated Gaussian monopoles towards a Lamb-like dipole is shown in Fig. 5. For more details on the formation process of a Lamb-like dipole from two interacting isolated Gaussian monopoles

in numerical setups and laboratory experiments, one may refer Refs. 82 and 83.

The Reynolds number of the flow is defined as $\text{Re} = \frac{UW}{\nu}$, with W being the domain half-width and U being the root-mean-square velocity of the flow field. It is important to connect this Reynolds number with the one introduced by Orlandi,⁵¹ which reads $\text{Re}_d = \frac{U_d a}{\nu}$ and is based on the dipole translation speed U_d and the dipole radius a . As shown in Fig. 5, the initial couple of isolated Gaussian monopoles start to shed their bands of oppositely signed vorticity, and a dipolar vortex emerges that travels in the negative y -direction. The dipole shown in Fig. 5(c) can in good approximation be modeled as a Lamb dipole travelling with a constant speed U_d . Evaluation of the dimensionless energy and enstrophy yields $E = \pi(U_d a/UW)^2$ and $Z = \pi(ka)^2(U_d/U)^2$ (using W and U as the characteristic length and velocity scales in our numerical experiments). We now assume that during the early stage of dipole formation, $E = 2$ and $Z = 800$ (the actual initial values). Thus we obtain $a/W \approx 0.2$ and $U_d/U \approx 4.2$, which then result in $\text{Re}_d = (U_d a/UW)\text{Re} \approx 0.8\text{Re}$. We can only find an approximate value for Re_d , so in the following, we will use the integral-scale Reynolds number Re instead of the definition used by Orlandi.⁵¹

Based on the snapshots shown in Fig. 4 for the dipole-wall collision at $\text{Re} = 2500$, we will illustrate the phenomenology of vortex-wall collisions. First, we introduce a schematic representation of some generic features of the dipole-wall collision, see Fig. 6(a). As the two cores of the dipole [one indicated by $V+$ in Fig. 6(a)] approach the rigid no-slip wall, vorticity with opposite sign is produced in the boundary layer (which we indicate here by $B-$). The closer the vortex core approaches the wall, the more the boundary layer increases in size and the amount of induced vorticity increases. At the same time,

FIG. 6. (a) Schematic illustration of vorticity patches at two instants during the collision process (compare to the right part of the dipole in Fig. 4). The positive half of the dipole (the primary vortex) is labeled with $V+$. The primary boundary layer, with negative vorticity, is labeled with $B-$. The secondary vortex and the secondary boundary layer are labeled with $v-$ and $b+$, respectively. (b) The time evolution of the total kinetic energy $E(t)$ of the flow during the dipole-wall collision and (c) the time evolution of the total enstrophy $Z(t)$ under similar conditions. The Reynolds number of the flow is $\text{Re} = 2500$. Reproduced with permission from W. Kramer, H. J. H. Clercx, and G. J. F. van Heijst, “Vorticity dynamics of a dipole colliding with a no-slip wall,” *Phys. Fluids* **19**, 126603 (2007). Copyright 2007 AIP Publishing LLC.

TABLE I. Based on the data from Kramer *et al.*,⁵⁰ we have summarized data on the enstrophy contained by the primary vortex (Z_{V+}), the primary boundary layer (Z_{B-}), and the secondary part of the boundary layer (Z_{b+}) all measured at t_{min} , the time the dipole halves are closest to the wall. See Fig. 6(a) for more details. As a comparison, the time t_{max} is given when the total enstrophy has reached a maximum (Z_{max}).

Re	t_{min}	Z_{V+}	Z_{B-}	Z_{b+}	t_{max}	Z_{max}
1 250	0.342	247	671	10	0.341	950
2 500	0.328	300	1283	50	0.328	1656
5 000	0.321	331	2177	220	0.323	2770
10 000	0.318	348	3559	748	0.320	4713
20 000	0.316	357	5544	1102	0.313	7385

the dipole halves separate from each other, see Fig. 4. These primary cores tend to move for a while parallel to the walls and boundary-layer vorticity is advected by the dipole. The boundary layer will detach from the wall and roll up to form a secondary vortex, which is denoted by $v-$. The resulting coherent structure is basically an asymmetric dipole that will move along a strongly curved trajectory as the vorticity contained by the primary vortex is higher than that in the secondary vortex. Also in between the boundary layer with negative vorticity ($B-$) and the rigid wall, once again vorticity of opposite sign will be generated, which is denoted with $b+$.

A nice illustration is provided by Kramer *et al.*⁵⁰ in their Fig. 11, where snapshots of the generation of boundary layers for dipole-wall collisions with $Re = 2 \times 10^4$ are shown. The enstrophy contained in the different vorticity patches $V+$, $B-$, and $b+$ is summarized in Table I (adapted from Ref. 50). From this table, we can conclude that for the range of Reynolds numbers considered, the enstrophy content of the primary vortex remains almost independent of Re and is relatively small compared with the enstrophy content of the (primary) boundary layer ($B-$); for $Re \gtrsim 10^4$, it is already more than an order of magnitude larger than that in the primary vortex. Also the enstrophy content of the secondary boundary layer ($b-$) is rapidly growing with increasing Re . These are all indications that vorticity generation at no-slip walls plays a potentially dominant role in 2D bounded flows.

