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A reinsurance risk model with a threshold coverage policy – The Gerber-Shiu penalty function

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Abstract

We consider a Cramér-Lundberg insurance risk process with the added feature of reinsurance. If an arriving claim finds the reserve below a certain threshold γ , or if it would bring the reserve below that level, then a reinsurer pays part of the claim. Using fluctuation theory and the theory of scale functions of spectrally negative Lévy processes, we derive expressions for the Laplace transform of the time to ruin and of the joint distribution of the deficit at ruin and the surplus before ruin. We specify these results in much more detail for the threshold set-up with proportional reinsurance.

1 Introduction

Let $X(t)$ be the surplus at time t of the classical *Cramér-Lundberg* risk process,

$$X(t) = u + ct - \sum_{i=1}^{N_t} Z_i \quad (1.1)$$

In this model the company earns premium at a fixed rate c , the claim arrival process $\{N_t : t \geq 0\}$ is a Poisson process at rate λ , $\{Z_i : i = 1, 2, \dots\}$ are the successive claim amounts indexed by their appearance, and are i.i.d. positive random variables and $u = X(0)$.

In models like this it is of interest to study the distribution of the time to ruin, the joint distribution of the time to ruin, the deficit at ruin and the surplus before ruin. For a comprehensive overview of the state of the art of the classical Cramér-Lundberg model see the book of Asmussen and Albrecher (2010).

In the last decade the classical Cramér-Lundberg model in (1.1) was modified to capture dividend payments to the shareholders. Under the threshold dividend policy, dividends at rate $\tilde{c} < c$ are paid whenever the reserve is above a threshold γ . This process has a 'bent' at γ and it is called refracted Lévy risk process; see *Gerber and Shiu* (2006), *Zhang et al.* (2006), *Dickson and Drešćić* (2006), *Lin and Pavlova* (2006). *Wan* (2007) considered the more general model where the compound Poisson risk model is perturbed by a Brownian motion.

Lately, *Kyprianou and Loeffen* (2010) considered such a state dependent premium rate model for the general spectrally negative Lévy risk process. They used fluctuation theory and the theory of scale functions for spectrally negative Lévy processes to obtain the Laplace transform of the exit time, of the time to ruin, and of the joint probability for the surplus before and at ruin. In a more recent paper Loeffen (2015) presented a more elegant analysis to obtain the results of the previous paper.

In order to reduce risk, the insurer insures part of the risk. The insurer pays a premium to the reinsurer, while the reinsurer pays a part of each claim. Motivated by the threshold dividend policy, we consider the reinsurance threshold policy; the insurer pays a constant premium to the reinsurer and the reinsurer pays part of the claim that falls below a threshold γ .

We apply the methods of Loeffen (2015) to obtain quantities of interest for the reinsurance model. Assume that the company has a reinsurance contract. To describe it define a function $I(x)$, where $I(x) \leq x$, and $I(x)$ is non-decreasing in x , $I(0) = 0$. The reinsurance pays part of the claim when the claim is below a given threshold γ . Let $\tilde{I}(x, y)$ denote the part that the insurer pays for a claim of size x occurring when the reserve level is y . The reinsurer pays $x - \tilde{I}(x, y)$. $\tilde{I}(x, y)$ is given by:

$$\tilde{I}(x, y) = \begin{cases} x & \text{if } y > \gamma, x \leq y - \gamma \\ y - \gamma + I(x - (y - \gamma)) & \text{if } y > \gamma, x \geq y - \gamma \\ I(x) & \text{if } y \leq \gamma \end{cases} \quad (1.2)$$

Examples for $I(x)$ are $I(x) = \min(a, x)$ for a given constant a , and $I(x) = \alpha x$, $0 < \alpha < 1$.

Throughout we will not specify the reserve level y in $\tilde{I}(x, y)$ but it will be clear from the context. We consider the following risk process: The premium rate, the claim arrival process and the claim amounts are as per (1.1). When an arrival of a claim of size x finds the reserve below γ the insurer pays only $I(x)$. When a claim of size x finds the reserve at level $y > \gamma$, and $x > y - \gamma$ the insurer pays $y - \gamma + I(x - (y - \gamma))$, i.e. he pays only part of the claim that falls below γ . We denote by U_t the reserve level at time t under this policy.

In a companion paper (Boxma et al. 2016) we analyze a risk process with state dependent premium rate and state dependent claim payments assuming a barrier dividend policy. Under this policy all the premium income is paid as dividends when the reserve level is bigger than a barrier b . In that paper we applied different tools to find the distribution of the deficit at ruin and the amount of dividends until ruin. In the present paper we consider a special case of state dependent claim payments and consider the expected discounted time to ruin and the joint distribution of the deficit at ruin and the reserve just before ruin.

The paper is organized as follows. Section 2 introduces some notations and a few identities related to exit times of spectrally negative Lévy processes which play a crucial role in the remainder of the paper. Section 3 presents expressions for the Laplace transform of the exit time from an upper barrier, the time to ruin and the joint probability for the surplus before and at ruin for general $I(x)$. In Section 4 these results are specified in much more detail for the case that $I(x) = \alpha x$.

2 Notations

Above level γ the process U behaves as a risk process X_1 , with premium rate c , and i.i.d. claims distributed as Z_1 with distribution F_1 , arriving according to a Poisson process at rate λ . Below the level γ it behaves as a risk process X_0 , with premium rate c , and i.i.d. claims distributed as Z_0 with distribution F_0 , arriving according to a Poisson process at rate λ , where $F_0(x) = \mathbb{P}(I(Z_1) \leq x)$.

When the process U is above γ it evolves as X_1 and when it is below γ it evolves as X_0 .

Let

$$\psi_i(s) = \mathbb{E}[e^{sX_i(1)}] = cs - \lambda + \lambda \tilde{F}_i(s), \quad i = 0, 1,$$

where $\tilde{F}_i(s) = \mathbb{E}[e^{-sZ_i}]$. Let

$$\Phi_i(v) = \sup\{y \geq 0 : \psi_i(y) = v\}.$$

Definition 2.1. For a given spectrally negative Lévy process X , with Laplace exponent ψ , and $q \geq 0$, there is a unique q scale function associated with X , $W^{(q)} : \mathbb{R} \rightarrow [0, \infty)$ such that $W^{(q)}(x) = 0$ for $x < 0$, and on $(0, \infty)$, $W^{(q)}$ is the unique continuous function with Laplace transform

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) dx = \frac{1}{\psi(\beta) - q}. \quad (2.1)$$

We will denote $W^{(0)}$ by W . Consider a spectrally negative Lévy risk processes X_i , with q -scale function $W_i^{(q)}$ $i = 0, 1$. For $b > 0$ let

$$\tau_{i,b}^+ = \inf\{t > 0 : X_i(t) \geq b\}$$

and

$$\tau_{i,a}^- = \inf\{t > 0 : X_i(t) < a\}.$$

For the modified process U define

$$\kappa_b^+ = \inf\{t > 0 : U_t \geq b\},$$

and

$$\kappa_a^- = \inf\{t > 0 : U_t < a\}.$$

We denote by $\mathbb{P}_x$ and by $\mathbb{E}_x$ the conditional probability and expectation given that $X(0) = x$. In the sequel we will apply the following identities, (Kyprianou (2006), Chapter 8):

Theorem 2.1. Let X be a spectrally negative Lévy process. Then

(i) For $q \geq 0$, and $x \leq b$

$$\mathbb{E}_x[e^{-q\tau_b^+} 1_{\tau_b^+ < \tau_0^-}] = \frac{W^{(q)}(x)}{W^{(q)}(b)}. \quad (2.2)$$

(ii) Let $b > 0$, $x \in [0, b]$, $q \geq 0$ then

$$\int_0^\infty e^{-qt} \mathbb{P}_x(X_t \in dy, t < \tau_b^+ \wedge \tau_0^-) dt = \left(\frac{W^{(q)}(x)W^{(q)}(b-y)}{W^{(q)}(b)} - W^{(q)}(x-y) \right) dy. \quad (2.3)$$

3 The Laplace transform of the time to ruin-General $I(x)$

To derive the Laplace transform of the time to ruin we need to obtain some quantities. Let $b > \gamma$, and let $B \subset \mathbb{R}$. For $0 \leq x \leq b$ let

$$V^{(q)}(x, \gamma, b, B) = \int_{t=0}^{\infty} e^{-qt} \mathbb{P}_x(U_t \in B, t < \kappa_0^- \wedge \kappa_b^+) dt$$

$V^{(q)}(x, \gamma, b, B)$ is the discounted time that the process U_t spends in B , given that $U_0 = x$.

