Cooperative Resource Pooling Games

with applications to the railway sector

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PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de Technische Universiteit Eindhoven, op gezag van de rector magnificus prof. dr. ir. F.P.T. Baaijens, voor een commissie aangewezen door het College voor Promoties, in het openbaar te verdedigen op maandag 16 oktober 2017 om 16:00 uur

door

Leonardus Peter Johannes Schlicher

geboren te Roermond

Dit proefschrift is goedgekeurd door de promotoren en de samenstelling van de promotiecommissie is als volgt:

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Somewhere in the middle of 2016, my roommate and I came up with the ingenious idea to raise some courgettes in our office. Although our office is not (so) suitable for such a process, we put in much effort –we watered our seeds on a daily basis, positioned them close to the window, and sometimes even talked to them. I have to say, this careful treatment showed off –a courgette plant with several elegant (yellow) flowers appeared. Unfortunately, we could not harvest any courgette at all, which was no big surprise after we realized that pollination was essential for the growth of these courgettes.

The essence of pollination is not limited to the growth of courgettes only. It can be essential (both literally and metaphorically) in many other situations. My own Ph.D. research is a good example. In this traject, I struggled with formulating interesting mathematical models, proving appealing properties of these models, and getting them published in (top) journals. I claim that without the help and support of many people around me, i.e., the pollinators, it was not possible to deliver this piece of work. At this place, I would like to express my deepest gratitude to all these pollinators.

First of all, I would like to thank my copromotor and daily supervisor Marco Slikker. Although our very first meeting was not a big success (I just forgot it), many good ones followed. You (gently) pushed me to my own limits and though I did not always like that, I now appreciate it much. One excellent example goes back to the start of 2016. I just completed a proof that showed that a specific type of game had a non-empty core for at most three players and, in addition to that, I found a counterexample that showed that a game with four players could have an empty core. Proud as I was, I showed you my results on the whiteboard. You complimented me with this proof, but immediately asked me what would happen if we would focus on a subclass of those games. In fact, you showed me that there is no final destination in research –just curiosity is the engine of research. In addition, I want to thank you for the freedom you gave me in developing my teaching skills. You allowed me to take over some lectures of your own (favorite) course on game theory with applications to supply chain management and, in fact, this was the starting point of an interesting teaching journey, which finally led to being a lecturer for a second year course with more than 350 students. Finally, I want to let you

know that I also enjoyed our social activities, e.g., watching the European championship soccer final in a lost bar somewhere in Odense (Denmark), making fun during lunch breaks, and, of course, our frequent walks to the railway station (and your standard complaint about my walking speed)!

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Then, a special word for Willem van Jaarsveld. I met Willem (in real life) for the first time at a conference dinner in Glasgow (Scotland), after we had some e-mail conversations about the existence of deterministic optimal policies for some class of Markov decision processes. During this dinner, I discussed with Willem my most recent work and our problem with getting a proof completed for a special type of game. At that time, I realized that Willem got fascinated by the problem as well. A couple of months later, Willem contacted me and shared with me his idea to slightly reformulate the original model in such a way that our envisioned proof could work out. This brilliant idea has completed the work in chapter 6 and I have to thank Willem deeply for that. Besides, I want to thank Willem for being part of my doctoral committee.

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Contents

Notation

The following symbols and notations are used throughout this monograph.

$\mathbb{N}$	the set of positive integers
$\mathbb{Z}$	the set of all integers
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^N$	the set of vectors whose coordinates are indexed by the elements of N
$\mathbb{R}_+$	the set of non-negative real numbers
$\emptyset$	the empty set
$n!$	$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$
$\binom{n}{k}$	$\frac{n!}{k!(n-k)!}$
$M \subseteq K$	M is a subset of K
$M \subset K$	M is a subset of K and M is not equal to K
2^N	the set of all subsets of N
$\lvert M \rvert$	the number of elements of M
$\mathrm{conv}\{A\}$	the convex hull of A
$\exists a$	there exists an a
$\forall a$	for all a
$\diamond$	the end of an example
$\square$	the end of a proof

Furthermore, we define the empty sum to be equal to zero, define the number of elements of the empty set to be equal to zero, and define $\mathrm{conv}\{\emptyset\} = \emptyset$.

