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Hybrid Fourier pseudospectral / discontinuous Galerkin time-domain method for wave propagation [☆]

Raúl Pagán Muñoz^{a,*}, Maarten Hornikx^a

^a*Building Physics and Services, Eindhoven University of Technology, P.O.Box 513, 5600
MB, Eindhoven, the Netherlands*

Abstract

The Fourier Pseudospectral time-domain (Fourier PSTD) method was shown to be an efficient way of modelling acoustic propagation problems as described by the linearized Euler equations (LEE), but is limited to real-valued frequency independent boundary conditions and predominantly staircase-like boundary shapes. This paper presents a hybrid approach to solve the LEE, coupling Fourier PSTD with a nodal Discontinuous Galerkin (DG) method. DG exhibits almost no restrictions with respect to geometrical complexity or boundary conditions. The aim of this novel method is to allow the computation of complex geometries and to be a step towards the implementation of frequency dependent boundary conditions by using the benefits of DG at the boundaries, while keeping the efficient Fourier PSTD in the bulk of the domain. The hybridization approach is based on conformal meshes to avoid spatial interpolation of the DG solutions when transferring values from DG to Fourier PSTD, while the data transfer from Fourier PSTD to DG is done utilizing spectral interpolation of the Fourier PSTD solutions. The accuracy of the hybrid approach is presented for one- and two-dimensional acoustic problems and the main sources of error are investigated. It is concluded that the hybrid methodology does not intro-

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*Corresponding author

Email addresses: `r.pagan.munoz@tue.nl` (Raúl Pagán Muñoz), `m.c.j.hornikx@tue.nl` (Maarten Hornikx)

duce significant errors compared to the Fourier PSTD stand-alone solver. An example of a cylinder scattering problem is presented and accurate results have been obtained when using the proposed approach. Finally, no instabilities were found during long-time calculation using the current hybrid methodology on a two-dimensional domain.

Keywords: Hybrid time-domain method, Linearized Euler equations, Fourier pseudospectral, Discontinuous Galerkin, Acoustic propagation problems, Complex geometries

1. Introduction

The benefits of using high order methods when solving time dependent wave propagation problems have been identified, for instance, by Hesthaven et al. [1]. Among high order methods, Fourier pseudospectral techniques have shown
5 to be an effective way of modelling wave propagation [2] and, particularly, the Fourier pseudospectral time-domain (Fourier PSTD) technique has shown to be suitable for acoustic applications [3]. However, although developments have been presented by Hornikx et al. to locally refine the grid using multidomain implementations [4] and to apply Fourier PSTD to orthogonal curvilinear coordinates for near-rigid moderately curved surfaces [5], the method is limited to
10 predominantly staircase-like boundary shapes. Regarding boundary conditions, non-reflecting terminations can be solved in Fourier PSTD, e.g. by using perfectly match layers (PML) [6]. In case of rigid boundaries and boundary media with a different density, solutions have been successfully presented, for instance
15 by Hornikx et al. [2], and an approximation for impedance boundary conditions was introduced by Spa et al. [7]. However, no accurate solution for frequency dependent boundary conditions has been presented thus far [8].

The present paper introduces a hybrid approach to solve the linearized Euler equations (LEE) to handle arbitrary boundary shapes. The idea of spatially
20 coupling numerical methodologies in order to get the benefits of each solver has already been presented by many authors [9], [10], [11], [12], [13]. In particular, Platte and Gelb proposed a hybrid method for the solution of partial differential equations for non-periodic problems, combining Fourier and Chebyshev spectral methods [14], but the problem of handling complex geometries
25 remained unsolved. Attempts to combine structured and unstructured mesh types in order to tackle problems with complex boundary shapes have been considered before, e.g. [9] [11] [15] [16], but none of the approaches made use of the efficient Fourier PSTD for propagation in the main regions.

The aim of the novel hybrid methodology presented in this work is to allow
30 the computation of arbitrary boundary shapes by using the benefits of the nodal

Discontinuous Galerkin (DG) method at the boundaries while keeping Fourier PSTD in the bulk of the domain. DG exhibits almost no restrictions with respect to geometrical complexity or boundary conditions, and has been successfully implemented for acoustic applications, e.g. [17] and [18]. The method
35 allows to locally refine the polynomial order (p_o) and/or the element size (h) and it is well suited for parallel computing. Overall, DG is a suitable option to complement Fourier PSTD. However, in contrast with the spectral accuracy of Fourier PSTD, the DG error typically converges with order $h^{(p_o+1)}$ given that more than π points per wavelength are present [19], [20]. Moreover, polyno-
40 mial based methods as DG lead to small elements to resolve geometrical details of boundaries what imposing restrictions to the time step. Due to its higher demand on the spatial discretization and time step, DG is only used for the computation of the regions near the boundaries in the hybrid approach. In particular, this approach would be beneficial for atmospheric sound propagation
45 in urban scenarios, with DG near the ground surface and Fourier PSTD in the (moving and inhomogeneous) propagation domain above it. The present work is framed as part of the further developments of the open-source implementation of the Fourier PSTD method for acoustic propagation openPSTD [21]. In that line, directivity has been incorporated in Fourier PSTD using spherical harmon-
50 ics by Georgiou et al. [22] and the present work aims to be a step towards the implementation of arbitrarily shaped frequency dependent boundaries.

The accuracy of the hybrid method for one- and two-dimensional cases is investigated and reported in this paper for different parameters of influence. This paper is structured as follows. Section 2 presents the main features of
55 the physical equations and the stand-alone numerical methods. In section 3, the hybridization process is detailed. The main sources of error of the hybrid method are investigated in section 4, where results from the hybrid method are compared with the results from a Fourier PSTD stand-alone solver and analytical solutions. Section 4 also includes a test for stability of the method.
60 Conclusions are drawn in section 5.

2. Physical and numerical methods

2.1. The linearized Euler equations

Acoustic propagation investigated in this paper is governed by the LEE. To simplify the model, in this work the propagation medium is at rest and its temperature is constant in space and time. Therefore, the LEE are reduced
65 to the coupled linear acoustic equations 1 where, the acoustic variables are the velocity components $\mathbf{u}$ and the pressure p while, the properties of the medium are defined by the density ρ_0 and the adiabatic speed of sound c_0 .

$$\begin{aligned}\rho_0 \frac{\partial \mathbf{u}}{\partial t} &= -\nabla p, \\ \frac{1}{\rho_0 c_0^2} \frac{\partial p}{\partial t} &= -\nabla \cdot \mathbf{u}.\end{aligned}\tag{1}$$

2.2. Fourier PSTD method

70 Fourier PSTD discretizes the computational domain on an orthogonal equidistant mesh with a grid spacing Δx_{PS} determined by the smallest acoustic wavelength of interest. The spatial derivatives are computed separately on a one-dimensional basis in the wavenumber domain using the Fourier pseudospectral method [2]. For instance, equation 2 is used to compute the spatial derivatives
75 in x-direction for the pressure and the horizontal velocity component in a one-dimensional domain $\Omega_1 = \{x_l \in [0, L_1]\}$ where, $x_l = l\Delta x_{PS}$ and $L_1 = (N_{PS} - 1)\Delta x_{PS}$, with N_{PS} the total number of grid points. The transformation of the discrete acoustic variables is done by using Fourier analysis and synthesis, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the forward and inverse discrete Fourier trans-
80 form. The derivatives are calculated by multiplying the transformed discrete variables by the derivative operator $ik_{x,n}$, with i the imaginary number and $k_{x,n}$ the x-wavenumber vector defined in equation 3. Sampling the wavenumber domain with Δk_x leads to periodicity of the acoustic variables in the spatial domain, and sampling the spatial domain leads to the requirement of two spatial
85 points to solve the minimum wavelength. Derivatives in the y- and z-direction

can be found similarly.