Proposition 3.1. (i) For $\gamma \leq x < b$:

$$\begin{aligned} V^{(q)}(x, \gamma, b, B) &= \int_{y \in B \cap [\gamma, b]} \left(\frac{W_1^{(q)}(x - \gamma) W_1^{(q)}(b - y)}{W_1^{(q)}(b - \gamma)} - W_1^{(q)}(x - y) \right) dy \\ &+ \int_{y=0}^{b-\gamma} \int_{\theta \geq y} \left(\frac{W_1^{(q)}(x - \gamma) W_1^{(q)}(b - \gamma - y)}{W_1^{(q)}(b - \gamma)} - W_1^{(q)}(x - \gamma - y) \right) \lambda dF_1(\theta) dy \\ &\cdot \left(\int_{z \in B \cap [0, \gamma]} \left(\frac{W_0^{(q)}(\gamma - I(\theta - y)) W_0^{(q)}(\gamma - z)}{W_0^{(q)}(\gamma)} - W_0^{(q)}(\gamma - I(\theta - y) - z) \right) dz \right. \\ &\left. + \frac{W_0^{(q)}(\gamma - I(\theta - y))}{W_0^{(q)}(\gamma)} V^{(q)}(\gamma, \gamma, b, B) \right). \end{aligned} \quad (3.1)$$

(ii) For $0 < x < \gamma$:

$$\begin{aligned} V^{(q)}(x, \gamma, b, B) &= \int_{y \in B \cap [0, \gamma]} \left(\frac{W_0^{(q)}(x) W_0^{(q)}(\gamma - y)}{W_0^{(q)}(\gamma)} - W_0^{(q)}(x - y) \right) dy \\ &+ \frac{W_0^{(q)}(x)}{W_0^{(q)}(\gamma)} V^{(q)}(\gamma, \gamma, b, B). \end{aligned} \quad (3.2)$$

Proof. (i) Let $\gamma \leq x < b$, then

$$V^{(q)}(x, \gamma, b, B) = \int_{t=0}^{\infty} e^{-qt} \int_{y \in B \cap [\gamma, b]} \mathbb{P}_{x-\gamma}(X_1(t) \in dy - \gamma, t < \tau_{1,0}^- \wedge \tau_{1,b-\gamma}^+) dy dt \quad (3.3)$$

$$+ \int_{t=0}^{\infty} e^{-qt} \int_{y=\gamma}^b \int_{\theta \geq y-\gamma} \mathbb{P}_{x-\gamma}(X_1(t) \in dy - \gamma, t < \tau_{1,0}^- \wedge \tau_{1,b-\gamma}^+) \lambda dF_1(\theta) dy dt \quad (3.4)$$

$$\cdot \left(\int_{s=0}^{\infty} \int_{z \in B \cap [0, \gamma]} e^{-qs} \mathbb{P}_{(\gamma - I(\theta - (y - \gamma)))}(X_0(s) \in dz, s < \tau_{0,0}^- \wedge \tau_{0,\gamma}^+) ds \right) \quad (3.5)$$

$$+ \mathbb{E}_{\gamma - I(\theta - (y - \gamma))}(e^{-q\tau_{0,\gamma}^+}, 1_{\tau_{0,\gamma}^+ < \tau_{0,0}^-}) V^{(q)}(\gamma, \gamma, b, B) \quad (3.6)$$

Above γ , $U(t)$ behaves as $X_1(t)$. (3.3) describes the discounted time that U (or X_1) spends in $B \cap [\gamma, b]$ before it down-crosses the level γ .

(3.4) describes the discounted time $U(t)$ is above γ until it downcrosses γ before hitting b . It is the same as the discounted time that $X_1 - \gamma$ is above 0 before hitting $b - \gamma$.

Below γ , U evolves as X_0 . Thus (3.5) is the expected discounted time that X_0 is in $B \cap [0, \gamma)$ given it started at $\gamma - I(\theta - (y - \gamma))$.

Thus, (3.4) multiplied by (3.5) is the discounted time that U spends in $B \cap [0, \gamma)$ from the moment it down-crosses γ until it exits $[0, \gamma)$. Similarly, (3.4) multiplied by (3.6) is the expected discounted time that U spends in B from the moment the process first hits γ after the first down-crossing of the level γ .

Using the scale function as in (2.3), and with $W_i^{(q)}(x)$ the scale function associated with X_i , $i = 0, 1$, we obtain (3.1).

(ii) For $0 < x < \gamma$: Similar arguments to (i) leads to:

$$\begin{aligned} V^{(q)}(x, \gamma, b, B) &= \int_{t=0}^{\infty} e^{-qt} \int_{y \in B \cap [0, \gamma)} \mathbb{P}_x(X_0(t) \in dy, 1_{t < \tau_{0,0}^- \wedge \tau_{0,\gamma}^+}) dt \\ &+ \int_{t=0}^{\infty} e^{-qt} \int_{y \in [0, \gamma)} \mathbb{P}_x(X_0(t) \in dy, 1_{t < \tau_{0,\gamma}^+ < \tau_{0,0}^-}) dt V^{(q)}(\gamma, \gamma, b, B). \end{aligned} \quad (3.7)$$

Applying (2.2) and (2.3) we obtain (3.2).

To obtain $V^{(q)}(\gamma, \gamma, b, B)$ we substitute $x = \gamma$ in (3.1) and solve the associated equation. □

Next we obtain

$$V^{(q)}(x, \gamma, B) = \int_{t=0}^{\infty} e^{-qt} \mathbb{P}_x(U_t \in B, t < \kappa_0^-) dt = \lim_{b \rightarrow \infty} V^{(q)}(x, \gamma, b, B)$$

For a Lévy process $X(t)$ with Lévy exponent ψ , adapted to a σ -field $\mathcal{F}_t$ let

$$M_t(\beta) = e^{\beta X(t) - \psi(\beta)t}$$

be the Wald martingale associated with X . Define the measure $\mathbb{P}^\beta$ as follows: For $\mathcal{A} \in \mathcal{F}_t$ define

$$\mathbb{P}^\beta(\mathcal{A}) = \mathbb{E}[e^{M_t(\beta)} 1_{\mathcal{A}}]$$

Denote by $W_{(\beta)}^{(q)}$ the scale function associated with X under $\mathbb{P}^\beta$. Apply (Kyprianou 2006) Chapter 8.2, or eq. (53) in Kuznetsov et al. (2013):

$$W^{(q)}(x) = e^{\Phi(q)x} W_{(\Phi(q))}(x) = e^{\Phi(q)x} \frac{1}{\psi'_{\Phi(q)}(0^+)} \mathbb{P}_x^{\Phi(q)}(\underline{X}(\infty) \geq 0), \quad (3.8)$$

where $\underline{X}(t) = \inf_{s \leq t} X(s)$. Under the measure $\mathbb{P}_x^{\Phi(q)}$, X drifts to ∞ .

Proposition 3.2. (i) For $x \geq \gamma$,

$$\begin{aligned}
V^{(q)}(x, \gamma, B) &= \int_{y \in B \cap [\gamma, \infty)} \left(W_1^{(q)}(x - \gamma) e^{-\Phi_1(q)(y - \gamma)} - W_1^{(q)}(x - y) \right) dy \\
&+ \int_{y=\gamma}^{\infty} \int_{\theta \geq y - \gamma} \left(W_1^{(q)}(x - \gamma) e^{-\Phi_1(q)(y - \gamma)} - W_1^{(q)}(x - y) \right) \lambda dF_1(\theta) dy \\
&\cdot \left(\int_{z \in B \cap [0, \gamma)} \left(\frac{W_0^{(q)}(\gamma - I(\theta + \gamma - y)) W_0^{(q)}(\gamma - z)}{W_0^{(q)}(\gamma)} - W_0^{(q)}(\gamma - I(\theta + \gamma - y) - z) \right) dz \right. \\
&\left. + \frac{W_0^{(q)}(\gamma - I(\theta + \gamma - y))}{W_0^{(q)}(\gamma)} V^{(q)}(\gamma, \gamma, B) \right) \quad (3.9)
\end{aligned}$$

ii) For $0 < x < \gamma$,

$$\begin{aligned}
V^{(q)}(x, \gamma, B) &= \int_{y \in B \cap [0, \gamma)} \left(\frac{W_0^{(q)}(x) W_0^{(q)}(\gamma - y)}{W_0^{(q)}(\gamma)} - W_0^{(q)}(x - y) \right) dy \\
&+ \frac{W_0^{(q)}(x)}{W_0^{(q)}(\gamma)} V^{(q)}(\gamma, \gamma, B). \quad (3.10)
\end{aligned}$$

(iii)

$$\begin{aligned}
V^{(q)}(\gamma, \gamma, B) &= \frac{1}{1 - \int_{y=\gamma}^{\infty} \int_{\theta \geq y - \gamma} W_1^{(q)}(0) e^{-\Phi_1(q)(y - \gamma)} \lambda dF_1(\theta) \frac{W_0^{(q)}(\gamma - I(\theta + \gamma - y))}{W_0^{(q)}(\gamma)} dy} \cdot \\
&\left[\int_{y \in B \cap (\gamma, \infty]} W_1^{(q)}(0) e^{-\Phi_1(q)(y - \gamma)} dy \right. \\
&+ \left(\int_{y=\gamma}^{\infty} \int_{\theta \geq y - \gamma} W_1^{(q)}(0) e^{-\Phi_1(q)(y - \gamma)} \lambda dF_1(\theta) dy \cdot \right. \\
&\cdot \left. \left. \int_{z \in B \cap (0, \gamma]} \left(\frac{W_0^{(q)}(\gamma - I(\theta + \gamma - y)) W_0^{(q)}(\gamma - z)}{W_0^{(q)}(\gamma)} - W_0^{(q)}(\gamma - I(\theta + \gamma - y) - z) \right) dz \right) \right] \quad (3.11)
\end{aligned}$$

Proof. (i)-(ii) Applying (3.8) to X_1 , we get that for $\gamma \leq y < b$

$$\frac{W_1^{(q)}(b - y)}{W_1^{(q)}(b - \gamma)} = \frac{e^{\Phi_1(q)(b - y)} \mathbb{P}_{b - y}^{\Phi_1(q)}(\underline{X}_1(\infty) \geq 0)}{e^{\Phi_1(q)(b - \gamma)} \mathbb{P}_{b - \gamma}^{\Phi_1(q)}(\underline{X}_1(\infty) \geq 0)} \xrightarrow{b \rightarrow \infty} e^{-\Phi_1(q)(y - \gamma)} \quad (3.12)$$

Thus, taking the limit as $b \rightarrow \infty$ in (3.1) we get (3.9). Similarly, taking the limit in (3.2) we obtain (3.10).