The above discussion concerns the enstrophy in several vorticity patches at a fixed moment in time (when the halves of the primary dipole are closest to the wall). Let us now consider the time evolution of the total energy and enstrophy of the flow. The evolution of the total (dimensionless) kinetic energy $E(t)$ of the flow for the dipole-wall collision shown in Fig. 4 (with $Re = 2500$) is displayed in Fig. 6(b), and in Fig. 6(c), the total enstrophy $Z(t)$ of this simulation is shown. The total enstrophy slowly decays from its initial value at $t = 0$ as long as the dipole does not interact with the rigid wall; in other words, enstrophy is not yet being produced. However, during the primary (and secondary) collision of the dipole with the wall, at $t \approx 0.32$ and $t \approx 0.65$ (for the secondary collision), we observe a strong increase in total enstrophy. To better appreciate these enstrophy increases, compare these instants with the vorticity contour plots in Fig. 4 for $t = 0.30$, $t = 0.3278$, and $t = 0.6$. Together with the first peak in enstrophy, due to vorticity production in the boundary layer during the collision process, we also observe a steeper decay of the kinetic energy $E(t)$ of the flow: vortex-wall collisions enhance dissipation. If the Reynolds number is increased, it turns out that the vorticity filaments that detach from the no-slip walls are thinner but also stronger in amplitude. Two examples to illustrate these phenomena are shown in Fig. 7. Both series of snapshots are taken after the first collision of the dipole with the wall. The Reynolds numbers are $Re = 5 \times 10^3$ (top panels) and $Re = 10^4$ (bottom panels). The initial conditions of both simulations were exactly the same as discussed above. In particular, the $Re = 10^4$ case (bottom row) shows much more fine-scale structures. The maximum of the total enstrophy is also increasing substantially with the Reynolds number: for $Re = 5 \times 10^3$, we have $Z_{max} = 2.77 \times 10^3$, and for $Re = 10^4$, we have $Z_{max} = 4.71 \times 10^3$ (compared with $Z_{max} = 1.66 \times 10^3$ for $Re = 2500$), see also Table I for additional data.

The first detailed simulations on dipole-wall collisions at large Reynolds numbers (Re in the range 10^4 – 10^5) and computation of the maximum values for enstrophy and palinstrophy during the collision process, denoted by Z_{max} and P_{max} , respectively, were reported by Clercx and van Heijst.⁴⁹ Two scaling regimes have been observed in this investigation: $Z_{max} \propto Re^{0.8}$ and $P_{max} \propto Re^{2.25}$ for $Re \lesssim 2 \times 10^4$ and $Z_{max} \propto Re^{0.5}$ and

FIG. 7. Sequence of vorticity contour plots illustrating the flow evolution during a dipole-wall collision for $Re = 5 \times 10^3$ (top panel) and $Re = 10^4$ (bottom panel). The plots only show the right-hand side part of the domain, $[1.0, 1.5] \times [-1, -0.5]$ (note that the dipole-wall collision is symmetric with regard to $x = 1$ for this channel geometry). Reproduced with permission from W. Kramer, H. J. H. Clercx, and G. J. F. van Heijst, "Vorticity dynamics of a dipole colliding with a no-slip wall," *Phys. Fluids* **19**, 126603 (2007). Copyright 2007 AIP Publishing LLC.

$P_{max} \propto \text{Re}^{1.5}$ for $\text{Re} \gtrsim 2 \times 10^4$ (but $\text{Re} \leq 1.6 \times 10^5$). In this study, also the difference in dissipation Δ between runs with no-slip walls and periodic boundary conditions (with exactly the same initial conditions), integrated from $t = 0$ to $t = 0.5$ (thus sufficiently far beyond the primary dipole-wall collision in the no-slip runs), has been computed. This difference is actually the effective total enstrophy production in the case with no-slip walls [as for the periodic run, enstrophy is always decreasing with time as in that case $Z(t) < Z(t = 0)$ for finite viscosity]. It turns out to scale in a similar way as the maximum enstrophy Z_{max} , i.e., $\Delta \propto \text{Re}^{0.8}$ for $\text{Re} \leq 2 \times 10^4$ and $\Delta \propto \text{Re}^{0.5}$ for larger Re . The scaling of both Z_{max} and Δ clearly suggests that also the dissipation of kinetic energy is affected. Suppose that $Z \propto \text{Re}^\alpha$, then we find for the time rate of change of kinetic energy,

$$\frac{dE}{dt} = -\frac{2}{\text{Re}} Z \propto \text{Re}^{\alpha-1}. \quad (11)$$

In the two regimes discussed in Ref. 49, we may conjecture that $\frac{dE}{dt} \propto \text{Re}^{-0.2}$ for $\text{Re} \lesssim 2 \times 10^4$ and $\frac{dE}{dt} \propto \text{Re}^{-0.5}$ for $\text{Re} \gtrsim 2 \times 10^4$, instead of the relation $\frac{dE}{dt} \propto \text{Re}^{-1}$ for the unbounded case. Similar scaling relations have been obtained for oblique dipole-wall collision experiments,⁴⁹ so we expect it is generally valid. The presence of rigid no-slip walls has a significant effect on the dissipation of kinetic energy of the flow. Open questions are as follows: what happens eventually in the limiting situation of $\text{Re} \rightarrow \infty$ and what will be the limiting value of α ? Is the limiting value $\alpha < 1$, is the limit $\alpha \uparrow 1$ still possible (finite dissipation), or is $\alpha \rightarrow 0$ more likely (no dissipation)?