$$\begin{aligned} \left. \frac{\partial p_1}{\partial x} \right|_{l\Delta x_{PS}} &= \mathcal{F}^{-1}(ik_{x,n}\mathcal{F}(p_1)), \quad 0 \leq l \leq N_{PS} - 1, \\ \left. \frac{\partial u_{x,1}}{\partial x} \right|_{l\Delta x_{PS}} &= \mathcal{F}^{-1}(ik_{x,n}\mathcal{F}(u_{x,1})), \quad 0 \leq l \leq N_{PS} - 1, \end{aligned} \quad (2)$$

$$k_{x,n} = \frac{2\pi n}{N_{PS}\Delta x_{PS}}, \quad n \in \left[-\frac{N_{PS}}{2} + 1, -\frac{N_{PS}}{2} + 2, \dots, \frac{N_{PS}}{2}\right]. \quad (3)$$

Additionally, to solve a time-domain problem as defined by the LEE, the pseudospectral method needs to be complemented with a time marching scheme. In this work, unless otherwise indicated, the time marching for the Fourier PSTD method is computed using the optimized low-storage six-stage Runge-Kutta scheme presented by Bogey and Bailly [23], referred to as RKo6s, as used in previous Fourier PSTD applications, for instance in [2] and [4]. Runge-Kutta methods compute a single time step by intermediate steps to obtain a higher order of accuracy. The stability condition is established by the Courant number ν_{PS} and the Fourier PSTD time step is computed as $\Delta t_{PS} = \nu_{PS}\Delta x_{PS}/c_0$.

2.3. Nodal DG time-domain method

The nodal DG method is used for solving the simplified conservative form of the LEE as presented in equation 1. Discontinuous Galerkin methods are based on local polynomial approximations of degree p_o . The computational domain Ω_{DG} is divided into a subset of T non-overlapping conforming elements. The method approximates the variables by a linear combination of locally defined base functions ϕ in each element and uses the so-called numerical flux at the element interfaces ∂T . Additionally, the numerical flux allows to prescribe physical boundary conditions where ∂T is a boundary of Ω_{DG} . The number of base functions N_p is determined by the selected polynomial order p_o and the dimensionality of the problem d . For simplex elements of the type segments, quadrilaterals or hexahedra, as used in this work, the number of local base functions, equivalent to the number of nodes in each element, can be computed as $N_p = (p_o + 1)^d$. The DG method requires for each element, $T \in \Omega_{DG}$, that

110 the residual of equation 1 is orthogonal to all test functions $\phi \in \mathcal{P}_{p_o}$ where, $\mathcal{P}_{p_o} = \{\phi_k, k = 1, 2, \dots, N_p\}$ is the space of polynomial functions of degree at most p_o . The approximate DG solution is expressed using a nodal interpolating approach

$$u_h^T(\mathbf{x}, t) = \sum_{l=1}^{N_p} u_h^T(\mathbf{x}_l^T, t) \varphi_l^T(\mathbf{x}), \quad (4)$$

with the basis $\mathcal{Q}_{p_o}^T = \{\varphi_l^T, l = 1, 2, \dots, N_p\}$, $u_h^T(\mathbf{x}, t)$ represents any of the un-
 115 knowns on element T : the acoustic variables $\mathbf{q}_h$, $\mathbf{f}_h = \mathbf{A}_j(\mathbf{q}_h)$ or the numerical flux $\hat{\mathbf{f}}_h$ with $\mathbf{A}_j$ the Jacobian flux of the LEE. The spatial discretization scheme of DG is derived for each element, multiplying equations 1, by test function ϕ and integrating it over the element T . Finally, integration by parts twice leads to the strong formulation of DG

$$\int_T \left[\frac{\partial \mathbf{q}_h}{\partial t} + \nabla \cdot \mathbf{f}_h \right] \phi(\mathbf{x}) d\mathbf{x} - \int_{\partial T} \mathbf{n} \cdot [\mathbf{f}_h - \hat{\mathbf{f}}_h] \phi(\mathbf{x}) d\mathbf{x} = 0, \quad (5)$$

120 with $\mathbf{n}$ the outgoing normal of ∂T . In this work, the numerical DG methodology is based on the quadrature-free approach introduced by Atkins and Shu [24] and following the algorithms for the nodal approach presented by Hesthaven and Warburton [19]. In our implementation, the choice of the location of the interpolation nodes follows the indications of section 3.1 in [19].

125 For acoustics applications, various types of fluxes have been proposed in literature. In this work, the commonly used numerical upwind flux has been used because of its optimal dissipation properties. The derivation of the upwind flux can be found in [19].

As for the pseudospectral method, a time integration scheme is needed in
 130 combination with the Discontinuous Galerkin methodology to compute the time derivatives of the LEE. Again, an explicit Runge Kutta method has been selected and the acoustic problem is discretized in time using the optimized forth-order eight-stage Runge Kutta scheme, referred to as RKF84 as derived by Toulorge and Desmet [25]. In that paper, the stability conditions for unstructured triangular grids are reported. The same conditions are used in this work for the
 135

calculation of the maximum allowed time step in DG. The calculation of Δt_{DG} for the hybrid methodology is described in section 3.2.

3. Hybrid methodology

There is a number of alternatives on how to spatially couple two numerical
140 methods. One approach is to use a non-overlapping domain decomposition,
coupling the domains at the interface, e.g. [16] [26]. The second methodology is
to couple them in a region where the methods overlap, for instance [11] [14]. The
second approach has been followed in this work, mainly because the coupling
algorithm needs to include a Gaussian window function to impose periodicity
145 of the Fourier PSTD solution. The methodology is summarized in figure 2.
Basically, the results of both solvers is equal in the overlapping area. In one part
of the overlapping area the values of the Fourier PSTD solutions are transferred
to DG, while a second part is where the values of DG are copied to Fourier
PSTD. In the later part, the Gaussian window is applied, i.e. the window size
150 is coincident with the size of the copy zone from DG to Fourier PSTD. The
approach is based on conformal meshes to avoid spatial interpolation of the DG
solutions, however the interpolation of the Fourier PSTD solutions is needed
before copying values to the DG solver. Since the solutions are updated over
a zone rather than at a single point, the method uses local time stepping and
155 the data update is only done after the larger time step as indicated in the time
diagram of figure 3.

In the next sections, the hybrid method is described in detail. Specifically,
the spatial and time discretization processes are presented together with the
data processing and exchange between the numerical solvers. The data process-
160 ing includes the spectral interpolation of the Fourier PSTD solutions and the
windowing of the Fourier PSTD variables. Finally, a description of the filtering
approach to minimize numerical instabilities is included.

3.1. Physical domain and spatial discretization

The hybridization process is developed for a non-staggered or collocated grid.

165 The physical domain $\Omega = \{x \in [x_{left}, x_{right}]\}$ is divided into two numerical subdomains with an overlapping area or copy zone, as shown in figure 1 for a one dimensional case, with $x_{left,DG} = x_{left}$ and $x_{right,PS} = x_{right}$: $\Omega_{PS} = \{x_{PS} \in [x_{left,PS}, x_{right,PS}]\}$, $\Omega_{DG} = \{x_{DG} \in [x_{left,DG}, x_{right,DG}]\}$ and $\Omega_{cz} = \{x_{cz} \in [x_{left,PS}, x_{right,DG}]\}$.