(iii) $V^{(q)}(\gamma, \gamma, B)$ is obtained by substituting $x = \gamma$ in (3.9) .

□

Next we obtain the Laplace transform of the time to ruin. Introduce $\mathcal{E}_q$, an Exponentially(q) distributed random variable. The Laplace transform of the time to ruin is:

$$\begin{aligned} \mathbb{E}_x[e^{-q\kappa_0^-} 1_{\kappa_0^- < \infty}] &= \mathbb{P}_x[\mathcal{E}_q > \kappa_0^-] \\ &= 1 - \mathbb{P}_x[\mathcal{E}_q \leq \kappa_0^-] = 1 - q \int_0^\infty e^{-qt} \mathbb{P}_x(U_t \in [0, \infty)) dt \\ &= 1 - qV^{(q)}(x, \gamma, [0, \infty)) \end{aligned}$$

Next we consider the Gerber-Shiu penalty function which is a non-negative function of the deficit at ruin $|U_{\kappa_0^-}|$ and the surplus just before ruin $U_{\kappa_0^-}$.

$$m(x, q) = \mathbb{E}_x[e^{-q\kappa_0^-} h(U_{\kappa_0^-}, |U_{\kappa_0^-}|)],$$

where h is a non-negative function.

Proposition 3.3.

$$\begin{aligned} m(x, q) &= \mathbb{E}_x[e^{-q\kappa_0^-} h(U_{\kappa_0^-}, |U_{\kappa_0^-}|)] \\ &= \int_{y=0}^{\gamma} V^{(q)}(x, \gamma, dy) \int_{\theta \geq y} \lambda dF_0(\theta) h(y, \theta - y) d\theta \\ &\quad + \int_{y=\gamma}^{\infty} V^{(q)}(x, \gamma, dy) \int_{I(\theta - (y - \gamma)) > \gamma} \lambda dF_1(\theta) h(y, I(\theta - (y - \gamma)) - \gamma) d\theta \end{aligned}$$

Proof. The discounted time in $(y, y + dy)$ is $V(x, \gamma, dy)$. When $y < \gamma$ ruin occurs when the downwards jump θ is bigger than y , in this case the deficit at ruin is $\theta - y$. When the surplus just before ruin is above γ , ruin occurs when the part that the insurer pays $I(\theta - (y - \gamma))$ is bigger than γ . Since $I(x) \leq x$, $I(\theta - (y - \gamma)) \geq \gamma$ implies that $\theta \geq y$. In this case the deficit is $I(\theta - (y - \gamma)) - \gamma$. $\square$

4 $I(x) = \alpha x$

In this section we consider the case where $I(x) = \alpha x$, $0 < \alpha < 1$. In this case we will obtain simpler expressions, especially for equations (3.1)-(3.11). We will get expressions involving only one integral instead of two. We apply a fundamental identity introduced by Loeffen (2015). Throughout we put an index 1 for quantities related to the risk process $X_1(t)$ with premium rate c and claim distribution F_1 , and an index 0 for quantities related to the risk process $X_0(t)$ with premium rate c and claim distribution F_0 , where $F_0(x) = F_1(x/\alpha)$.

Let $\mathcal{A}_j$ be the generator of X_j , $j = 0, 1$. Let h be a locally bounded function satisfying the smoothness and boundedness conditions (i),(ii) and (iv) of definition 1 in Loeffen (2015),

$$\mathcal{A}_j h(x) = ch'_-(x) + \int_0^\infty [h(x - \theta) - h(x)] \lambda dF_j(x), \quad (4.1)$$

where h'_- denotes the left derivative of h .

When $I(x) = \alpha x$, $\tilde{F}_1(s) = \tilde{F}_0(s/\alpha)$, thus

$$\psi_1(s) = \mathbb{E}[e^{sX_1(1)}] = \psi_0(s/\alpha) - \frac{cs}{\alpha}(1 - \alpha). \quad (4.2)$$

This section is organized as follows. In section 4.1 we apply Loeffen's (2015) result and present a key identity which is applied in the remainder of this section. In Section 4.2 we obtain an expression for the discounted time that $U_t \in B$ before exiting $[0, b]$, where $B \subset \mathbb{R}$. In Sections 4.3 and 4.4 we obtain expressions for the potential measure for U . Section 4.5 presents the Laplace transform of the time to ruin, and in Section 4.6 we derive the ruin probability. Section 4.7 presents an expression for the Gerber-Shiu penalty function and the joint probability of the surplus before and at ruin.

4.1 A key identity

The following key identity is a consequence of Theorem 2 in Loeffen (2015) (see also equation (19) in that paper):

Proposition 4.1. *For $x \in [\gamma, b)$,*

$$\begin{aligned} & \mathbb{E}_x \left[e^{-q\tau_{1,\gamma}^-} W_0^{(q)}(\alpha X_1(\tau_{1,\gamma}^-) + (1 - \alpha)\gamma) 1_{\tau_{1,\gamma}^- < \tau_{1,b}^+} \right] \\ &= W_0^{(q)}(\alpha x + \gamma(1 - \alpha)) - \frac{W_1^{(q)}(x - \gamma)}{W_1^{(q)}(b - \gamma)} W_0^{(q)}(\alpha b + \gamma(1 - \alpha)) \\ & \quad - (1 - \alpha)c \int_{\gamma}^b W_0^{(q)'}(\alpha y + \gamma(1 - \alpha)) \left[\frac{W_1^{(q)}(x - \gamma)}{W_1^{(q)}(b - \gamma)} W_1^{(q)}(b - y) - W_1^{(q)}(x - y) \right] dy. \end{aligned}$$

Proof. By Theorem 2 in Loeffen (2015) (with $\sigma = 0$)

$$\begin{aligned} & \mathbb{E}_x \left[e^{-q\tau_{1,\gamma}^-} W_0^{(q)}(\alpha X_1(\tau_{1,\gamma}^-) + (1 - \alpha)\gamma) 1_{\tau_{1,\gamma}^- < \tau_{1,b}^+} \right] \\ &= W_0^{(q)}(\alpha x + \gamma(1 - \alpha)) - \frac{W_1^{(q)}(x - \gamma)}{W_1^{(q)}(b - \gamma)} W_0^{(q)}(\alpha b + \gamma(1 - \alpha)) \\ & \quad + \int_{\gamma}^b (\mathcal{A}_1 - q) W_0^{(q)}(\alpha y + (1 - \alpha)\gamma) \left[\frac{W_1^{(q)}(x - \gamma + q)}{W_1^{(q)}(b - \gamma)} W_1^{(q)}(b - y) - W_1^{(q)}(x - y) \right] dy. \quad (4.3) \end{aligned}$$

Next we show that

$$(\mathcal{A}_1 - q) W_0^{(q)}(\alpha y + (1 - \alpha)\gamma) = -c(1 - \alpha) W_0^{(q)'}(\alpha y + \gamma(1 - \alpha)) \quad (4.4)$$

By (4.1)

$$\begin{aligned} & (\mathcal{A}_1 - q) W_0^{(q)}(\alpha y + \gamma(1 - \alpha)) \\ &= \alpha c W_0^{(q)'}(\alpha y + \gamma(1 - \alpha)) + \int_0^{\infty} (W_0^{(q)}(\alpha(y - \theta) + \gamma(1 - \alpha)) \lambda dF_1(\theta) - (\lambda + q) W_0^{(q)}(\alpha y + \gamma(1 - \alpha))). \end{aligned} \quad (4.5)$$

(4.5) is defined for $y \geq -\frac{\gamma(1-\alpha)}{\alpha}$. To prove (4.4) we take Laplace transforms of both sides of (4.4) and show that they are equal.