Chapter 1

Introduction

1.1 The Dutch railway network

The Netherlands has one of the most intensively used railway networks of the world. Every day, more than one million passengers are transported on a compact, capacitated railway network of 6830 kilometers length (Ramaekers et al. [46]). To support such a service, a highly reliable and available railway network is needed. However, there are disruptions too, which may affect the availability of the railway network negatively. In that perspective, it is important that disruptions are repaired as soon as possible. This requires bringing the right service engineers, the right equipment, and the right spare parts to the disruption as quickly as possible. The current administrator of the Dutch railway network, i.e., ProRail, regulates this by outsourcing maintenance to several competitive contractors (Schlicher et al. [48]). Via performance based contracts, these contractors are held responsible for specific regions of the Dutch railway network. In particular, these contractors each hold an individual set of maintenance resources, dedicated for the execution of the maintenance in their own region. Looking from a national perspective, these dedicated maintenance resources can be used more efficiently. For instance, contractors can set up an arrangement in which a group of common maintenance resources are pooled and, as a consequence, can obtain some interesting cost savings. In what follows, we identify three (Dutch) railway oriented settings in which pooling of maintenance resources may be of interest.

1.1.1 Pooling of tamping machines

Due to the intensive use of the Dutch railway network, railway tracks may prolapse throughout the years. As a consequence, contractors have to stabilize or in rail jargon, tamp, these prolapsed railway tracks. This will be executed by tamping machines. In general, each contractor owns one tamping machine. Typically, these machines have the following properties. First of all, their utilization rate is low. These machines are only used a few times a year and tamping takes a few hours only. Moreover, tamping machines are critical resources. In case of a railway segment prolapsing, tamping is required immediately as transportation (on this specific railway segment) is not possible anymore. Finally, these machines are also subject to failures, repair, and so on, i.e., they are sometimes unavailable. So, it may happen that tamping is required in a certain contractor's region although the tamping machine (of this contractor) is unavailable. However, tamping machines are low-utilized and so, it is likely that another contractor has a tamping machine available, which is not being used at that time. As performance based contracts penalize contractors for situations in which failures are repaired too late, pooling of tamping machines may be beneficial here as more prolapsed railway tracks can be tamped without (long) delays. This may lead to interesting cost savings.

1.1.2 Spare parts pooling

In the Dutch railway network, every contractor has its own service network per region, which consists of one, two, or multiple local warehouses. At every local warehouse, spare parts for a specific sub-region are stored. Individual decisions on the type and number of spare parts to keep in stock are made per contractor. Due to such control of spare parts, it is likely that at some moment in time a contractor is out-of-stock, while another contractor may have a required spare part available. By spare parts pooling, contractors can prevent these stock-out situations to a certain extent. In particular, spare parts are not longer dedicated to one contractor only and in this way, more failures can be repaired in time. This may lead to significant cost savings.

In some situations, this simple form of spare parts pooling may be somehow naive. As an example, think of two neighbouring contractors, one with a low train density (and low penalty costs) and one with a high train density (and high penalty costs). When the total number of spare parts is low, for instance, one, then it may be smart to reserve this last spare part for the contractor with the higher penalty costs. In principle, under such form of pooling, the contractor with the low penalty costs takes (most of) the penalties and, consequently, joint penalty costs decrease in the long-run. Especially in situations in which penalty costs may differ significantly, such type of pooling can pay off.

1.1.3 Pooling of repair vans

As already mentioned in the previous sections, contractors are penalized for situations in which failures are repaired too late. Contractors can avoid (a part of) these penalty costs by positioning their fleet of repair vans strategically. Currently, strategic positioning decisions are made locally per contractor. This implies that some railway track segments can be reached in time by several repair vans, while other railway track segments cannot be reached in time at all. Seen from a national perspective, this is suboptimal and, indeed, pooling of the repair vans may be a logical option here. Under such a pooling arrangement, contractors can respond to more failures in time, and, consequently, this may lead to interesting cost savings again.