170 The grids are conformal in the coupling zone in order to avoid interpolation of the DG solutions. The spatial discretization is determined by the smallest wavelength or maximum frequency of interest. The maximum frequency in Fourier PSTD $f_{max,PS}$ is limited by the spatial Nyquist condition and is related to the grid spacing according to the following expression $f_{max,PS} = c_o/(2\Delta x_{PS})$. This condition is known as the two points per wavelength condition. The size of 175 the one-dimensional elements in DG (h) is equal to the element size in Fourier PSTD. Therefore, the two points per wavelength condition in Fourier PSTD corresponds with two elements per wavelength in DG. Following the indications by Chevaugeon et al. [27], typically, 2 or 3 elements per wavelength are sufficient for polynomial orders 4 or 5 in aero-acoustics applications. In this work, 180 $p_o = 4$ was the minimum polynomial order considered in the analysis of the hybrid method and the results in section 4 are according to Chevaugeon et al. indications. Moreover, as shown in figure 2 for a one-dimensional case, the data exchange area has T_{cz} elements and it is divided in one area where the values of the Fourier PSTD solutions are copied to DG and a second zone where the 185 DG values are transferred to Fourier PSTD. For higher dimensions, the DG grid is divided in a structured area, corresponding to the coupling zone, and an unstructured part that fits with the irregular boundary, as shown for instance in the grid detail in figure 10 for the scattering problem. All two-dimensional 190 problems presented in this work are built with quadrilateral elements but the extension to triangles is straightforward.

Figure 1: Sketch of a one-dimensional example of the hybrid mesh. The circles represent DG nodes and the crosses Fourier PSTD nodes (for clarity, the internal nodes of the DG elements are not represented in the figure).

Figure 2: Post-processing and data exchange in the coupling zone of the hybrid method (for DG polynomial order $p_o = 4$).

3.2. Time discretization

Both solvers can use local time stepping without transferring data at intermediate steps. In this paper, following the work by Bogey and Bailly [23], the accuracy constraints from the Runge-Kutta method RKo6s have been used as given by the Courant number $\nu_{PS} \leq 0.5$ for the Fourier PSTD calculations. For DG, ν_{DG} is initially taken from the optimal working conditions for a dissipation error $E_{mag} = 0.0001$ dB as defined by Toulorge et al. [25] for the RKF84 method. Courant numbers ν_{DG} are given as a function of the polynomial order of DG and are more restrictive than ν_{PS} when $p_o > 2$. The differences between Fourier PSTD and DG time steps are related to the Courant numbers and the

minimum spatial resolution.

In the hybrid process, the data are post-processed and exchanged after every Fourier PSTD time step. The final DG time step Δt_{DG} is calculated after computing the minimum integer number of DG time steps s_{hyb} that fits one Fourier PSTD time step, such that the conditions given in [25] are fulfilled. In this work, s_{hyb} is referred to as time step factor. This time process is schematically shown in Figure 3.

Figure 3: Time diagram of the hybridization process.

3.3. Data processing and exchange

The hybridization approach follows three main steps indicated in figure 2: Step 1) consists of performing spectral interpolation of the Fourier PSTD solutions in order to find the values at the DG internal nodes in the copy zone. In step 2), data are exchanged between solvers in the data-exchange areas. Finally, the acoustic Fourier PSTD variables are multiplied by a spatial Gaussian window in step 3) before computing the next time iteration.

3.3.1. Spectral interpolation

To obtain the values of the Fourier PSTD variables at the DG nodes positions, spectral interpolation in the x-direction of the Fourier PSTD solutions $\mathbf{q}$ of the whole domain is performed using

$$\mathbf{q}(x_l - \Delta x_{int,j}) = \mathcal{F}_x^{-1}[e^{ik_{x,n}\Delta x_{int,j}} \mathcal{F}_x[\mathbf{q}(x_l)]], \quad (6)$$

$$\Delta x_{int,j} = (r_{p,j} - 1) \frac{\Delta x_{PS}}{2}, \quad j = [2, \dots, Np - 1], \quad (7)$$

220 with, $\Delta x_{int,j}$ representing the distances between the Fourier PSTD nodes to the DG internal nodes calculated using equation 7, where $r_{p,j}$ contains the j coordinates of the DG nodes in the one-dimensional reference element, i.e. $r_{p,j} \in [-1, 1]$. The interpolated solutions $\mathbf{q}(x_l - \Delta x_{int,j})$ can be interpolated again in the y- and z-direction in a similar way for higher dimensional problems.

225 The interpolation is done by a transformation of the Fourier PSTD solutions to the wavenumber domain, where the transformed variables are multiplied with an exponential that shifts the location of the PSTD nodes to the location of the DG nodes. Since the grids of both solvers are conformal, the interpolation is only needed to the internal nodes of the DG elements and the distance from the

230 Fourier PSTD nodes to each internal node is constant for every element. Therefore, for simplex elements of the type segments, quadrilaterals or hexahedra, $N_p - 2^d$ interpolations of the Fourier PSTD solutions are needed.

3.3.2. Data exchange

Since the grids are conformal in the coupling zone, the values of the DG

235 solutions are already collocated at the Fourier PSTD nodes and, therefore, can directly be mapped to the Fourier PSTD solutions. In this work, the copied value from DG to each Fourier PSTD node is the average of the solutions of all conformal DG nodes, i.e. from the elements that share the same node. The other data transfer occurs from Fourier PSTD to DG, following the interpolation

240 according to the previous section.

3.3.3. Windowing to the Fourier PSTD solutions

As part of the coupling algorithm, the field Fourier PSTD variables are multiplied by a Gaussian window to obtain spatial periodicity and minimize the wrap-around effects due to the used discrete Fourier Transform for the spatial

245 derivative operator. The window is built using equation 8 and following the

indications from Hornikx et al. [4].

$$w(l_w, N_w) = e^{-\alpha_w \ln(10) \left(\frac{l_w - N_w}{N_w} \right)^{2\beta_w}}, \quad 0 \leq l_w \leq N_w - 1. \quad (8)$$

The single sided exponential part of the window has N_w number of nodes and is coincident with the coupling area from DG to Fourier PSTD. The coefficient α_w and β_w are selected from the work by Hornikx et al. [4]. The values of α_w are calculated as a function of the window size using $\alpha_w = (N_w + 3)/14$ for $N_\lambda = 2.5$ points per wavelength, while $\beta_w = 3$ in all cases.

3.4. Filtering

The coupling algorithm includes a low-pass filtering approach to minimize numerical instabilities arising from the Fourier PSTD solver, similar as reported by Hornikx et al. [3]. In this work, a low-pass Gaussian frequency filter is used to filter the high frequency components of the Fourier PSTD solutions after every time step. For two dimensional cases, the low-pass filter is built using equation

$$f_{n_x, n_y} = \begin{cases} e^{-\alpha_f \ln(10) \left(\frac{|n_x| \Delta k_x - k_c}{k_{\max} - k_c} \right)^{2\beta_f}} e^{-\alpha_f \ln(10) \left(\frac{|n_y| \Delta k_y - k_c}{k_{\max} - k_c} \right)^{2\beta_f}} & \text{for } k_c / \Delta k_x < |n_x|, \\ & |n_y| \leq k_{\max} / \Delta k_y; \\ e^{-\alpha_f \ln(10) \left(\frac{|n_x| \Delta k_x - k_c}{k_{\max} - k_c} \right)^{2\beta_f}} & \text{for } k_c / \Delta k_x < |n_x| \leq k_{\max} / \Delta k_x, \\ & |n_y| \leq k_c / \Delta k_y; \\ e^{-\alpha_f \ln(10) \left(\frac{|n_y| \Delta k_y - k_c}{k_{\max} - k_c} \right)^{2\beta_f}} & \text{for } k_c / \Delta k_y < |n_y| \leq k_{\max} / \Delta k_y, \\ & |n_x| \leq k_c / \Delta k_x; \\ 1 & \text{for } |n_x| \leq k_c / \Delta k_x, \\ & |n_y| \leq k_c / \Delta k_y, \end{cases} \quad (9)$$