$$\begin{aligned}
\int_{-\frac{\gamma(1-\alpha)}{\alpha}}^{\infty} e^{-sy} W_0^{(q)'}(\alpha y + \gamma(1-\alpha)) dy &= \frac{1}{\alpha} e^{\frac{s\gamma(1-\alpha)}{\alpha}} \int_{z=0}^{\infty} e^{-\frac{s}{\alpha}z} W_0^{(q)'}(z) dz \\
&= \frac{1}{\alpha} e^{\frac{s\gamma(1-\alpha)}{\alpha}} \left(-\frac{1}{c} + \frac{s}{\alpha(\psi_0(s/\alpha) - q)} \right).
\end{aligned} \tag{4.6}$$

We apply integration by parts and the identity $W_0^{(q)}(0) = 1/c$ (Lemma 8.6 in Kyprianou (2006)) in the first equality and (2.1) in the second one. Similarly, by (2.1)

$$\int_{-\frac{\gamma(1-\alpha)}{\alpha}}^{\infty} e^{-sy} W_0^{(q)}(\alpha y + \gamma(1-\alpha)) dy = e^{\frac{s\gamma(1-\alpha)}{\alpha}} \frac{1}{\alpha(\psi_0(s/\alpha) - q)}. \tag{4.7}$$

Integration by parts and change of variables yield:

$$\begin{aligned}
&\lambda \int_{-\frac{\gamma(1-\alpha)}{\alpha}}^{\infty} e^{-sy} \int_{\theta=0}^{\infty} (W_0^{(q)}(\alpha(y-\theta) + \gamma(1-\alpha)) dF_1(\theta) \\
&= \lambda e^{\frac{s\gamma(1-\alpha)}{\alpha}} \int_{z=0}^{\infty} e^{-sz} \int_{\theta=0}^z W_0^{(q)}(\alpha(z-\theta)) dF_1(\theta) \\
&= \frac{\lambda}{\alpha} e^{\frac{s\gamma(1-\alpha)}{\alpha}} \int_{\theta=0}^{\infty} e^{-s\theta} dF_1(\theta) \int_{z=0}^{\infty} e^{-\frac{s}{\alpha}z} W_0^{(q)}(z) dz \\
&= e^{\frac{s\gamma(1-\alpha)}{\alpha}} \frac{\lambda \tilde{F}_1(s)}{\alpha(\psi_0(s/\alpha) - q)} = e^{\frac{s\gamma(1-\alpha)}{\alpha}} \frac{\lambda \tilde{F}_0(s/\alpha)}{\alpha(\psi_0(s/\alpha) - q)}.
\end{aligned} \tag{4.8}$$

In the last line we applied (2.1) and that $Z_0 = \alpha Z_1$. (4.5), (4.6), (4.7) and (4.8) yield that the Laplace transform of $(\mathcal{A}_1 - q)W_0^{(q)}(\alpha x + \gamma(1-\alpha))$ is:

$$\begin{aligned}
&e^{\frac{s\gamma(1-\alpha)}{\alpha}} \left(-1 + \frac{1}{\alpha} + (1-\alpha) \frac{cs}{\alpha^2(\psi_0(s/\alpha) - q)} \right) \\
&= -c(1-\alpha) \frac{1}{\alpha} \left(-\frac{1}{c} + \frac{s}{\alpha(\psi_0(s/\alpha) - q)} \right).
\end{aligned}$$

Thus, (4.6) yields (4.4). □

For $x \geq \gamma$ define

$$w_{\alpha}^{(q)}(x, z) = W_0^{(q)}(\alpha x - z + \gamma(1-\alpha)) + (1-\alpha)c \int_{\gamma}^x W_1^{(q)}(x-y) W_0^{(q)'}(\alpha y - z + \gamma(1-\alpha)) dy. \tag{4.9}$$

and for $x < \gamma$ define

$$w_{\alpha}^{(q)}(x, z) = W_0^{(q)}(x - z). \tag{4.10}$$

Then for $\gamma \leq x < b$,

$$\mathbb{E}_x \left[e^{-q\tau_{1,\gamma}^-} W_0(\alpha x + (1-\alpha)\gamma) 1_{\tau_{1,\gamma}^- < \tau_{1,b}^+} \right] = w_{\alpha}^{(q)}(x, 0) - \frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)} w_{\alpha}^{(q)}(b, 0). \tag{4.11}$$

4.2 Exit time for U

Let $\gamma < b$. We obtain an expression for $p(x, \gamma, b, q)$, where

$$p(x, \gamma, b, q) = \mathbb{E}_x(e^{-q\kappa_b^+} \mathbf{1}_{\{\kappa_b^+ < \kappa_0^-\}} | U_0 = x).$$

Proposition 4.2.

$$p(x, \gamma, b, q) = \frac{w_\alpha^{(q)}(x, 0)}{w_\alpha^{(q)}(b, 0)}.$$

Proof. Let $\gamma \leq x < b$. Either b is reached before γ or the process downcrosses γ before reaching b . $X_1(\tau_{1,\gamma}^-)$ is the state of X_1 after downcrossing γ . Since the insurer pays $\alpha(\gamma - X_1(\tau_{1,\gamma}^-))$ (instead of $(\gamma - X_1(\tau_{1,\gamma}^-))$), the state of the process U after undershooting γ is

$$\gamma - \alpha(\gamma - X_1(\tau_{1,\gamma}^-)) = \alpha X_1(\tau_{1,\gamma}^-) + (1 - \alpha)\gamma.$$

Hence, the discounted time until reaching b before ruin is the discounted time to reach γ before ruin, and then the discounted time to reach b before ruin starting at γ . Applying the strong Markov property at γ , (2.2) and (4.9):

$$\begin{aligned} p(x, \gamma, b, q) &= \frac{W_1^{(q)}(x - \gamma)}{W_1^{(q)}(b - \gamma)} + \frac{\mathbb{E}_x \left[e^{-q\tau_{1,\gamma}^-} W_0^{(q)}(\alpha X_1(\tau_{1,\gamma}^-) + (1 - \alpha)\gamma) \mathbf{1}_{\tau_{1,\gamma}^- < \tau_{1,b}^+} \right]}{W_0^{(q)}(\gamma)} p(\gamma, \gamma, b, q) \\ &= \frac{W_1^{(q)}(x - \gamma)}{W_1^{(q)}(b - \gamma)} + \frac{w_\alpha^{(q)}(x, 0) - \frac{W_1^{(q)}(x - \gamma)}{W_1^{(q)}(b - \gamma)} w_\alpha^{(q)}(b, 0)}{W_0^{(q)}(\gamma)} p(\gamma, \gamma, b, q) \end{aligned} \quad (4.12)$$

We derive $p(\gamma, \gamma, b, q)$ by substituting $x = \gamma$ in (4.12) and applying (4.11).

$$p(\gamma, \gamma, b, q) = \frac{W_1^{(q)}(0)}{W_1^{(q)}(b - \gamma)} + \frac{w_\alpha^{(q)}(\gamma, 0) - \frac{W_1^{(q)}(0)}{W_1^{(q)}(b - \gamma)} w_\alpha^{(q)}(b, 0)}{W_0^{(q)}(\gamma)} p(\gamma, \gamma, b, q) \quad (4.13)$$

Since $w_\alpha^{(q)}(\gamma, 0) = W_0^{(q)}(\gamma)$, we obtain that

$$p(\gamma, \gamma, b, q) = \frac{W_0^{(q)}(\gamma)}{w_\alpha^{(q)}(b, 0)} = \frac{w_\alpha^{(q)}(\gamma, 0)}{w_\alpha^{(q)}(b, 0)} \quad (4.14)$$

Substituting (4.14) in (4.12) we obtain that

$$p(x, \gamma, b, q) = \frac{w_\alpha^{(q)}(x, 0)}{w_\alpha^{(q)}(b, 0)}.$$

For $0 < x < \gamma$, the strong Markov property, (2.2) and (4.14) yield:

$$p(x, \gamma, b, q) = \frac{W_0^{(q)}(x)}{W_0^{(q)}(\gamma)} p(\gamma, \gamma, b, q) = \frac{W_0^{(q)}(x)}{w_\alpha^{(q)}(b, 0)} = \frac{w_\alpha^{(q)}(x, 0)}{w_\alpha^{(q)}(b, 0)}. \quad (4.15)$$

□

4.3 $V^{(q)}(x, \gamma, b, B)$

For $B \subset (0, b)$ we obtain the discounted time the process is in B before exiting $(0, b)$. Define

$$V^{(q)}(x, \gamma, b, B) = \int_0^\infty e^{-qt} \mathbb{P}_x(U_t \in B, t < \kappa_0^- \wedge \kappa_b^+) dt$$

Proposition 4.3.

$$\begin{aligned} V^{(q)}(x, \gamma, b, B) &= \int_{y \in B \cap [\gamma, b)} \left(\frac{w_\alpha^{(q)}(x, 0)}{w_\alpha^{(q)}(b, 0)} W_1^{(q)}(b - y) - W_1^{(q)}(x - y) \right) dy \\ &+ \int_{y \in B \cap [0, \gamma)} \left(\frac{w_\alpha^{(q)}(x, 0)}{w_\alpha^{(q)}(b, 0)} w_\alpha^{(q)}(b, y) - w_\alpha^{(q)}(x, y) \right) dy \end{aligned} \quad (4.16)$$