V. SCALING ANALYSIS OF VORTICITY PRODUCTION

As a first approach to understanding the scaling of the (first) maxima of enstrophy and palinstrophy, Z_{max} and P_{max} , respectively, we apply the Prandtl boundary-layer theory, see Ref. 49. Consider the collision shown in Fig. 4. As the collision process is fully symmetric with regard to the axis $x = 1$, we only consider the right half of the dipole, a vortex with size a (the dipole radius), and circulation Γ_{V+} . This

dipole half induces a boundary layer with thickness δ and width a , see also the schematic in Fig. 6(a). We assume that the circulation in the boundary layer, Γ_{B-} , is independent of Re (but not necessarily $\Gamma_{V+} = -\Gamma_{B-}$), which is confirmed by Kramer *et al.*⁵⁰ At the rigid no-slip wall, the following balance from the Navier-Stokes equations can be obtained:

$$-v \frac{\partial \omega}{\partial y} \Big|_{wall} = \frac{1}{\rho} \frac{\partial p}{\partial x} \Big|_{wall}. \quad (12)$$

By assuming that the pressure distribution along the boundary should be finite, we then obtain $\frac{\partial \omega}{\partial y} \Big|_{wall} = \mathcal{O}(\text{Re})$, for $\text{Re} \rightarrow \infty$. With the well-known Prandtl scaling of the boundary-layer thickness according to $\delta \propto \text{Re}^{-1/2}$, we immediately obtain $\omega|_{wall} \propto \sqrt{\text{Re}}$ (which is consistent with the alternative estimate $\omega|_{wall} \propto \frac{\Gamma_{B-}}{a\delta}$). For sufficiently high Reynolds numbers, here $\text{Re} \gtrsim 2 \times 10^4$, both enstrophy and palinstrophy are dominated by vorticity and vorticity gradients generated in the boundary layer (see also Table I for the highest Reynolds numbers), thus $Z \propto a\delta\omega_{wall}^2 \sim a\sqrt{\text{Re}}$ and $P \propto a\delta\left(\frac{\partial \omega}{\partial y}\Big|_{wall}\right)^2 \sim a\text{Re}\sqrt{\text{Re}}$. These scaling relations turn out to be consistent with the data reported by Clercx and van Heijst,⁴⁹ provided $\text{Re} \gtrsim 2 \times 10^4$.

The validity of the Prandtl boundary-layer scaling and the associated scaling behaviour of Z and P might be somewhat surprising and in contradiction with the contour plots shown in Fig. 7, which clearly indicate the detachment and roll up of boundary layers, suggesting that these boundary-layer processes cannot be described with the idealized flat-plate boundary-layer theory according to Prandtl. However, the measurement of Z_{max} and P_{max} takes place during the initial collision of the dipole with the wall, and at that stage, the thin boundary layer is mostly not yet detached, see Fig. 8. On the contrary, the dipole half pushes the boundary layer better to the rigid wall with increasing Re , a situation that seems favorable to obtain the traditional scaling at higher Reynolds numbers.

In the intermediate Reynolds number regime, the enstrophy scaling and palinstrophy scaling with Re are not compatible with the Prandtl-based scaling discussed in the previous paragraph, thus another approach is needed. There have been

FIG. 8. The vorticity field near the no-slip wall at the moment the dipole halves are closest to the wall. The Reynolds number is ranging from $\text{Re} = 625$ to 2×10^4 . Recirculation cells near the wall are indicated by the contours for zero stream function values. Reproduced with permission from W. Kramer, H. J. H. Clercx, and G. J. F. van Heijst, “Vorticity dynamics of a dipole colliding with a no-slip wall,” *Phys. Fluids* **19**, 126603 (2007). Copyright 2007 AIP Publishing LLC.