with, k_c the cut-off wave number, $k_{\max} = \pi / \Delta x_{PS}$ and $[\Delta k_x, \Delta k_y]$ the resolution of the wave number vector for each dimension. After every time step, the spatial Fourier PSTD variables $\mathbf{q}_{PS}(\mathbf{x})$ are transformed to the wavenumber domain $\mathbf{Q}_{PS}(\mathbf{k})$ by using a forward Fourier transform. The transformed variables are filtered through multiplication by f_{n_x, n_y} and transformed back to the spatial

domain using the inverse Fourier transform before computing the next time
 265 interaction. The filter has zero dissipation up to the cut-off wave number and a
 Gaussian decay from that point. In all calculations presented in this work the
 following parameters of the Gaussian filter have been used: $k_c = 1/(2.3\Delta x_{PS})$,
 $\alpha_f = 4$ and $\beta_f = 4$. The consequence of the filter is that the minimum numbers
 of spatial points per wavelength increases, see also results of section 4.

270 3.5. DG boundary conditions in the coupling zone

The boundary condition imposed in DG at the end of the coupling zone
 is the characteristic non-reflective boundary condition, i.e. the non-reflective
 boundary condition for waves incident normal to the boundary. After a full
 time step in the Fourier PSTD method, the wave has traveled a half element
 275 size, while the values from Fourier PSTD are imposed in the whole overlapping
 DG element. Therefore, the boundary condition at this termination of DG is not
 influencing the results, i.e. it could be a rigid or impedance boundary condition
 as well.

4. Accuracy of the method

280 In this section, different sources of error of the hybrid methodology are
 discussed for a one-dimensional implementation. For the errors investigated
 in the subsequent sections, the other sources are kept constant. Additionally,
 this section includes the extension to two-dimensional cases for a scattering
 problem. The scattering problem is compared with analytical solutions in order
 285 to establish the accuracy of the two-dimensional implementation. For the speed
 of sound and medium density, $c_0 = 343 [m/s]$ and $\rho_0 = 1.2 [kg/m^3]$, have
 been used in all calculations. The features of all the models are summarized in
 AppendixA.

In the following subsections, the error will be represented or expressed as
 290 a function of points per wavelength (N_λ). In this work, N_λ is computed by
 only taking into account the spatial discretization of the Fourier PSTD domain

Δx_{PS} determined by the smallest acoustic wavelength of interest λ_{min} , i.e. $N_\lambda = \lambda_{min}/\Delta x_{PS}$. It is considered that the main application of the hybrid methodology is computing scenarios where the Fourier PSTD domain is much
295 bigger than the DG domain and, the latter, restricted to the complex boundaries as, for instance, for atmospheric sound propagation in urban scenarios. Clearly, the bigger the Fourier PSTD domain is, compared with the DG domain, the more similar the degrees of freedom of the hybrid method are to the points per wavelength of the Fourier PSTD domain.

300 4.1. Initial and boundary conditions

In this work, all the calculations are initiated with a broadband pressure distribution in the Fourier PSTD domain.

$$p(\mathbf{x}, t_0) = Ae^{-b_s(|\mathbf{x}-\mathbf{x}_s|)^2}, \quad (10)$$

where, A is the amplitude, b_s determines the bandwidth of the spectrum, $|\mathbf{x} - \mathbf{x}_s| = \sqrt{(x - x_s)^2 + (y - y_s)^2}$ and $\mathbf{x}_s = [x_s, y_s]$ is the source location. In
305 the domain, a zero-valued velocity distribution $u(\mathbf{x}, t_0) = 0$ is used for the two-dimensional cases and the velocity distribution $u(x, t_0) = -p(x, t_0)/\rho_0 c_0$ for the one-dimensional problems. The time-domain computation in DG is initialized with a zero-valued pressure and velocity distribution.

The computational domain needs to be truncated by an artificial boundary in
310 order to have a finite domain. For the experiments, two types of boundaries are implemented: rigid and non-reflecting surfaces. Acoustically rigid boundaries, i.e. boundaries where the acoustic waves are totally reflected, are modeled by imposing a zero velocity perturbation normal to the boundary and a zero-valued normal derivative of the pressure.

315 The second type of boundaries used in this work is a non-reflecting surface. The perfectly match layer (PML) technique and the characteristic non-reflective boundary, as described in section 3.5, are employed here in order to obtain a reflection-free implementation. The PML was first presented for electromagnetic problems by Berenger [28] and has been widely used in acoustic numerical

320 computations, for instance by Echevarria et al. [29], Hornikx et al. [30] and Liu [6]. The main drawback of the PML is the increase of the computational cost since the thickness of the layer influences the accuracy of the method [31]. The PML factors $\sigma = [\sigma_x, \sigma_y, \sigma_z]^T$ are computed using equation 11 for σ_x and can be calculated similarly for the y- and z- components.

$$\sigma_x = \beta \left(\frac{x - x_{PML}}{D} \right)^m, \quad x_{PML} \leq x \leq x_{PML} + D, \quad (11)$$

325 where, β is the maximum amplitude of the PML function, m an exponent and D the thickness of the layer.

4.2. Errors in one-dimensional implementations

In this section, the main sources of error of the hybrid methodology are investigated: section 4.2.1 deals with the error of the temporal scheme whereas, section 4.2.2 present the errors of the coupling between DG and Fourier PSTD.

330 The one-dimensional computational domain used to compute the errors is shown in figure 4 and has a total length of 190 [m]. The number of DG elements is fixed in all computations $T_{DG} = 102$, whereas the number of nodes in Fourier PSTD N_{PS} changes as a function of the size of the copy zone. In this way, 335 the travelling time of the wave in the DG domain is kept constant for all cases and thereby the error contribution from the DG solver. However, in an efficient implementation of the hybrid methodology, the number of DG elements should be minimized. The DG polynomial order (p_o) employed in each problem will be indicated in the subsequent sections.

340 In all one-dimensional scenarios, the grid spacing $\Delta x_{PS} = h = 0.1$ [m] and the spatial sampling frequency is computed as $f_s = \frac{c_0}{\Delta x_{PS}}$. The initial conditions imposed in Fourier PSTD are a broadband pressure distribution as given in equation 10 with, $A = 1$, $b_s = 0.0706(\frac{1}{\Delta x})^2 [m^{-2}]$ and $x_s = 30$ [m], and a velocity distribution $u(x, t_0) = -p(x, t_0)/\rho_0 c_0$. To minimize the error contribution from 345 the PML, a 100-elements layer has been used together with PML parameters $\beta = 20000$ and $m = 4$ (an error evaluation from the PML can be found in [4]). The left side of the DG domain is computed using an acoustically hard boundary

Figure 4: Sketch of the one-dimensional hybrid domain. The number elements in the DG domain Ω_{DG} (brown) is fixed in all computations, whereas the number of nodes in the Fourier PSTD domain $\Omega_{PS,n}$ (green) changes as a function of the size of the copy zone in each experiment. The right end of the computational domain Ω (blue) is terminated using a PML, while the left side is computed in DG using an acoustically hard boundary. A left-travelling Gaussian pressure distribution is imposed at x_s and recorded at positions x_{r1} and x_{r2} after being reflected. Additionally, the pressure distribution is recorded at a reference position x_{ref} close to the source.

whereas the right end is where the values from Fourier PSTD are copied and the characteristic non-reflective boundary condition is imposed there. The reflected
350 wave is then recorded at positions $x_{r1} = 40$ [m] and $x_{r2} = 150$ [m] in the Fourier PSTD domain. Additionally, the same scenario is computed using a Fourier PSTD stand-alone solver, where the hard left boundary of the domain is modeled using an image source mirror technique [2]. The same spatial and temporal discretization as in the hybrid model is used in the stand-alone solver.
355 The main settings in all the one-dimensional calculations underlying the results of this section are summarized in table A.2.