Proof. First we consider the case $\gamma \leq x < b$. By (2.3)

$$V^{(q)}(x, \gamma, b, B) = \int_{y \in B \cap [\gamma, b)} \left(\frac{W_1^{(q)}(x - \gamma)}{W_1^{(q)}(b - \gamma)} W_1^{(q)}(b - y) - W_1^{(q)}(x - y) \right) dy \quad (4.17)$$

$$\begin{aligned} &+ \int_{y \in B \cap (0, \gamma]} \mathbb{E}_x \left[e^{-q\tau_{1,\gamma}^-} 1_{\tau_{1,\gamma}^- < \tau_{1,b}^+} \left(\frac{W_0^{(q)}(\alpha X_1(\tau_{1,\gamma}^-) + (1 - \alpha)\gamma)}{W_0^{(q)}(\gamma)} W_0^{(q)}(\gamma - y) \right. \right. \\ &\left. \left. - W_0^{(q)}(\alpha X_1(\tau_{1,\gamma}^-) + (1 - \alpha)\gamma - y) \right) \right] dy \end{aligned} \quad (4.18)$$

$$+ \frac{\mathbb{E}_x[e^{-q\tau_{1,\gamma}^-} 1_{\tau_{1,\gamma}^- < \tau_{1,b}^+} W_0^{(q)}(\alpha X_1(\tau_{1,\gamma}^-) + \gamma(1 - \alpha))]}{W_0^{(q)}(\gamma)} V^{(q)}(\gamma, \gamma, b, B) \quad (4.19)$$

(4.17) is the discounted time spent in $B \cap [\gamma, b)$ until exit from $[\gamma, b)$. (4.18) is the discounted time spent in $B \cap (0, \gamma)$ until exit from $(0, \gamma)$. Since the process starts at $x > \gamma$ it first enters $(0, \gamma)$ at $\tau_{1,\gamma}^-$ and then falls to $\alpha X_1(\tau_{1,\gamma}^-) + (1 - \alpha)\gamma$ (instead to $X_1(\tau_{1,\gamma}^-)$ due to the partial coverage by the reinsurer). Thus (4.18) follows from (2.3). Substituting (4.11) in (4.18) and (4.19) we obtain:

$$\begin{aligned} V^{(q)}(x, \gamma, b, B) &= \int_{y \in B \cap [\gamma, b)} \left(\frac{W_1^{(q)}(x - \gamma)}{W_1^{(q)}(b - \gamma)} W_1^{(q)}(b - y) - W_1^{(q)}(x - y) \right) dy \\ &+ \int_{y \in B \cap (0, \gamma]} \left(\frac{w_\alpha^{(q)}(x, 0) - \frac{W_1^{(q)}(x - \gamma)}{W_1^{(q)}(b - \gamma)} w_\alpha^{(q)}(b, 0)}{W_0^{(q)}(\gamma)} W_0^{(q)}(\gamma - y) \right. \\ &\left. - (w_\alpha^{(q)}(x, y) - \frac{W_1^{(q)}(x - \gamma)}{W_1^{(q)}(b - \gamma)} w_\alpha^{(q)}(b, y)) \right) dy \end{aligned} \quad (4.20)$$

$$+ \frac{w_\alpha^{(q)}(x, 0) - \frac{W_1^{(q)}(x - \gamma)}{W_1^{(q)}(b - \gamma)} w_\alpha^{(q)}(b, 0)}{W_0^{(q)}(\gamma)} V^{(q)}(\gamma, \gamma, b, B) \quad (4.21)$$

Substituting $x = \gamma$ and remembering that $w_\alpha^{(q)}(\gamma, y) = W_0^{(q)}(\gamma - y)$ for $0 \leq y < \gamma$ we obtain that

$$\begin{aligned}
V^{(q)}(\gamma, \gamma, b, B) &= \frac{W_0^{(q)}(\gamma)}{w_\alpha(b, 0)} \int_{y \in B \cap [\gamma, b]} W_1^{(q)}(b - y) dy \\
&\quad \int_{y \in B \cap (0, \gamma]} \left(-W_0^{(q)}(\gamma - y) + \frac{W_0^{(q)}(\gamma)}{w_\alpha^{(q)}(b, 0)} w_\alpha^{(q)}(b, y) \right) dy
\end{aligned} \tag{4.22}$$

By substituting (4.22) in (4.21) we obtain (4.16) for $\gamma \leq x < b$.

For $x < \gamma$ we obtain from (3.2) and by substituting (4.22) there,

$$\begin{aligned}
V^{(q)}(x, \gamma, b, B) &= \int_{y \in B \cap [0, \gamma)} \left(\frac{W_0^{(q)}(x)}{W_0^{(q)}(\gamma)} W_0^{(q)}(\gamma - y) - W_0^{(q)}(x - y) \right) dy \\
&\quad + \frac{W_0^{(q)}(x)}{W_0^{(q)}(\gamma)} V^{(q)}(\gamma, \gamma, b, B) \\
&= \int_{y \in B \cap [0, \gamma)} \left(\frac{W_0^{(q)}(x)}{W_0^{(q)}(\gamma)} W_0^{(q)}(\gamma - y) - W_0^{(q)}(x - y) \right) dy \\
&\quad + \frac{W_0^{(q)}(x)}{W_0^{(q)}(\gamma)} \left(\frac{W_0^{(q)}(\gamma)}{w_\alpha^{(q)}(b, 0)} \int_{y \in B \cap [\gamma, b]} W_1^{(q)}(b - y) dy \right. \\
&\quad \left. + \int_{y \in B \cap [0, \gamma]} \left(-W_0^{(q)}(\gamma - y) + \frac{W_0^{(q)}(\gamma)}{w_\alpha^{(q)}(b, 0)} w_\alpha^{(q)}(b, y) \right) dy \right) \\
&= \int_{y \in B \cap [0, \gamma)} \left(\frac{W_0^{(q)}(x)}{w_\alpha^{(q)}(b, 0)} w_\alpha^{(q)}(b, y) - W_0^{(q)}(x - y) \right) dy \\
&\quad + \frac{W_0^{(q)}(x)}{w_\alpha^{(q)}(b, 0)} \int_{y \in B \cap [\gamma, b]} W_1^{(q)}(b - y) dy
\end{aligned} \tag{4.23}$$

Since for $x < \gamma$, $w_\alpha^{(q)}(x, z) = W_0^{(q)}(x - z)$, cf. (4.10), we get a similar expression to (4.16). $\square$

4.4 $b \rightarrow \infty$

In this subsection we obtain

$$V^{(q)}(x, \gamma, B) = \mathbb{E}_x \left[\int_{t=0}^{\infty} e^{-qt} 1_{(U_t \in B, t < \kappa_0^-)} \right] \tag{4.24}$$

By letting $b \rightarrow \infty$ in $V^{(q)}(x, \gamma, b, B)$:

$$V^{(q)}(x, \gamma, B) = \lim_{b \rightarrow \infty} V^{(q)}(x, \gamma, b, B)$$

Proposition 4.4. *For all $x \geq 0$:*

$$\begin{aligned}
V^{(q)}(x, \gamma, dy) &= \left(\frac{w_\alpha^{(q)}(x, 0)}{c(1 - \alpha)A} e^{-\Phi_1(q)y} - W_1^{(q)}(x - y) \right) 1_{y \in [\gamma, \infty)} dy \\
&\quad + \left(\frac{w_\alpha^{(q)}(x, 0)}{A} \int_{z=\gamma}^{\infty} e^{-\Phi_1(q)z} W_0^{(q)'}(\alpha z - y + \gamma(1 - \alpha)) dz - w_\alpha^{(q)}(x, y) \right) 1_{y \in [0, \gamma)} dy,
\end{aligned} \tag{4.25}$$

where

$$A = \int_{\gamma}^{\infty} e^{-\Phi_1(q)y} W_0^{(q)'}(\alpha y + \gamma(1 - \alpha)) dy. \quad (4.26)$$

Proof. Consider the expression for $w_{\alpha}^{(q)}(b, z)$ given in (4.9), and apply (3.8) to obtain the limit as $b \rightarrow \infty$ of $\frac{w_{\alpha}^{(q)}(b, z)}{W_1^{(q)}(b)}$.

$$\begin{aligned} \frac{w_{\alpha}^{(q)}(b, z)}{W_1^{(q)}(b)} &= \frac{W_0^{(q)}(\alpha b - z + \gamma(1 - \alpha)) + c(1 - \alpha) \int_{\gamma}^b W_1^{(q)}(b - y) W_0^{(q)'}(\alpha y - z + \gamma(1 - \alpha)) dy}{W_1^{(q)}(b)} \\ &= \frac{e^{\Phi_0(q)(\alpha b - z + \gamma(1 - \alpha))} W_{(\Phi_0(q))}(\alpha b - z + \gamma(1 - \alpha))}{e^{\Phi_1(q)b} W_{(\Phi_1(q))}(b)} \end{aligned} \quad (4.27)$$

$$+ c(1 - \alpha) \int_{\gamma}^b \frac{e^{\Phi_1(q)(b - y)} W_{(\Phi_1(q))}(b - y)}{e^{\Phi_1(q)b} W_{(\Phi_1(q))}(b)} W_0^{(q)'}(\alpha y - z + \gamma(1 - \alpha)) dy \quad (4.28)$$

Since $X_0(1) > X_1(1)$, $\psi_0(s) > \psi_1(s)$, and $\Phi_0(q) < \Phi_1(q)$. By Kuznetsov et al. (2013) Section 3.1, $W_{(\Phi(q))}(+\infty) = 1/\psi'(\Phi(q))$.