many detailed studies to explore the evolution of the boundary layer at a rigid (flat) no-slip wall under the influence of a vortex or a vortex couple, see Refs. 51, 55, 62, 65, 66, 68, and 71. To start with the basic processes, it is instructive to consider viscous vorticity production at no-slip boundaries first. Morton investigated the mechanism of vorticity production at rigid no-slip walls itself by analyzing various Stokes problems.⁸⁴ By exploring a few exact solutions of viscous diffusion problems, these can, by exploiting several analogies, eventually be used to understand the basic phenomena of more-complicated flow-wall interaction problems such as the dipole-wall collision. From such basic viscous solutions, some heuristic scaling arguments can be introduced to obtain proper scaling of the energy, enstrophy, palinstrophy, and their time derivatives. An example concerns the Stokes problem of a flat plate with in-plane oscillation embedded in a viscous fluid that turns out to be relevant for our dipole-wall collision problem. In Keetels *et al.*,⁸¹ it is shown that for this Stokes problem, the following expressions can be obtained for the enstrophy and palinstrophy in the boundary layer nearby the oscillating plate:

$$Z \propto \frac{\mathcal{L}V^2}{\delta_p} \propto \frac{\mathcal{L}V^2}{(T_p\nu)^{1/2}}, \quad P \propto \frac{\mathcal{L}V^2}{\delta_p^3} \propto \frac{\mathcal{L}V^2}{(T_p\nu)^{3/2}}, \quad (13)$$

where the Stokes boundary-layer thickness is defined as $\delta_p = \sqrt{\nu/\omega_p} \propto \sqrt{T_p\nu}$, ω_p is the oscillation frequency, T_p is the oscillation period, and $\mathcal{L}$ and V are the typical length and velocity scales, respectively. The area of the boundary layer is estimated according to $\mathcal{A} \approx \delta_p \mathcal{L}$ as all extreme values of vorticity and vorticity gradients can be found almost exclusively in this tiny area. Scaling expressions for the time derivatives like, for example, $\frac{dP}{dt}$ are obtained by including the factor T_p^{-1} .

As already mentioned, Peridier *et al.* investigated processes in the boundary layer for a single vortex above a flat no-slip plate.^{65,66} Besides the presence of the vortex, the fluid above the plate was assumed to be stagnant. In a more recent study, Obabko and Cassel considered a thick-core vortex that was similar to one half of a Lamb dipole.⁷¹ In this numerical experiment, they assumed that the self-induced velocity of the vortex was balanced by the free-stream velocity (and the vortex remained at a fixed position with respect to the rigid wall). However, the real dipole-wall collision is better characterized by an unsteady interaction of the vortex with the emerging boundary layer. This interaction has indeed some similarities with the Stokes problem of a flat plate with in-plane oscillation embedded in a viscous fluid as in the reference frame of the approaching and rebounding dipole half, the wall is seen as making an oscillatory motion corresponding with a half period of the Stokes problem. Keeping this observation in mind, alternative scaling relations for boundary-layer vorticity are proposed by Keetels *et al.*⁸¹ Here we will briefly outline their analysis.

The dipole-wall collision is characterized by two typical time scales. The first one, denoted by T_a , is associated with the approach of the dipole towards the rigid wall and can be considered as a measure of the time between the start of the formation of the boundary layer and the moment the nose of the dipole hits the newly formed boundary layer. For almost inviscid flows (thus at sufficiently large Re), the dissipation of

the kinetic energy of the dipolar vortex is negligible in the interior of the domain; thus, the time scale T_a is estimated by the dipole translation speed U_d and the size a of the dipole half (or dipole radius), thus $T_a \propto \frac{a}{U_d}$. The other time scale is based on the duration of the dipole-wall collision and rebound process. It is also called the contact time with the viscous boundary layer. As the dipole approaches the wall, a boundary layer is formed due to the induction of velocity in the near-wall region, and as the dipole halves move back into the interior of the flow domain, the magnitude of the velocity in the near-wall region will decrease again, a process with a total duration (as mentioned above) similar to a half period ($\frac{1}{2}T_p$) of the oscillating plate problem, but now the fluid exhibits an oscillatory motion above a fixed boundary. The first part of the collision is determined by the instant the nose of the dipolar vortex hits the emerging boundary layer for the first time until the moment maximum enstrophy is generated near the no-slip wall. The duration is estimated to be $\frac{1}{4}T_p$ and similar for the subsequent rebound period.

According to the approach by Keetels *et al.*,⁸¹ two cases can be distinguished for the time scale T_p . Below a critical Reynolds number Re_c , the boundary layer is sufficiently thick that it will play a role in the destruction of the wall-normal component of the translation velocity of the vortex, see, for example, Fig. 8. Therefore, besides the dipole radius, a , and its velocity, U_d , also the kinematic viscosity ν must play a role. With a force balance, it can be shown (see Ref. 81 for details) that $T_p \propto \frac{\delta}{U_d}$, and for $Re \lesssim Re_c$, with the critical Reynolds number $Re_c \approx 2 \times 10^4$, the boundary-layer thickness is expressed as $\delta \propto \sqrt{\nu T_a} = \sqrt{\nu a/U_d}$ or

$$T_p \propto \frac{\sqrt{\nu a}}{U_d^{3/2}}. \quad (14)$$

For large Reynolds numbers, the thickness of the boundary layer will decrease according to $\delta \propto Re^{-1/2}$ and becomes so thin that above Re_c , one can reasonably assume that it will hardly be deformed anymore by the approaching dipole, see the bottom panels in Fig. 8, and the kinematic viscosity does not play a role in the rebound process. The destruction of the wall-normal component of the translation velocity of the dipole will now only depend on a and U_d , and for $Re \gtrsim Re_c$, we can derive