The errors of the time recorded signals, $\epsilon_{amp}(f_i)$ and $\epsilon_{\phi}(f_i)$, are calculated from the sound pressure solutions and the ends of the signals are tapered by a window to avoid aliasing. The errors are computed by transforming the
360 recorded acoustic variables to the frequency domain ($Q(f_i)$) using a forward Fourier transform. The amplitude and dispersion errors are calculated using equation 12 for representing the errors as a function of frequency, where $\Delta_{...}(f_i) = |Q_{...,ref}(f_i)| - |Q_{...}(f_i)|$ and $\Delta\phi_{...}(f_i) = |\phi[Q_{...,ref}(f_i)] - \phi[Q_{...}(f_i)]|$.

365 Additionally, for presenting the errors as the maximum value in a certain frequency range, equation 13 is used.

$$\begin{aligned}\epsilon_{amp}(f_i) &= \left| \frac{|\Delta_{ana}(f_i)| - |\Delta_{calc}(f_i)|}{|Q_{ref}(f_i)|} \right|, \\ \epsilon_{\phi}(f_i) &= \frac{|\Delta\phi_{ana}(f_i) - \Delta\phi_{calc}(f_i)|}{\pi},\end{aligned}\tag{12}$$

$$\begin{aligned}\epsilon_{amp}(f_1, f_2) &= \sup(\epsilon_{amp}(f_i)); f_i \in [f_1, f_2], \\ \epsilon_{\phi}(f_1, f_2) &= \sup(\epsilon_{\phi}(f_i)); f_i \in [f_1, f_2],\end{aligned}\tag{13}$$

where, $Q_{ana,ref}$ is the analytical and $Q_{calc,ref}$ the numerical solution of a recorded signal at a receiver located close to the source ($x_{ref} = 20$ [m]), and Q_{ana} and Q_{calc} are the analytical and the numerical solution, respectively, at the different recording positions, x_{r1} and x_{r2} .

370 Additionally, the errors of the spatial solutions, $\hat{\epsilon}_{amp}(f_i)$ and $\hat{\epsilon}_{\phi}(f_i)$, from the numerical methods are computed by transforming the spatial variables to the wavenumber domain at every time step using a forward Fourier transform and a conversion from the wavenumber domain to the frequency domain ($\hat{Q}(f_i)$) by using an adiabatic speed of sound, i.e. $f_i = \frac{k_i c_a}{2\pi}$. The errors and their maximum
375 values in a certain frequency range are computed using the following equations

$$\begin{aligned}\hat{\epsilon}_{amp}(f_i) &= \left| \frac{|\hat{Q}_{ana}(f_i)| - |\hat{Q}_{calc}(f_i)|}{|\hat{Q}_{ana}(f_i)|} \right|, \\ \hat{\epsilon}_{\phi}(f_i) &= \frac{|\phi[\hat{Q}_{ana}(f_i)] - \phi[\hat{Q}_{calc}(f_i)]|}{\pi},\end{aligned}\tag{14}$$

$$\begin{aligned}\hat{\epsilon}_{amp}(f_1, f_2) &= \sup(\hat{\epsilon}_{amp}(f_i)); f_i \in [f_1, f_2], \\ \hat{\epsilon}_{\phi}(f_1, f_2) &= \sup(\hat{\epsilon}_{\phi}(f_i)); f_i \in [f_1, f_2],\end{aligned}\tag{15}$$

where, $\hat{Q}_{ana}$ is the analytical spatial solution of the one-dimensional problem and $\hat{Q}_{calc}$ is the spatial numerical solution.

Since in the hybrid method both solvers are overlapping in the copy zone, the spatial solution in this area needs to be built at each node from the solution
380 of one of the solvers. The spatial solutions of the whole domain are constructed from the DG solutions in the interval $[x_{left,DG}, (x_{right,DG} - T_{czPSDG}\Delta x_{PS})]$

and from the Fourier PSTD solutions in the interval $[(x_{right,DG} - (T_{czPSDG} - 1)\Delta x_{PS}), x_{right,PS}]$, where T_{czPSDG} is the number of elements of the copy area from Fourier PSTD to DG. For DG, only solutions at the boundary nodes of each
385 element are considered. In this way, the spatial solution of the whole domain is built at equally space points and the transformation to the wavenumber domain can be computed.

4.2.1. The error related to the temporal scheme

This section presents the time evolution of the dispersion and dissipation
390 errors of a one-dimensional implementation of the hybrid approach. The combination of temporal schemes is used in this section.

The Runge-Kutta dispersion and dissipation errors can be computed analytically per time step following the work by Bogey and Bailly [23]. Additionally, the errors from the numerical methods are computed using equations 14 and
395 15. The dispersion and dissipation errors are presented in the frequency range corresponding with $(0, \frac{f_s}{5})$ points per wavelength for the hybrid method and the Fourier PSTD stand-alone solver, to be in the range where the Runge-Kutta error dominates the total error. In these evaluations, the order of the polynomial approximation in DG is set to $p_o = 5$, with a Courant number $\nu_{PS} = 0.5$ and
400 a time step factor $s_{hyb} = 3$. The amplitude and phase errors are presented in figure 5 for a window size of 30 (figures 5a and 5b) and 100 elements (figures 5c and 5d), where the vertical lines represent the time instances when the centre of the pressure distribution enters/leaves the DG area. The errors are computed up to 2000 time steps from the sound pressure solutions.

Figure 5 shows a good agreement between the numerical and the analytical
405 RK errors for the hybrid method. Clearly, Runge-Kutta errors are dominating the global error of the hybrid approach except at local time instances when the centre of the Gaussian distribution is travelling through the copy zone. Additionally, there is an error increase when the pressure distribution reaches
410 the boundary, corresponding with the lack of periodicity of the spatial variables when computing the Fourier transform at those time instances. The comparison

between the Runge-Kutta amplitude and phase error, i.e. graphs 5a and 5b or
5c and 5d, shows that dispersion errors are always higher. This is in line with
the findings in [4]. Moreover, in general, local errors are reduced when increasing
415 the window size. Finally, it is clear, from the analysis of figure 5, that the error
from the hybrid method tends to converge with the error of the stand-alone
solver for long time propagation scenarios.

Figure 5: Time evolution of the amplitude $\hat{\epsilon}_{amp}(0, \frac{f_s}{5})$ and phase $\hat{\epsilon}_\phi(0, \frac{f_s}{5})$ errors of the hybrid method (solid black line) together with the analytical Runge-Kutta error of the hybrid (red 'o' markers) and the stand-alone Fourier PSTD solver (broken green line). Figures (a) and (b) present the amplitude and phase error, respectively, for a window size $N_w = 30$. Figures (c) and (d) represent amplitude and phase errors, respectively, for $N_w = 100$. The vertical lines represent the time instances when the centre of the pressure distribution enters/leaves the DG area.

4.2.2. The error related to the coupling: DG to Fourier PSTD

As mentioned in section 3.3.3, the hybrid methodology uses a Gaussian win-
420 dow to keep spatial periodicity in the Fourier PSTD domain after transferring
values from DG. In this section, the error contribution of this coupling includes
both actions, copying values from DG to PSTD and windowing. The influence
of this error is investigated in this section.