Thus

$$e^{(\Phi_0(q)\alpha - \Phi_1(q))b} \xrightarrow{b \rightarrow \infty} 0$$

and the limit as $b \rightarrow \infty$ of the expression in (4.27) is 0. The limit of (4.28) is as follows:

$$\frac{\int_{\gamma}^b W_1^{(q)}(b - y) W_0^{(q)'}(\alpha y - z + \gamma(1 - \alpha)) dy}{W_1^{(q)}(b)} \xrightarrow{b \rightarrow \infty} \int_{\gamma}^{\infty} e^{-\Phi_1(q)y} W_0^{(q)'}(\alpha y - z + \gamma(1 - \alpha)) dy \quad (4.29)$$

By (3.12) and (4.26)-(4.29)

$$\frac{W_1^{(q)}(b - y)}{w_{\alpha}^{(q)}(b, 0)} = \frac{\frac{W_1^{(q)}(b - y)}{W_1^{(q)}(b)}}{\frac{w_{\alpha}^{(q)}(b, 0)}{W_1^{(q)}(b)}} \xrightarrow{b \rightarrow \infty} \frac{e^{-\Phi_1(q)y}}{c(1 - \alpha)A}, \quad (4.30)$$

and

$$\frac{w_{\alpha}^{(q)}(b, y)}{w_{\alpha}^{(q)}(b, 0)} = \frac{\frac{w_{\alpha}^{(q)}(b - y)}{W_1^{(q)}(b)}}{\frac{w_{\alpha}^{(q)}(b, 0)}{W_1^{(q)}(b)}} \xrightarrow{b \rightarrow \infty} \frac{\int_{\gamma}^{\infty} e^{-\Phi_1(q)y} W_0^{(q)'}(\alpha y - z + \gamma(1 - \alpha)) dy}{A} \quad (4.31)$$

Letting $b \rightarrow \infty$ in (4.16) we obtain from (4.30) and (4.31):

$$\begin{aligned} V^{(q)}(x, \gamma, B) &= \int_{y \in B \cap [\gamma, \infty)} \left(\frac{w_{\alpha}^{(q)}(x, 0)}{c(1 - \alpha)A} e^{-\Phi_1(q)y} - W_1^{(q)}(x - y) \right) dy \\ &+ \int_{y \in B \cap [0, \gamma)} \left(\frac{w_{\alpha}^{(q)}(x, 0)}{A} \int_{z=\gamma}^{\infty} e^{-\Phi_1(q)z} W_0^{(q)'}(\alpha z - y + \gamma(1 - \alpha)) dz - w_{\alpha}^{(q)}(x, y) \right) dy \end{aligned} \quad (4.32)$$

Thus we can write for all $x \geq 0$:

$$\begin{aligned} V^{(q)}(x, \gamma, dy) &= \left(\frac{w_\alpha^{(q)}(x, 0)}{c(1-\alpha)A} e^{-\Phi_1(q)y} - W_1^{(q)}(x-y) \right) 1_{y \in [\gamma, \infty)} dy \\ &+ \left(\frac{w_\alpha^{(q)}(x, 0)}{A} \int_{z=\gamma}^{\infty} e^{-\Phi_1(q)z} W_0^{(q)'}(\alpha z - y + \gamma(1-\alpha)) dz - w_\alpha^{(q)}(x, y) \right) 1_{y \in [0, \gamma)} dy \end{aligned} \quad (4.33)$$

□

4.5 The Laplace transform of the time to ruin

Proposition 4.5. *The Laplace transform of the time to ruin is as follows:*

(i) For $x \geq \gamma$,

$$\begin{aligned} &\mathbb{E}_x[e^{-q\kappa_0^-} 1_{\kappa_0^- < \infty}] \\ &= 1 + q \int_0^{\alpha x + \gamma(1-\alpha)} W_0^{(q)}(y) dy + qc(1-\alpha) \int_{y=\gamma}^x W_1^{(q)}(x-y) W_0^{(q)}(\alpha y + \gamma(1-\alpha)) dy \\ &\quad - q \frac{w_\alpha^{(q)}(x, 0) \int_\gamma^\infty e^{-\Phi_1(q)y} W_0^{(q)}(\alpha y + \gamma(1-\alpha)) dy}{\int_\gamma^\infty e^{-\Phi_1(q)y} W_0^{(q)'}(\alpha y + \gamma(1-\alpha)) dy} \end{aligned} \quad (4.34)$$

(ii) For $x < \gamma$,

$$\mathbb{E}_x[e^{-q\kappa_0^-} 1_{\kappa_0^- < \infty}] = 1 - q \frac{W_0^{(q)}(x) \int_\gamma^\infty e^{-\Phi_1(q)y} W_0^{(q)}(\alpha y + \gamma(1-\alpha)) dy}{A} + q \int_{z=0}^\gamma W_0^{(q)}(x-z) dz \quad (4.35)$$

Proof. Let $\mathcal{E}(q)$ be an Exponentially distributed random variable with parameter q .

$$\begin{aligned} &\mathbb{E}_x[e^{-q\kappa_0^-} 1_{\kappa_0^- < \infty}] = \mathbb{P}_x[\mathcal{E}(q) > \kappa_0^-] = 1 - \mathbb{P}_x[U_s > 0, s < \mathcal{E}(q)] \\ &= 1 - q \int_0^\infty e^{-qt} 1_{(U_s \in (0, \infty), 0 < s < t)} dt = 1 - q \int_{y=0}^\infty V^{(q)}(x, \gamma, dy) \end{aligned} \quad (4.36)$$

In the last equality of (4.36) we applied (4.24). To find the last integral we have to integrate (4.33) between 0 and ∞ . Notice that

$$\begin{aligned} &\int_{y=0}^\gamma \int_{z=\gamma}^\infty e^{-\Phi_1(q)z} W_0^{(q)'}(\alpha z - y + \gamma(1-\alpha)) dz dy \\ &= \int_{z=\gamma}^\infty e^{-\Phi_1(q)z} \int_{y=0}^\gamma W_0^{(q)'}(\alpha z - y + \gamma(1-\alpha)) dy dz \\ &= \int_{z=\gamma}^\infty e^{-\Phi_1(q)z} W_0^{(q)}(\alpha z + \gamma(1-\alpha)) dz - \int_{z=\gamma}^\infty e^{-\Phi_1(q)z} W_0^{(q)}(\alpha(z-\gamma)) dz \end{aligned} \quad (4.37)$$

and

$$\begin{aligned}
& \int_{z=\gamma}^{\infty} e^{-\Phi_1(q)z} W_0^{(q)}(\alpha(z-\gamma)) dz \\
&= \frac{e^{-\Phi_1(q)\gamma}}{\alpha} \int_{z=0}^{\infty} e^{-\frac{\Phi_1(q)}{\alpha}z} W_0^{(q)}(z) dz \\
&= \frac{e^{-\Phi_1(q)\gamma}}{\alpha(\psi_0(\Phi_1(q)/\alpha) - q)} = \frac{e^{-\Phi_1(q)\gamma}}{c(1-\alpha)\Phi_1(q)} \tag{4.38}
\end{aligned}$$

We applied (4.2) in the last equality. Thus, by (4.33), (4.37) and (4.38),

$$\begin{aligned}
& 1 - q \int_{y=0}^{\infty} V^{(q)}(x, \gamma, dy) \\
&= 1 - q \left[w_{\alpha}^{(q)}(x, 0) \frac{e^{-\Phi_1(q)\gamma}}{\Phi_1(q)c(1-\alpha)A} - \int_{\gamma}^{\infty} W_1^{(q)}(x-z) dz + \frac{w_{\alpha}^{(q)}(x, 0) \int_{\gamma}^{\infty} e^{-\Phi_1(q)y} W_0^{(q)}(\alpha y + \gamma(1-\alpha)) dy}{A} \right. \\
&\quad \left. - \frac{w_{\alpha}^{(q)}(x, 0) e^{-\Phi_1(q)\gamma}}{\Phi_1(q)Ac(1-\alpha)} - \int_{z=0}^{\gamma} w_{\alpha}^{(q)}(x, z) dz \right] \tag{4.39}
\end{aligned}$$

(i) Let $x \geq \gamma$. Substituting (4.9) for $w_{\alpha}^{(q)}(x, z)$ in (4.39) and adding and subtracting $\int_0^{\alpha x + \gamma(1-\alpha)} W_0^{(q)}(y) dy$