$$T_p \propto \frac{a}{U_d}. \quad (15)$$

Substitution of this large-Reynolds number expression for T_p in Eq. (13), with $\mathcal{L} = a$ and $V = U_d$ (both scales are fixed in our numerical experiments), yields $Z \propto Re^{1/2}$ and $P \propto Re^{3/2}$ (as before). However, for $Re \lesssim Re_c$, we obtain, with substitution of Eq. (14) in (13), $Z \propto Re^{3/4}$ and $P \propto Re^{9/4}$. These scaling relations are in close agreement with those reported in Ref. 49, see also the first paragraph of this section. Finally, for the time rate of change of enstrophy, we find $\frac{dP}{dt} \propto \frac{U_d a^2}{T_p^{5/2} \nu^{3/2}} \propto Re^{11/4}$. These scaling relations are supported by data from our numerical simulations with $Re \lesssim 2 \times 10^4$, see Fig. 9, and the scaling relations for higher Reynolds numbers are supported by the data reported in Ref. 49.

FIG. 9. (a) Computed maxima of the enstrophy Z (open circles) as a function of Re with the reference line $Re^{3/4}$, and (b) the computed maxima of the palinstrophy P (open circles) and the maxima of $\frac{dP}{dt}$ (plusses) with the reference lines $Re^{9/4}$ and $Re^{11/4}$ for P and $\frac{dP}{dt}$, respectively. Reproduced with permission from G. H. Keetels, W. Kramer, H. J. H. Clercx, and G. J. F. van Heijst, “On the Reynolds number scaling of vorticity production at no-slip walls during vortex-wall collisions,” *Theor. Comput. Fluid Dyn.* **25**, 293–300 (2011). Copyright 2011 Springer.

It is remarkable that we can apply relatively simple scaling arguments to quite complicated dipole-wall collision processes. As can be seen in Fig. 7, the formation of secondary boundary layers, the development of shear layer instabilities, and the formation of several small-scale secondary vortices for the case $Re = 10^4$ do not strongly affect the predicted scaling behavior at vortex impact. A simplified unsteady boundary-layer model is thus able to explain the anomalous scaling of both enstrophy and palinstrophy around the moment of impact, and that these scaling predictions show satisfactory agreement with numerical data, also for $Re \gtrsim Re_c$. An important aspect might be, once again, that at the moment of impact, the boundary layer remains almost fully attached to the wall, and the complex flow features emerge later on and do not immediately affect the scaling behavior of Z_{max} and P_{max} .

These scaling regimes are expected to affect both forced and decaying 2D turbulence in confined domains (with no-slip sidewalls). We have already seen in Sec. II that substantial vorticity is produced at the rigid no-slip walls. Current numerical studies are limited to integral-scale Reynolds numbers of $\mathcal{O}(10^5\text{--}10^6)$. This implies that the Reynolds number based on the larger vortices (that eventually collide with the boundaries) is in the range $10^3 \lesssim Re \lesssim 10^5$ and will indeed show strong vortex-wall interactions including strong and possibly enhanced dissipation due to substantial enstrophy production. In Ref. 85, evidence is reported supporting an eventual anomalous scaling regime. Keeping the accessible range of Reynolds numbers in mind, we may expect that either of the scaling regimes will be applicable. This will thus seriously enhance the dissipation of the turbulent flow although we do not yet know which regime eventually dominates. Moreover, we also do not know how robust the scaling for $Re \gtrsim Re_c$ will be. For example, the boundary layer emerging during the dipole-wall collision in our numerical experiments is still laminar (Prandtl-Blasius like), and a transition to a turbulent boundary layer is to be expected at sufficiently high Reynolds numbers. Does the simple scaling relation $Z \propto Re^{1/2}$ survive at even much larger Reynolds numbers or does the scaling exponent change once again? And what might be the effect on the kinetic energy dissipation? Two recent studies have addressed some

of these aspects, and we will briefly discuss these results in Sec. VI.

VI. DIPOLE-WALL COLLISION AND THE VANISHING VISCOSITY LIMIT

In recent years, the issue of energy dissipating structures generated by walls in 2D flows has been explored from different perspectives that eventually might shed some more light on the interaction of coherent structures with no-slip walls in the limit of vanishing viscosity. It basically builds on the studies reported in Refs. 49, 50, and 81 but explores a few different aspects of the problem. We will discuss two recent studies^{86,87} that address the possible existence of extremely thin dissipation layers near the boundaries and the role of slip velocities at the wall. None of these studies actually extend the range of Reynolds numbers compared with previous studies, so any conclusion applicable for the vanishing viscosity limit is still open to debate.