In order to minimize the errors arising from the temporal scheme, the time
425 steps in Fourier PSTD and DG are equal, i.e. the time step factor $s_{hyb} = 1$,
and calculated using a Fourier PSTD Courant number $\nu_{PS} = 0.05$, following
section 3.2. Moreover, the Runge-Kutta time method used in Fourier PSTD is
RKF84, to suppress errors arising from a mismatch in the errors from different
RK methods. Under these conditions, the acoustic variables have been recorded
430 in all cases for a duration of 53200 time samples. Coupling errors from window
sizes N_w from 10 to 100 nodes are investigated, while keeping the size of the copy
zone from Fourier PSTD to DG $T_{czPSDG} = 1$ element for all cases. Additionally,
the hybrid method has been evaluated for different DG polynomial orders, $p_o =$
[4, 5, 6, 7, 10], in order to analyse the influence of this parameter.

435 The same scenario has been solved with a Fourier PSTD stand-alone solver
using a mirror technique to model the hard reflecting surface. In this section,
the stand-alone solver has been computed using the same Runge-Kutta method
as in the hybrid case, i.e. RKF84, as well as the filter described in section 3.4.
All other parameters, e.g. time step and spatial resolution, are the same as in
440 the hybrid case.

Errors ϵ_{amp} and ϵ_ϕ of the hybrid method ($p_o = 10$) together with the error
from the stand-alone Fourier PSTD solver as a function of points per wavelength
(N_λ) are shown in figure 6. The errors are presented in figures 6a and 6b for
a window size $N_w = 30$, and in graphs 6c and 6d for $N_w = 100$. All graphs
445 include the analytical Runge-Kutta errors of the calculated scenario. Figure 6
shows that a smaller window size increases the overall error at all frequencies
of the hybrid methodology, i.e. the window error is, in general, dominating the

total error of the hybrid method for small window sizes down to 2.5 points per wavelength. On the other hand, the stand-alone solver is mainly dominated by the Runge-Kutta error except for frequencies close to 2.5 points per wavelength
450 where other sources of error take over, as described by Hornikx et al. [4]. This latter feature is found as well in the hybrid method (graphs 6c and 6d), for frequencies $N_\lambda < 3$ for the amplitude error and $N_\lambda < 2.5$ for the phase error, corresponding with the same trend found in the stand-alone solver. Moreover,
455 for large N_λ values, the errors of the hybrid method are larger than the errors from the stand-alone solver but they are, in this frequency range, already very low and, therefore, of less relevance. Additionally, the influence of filtering, as described in section 3.4, is visible in figure 6 for frequencies between 2.5 and 2 points per wavelength. In this figure, the cut-off frequency of the filter is
460 represented by a vertical dotted line.

The amplitude $\epsilon_{amp}(0, \frac{f_s}{2.5})$ and phase $\epsilon_\phi(0, \frac{f_s}{2.5})$ errors from coupling DG to Fourier PSTD are presented in figure 7, showing the analysis for different DG polynomial orders ($p_o = [4, 5, 6, 7, 10]$) and together with the results of the stand-alone solver. Additionally, the figure includes the analytical error
465 for the window function reported by Hornikx et al. in [4] for 2.5 points per wavelength. In that work, the analytical window error was found from the largest errors, either amplitude or phase. In all cases the phase error was found to be smaller than the amplitude error, hence, the analytical error was built from the amplitude error solutions. The same tendency is found here for the hybrid
470 method with amplitude errors always higher than the phase error. Moreover, a good agreement between the amplitude and the analytical window error for window lengths $N_w < 50$ is found in 7a, i.e. the window function dominates the errors in that range for $p_o \geq 7$, and the same conclusions can be drawn for the phase error in figure 7b. Furthermore, figure 7 shows how, for larger N_w
475 values and DG polynomial order $p_o \geq 7$, the error from the Fourier PSTD solver dominates the hybrid error for both, amplitude and phase.

Finally, figure 8 presents the errors, $\epsilon_{amp}(0, \frac{f_s}{3})$ and $\epsilon_\phi(0, \frac{f_s}{3})$, for two different wave-travelling distances, recorded at x_{r1} and x_{r2} , for DG polynomial

order $p_o = 10$. In this frequency range, the error is always dominated by the
 480 Runge-Kutta scheme when the window size is big enough, as shown in graphs
 6c and 6d. The influence of the Runge-Kutta error is shown in figure 8, where
 the longest travelled-distance (recorded at x_{r2}) shows a higher error. In all
 cases, the error of the hybrid method converges with the error of the stand-
 alone solver for high enough window sizes. Nevertheless, for smaller copy zone
 485 areas the window error takes over and, therefore, the travelling distance is not
 of influence anymore in this scenario.

Figure 6: Amplitude ϵ_{amp} and phase ϵ_ϕ errors as defined in equation 12 of the hybrid method (solid black line) together with the error from the stand-alone Fourier PSTD solver (broken green line) as a function of points per wavelength (N_λ). Figures (a) and (b) present the amplitude and phase error, respectively, for a window size $N_w = 30$. Figures (c) and (d) represent amplitude and phase errors, respectively, for $N_w = 100$. The graph includes the analytical Runge-Kutta error of the calculated scenario (red line with 'o' markers). The vertical dotted line represents the cut-off frequency of the filter.

Figure 7: Error from the hybrid method (darker solid lines represent lower DG polynomial orders corresponding to higher errors for large window sizes) together with the error from the stand-alone Fourier PSTD solver (broken line) as a function of the window size according to equation 13. The figure shows, up to 2.5 points per wavelength, the (a) amplitude $\epsilon_{amp}(0, \frac{f_s}{2.5})$ and (b) phase $\epsilon_{\phi}(0, \frac{f_s}{2.5})$ errors, for different DG polynomial order ($p_o = [4, 5, 6, 7, 10]$) at recording point x_{r2} . Additionally, the figures include the analytical window error reported by Hornikx et al. in [4] (red line with 'o' markers) and the analytical Runge-Kutta error (green line with 'x' markers), both for 2.5 points per wavelength.

Figure 8: Error from the hybrid method (red solid line and green broken line) together with the error from the stand-alone Fourier PSTD solver (black and grey dotted lines with 'o' marker) and the Runge-Kutta error (black and grey broken lines with '.' marker) as a function of the window size according to equation 13. The figure shows, up to 3 points per wavelength, the (a) amplitude $\epsilon_{amp}(0, \frac{f_s}{3})$ and (b) phase $\epsilon_\phi(0, \frac{f_s}{3})$ errors, for two different wave-travelling distances recorded at points x_{r1} and x_{r2} and for DG polynomial order $p_o = 10$.

4.3. Extension to two-dimensional problems

The hybrid methodology has been tested in two-dimensional problems, with the objective of demonstrating the accuracy and applicability of the novel approach. The analysis in this section focusses on the amplitude error, since the dispersion error has proven to be smaller for the one-dimensional results. One case is posed in this section: a scattering problem where analytical solutions are available. In all scenarios, the order of the polynomial approximation in DG is set to $p_o = 5$, with a Fourier PSTD grid spacing $\Delta x_{PS} = 0.1$ [m], a Courant number $\nu_{PS} = 0.5$ and a time step factor $s_{hyb} = 13$, unless otherwise indicated. As for the one-dimensional problems, the main features of the two-dimensional models are summarized in table A.3.