1.2 Research question

In the preceding sections, several types of railway oriented settings are introduced in which cost savings can be realized by contractors who pool their resources. Although such cost savings can be considerably large, establishing a pooling arrangement between several independent, self-interested contractors is not easy! How can we, for instance, make sure that no individual contractor, nor any subgroup of contractors has reasons to split off from the collaboration? These fairness requirements are crucial for a sustainable cooperation and, in that perspective, the development of an allocation or rule that allocates the cost savings in such a way that no (group of) contractor(s) wants to split off is a necessity. In this monograph, we focus on this cost savings allocation problem for several types of situations in which service providers, e.g., contractors, can pool their resources. All situations are inspired by the railway settings of section 1.1. In particular, for each of them, we pose the following research question:

> *When several service providers collaborate by pooling their resources, is it possible to allocate the joint cost savings in such a way that no individual service provider, nor any subgroup of service providers wants to split off from the collaboration, and can we formulate allocation rules with such, and possibly other, fairness properties?*

The existing literature does not provide answers to the research question posed for the various resource pooling situations of our interest. In this monograph, we aim to find satisfactory answers by making use of a quantitative modelling approach.

1.3 Research methodology

In this work, we apply a quantitative modelling approach in order to tackle the cost savings allocation problem for several resource pooling situations. We use the theory of cooperative games and several operations research modelling techniques, including graph theory and Markov decision processes. We now explain them in more detail.

In cooperative game theory, one primarily deals with the modelling and analysis of situations in which groups of players, i.e., coalitions, can benefit from coordinating their actions. In this monograph, we focus on a specific class of cooperative games only, namely those in which binding agreements are made between players and side payments are allowed. For such a cooperative game, one lists for each possible coalition a single number, representing, for instance, the associated costs, cost savings, or joint profit for this group of players when they coordinate their actions. In the theory of cooperative games, an important question is how to allocate this associated amount when all players decide to cooperate. An allocation rule identifies how to divide this amount amongst the participating players for a class of cooperative games. In the theory of cooperative games, one is interested in allocation rules with appealing fairness properties. A well-known concept within the theory of cooperative games is the so-called core. The core is the set of all allocations that are efficient, i.e., for which the total amount is divided completely, and stable, i.e., for which it is not profitable (with respect to the allocation) for any coalition to leave the others and to work on its own. Indeed, under such an efficient and stable allocation, no individual player, nor any subgroup of players wants to split off from the collaboration. Unfortunately, the existence of such efficient and stable allocations cannot be guaranteed in general. In this monograph, we primarily focus on core non-emptiness for various types of cooperative games originating from underlying resource pooling situations. We formulate these resource pooling games in terms of costs, (additional) profit, or cost savings directly. Note that cost savings and profit can be treated similarly. Besides our interest for core non-emptiness of various types of resource pooling games, we also study several allocation rules and investigate whether these satisfy interesting fairness properties.

In addition to the theory of cooperative games, we make use of several operations research modelling techniques to describe the underlying resource pooling situations. For instance, we make use of graph theory to describe the positioning of resources in a geographical network and Markov decision processes in order to formulate various spare parts pooling strategies. Moreover, we use these operations research modelling techniques to analyse (various properties of) these cooperative games.

1.4 Overview and contribution of the monograph

In this section, we give an overview of this monograph and simultaneously describe our main contributions. We start in chapter 2 with mathematical preliminaries to make this monograph self-contained. In particular, these preliminaries are required for the analysis of the cooperative games that arise from our resource pooling situations. In chapter 3, we give an overview of operations research games. These games originate from underlying operations research situations. The cooperative games that will be introduced and analysed in this monograph fit within this class of games.

In chapter 4, we investigate a situation in which several service providers may increase profit by pooling their critical, low-utilization resources that are subject to unavailability. Note that these service providers may represent contractors who each keep a single tamping machine. For the associated game, which we call an availability game, we show that there always exists an allocation of the joint profit that cannot be improved upon by any coalition. This implies that the so-called core of the associated game is non-empty. Moreover, we present an allocation of the joint profit for every possible coalition such that each player's payoff increases as the coalition to which the player belongs to grows larger. This implies that we formulate a so-called population monotonic allocation scheme. In addition, we present conditions that ensure that each player's marginal contribution increases as the coalition to which this player belongs to grows larger. Under these conditions, availability games are so-called convex. We also introduce four different allocation rules and investigate whether the allocations resulting from those allocation rules are increasing in the availability and in the profit function. Furthermore, we investigate whether the allocations resulting from those allocation rules are the same for players who are similar in the underlying situation. A similar approach is taken for players who are equivalent in terms of the associated availability game. Finally, we also investigate whether the allocations resulting from those allocation rules are members of the core.