Nguyen van yen, Farge, and Schneider⁸⁶ did a first attempt to show the existence of a Re -independent energy dissipation rate based on the dipole-wall collision experiment proposed by Clercx and van Heijst.⁴⁹ Keeping the scalings in mind for the enstrophy in 2D flows with no-slip boundaries, which are reported to scale as $Z \propto Re^\alpha$, with $\alpha \approx 0.5$ for high Reynolds number cases and $\alpha \approx 0.8$ for intermediate Reynolds number cases (we cannot yet draw any conclusion from the currently obtained values of α for the vanishing viscosity limit), they hypothesize on the possibility that α eventually approaches unity in the inviscid limit. Although for intermediate Reynolds numbers, α turns out to be somewhat larger than that for higher Reynolds numbers, see also Ref. 81, the fact that in current measurements, $\alpha \gtrsim 0.5$ could eventually imply the existence of a finite dissipation of decaying high-Reynolds number flows with no-slip boundaries, as $\frac{dE}{dt} = -\frac{2}{Re}Z \propto Re^{\alpha-1}$, thus potentially approaching a constant value when $\alpha \rightarrow 1$. This is in sharp contrast to the unbounded (or periodic) case where the enstrophy is bounded by its initial value and in the inviscid limit $\frac{dE}{dt} \rightarrow 0$. To support their hypothesis, Nguyen van yen *et al.* complement Prandtl’s classical boundary layer argument, which states that both the boundary-layer thickness and dissipation rate are proportional to $Re^{-1/2}$ in the vanishing viscosity limit,⁸⁸ with some theorems on the dissipation rate in the vanishing viscosity limit by Kato.⁸⁹ This is particularly important as Prandtl’s theory does not apply for the generic case with detaching boundary layers, such as in our dipole-wall collision experiments. Kato addressed the energy dissipation rate and proved that the dissipation rate indeed tends to zero if and only if the flow solution of the Navier-Stokes equation converges to the flow solution of the Euler equations provided the same initial conditions and flow geometry are used. Although this conclusion is often implicitly assumed, another aspect is less standard, as Kato proved a condition for dissipation to occur anywhere in the flow. In that case, at least some dissipation should take place in the neighborhood of a rigid wall within a thin boundary layer of thickness proportional to Re^{-1} , a non-standard result. Such a thin boundary layer, where substantial dissipation will take place, with thickness proportional to Re^{-1} , is orders of magnitude thinner than the classical Prandtl

FIG. 10. Dipole collision with the vertical rigid wall at $Re = 7880$. The vorticity is shown at $t = 0.36, 0.40, 0.45$, and 0.495 (from left to right). The white dotted box at $t = 0.495$ indicates the core of the secondary vortex (see the main text). Black pixels correspond to $\omega = \pm 300$ in all panels. Reproduced with permission from R. Nguyen van yen, M. Farge, and K. Schneider, “Energy dissipating structures produced by walls in two-dimensional flows at vanishing viscosity,” *Phys. Rev. Lett.* **106**, 184502 (2011). Copyright 2011 American Physical Society.

boundary layer, in particular for high Reynolds numbers. By carrying out numerical simulations of a dipole-wall collision, it is suggested by Nguyen van yen *et al.* that for detaching boundary layers, Kato’s scaling turns out to be more appropriate than Prandtl’s scaling,^{86,90} thus also requiring much higher resolutions as used in numerical dipole-wall collision studies before.^{49–51,73}

Nguyen van yen *et al.*⁸⁶ used the initial conditions and flow geometry introduced in Ref. 49 and employed the volume penalization method^{34,35} (and see Refs. 22, 28, and 33 for application of such methods to 2D turbulence simulations in complex domains) allowing to resolve scales as fine as Re^{-1} . However, by doing so, the no-slip boundary conditions are replaced by the Navier boundary conditions with a certain slip length. This slip length reduces to zero in the vanishing viscosity limit ($Re \rightarrow \infty$). For further numerical details, the implementation of the initial condition, details of the boundary conditions, and contribution of penalization to the energy dissipation rate, we refer to Ref. 86. Four runs with an identical initial flow field but different Reynolds numbers have been performed, with Reynolds number $Re = 985, 1970, 3940$, and 7880 . Snapshots of the vorticity field of the dipole-wall collision at $Re = 7880$ are shown in Fig. 10. The resolution is increased per direction by a factor of two, $N = 2048$ for the lowest Reynolds number run up to $N = 16\,384$ for the highest Reynolds number case (the penalization parameter η is reduced by a factor of two when doubling the Reynolds number and ranges from $\eta = 4 \times 10^{-5}$ to 5×10^{-6} for increased Reynolds numbers).

The main observations were the following:⁸⁶ the wall normal velocity at the boundary is small (about 0.1% of the root-mean-square velocity) and independent of Re as expected for this kind of volume penalization methods, and a small slip