4.3.1. Scattering problem

In order to assess the impact of irregular geometries, an acoustical scattering problem is studied. The computational Fourier PSTD domain is the rectangle, with $N_{PS,x} \times N_{PS,y} = 240 \times 600$ grid points, shown in figure 9. A detail of the two-dimensional hybrid grid around the scatterer is shown in figure 10, where the cylinder is in the centre of the domain with a radius $a_{cyl} = 10\Delta x$ [m]. In figure 9, the grey area represents the DG domain with a dimension of $T_a\Delta x \times T_a\Delta x$. This area is shown in more detail in figure 10, where the black dots represent the Fourier PSTD grid nodes. The DG grid is built with quadrilateral elements and is divided in a structured area, corresponding with the coupling zone, and an unstructured part surrounding the scatterer. Therefore, the coupling zone for the scattering problem is a square with a square hole, centered at the origin of the domain, where the coupling zone length is $T_{cz}\Delta x = (T_{czPSDG} + T_{czDGPS})\Delta x$ [m]. The details of the computed grids are given in table 1, where, in all cases, $T_{czPSDG} = 1$ element.

In figure 9, the circle of points around the scatterer represents 315 recording positions at a distance of $a_r = 71\Delta x$ [m] from the origin of the domain at angles $\xi_{rn} = [0 : 0.02 : 2\pi]$ radians, while the broken line represents the center of the plane wave excitation located at $x_s = 100\Delta x$ [m] from the origin. The initial

Grid name	Elements					Vertices
	Total	Structured	Unstructured	T_a	T_{cz}	On scatterer
SC1	9740	9561	224	100	39	128
SC2	3584	3360	224	62	20	128
SC3	9930	9516	414	100	39	256

Table 1: Characteristics of the DG grids and windows used for the scattering problem.

conditions, imposed in Fourier PSTD, are a broadband pressure and velocity distribution as given in equation 10 with, $A = 1$, $b_s = \frac{3}{16}(\frac{1}{\Delta x})^2 [m^{-2}]$ and $|\mathbf{x} - \mathbf{x}_s| = \sqrt{(x - x_s)^2}$. In DG, the time-domain computation is initialized with a zero-valued pressure and velocity distribution. The left and right boundaries of the Fourier PSTD domain are terminated with a PML using equation 11, while periodic boundary condition applies for the upper and lower limits. The main parameters of the PML are $\beta = 20000$, $m = 4$ and $D = 50\Delta x$ [m]. Regarding the DG domain, the boundary of the scattering object is acoustically hard whereas the external DG boundary is where the values from Fourier PSTD are copied and no boundary conditions are imposed at those limits of the domain.

The results presented in this section are computed from the acoustic pressure scattered by the cylinder. The total wave field consists of an incident and a scattered field, of which the incident field is the incident plane wave. In order to obtain only the scattered solutions, the same scenarios are computed without the scatterer, i.e. the wave field is built only with the incident plane wave and this field is subtracted from the total wave field. The acoustic variables are recorded in both cases (with and without the scatterer) at the receiver positions shown in figure 9 during a recording time $t_{rec} = 1200\Delta t_{PS}$ [s].

The scattered wave field represents an outward travelling wave from the cylinder and can analytically be written as a composition of cylindrical waves. To solve such a problem, it is convenient to use cylindrical coordinates with the center on the axis of the cylinder (the z -axis is the cylinder axis). The incident and total pressure field can be computed with

Figure 9: Computational hybrid domain for the calculation of the scattering problem. The scatterer is in the origin of the domain with a radius of $10\Delta x$ [m]. The grey area inside the broken line square represents the DG domain with a side length of $T_a\Delta x$ [m]. The circle of points surrounding the scatterer are the recording positions at a distance $a_r = 71\Delta x$ [m] from the centre of the scatterer. The broken line represents the center of the plane wave excitation located at $100\Delta x$ [m] from the origin. The left and right boundaries are terminated with a PML, while periodic boundary condition applies for the upper and lower limits.

$$P_{inc}(r, \phi) = \sum_{n=-\infty}^{n=\infty} A i^n J_n(kr) e^{in\phi}. \quad (16)$$

$$P_{tot}(r, \theta) = \sum_{n=-\infty}^{n=\infty} \left(A i^n J_n(kr) + A_n H_n^{(1)}(kr) \right) e^{in\theta}. \quad (17)$$

$$A_{a_{cyl}, n} = -A i^n \frac{J_n(ka_{cyl}) + \frac{Z_n}{i\omega\rho} J'_n(ka_{cyl})}{H_n^{(1)}(ka_{cyl}) + \frac{Z_n}{i\omega\rho} H_n^{(1)'}(ka_{cyl})}. \quad (18)$$

540 where r and ϕ are the radial and angular coordinates, $H_n^{(1)}(kr)$ the Hankel function of order n and first kind, $J_n(kr)$ the Bessel function of first kind and

Figure 10: Detail of the conformal grids close to the scattering object. The grey area corresponds with the DG domain with a side length of $T_a \Delta x$ [m] while the black dots represent the Fourier PSTD nodes. The DG grid is built with quadrilateral elements and is divided in a structured area, corresponding with the coupling zone, and an unstructured part surrounding the scatterer. The number of structured elements between the sides of the squares is T_{cz} .

order n , A the amplitude of the incident wave, k the wavenumber and Z_n the surface impedance. By using the former expressions, the analytical scattered pressure in the frequency domain $P_{ana,SC}(f_i)$ is computed and compared with
545 the results of the hybrid numerical method $P_{hyb,SC}(f_i)$. Since the problem has axial symmetry, only the results of receivers positions at angles from zero to π radians are presented in the directivity patterns shown in figure 11 for three different frequencies expressed as points per wavelength with respect to the PSTD grid, $N_\lambda = 2.5$, $N_\lambda = 5$ and $N_\lambda = 32$. Furthermore, figure 12 presents
550 the waveforms of the scattered pressure field for receivers located at $\pi/2$, $3\pi/4$ and π radians, comparing the numerical and analytical results. Figures 11 and 12 show a good agreement between the solutions. Additionally, an amplitude error comparison is presented in figure 13 for the different evaluated grids given in table 1. The amplitude error is calculated from

$$\epsilon_{amp,SC}(f_i) = \left| \frac{|P_{hyb,SC}(f_i)| - |P_{ana,SC}(f_i)|}{|P_{ana,SC}(f_i)|} \right|; f_i \in [f_1, f_2], \quad (19)$$

555 using the numerical scattered pressure recordings transformed to the frequency

domain by using a forward Fourier transform and the analytical solutions. Taking SC1 grid as the reference case, figure 13 shows how, reducing the copy zone size as in case SC2, mainly affects the precision of the method around 2.5 points per wavelength. On the other hand, by increasing the number of vertices on the cylinder (case SC3), reduces the error in a large part of the frequency range. Once more, the influence of filtering, as described in section 3.4, is visible in figure 13 for frequencies between 2.3 and 2 points per wavelength.

Figure 11: Directivity patterns of the scattered pressure field for a) $N_\lambda = 2.5$, b) $N_\lambda = 5$ and c) $N_\lambda = 32$ points per wavelength. A plane wave is incident at 0 radians. The graphs compare the solution of the computation using the hybrid method on grid SC1 with the analytical solution of the scattering problem. The solutions have been referred to the maximum value of the analytical solution.

Figure 12: Waveforms of the scattered pressure field for receivers located at a) $\pi/2$, b) $3\pi/4$ and c) π radians. A plane wave is incident at 0 radians. The graphs compare the solution of the computation using the hybrid method on grid SC1 with the analytical solution.

Figure 13: Amplitude error of the hybrid method when computing the scattering problem on different grids for receivers located at a) zero and b) π radians. A plane wave is incident at 0 radians. The error is computed from the scattered pressure and referred to the analytical solution for all grid cases. The vertical green line represents the cut-off frequency of the filter.