Chapter 5 is devoted to the analysis of an environment in which several service providers each keep a single spare part in stock to avoid downtime of their technical systems. The costs related to the downtime of these technical systems may differ per service provider. Note that these service providers may represent contractors who keep spare parts in stock for a specific railway segment and deal with different penalty costs specified in their performance based contracts. The service providers can reduce total downtime costs by pooling their spare parts according to a critical level policy. Under a critical level policy, the number of spare parts in the on-hand stock determines which service providers are allowed to satisfy demand. In chapter 5, we select a specific critical level policy: if k spare parts are left in the on-hand stock, the k players with highest downtime costs are allowed to satisfy demand only. So, players are added one-by-one to the group of players that are allowed to satisfy demand for an increasing number of spare parts in the on-hand stock. We refer to this form of pooling as one-by-one pooling. For the associated game, which we call a one-by-one pooling game, we first derive necessary and sufficient conditions for convexity. In addition, we show that the values of one-by-one pooling games can be recognized as convex combinations of values of Böhm-Bahwerk horse market games. Such a Böhm-Bahwerk horse market game is a game associated with a two-sided market for homogenous goods, e.g., horses. In this market, there are sellers that each have one good for sale and buyers that each want to buy one good. By cooperating amongst these two types of players, a joint profit can be realized. It is well-known that the core of a Böhm-Bawerk horse market game is non-empty. As a consequence, we can show that one-by-one pooling games have a non-empty core as well. In addition, we present a class of allocation rules for which the resulting allocations are core members. Last, we study a simple and intuitive allocation rule within this class of allocation rules that satisfies appealing fairness properties.

The results of chapter 5 show that pooling of resources, e.g., spare parts, according to a smart pooling strategy may pay off. First of all, it can decrease total downtime costs significantly. Secondly, it can create incentives for all service providers to collaborate. It is interesting to investigate whether this holds for other types of resource pooling strategies as well. In chapter 6, we investigate this aspect for a resource pooling situation that has quite some similarities with the one of chapter 5. However, this time, we assume that service providers collaborate by pooling their spare parts according to an optimal pooling policy, i.e., a pooling policy that minimizes the long-run average downtime costs! It turns out that this optimal pooling policy has the form of a critical level policy and thus, players are allowed to satisfy demand only if the number of spare parts in the on-hand stock is above a certain threshold. So, in fact, there is some kind of a stratification that determines when players are allowed to make use of the pool. For that reason, we refer to this form of optimal spare parts pooling as stratified pooling. We want to mention that, although spare parts are pooled in a stratified way, the optimal spare parts pooling strategy can change per situation and coalition which makes the mathematical analysis in chapter 6 quite challenging. For the associated cooperative game, which we call a stratified pooling game, we use that the underlying spare parts situation can be described by a Markov decision process and the optimal pooling strategy as a stationary decision rule in this Markov decision process. In particular, we use Markov decision processes to prove core non-emptiness of stratified pooling games.

In the next chapter of this monograph, chapter 7, we study a situation with several service providers who each may or may not own a single resource to cover their region completely. Covering a region (of a service provider) leads to a certain profit. The service providers can increase total covering (and so profit) by pooling their resources.

Note, these service providers may represent (a part of) a region of a contractor with or without a repair van. Covering a contractor's region implies that one can avoid penalty costs set by performance based contracts (and so increase profit). For the associated cooperative game, which we call a maximal covering location game, we show that the core may be empty. This implies that there exist situations for which the allocation of the joint profit can always be improved upon by at least one coalition. Although the core may be empty, we provide several types of sufficient conditions that ensure core non-emptiness. These conditions are in terms of the number of players, the type of graph, the number of resources, and an underlying integer linear program. Finally, for each condition, we provide an example showing that when the condition is not satisfied, core non-emptiness is not guaranteed.

In chapter 8, which is the last chapter of this monograph, we summarize all results and present managerial insights. Moreover, we identify directions for future research.