velocity exists which scales like Re^{-1} , see Figs. 11(a) and 11(b). The kinetic energy of the flow is computed at the early stage ($t \in [0, 0.2]$) when the dipole is still travelling to the no-slip wall and also later on during the collision with the wall ($t \in [0.39, 0.495]$). Here, results similar as in Ref. 49 are found: at the early stage, $\Delta E = E(t_2) - E(t_1) \propto Re^{-1}$ (wall-less scaling as for unbounded flow), and during the collision stage, they observed $\Delta E \propto Re^{-0.5}$. These data always include a combination due to the bulk flow and Prandtl boundary layer on the one hand and the Kato-like dissipation layer on the other hand, the former dominating the latter in this numerical experiment as long as the kinetic energy is considered.⁸⁶ The computation of the enstrophy for the early and collision stages shows $Z \propto Re^{0.5}$, thus satisfying the Prandtl scaling, and $\Delta Z = Z(t_2) - Z(t_1) \propto Re$, respectively, see Fig. 11(c). Clearly, during the collision stage, the Kato-like dissipation scaling is found, as claimed by Nguyen van yen *et al.*,⁸⁶ implying that the enstrophy production is sufficiently high to enable finite dissipation at ever increasing Reynolds number. This is in contrast with the result found for ΔE discussed above. Increasing the Reynolds number will change the balance in favor of enstrophy production in the boundary layer, which keeps increasing while the bulk enstrophy is bounded by its value from the initial flow field. Computation of the energy dissipation rate in a thin slab in the boundary layer, but also in the core of the emerging secondary vortex (due to boundary-layer detachment), indicated by the small white-dotted box in Fig. 10(d), reveals an almost constant value for $Re \gtrsim 4000$.⁸⁶

Summarizing, the numerical simulations by Nguyen van yen, Farge, and Schneider⁸⁶ suggest finite dissipation due to the presence of no-slip walls in the vanishing viscosity limit. However, a follow-up numerical investigation by Sutherland, Macaskill, and Dritschel⁸⁷ of the same collision problem (and

FIG. 11. (a) Scatter plot of the tangential velocity u_t at the wall versus the tangential strain rate $\frac{\partial u_t}{\partial n}$. The dashed lines are least squares linear fits, and (b) the slip length s_L , obtained from the relation $u_t + s_L \frac{\partial u_t}{\partial n} \approx 0$ as a function of Re . Note that s_L is also a function of the penalization parameter η and resolution N . (c) The enstrophy increase as a function of Re for $t \in [0, 0.2]$ (black line) and $t \in [0.39, 0.495]$ (red line) and as guide to the eye for the Prandtl (dotted line) and dissipative (dashed line) scaling. Reproduced with permission from R. Nguyen van yen, M. Farge, and K. Schneider, “Energy dissipating structures produced by walls in two-dimensional flows at vanishing viscosity,” *Phys. Rev. Lett.* **106**, 184502 (2011). Copyright 2011 American Physical Society.

FIG. 12. From left to right, the trajectories followed by the positive half of the dipole, based on the vortex maximum in the positive dipole half, while travelling to and colliding with the flat no-slip wall, are shown for (a) $Re = 1250$, (b) 2500 , and (c) 5×10^3 , respectively. Reproduced with permission from W. Kramer, H. J. H. Clercx, and G. J. F. van Heijst, “Vorticity dynamics of a dipole colliding with a no-slip wall,” *Phys. Fluids* **19**, 126603 (2007). Copyright 2007 AIP Publishing LLC.

a similar range of Reynolds numbers) cast doubt on the validity of this conjecture. They used an essentially similar numerical setup as Kramer *et al.*⁵⁰ with the main difference the application of a slip length at the rigid walls of the channel, thus vanishing normal velocity at the wall and $u = \mp s_L \frac{\partial u}{\partial y}$ at the bottom and top boundaries of the channel, respectively. Here, s_L is the slip length, and $s_L = 0$ gives the no-slip condition and $s_L \rightarrow \infty$ results in a stress-free ($\omega|_{wall} = 0$) boundary condition. The slip length s_L can be controlled independently from the Reynolds number, in contrast with Ref. 86 where it scales as $s_L \propto Re^{-1}$, so the slip length and viscosity are not treated independently in that particular investigation. For their simulations, Sutherland *et al.*⁸⁷ have used the 2D stream function-vorticity equation together with the influence matrix method⁷⁴ to specify the vorticity boundary condition at the no-slip walls (or a modified condition for the case when Navier boundary conditions, allowing slip but no normal velocity at the wall, are used). For comparison, they have also implemented the same volume penalization method as used by Nguyen van yen *et al.*⁸⁶ Sutherland and co-workers observed that for the no-slip boundary condition and for the Navier boundary condition with a fixed slip length s_L , no evidence is found for persistence of energy dissipating structures in the vanishing viscosity limit. Basically, the enstrophy scaling in the applied range of Reynolds numbers is similar as found in Refs. 49 and 81. On the other hand, on applying a slip length of the form $s_L \propto Re^{-1}$, results that are more close to those reported by Nguyen van yen, Farge, and Schneider⁸⁶ are found.