As indicated by Hamilton [32], numerical instabilities have been reported in the adaptation of spectral methods to non-trivial geometries. In this work,
565 a long-time calculation using the current hybrid methodology on a rectangular domain with a scatterer and with $N_{PS,x} \times N_{PS,y} = 240 \times 160$ grid points has been carried out to identify possible late-time instabilities. All Fourier PSTD boundaries are periodic in this scenario while keeping the rest of grid characteristics as shown in table 1 for SC1. The scenario is excited with a pressure
570 distribution as indicated in section 4.1 at coordinates $(x_s, y_s) = (8, 0)$ [m]. The total wave field pressure is recorded at position $(x_r, y_r) = (1, 6)$ [m] and the time evolution is shown in figure 14. No sign of instability is found from this results after 10 seconds of propagation.

Figure 14: Pressure responses computed with the hybrid method in a rectangular domain with a scatterer cylinder located at the centre with periodic boundary conditions in the outer limit of the domain and an acoustically hard boundary of the scatterer: a) whole computation and b) only first reflections.

5. Conclusions

575 A novel numerical hybrid approach is presented to solve the linearized Euler equations (LEE), coupling Fourier pseudospectral time-domain methodology (Fourier PSTD) with the nodal discontinuous Galerkin (DG) method. The aim of the hybrid approach is to allow the computation of arbitrary boundary conditions and complex geometries by using the benefits of the DG methodology
580 close to the boundaries while keeping the efficient Fourier PSTD method in the bulk of the domain. A clear application of the novel approach is atmospheric sound propagation in urban scenarios, with DG near the ground surface and Fourier PSTD in the (moving and inhomogeneous) propagation domain above it.

585 The methodology is presented for one- and two-dimensional cases where the computational domain is decomposed into Fourier PSTD and DG subdomains with overlapping areas. The solutions of the LEE are approximated in each subdomain by one of the numerical methods and the hybrid method couples the

results in the overlapping area. The approach is based on conformal meshes in
 the coupling zone to avoid spatial interpolation of the DG solutions, however a
 590 spatial spectral interpolation scheme is needed in the hybridization process for
 the Fourier PSTD solutions in order to obtain the values at the internal nodes
 of the DG elements. The coupling algorithm includes a Gaussian window func-
 tion to impose periodicity of the Fourier PSTD solutions. Hence, the hybrid
 595 approach couples the solutions over a zone rather than at the interface of the
 numerical methods. Therefore, the solvers can use local time stepping without
 transferring data at intermediate steps. Since, in this work, the Courant num-
 bers in DG are more restrictive than in Fourier PSTD the data update between
 the solvers is only done after Δt_{PS} time steps. Finally, the method includes a
 600 low-pass filtering approach to minimize numerical instabilities arising from the
 Fourier PSTD solver.

Overall, the novel hybrid methodology shows no significant additional error
 when compare with a Fourier PSTD stand-alone solver when using a suitable
 selection of the main parameters. The global error of the hybrid method is gen-
 605 erally dominated by the Gaussian window or by the Runge-Kutta time scheme.
 The window error is, in general, responsible for the total error for small windows
 up to 2.5 points per wavelength. For higher frequencies, around 2 points per
 wavelength, the aliasing error and the influence of the filtering take over the
 global error. However, if big enough windows and a suitable DG polynomial
 610 order are considered, the hybrid error is dominated by the time scheme. In the
 low frequency range, the hybrid error is, in general, higher than the stand-alone
 solver error but in this frequency range the errors are already very low and,
 therefore, of less relevance. Finally, a long-time calculation using the hybrid
 methodology on a rectangular domain with a scatterer has been carried out to
 615 identify possible late-time instabilities. The results show no sign of instability
 after 10 seconds of sound propagation.

The hybrid methodology has been applied to solve a two-dimensional scat-
 tering problem to test the accuracy and applicability of the approach. The
 results show, once more, the influence of the size of the copy area in the hybrid

620 error, mainly around 2.5 points per wavelength. For this case, the precision of
the results can be improved by increasing the resolution of the cylinder.

In general, the hybrid method has shown to be a suitable tool for computing sound propagation problems for domains with arbitrary boundary shapes, not having a significant additional error when compared with a Fourier PSTD
625 stand-alone solver, and with a clear application for urban atmospheric sound propagation problems. Additionally, the approach is a step towards the implementation of arbitrarily shaped frequency dependent boundaries in Fourier PSTD as well as a moving inhomogeneous atmosphere.

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Appendices

Appendix A. Main parameters used in the calculations

In this appendix, the main parameters used in the computations of the error
750 are summarized per figure of results for one- and two-dimensional cases.

Table A.2: Main parameters for the one-dimensional computations presented in figures 5, 6, 7 and 8

	Figure 5	Figure 6	Figure 7	Figure 8
Dimensions	1900 Δx_{PS} [m]			
N_w	[30,100]		[10:1:100]	
p_o	5	10	[4:1:10]	10
$\Delta x_{PS} = h$	0.1 [m]			
$p(x, t_0)$	$Ae^{-b_s(x-x_s)^2}$ [Pa]			
$u(x, t_0)$	$-p(x, t_0)/\rho_0 c_0$ [ms ⁻¹]			
(A, b_s, x_s)	(1, 0.0706($\frac{1}{\Delta x}$) ² [m ⁻²], 300 Δx_{PS} [m])			
$(x_{ref}, x_{r1}, x_{r2})$	(100, 400, 1500) Δx_{PS} [m]			
BC (left/right)	(rigid/PML)			
PML (D, β, m)	(100 Δx_{PS} [m], 20000, 4)			
RK Fourier PSTD	RKo6s	RKF84		
RK DG	RKF84			
ν_{PS}	0.5	0.05		
s_{hyb}	3	1		
Time steps	2000	53200		
Filter (hybrid/stand-alone)	(yes/−)	(yes/yes)		

Table A.3: Main parameters used in the two-dimensional problems presented in figures 11, 12, 13 and 14

	Figures 11 12	Figure 13	Figure 14
$N_{PS,x} \times N_{PS,y}$	240×600		240×160
T_{cz}	39	[20,39]	39
$T_{x,cz}$	—		
$T_{y,cz}$	—		
Vertices cylinder	128	[128,256]	128
a_{cyl}	$10\Delta x_{PS} [m]$		
p_o	5		
$\Delta x_{PS} = h$	$0.1 [m]$		
$p(\mathbf{x}, t_0)$	$Ae^{-b_s(\mathbf{x}-\mathbf{x}_s)^2} [Pa]$		
$u(\mathbf{x}, t_0)$	$0 [ms^{-1}]$		
(A, b_s)	$(1, \frac{3}{16}(\frac{1}{\Delta x})^2 [m^{-2}])$		
$\mathbf{x}_s = (x_s, y_s)$	$(10, -) [m]$		$(8, 0) [m]$
$\mathbf{x}_r = (x_r, y_r)$	—		$(1, 6) [m]$
a_r	$71\Delta x_{PS} [m]$		—
ξ_{rn}	$[0:0.02:2\pi] [rad]$		—
BC (left & right)	(PML)		(periodic)
BC (up & down)	(periodic)		
BC (cylinder)	(rigid)		
PML (D, β, m)	$(50\Delta x_{PS} [m], 20000, 4)$		
RK Fourier PSTD	RKo6s		
RK DG	RKF84		
ν_{PS}	0.5		
s_{hyb}	13		
Time steps	1200		$10/\Delta t_{PS}$
Filter (hybrid/openPSTD)	(Yes/—)		