The nonexistence of distance-regular graphs with intersection arrays \( \{27, 20, 10; 1, 2, 18\} \) and \( \{36, 28, 4; 1, 2, 24\} \)

Brouwer, A.E.; Sumalroj, S.; Worawannotai, C.

Published in:
Australasian Journal of Combinatorics

Published: 01/01/2016

Citation for published version (APA):
Brouwer, A. E., Sumalroj, S., & Worawannotai, C. (2016). The nonexistence of distance-regular graphs with intersection arrays \( \{27, 20, 10; 1, 2, 18\} \) and \( \{36, 28, 4; 1, 2, 24\} \). Australasian Journal of Combinatorics, 66(2), 330-332.
The nonexistence of distance-regular graphs with intersection arrays \( \{27, 20, 10; 1, 2, 18\} \) and \( \{36, 28, 4; 1, 2, 24\} \)

ANDRIES E. BROUWER

Department of Mathematics
Eindhoven University of Technology, Eindhoven
Netherlands
aeb@cwi.nl

SUPALAK SUMALROJ  CHALERMPONG WORAWANNOTAI

Department of Mathematics
Silpakorn University, Nakhon Pathom
Thailand
sumalroj_s@silpakorn.edu  worawannotai_c@silpakorn.edu

Abstract

Locally, a distance-regular graph with ‘\( \mu = 2 \)’ carries the structure of a partial linear space. Using this, we show that there are no distance-regular graphs with intersection array \( \{27, 20, 10; 1, 2, 18\} \) or \( \{36, 28, 4; 1, 2, 24\} \) (on, respectively, 448 or 625 vertices).

1 Introduction

Let \( \Gamma \) be a distance-regular graph (for definitions, see Section 2 below) such that any two vertices at distance 2 have precisely 2 common neighbours. Let \( \Delta \) be the subgraph of \( \Gamma \) induced on the neighbours of a fixed vertex. Then in \( \Delta \) any two nonadjacent vertices have at most one common neighbour. It follows that each edge of \( \Delta \) lies in a unique maximal clique, so that \( \Delta \) is the collinearity graph of a partial linear space of girth at least 5 (see [2, Section 2]).

In [2] the following theorem was proved.

Theorem 1.1 A connected partial linear space with girth at least 5 and more than one line in which every point has \( \lambda \) neighbours, contains \( k \geq \lambda(\lambda + 3)/2 \) points. Equality holds only in the case \( \lambda = 2, \ k = 5 \).
Applying this to a putative distance-regular graph with intersection array \( \{27, 20, 10; 1, 2, 18\} \), we see that the connected components of a local partial linear space have \( k \leq 27 \) points, where each point has \( \lambda = 6 \) neighbours, and we arrive at a contradiction: each connected component must be a single line of size 7, but 7 does not divide 27. We found

**Proposition 1.2** There is no distance-regular graph with intersection array \( \{27, 20, 10; 1, 2, 18\} \).

Next, consider the intersection array \( \{36, 28, 4; 1, 2, 24\} \). Here \( \lambda = 7 \), \( k = 36 \) and the inequality is not violated. But there is not much room, and a closer inspection rules out this case as well, see Section 3 below. We find

**Proposition 1.3** There is no distance-regular graph with intersection array \( \{36, 28, 4; 1, 2, 24\} \).

## 2 Distance-regular graphs

For the concept of distance-regular graph, and any unexplained notation, cf. [1]. Here we very briefly repeat the definitions of distance-regular graph and intersection array.

Let \( \Gamma \) be an undirected graph, without loops or multiple edges. Assume that \( \Gamma \) is connected, with diameter \( d \). If \( u, v \) are two vertices, then the distance \( d(u, v) = d_\Gamma(u, v) \) between \( u \) and \( v \) in \( \Gamma \) is the length of a shortest path from \( u \) to \( v \). Thus, \( d(u, v) = 0 \) means \( u = v \), and \( d(u, v) = 1 \) precisely when \( u, v \) are adjacent. Define \( \Gamma_i(x) = \{y \mid d(x, y) = i\} \) so that \( \Gamma_1(x) = \{y \mid d(x, y) = 1\} \) is the set of neighbours of \( x \). The graph \( \Gamma \) is called distance-regular when there are constants \( p_{hij} \) (0 ≤ \( h, i, j \) ≤ \( d \)) such that \( |\Gamma_i(x) \cap \Gamma_j(y)| = p_{hij} \) for every two vertices \( x, y \) with \( d(x, y) = h \). For a distance-regular graph with parameters \( p_{hij} \) one defines constants \( a_i := p_{1i}, b_i := p_{1,i+1}, c_i := p_{1,i-1} \), and further \( k := b_0, \lambda := a_1, \mu := c_2 \). Now \( \Gamma \) is regular of valency \( k \), and \( a_i + b_i + c_i = k \) for all \( i, 0 \leq i \leq d \). Every local graph \( \Gamma(x) \) is regular of valency \( \lambda \). Two vertices at distance 2 have \( \mu \) common neighbours. All parameters \( p_{hij} \) can be expressed in terms of the \( b_i \) and \( c_i \). The *intersection array* of \( \Gamma \) is by definition \( \{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\} \).

In the tables at the end of [1], the existence of distance-regular graphs with intersection arrays \( \{27, 20, 10; 1, 2, 18\} \) and \( \{36, 28, 4; 1, 2, 24\} \) is shown as undecided. Much happened since the publication of that book, and [3] is a survey of all that is new. But the tables at the end of this survey show no progress on these existence questions. In this note we show that there are no such graphs.

## 3 Nonexistence of a partial linear space

A *partial linear space* is a geometry with points and lines, where two lines meet in at most one point. Let \( xIL \) denote that the point \( x \) is incident with the line \( L \). The
girth $g$ of a partial linear space is the length of a shortest cycle $x_0 IL_1 IL_2 \ldots IL_g IL_0$ of mutually distinct points and lines. Thus, girth at least 5 means that the partial linear space does not contain triangles or quadrangles.

Consider a partial linear space of girth at least 5 on 36 vertices, where each point is collinear with 7 other points. We shall derive a contradiction.

Fix a point $x_0$, and let there be $m_i$ lines of size $i$ (say, $i$-lines, $i \geq 2$) on $x_0$. The valency condition says $\sum m_i (i - 1) = 7$. At distance 2 from $x_0$ there are $\sum m_i (i - 1)(8 - i)$ vertices, so this expression is not larger than $36 - 1 - 7 = 28$. Let us call a line long if it has size at least 5. If $x_0$ is not on any long lines, then we can bound $8 - i \geq 4$ and find $28 = 4 \sum m_i (i - 1) \leq \sum m_i (i - 1)(8 - i) \leq 28$. It follows that equality holds everywhere and $x_0$ is on 4-lines only. But 3 does not divide 7, this is impossible.

We proved that each point is on a (unique) long line, so that the vertex set is partitioned by the long lines.

Consider a long line $L$ of size $m$, $m = 5, 6, 7$. Then $L$ has $m(8 - m)$ neighbours, each on a different long line. So $5m(8 - m) + m \leq 36$, a contradiction. It follows that each point is on an 8-line. But 8 does not divide 36, and we reach the final contradiction.

References


(Received 7 Apr 2016; revised 16 July 2016)