(to obtain a simpler expression for (4.39)) we obtain that:

$$\begin{aligned}
& \mathbb{E}_x[e^{-q\kappa_0^-} 1_{\kappa_0^- < \infty}] \\
&= 1 - q \left[- \int_{\gamma}^{\infty} W_1^{(q)}(x - z) dz + \frac{w_{\alpha}^{(q)}(x, 0) \int_{\gamma}^{\infty} e^{-\Phi_1(q)y} W_0^{(q)}(\alpha y + \gamma(1 - \alpha)) dy}{A} \right. \\
&\quad - \int_{z=0}^{\gamma} W_0^{(q)}(\alpha x - z + \gamma(1 - \alpha)) dz \\
&\quad \left. - c(1 - \alpha) \int_{y=\gamma}^x W_1^{(q)}(x - y) \int_{z=0}^{\gamma} W_0^{(q)'}(\alpha y - z + \gamma(1 - \alpha)) dz dy \right] \\
&= 1 - q \left[- \int_0^{\alpha x + \gamma(1 - \alpha)} W_0^{(q)}(y) dy - \int_{\gamma}^{\infty} W_1^{(q)}(x - z) dz \right. \\
&\quad + \frac{w_{\alpha}^{(q)}(x, 0) \int_{\gamma}^{\infty} e^{-\Phi_1(q)y} W_0^{(q)}(\alpha y + \gamma(1 - \alpha)) dy}{A} \\
&\quad + \int_0^{\alpha x + \gamma(1 - \alpha)} W_0^{(q)}(y) dy - \int_{z=0}^{\gamma} W_0^{(q)}(\alpha x - z + \gamma(1 - \alpha)) dz \\
&\quad \left. - c(1 - \alpha) \int_{y=\gamma}^x W_1^{(q)}(x - y) \int_{z=0}^{\gamma} W_0^{(q)'}(\alpha y - z + \gamma(1 - \alpha)) dz dy \right] \\
&= 1 - q \left[- \int_0^{\alpha x + \gamma(1 - \alpha)} W_0^{(q)}(y) dy - \int_{\gamma}^{\infty} W_1^{(q)}(x - z) dz \right. \\
&\quad + \frac{w_{\alpha}^{(q)}(x, 0) \int_{\gamma}^{\infty} e^{-\Phi_1(q)y} W_0^{(q)}(\alpha y + \gamma(1 - \alpha)) dy}{A} \\
&\quad + \int_0^{\alpha x + \gamma(1 - \alpha)} W_0^{(q)}(y) dy - \int_{z=0}^{\gamma} W_0^{(q)}(\alpha x - z + \gamma(1 - \alpha)) dz \\
&\quad - c(1 - \alpha) \int_{y=\gamma}^x W_1^{(q)}(x - y) W_0^{(q)}(\alpha y + \gamma(1 - \alpha)) dy \\
&\quad \left. + c(1 - \alpha) \int_{y=\gamma}^x W_1^{(q)}(x - y) W_0^{(q)}(\alpha(y - \gamma)) dy \right]
\end{aligned}$$

Thus, for $x > \gamma$,

$$\begin{aligned}
\mathbb{E}_x[e^{-q\kappa_0^-} 1_{\kappa_0^- < \infty}] &= 1 + q \int_0^{\alpha x + \gamma(1 - \alpha)} W_0^{(q)}(y) dy - q \frac{w_{\alpha}^{(q)}(x, 0) \int_{\gamma}^{\infty} e^{-\Phi_1(q)y} W_0^{(q)}(\alpha y + \gamma(1 - \alpha)) dy}{A} \\
&+ qc(1 - \alpha) \int_{y=\gamma}^x W_1^{(q)}(x - y) W_0^{(q)}(\alpha y + \gamma(1 - \alpha)) dy + q\mathcal{C}(x), \tag{4.40}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{C}(x) &= \int_{\gamma}^{\infty} W_1^{(q)}(x - z) dz - \int_0^{\alpha x + \gamma(1 - \alpha)} W_0^{(q)}(y) dy \\
&+ \int_{z=0}^{\gamma} W_0^{(q)}(\alpha x - z + \gamma(1 - \alpha)) dz - c(1 - \alpha) \int_{y=\gamma}^x W_1^{(q)}(x - y) W_0^{(q)}(\alpha(y - \gamma)) dy
\end{aligned}$$

Notice that

$$\begin{aligned}\int_{\gamma}^{\infty} W_1^{(q)}(x-z)dz &= \int_0^{x-\gamma} W_1^{(q)}(y)dy, \\ \int_{z=0}^{\gamma} W_0^{(q)}(\alpha x - z + \gamma(1-\alpha))dz &= \int_{y=\alpha(x-\gamma)}^{\alpha x + \gamma(1-\alpha)} W_0^{(q)}(y)dy, \\ \int_{y=\gamma}^x W_1^{(q)}(x-y)W_0^{(q)}(\alpha(y-\gamma))dy &= \int_{y=0}^{x-\gamma} W_1^{(q)}(x-\gamma-y)W_0^{(q)}(\alpha y)dy.\end{aligned}$$

We thus obtain a simpler expression for $\mathcal{C}(x)$:

$$\mathcal{C}(x) = \int_0^{x-\gamma} W_1^{(q)}(y)dy - \int_0^{\alpha(x-\gamma)} W_0^{(q)}(y)dy - c(1-\alpha) \int_{y=0}^{x-\gamma} W_1^{(q)}(x-\gamma-y)W_0^{(q)}(\alpha y)dy$$

Thus $\mathcal{C}(x) = g(x-\gamma)$, where

$$g(x) = \int_0^x W_1^{(q)}(y)dy - \int_0^{\alpha x} W_0^{(q)}(y)dy - c(1-\alpha) \int_{y=0}^x W_1^{(q)}(x-y)W_0^{(q)}(\alpha y)dy$$

Taking Laplace transform $\int_0^{\infty} e^{-sx} g(x)dx$ we obtain that

$$\int_0^{\infty} e^{-sx} g(x)dx = \frac{1}{s(\psi_1(s) - q)} - \frac{1}{s(\psi_0(s/\alpha) - q)} - \frac{\frac{1}{\alpha}c(1-\alpha)}{(\psi_0(s/\alpha) - q)(\psi_1(s) - q)}$$

Applying (4.2) we conclude that the last expression equals 0. Thus $g(x) = 0$ and (4.34) is obtained from (4.40).

(ii) For $x < \gamma$ we obtain from (4.39) or directly from (4.33) that

$$\mathbb{E}_x[e^{-q\kappa_0^-} 1_{\kappa_0^- < \infty}] = 1 - q \frac{W_0^{(q)}(x) \int_{\gamma}^{\infty} e^{-\Phi_1(q)y} W_0^{(q)}(\alpha y + \gamma(1-\alpha))dy}{A} + q \int_{z=0}^{\gamma} W_0^{(q)}(x-z)dz \quad (4.41)$$

□

4.6 Ruin probability

Proposition 4.6. (i) For $x > \gamma$, the ruin probability is 1 when $\psi_1'(0) < 0$, and when $\psi_1'(0) \geq 0$ it is given by:

$$\begin{aligned}\mathbb{P}_x(\kappa_0^- < \infty) &= \lim_{q \rightarrow 0} \mathbb{E}_x[e^{-q\kappa_0^-} 1_{\kappa_0^- < \infty}] \\ &= 1 - \left(W_0(\alpha x + \gamma(1-\alpha)) + c(1-\alpha) \int_{\gamma}^x W_1(x-y)W_0'(\alpha y + \gamma(1-\alpha))dy \right) \frac{\alpha\psi_1'(0)}{1 - c(1-\alpha)W_0(\gamma)}\end{aligned} \quad (4.42)$$

(ii) For $x < \gamma$ the ruin probability is 1 when $\psi_1'(0) < 0$, and when $\psi_1'(0) \geq 0$ it is given by:

$$\mathbb{P}_x(\kappa_0^- < \infty) = 1 - W_0(x) \frac{\alpha\psi_1'(0)}{1 - c(1-\alpha)W_0(\gamma)} \quad (4.43)$$

Proof. (i) To obtain the ruin probability for $x \geq \gamma$, we take the limit of (4.34) as $q \rightarrow 0$. The limit of the last two terms in the first line of (4.34) is 0.