Sutherland and co-workers⁸⁷ also explored the role of the slip length s_L on the kinetic energy of the flow, the enstrophy, and the trajectories of the dipole halves for the case $Re = 1250$. For the kinetic energy $E(t)$, the enstrophy $Z(t)$ and the vortex trajectories, based on the position of the vortex maximum in the dipole half, they recovered for $s_L = 10^{-4}$ the no-slip results of Kramer *et al.*⁵⁰ For $s_L = 100$ (approaching stress-free dynamics), the trajectories of the dipole halves converge to the stress-free case briefly reported in Sec. IV A where a comparison with the modified point-vortex model (with a correction to include weak diffusive effects⁷⁸) was discussed. By studying a range of values for s_L , it was shown by Sutherland *et al.*⁸⁷ (see their Fig. 6) that a transition occurs from a regime with strong rebounds ($s_L \lesssim 4 \times 10^{-3}$) to weak, skipping behavior for $s_L \gtrsim 10^{-2}$ (and the classical stress-free trajectory for $s_L \gtrsim 100$). In Kramer *et al.*,⁵⁰ these trajectories have been investigated for $Re = 1250$, 2500 , and 5×10^3 , and are shown in Fig. 12. From these trajectories, in combination with the vorticity contour plots of the normal-wall

collision displayed in Fig. 7 for $Re = 5 \times 10^3$ and 10^4 , one may safely conclude that, after the first vigorous interaction of the dipole with the no-slip wall, we will not retrieve the smooth trajectories of both dipole halves expected based on the (modified) point-vortex model or based on the analysis reported by Saffman.⁵⁷ The dipole halves basically disintegrate for sufficiently high Reynolds numbers, see Fig. 7. These trajectories clearly illustrate the different results of dipole collisions with no-slip walls compared with collisions with stress-free walls (which mimics the inviscid case for high Reynolds numbers) in the vanishing viscosity limit. On the other hand, Kramer *et al.*⁵⁰ provide evidence that up to about the first impact of the dipole with the no-slip wall, the trajectories of both dipole halves are well-described by the modulated point-vortex model. The trajectories before impact shown in Fig. 12 are almost the same, whereas for increasing Reynolds numbers, the trajectory matches increasingly better with that of the modulated point-vortex model (before impact).

Although the results by Sutherland *et al.*⁸⁷ and those of Clercx and co-workers^{49,50,81} would indeed suggest that kinetic energy dissipation will disappear in the vanishing viscosity limit (but at a considerably slower rate, $\frac{dE}{dt} \propto \frac{1}{\sqrt{Re}}$, than for the unbounded case where $\frac{dE}{dt} \propto \frac{1}{Re}$), we still need to be careful with definitive conclusions as the boundary layers are still of the Prandtl-Blasius type (laminar), and a transition to turbulent boundary layers might be expected at higher Reynolds numbers. The simulations by Sutherland *et al.* (like those by Nguyen van yen *et al.*⁸⁶) are still conducted at intermediate Reynolds numbers (more than one order of magnitude smaller than those in Clercx and van Heijst⁴⁹), and strong conclusions should be avoided at this stage.

VII. CONCLUSION

We started this overview with the observation that the dynamics of 2D decaying turbulence in domains bounded by rigid no-slip walls is strongly affected by vorticity production in the boundary layers at the domain walls. Currently, computational power does not yet allow to explore the very high-Reynolds number limit to address the role of dissipation in the vanishing viscosity limit. Nevertheless, investigations during the last three decades have shown that much could already be learned on one of the key mechanisms responsible for additional dissipation in 2D turbulence in bounded domains: the dipole-wall collision and rebound processes. In particular, the high-Reynolds number regime could be explored with carefully designed numerical simulations. From these numerical

studies, we conjecture two main regimes with anomalous scaling of enstrophy at intermediate Reynolds numbers and regular scaling, by invoking Prandtl scaling in the boundary layers, at higher Reynolds numbers. Theoretical scaling regimes can be found with similar scaling exponents as in the simulations. An important observation for both the intermediate and the higher Reynolds number regimes is that the observed Reynolds number scaling of the enstrophy seriously increases the dissipation in 2D turbulence in bounded domains compared with its analogue in unbounded or periodic domains.

Although much has already been learned, several open questions remain for future investigations. First of all, we do not yet have an answer on the question about what scaling behavior can be expected for the enstrophy, palinstrophy, and other quantities in the limit $Re \rightarrow \infty$. This needs to be pursued to better understand the relation between Navier-Stokes solutions of wall-bounded flows with no-slip conditions and idealized solutions based on free-slip boundary conditions. A related issue concerns a better understanding of the role of slip lengths on the flow dynamics and scaling relations as, in particular, we expect to rely more on volume penalization techniques as one of the numerical tools to explore really high Reynolds number 2D turbulence in bounded domains. Finally, we have not addressed the most important challenge yet, the translation of what we have learned from simple vortex-wall interactions to high-Reynolds number 2D turbulence in bounded domains. Many challenges remain to be addressed, and a rapid growth of computing power and new simulation approaches will guarantee that useful answers will most likely become available in the next decades.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the financial support from the Dutch Foundation for Fundamental Research on Matter (FOM), which is financially supported by the Nederlandse Organisatie voor Wetenschappelijk Onderzoek (Netherlands Organization for Scientific Research, NWO). Part of this work was sponsored by the Stichting Nationale Computerfaciliteiten (National Computing Facilities Foundation, NCF) for the use of supercomputer facilities, with financial support from NWO. The authors thank Kai Schneider (CMI, Aix-Marseille Université, France) and Marie Farge (LMD, Ecole Normale Supérieure, Paris, France) for kindly giving permission to use their figures from Ref. 86, now shown in Figs. 10 and 11 of this review, and Leon Kamp for careful reading of the manuscript.

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