To calculate the limit of the last expression in (4.34) notice that:

$$\begin{aligned}
& q \frac{w_\alpha^{(q)}(x, 0) \int_\gamma^\infty e^{-\Phi_1(q)y} W_0^{(q)}(\alpha y + \gamma(1-\alpha)) dy}{\int_\gamma^\infty e^{-\Phi_1(q)y} W_0^{(q)'}(\alpha y + \gamma(1-\alpha)) dy} \\
&= q w_\alpha^{(q)}(x, 0) \frac{1}{\alpha} e^{\Phi_1(q)\gamma(1-\alpha)/\alpha} \frac{\int_0^\infty e^{-\Phi_1(q)y/\alpha} W_0^{(q)}(y) dy - \int_0^\gamma e^{-\Phi_1(q)y/\alpha} W_0^{(q)}(y) dy}{\int_\gamma^\infty e^{-\Phi_1(q)y} W_0^{(q)'}(\alpha y + \gamma(1-\alpha)) dy} \\
&= w_\alpha^{(q)}(x, 0) \frac{1}{\alpha} e^{\Phi_1(q)\gamma(1-\alpha)/\alpha} \frac{\frac{q}{\psi_0(\Phi_1(q)/\alpha) - q} - q \int_0^\gamma e^{-\Phi_1(q)y/\alpha} W_0^{(q)}(y) dy}{\int_\gamma^\infty e^{-\Phi_1(q)y} W_0^{(q)'}(\alpha y + \gamma(1-\alpha)) dy} \\
&= w_\alpha^{(q)}(x, 0) \frac{1}{\alpha} e^{\Phi_1(q)\gamma(1-\alpha)/\alpha} \frac{\frac{q}{\delta\Phi_1(q)} - q \int_0^\gamma e^{-\Phi_1(q)y/\alpha} W_0^{(q)}(y) dy}{\int_\gamma^\infty e^{-\Phi_1(q)y} W_0^{(q)'}(\alpha y + \gamma(1-\alpha)) dy}, \tag{4.44}
\end{aligned}$$

where we applied (4.2) in the last line of (4.44).

If $\psi_1'(0) < 0$ then $\Phi_1(0) > 0$, the limit of the last expression is 0 and thus the ruin probability is 1. If $\psi_1'(0) \geq 0$ then $\Phi_1(0) = 0$. In this case,

$$\lim_{q \rightarrow 0} \frac{q}{\Phi_1(q)} = \psi_1'(0) = \frac{\psi_0'(0)}{\alpha} - \frac{1}{\alpha} c(1-\alpha). \tag{4.45}$$

Next, we obtain a simpler expression for the denominator in (4.44). By integration by parts of the denominator, we obtain

$$\begin{aligned}
A &= \int_\gamma^\infty e^{-\Phi_1(q)y} W_0^{(q)'}(\alpha y + \gamma(1-\alpha)) dy = \frac{1}{\alpha} e^{\Phi_1(q)\gamma(1-\alpha)/\alpha} \int_\gamma^\infty e^{-\frac{\Phi_1(q)}{\alpha}y} W_0^{(q)'}(y) dy \\
&= \frac{1}{\alpha} e^{\Phi_1(q)\gamma(1-\alpha)/\alpha} \left(-W_0^{(q)}(\gamma) e^{-\frac{\Phi_1(q)}{\alpha}\gamma} \right. \\
&\quad \left. + \frac{\Phi_1(q)}{\alpha} \int_0^\infty e^{-\frac{\Phi_1(q)}{\alpha}y} W_0^{(q)}(y) dy - \frac{\Phi_1(q)}{\alpha} \int_0^\gamma e^{-\frac{\Phi_1(q)}{\alpha}y} W_0^{(q)}(y) dy \right) \\
&= \frac{1}{\alpha} e^{\Phi_1(q)\gamma(1-\alpha)/\alpha} \left(-W_0^{(q)}(\gamma) e^{-\frac{\Phi_1(q)}{\alpha}\gamma} \right. \\
&\quad \left. + \frac{\Phi_1(q)}{\alpha} \frac{1}{\frac{1}{\alpha} c(1-\alpha) \Phi_1(q)} - \frac{\Phi_1(q)}{\alpha} \int_0^\gamma e^{-\frac{\Phi_1(q)}{\alpha}y} W_0^{(q)}(y) dy \right) \tag{4.46}
\end{aligned}$$

We applied (2.1) and (4.2) in the last equality. The limit of the denominator of (4.44) as $q \rightarrow 0$ is

$$\frac{1}{\alpha} (-W_0(\gamma) + \frac{1}{c(1-\alpha)}) \tag{4.47}$$

Thus when $\Phi_1(0) = 0$, substitute (4.9) in (4.44), and applying (4.9) and (4.47) we obtain that the ruin probability for $x > \gamma$ is as in (4.42).

- (ii) For $x < \gamma$ taking the limit of (4.35) as $q \rightarrow 0$ we obtain that also in this case the ruin probability is 1 when $\psi_1'(0) < 0$ and it is given by (4.43) otherwise.

□

4.7 Gerber-Shiu penalty function

Let $-U_{\kappa_0^-}$ be the deficit at ruin and $U_{\kappa_0^-}$ the surplus just before ruin. We want to obtain

$$m(x, q) = \mathbb{E}_x[e^{-q\kappa_0^-} h(U_{\kappa_0^-}, |U_{\kappa_0^-}| | U_0 = x),$$

where $h(x, y)$ —the penalty function is a nonnegative function.

Proposition 4.7.

$$\begin{aligned} m(x, q) &= \int_{y=0}^{\gamma} V^{(q)}(x, \gamma, dy) \int_{z=0}^{\infty} h(y, z) \lambda dF_0(y+z) dy \\ &+ \int_{y=\gamma}^{\infty} V^{(q)}(x, \gamma, dy) \int_{z=0}^{\infty} h(y, z) \lambda dF_1(y-\gamma + \frac{\gamma+z}{\alpha}) dy \\ &= \int_{y=0}^{\gamma} \left(\frac{w_{\alpha}^{(q)}(x, 0)}{A} \int_{s=\gamma}^{\infty} e^{-\Phi_1(q)s} W_0^{(q)'}(\alpha s - y + \gamma(1-\alpha)) ds - w_{\alpha}^{(q)}(x, y) \right) \int_{z=0}^{\infty} h(y, z) \lambda dF_0(y+z) dy \\ &+ \int_{\gamma}^{\infty} \left(\frac{w_{\alpha}^{(q)}(x, 0)}{c(1-\alpha)A} e^{-\Phi_1(q)y} - W_1^{(q)}(x-y) \right) \int_{z=0}^{\infty} \lambda dF_1(y-\gamma + \frac{\gamma+z}{\alpha}) h(y, z) dy \end{aligned} \quad (4.48)$$

Proof. The proof is the same as the proof of proposition 3.3, and by substituting (4.33) for $V^{(q)}(x, \gamma, dy)$. $\square$

An application of the Gerber-Shiu penalty function is the derivation of the joint distribution of the reserve just before ruin and the deficit at ruin when $\psi_1'(0) \geq 0$, i.e. when $\Phi_1(0) = 0$.

Let $C, D \subset (0, \infty)$, and let $h^*(y, z) = 1_{(y \in C, z \in D)}$

In order to obtain $\mathbb{P}_x(U_{\kappa_0^-} \in C, |U_{\kappa_0^-}| \in D)$, we substitute h^* in (4.48) and take the limit as $q \rightarrow 0$.

By (4.46), the $\lim_{q \rightarrow 0} A$ is given by (4.47). Similarly

$$\lim_{q \rightarrow 0} \int_{s=\gamma}^{\infty} e^{-\Phi_1(q)s} W_0^{(q)'}(\alpha s - y + \gamma(1-\alpha)) ds = \frac{1 - c(1-\alpha)W_0(\gamma - y)}{\alpha c(1-\alpha)}$$

Denote by $w_{\alpha}^{(0)}(x, y) = w_{\alpha}(x, y)$. Since $\Phi_1(0) = 0$, substituting h^* in (4.48) and taking the limit as $q \rightarrow 0$ we obtain

$$\begin{aligned} \mathbb{P}_x(U_{\kappa_0^-} \in C, |U_{\kappa_0^-}| \in D) &= \int_{y \in C \cap (\gamma, \infty)} \left(\frac{\alpha w_{\alpha}(x, 0)}{1 - c(1-\alpha)W_0(\gamma)} - W_1(x-y) \right) \Pi_1(y - \gamma + \frac{\gamma+D}{\alpha}) dy \\ &+ \int_{y \in B \cap (0, \gamma)} \left(w_{\alpha}(x, 0) \frac{1 - c(1-\alpha)W_0(\gamma - y)}{1 - c(1-\alpha)W_0(\gamma)} - w_{\alpha}(x, y) \right) \Pi_0(y + D) dy \end{aligned}$$

where $\Pi_i(B)$ is the Lévy measure of the set B , in the compound Poisson case $\Pi_i(B) = \lambda F_i(B)$ and $F_i(B) = \mathbb{P}(Z^i \in B)$ and Z^i has distribution F_i , $i = 0, 1$.

5 Conclusions

In this paper we studied a compound Poisson risk process where the claims are "refracted" – i.e. only a part of the claim is paid when the reserve is under γ . We obtained expressions for the Laplace transform of the exit time from an upper barrier, the time to ruin and the joint probability for the surplus before and at ruin, for a general function $I(x)$ as defined in the Introduction. We obtained relatively simple expressions for the special but important case that $I(x) = \alpha x$. In this case the results are in the same flavor as for spectrally negative refracted Lévy processes, where the premium income rate when the reserve is above γ is αc , as was studied by Kyprianou and Loeffen (2010).

We analyzed the model for the compound Poisson risk process but the same analysis goes through for the more general spectrally negative bounded variation Lévy risk process. It would be worthwhile to consider more general reinsurance policies, for example $I(x) = \min(a, x)$ where a is a positive constant.

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