1.5 Readers guide

Although this monograph is foremost written for our colleagues within the field of cooperative game theory and operations research, at least parts of it can be read by others as well. In particular, we advise them to first focus on chapter 1, chapter 3, and chapter 8 of this monograph. In this way, they can obtain a broad overview of what has been realized during the last couple of years. We advise our colleagues and all others interested in the (detailed) mathematical analysis to read the complete monograph. For the main body of this monograph, which consists of chapter 4, chapter 5, chapter 6, and chapter 7, we have some recommendations as well. From a topic perspective, we advise them to read these four chapters in the order in which they are presented. However, from a complexity perspective, we advise them to first read chapter 4 and chapter 7. These two chapters are relatively easy to understand and can be read separately from the rest. Thereafter, we advise them to first read chapter 5 and subsequently chapter 6. These chapters can be seen as quite challenging. In particular, chapter 6 can be seen as the most challenging one of this monograph.

Chapter 2

Preliminaries

2.1 Introduction

In this chapter, we introduce some basics of cooperative game theory. Thereafter, we discuss bipartite undirected graphs, integer linear programming problems, and linear programming problems. Finally, we discuss discrete time Markov decision processes.

2.2 Cooperative game theory

In this section, we introduce some basic elements of cooperative game theory. We start in section 2.2.1 with basics of cooperative games. Thereafter, we introduce in section 2.2.2 the concept of the core and balancedness. In section 2.2.3, we discuss allocation rules and allocation schemes and subsequently give an overview of implications in section 2.2.4. Finally, we describe cooperative cost games in section 2.2.5.

2.2.1 Cooperative games

A *cooperative game with transferable utility*, which we will refer to as a *game*, is a pair (N, v), where N is a non-empty, finite *player set* and $v : 2^N \rightarrow \mathbb{R}$ a *characteristic function* with $v(\emptyset) = 0$. A non-empty subset $M \subseteq N$ is a *coalition* and $v(M)$ is the *value*[1] that coalition M can obtain by itself. This value can be transferred freely among the players. The set N is sometimes called the *grand coalition*. For a given coalition $M \subseteq N$, the *subgame* (M, v_M) is the game with player set M and characteristic function v_M such that $v_M(K) = v(K)$ for all $K \subseteq M$. A game (N, v) is *superadditive* if the value of the union of any two disjoint coalitions is larger than or equal to the sum of the values of these disjoint coalitions, i.e., $v(M) + v(K) \leq v(M \cup K)$ for all $M, K \subseteq N$ with $M \cap K = \emptyset$ and *monotonic* if the value of every coalition is at least the value of any of its subcoalitions, i.e., $v(M) \leq v(K)$ for all $M, K \subseteq N$ with $M \subseteq K$. A game (N, v) is *convex* if the marginal contribution of any player

[1] This value may, for instance, represent a (joint) profit or a (joint) cost saving.

to any coalition is less than or equal to his marginal contribution to a larger coalition, i.e., $v(M \cup \{i\}) - v(M) \leq v(K \cup \{i\}) - v(K)$ for all $i \in N$ and all $M \subseteq K \subseteq N \backslash \{i\}$ and *zero-normalized* if $v(\{i\}) = 0$ for all $i \in N$.

Example 2.1. *Consider three employees (Janneke, Tim, and Rick) working at an old cheese factory. They all live in the same neighbourhood, but all travel by car individually. Janneke initiates to carpool to save some kilometers (€1/km). Carpooling means that one employee uses his (or her) car to pick up and bring home the other one(s). The one who picks up and brings home everyone in the least number of kilometers will be the driver. In figure 2.1, an overview of the situation (with the position of all employees and the cheese factory) is presented.*

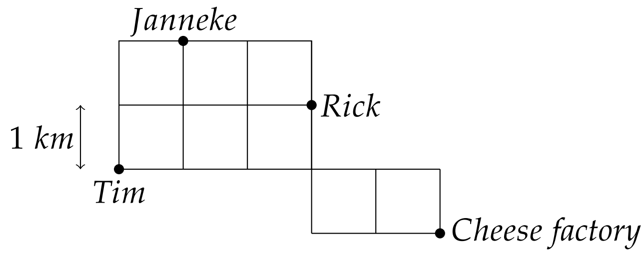

Figure 2.1: Graphical representation of the situation.

In the situation without carpooling, Janneke pays €14 euro per day, Tim pays €12 per day, and Rick pays €8 per day. If Janneke and Tim carpool, the costs are €18 per day (with Janneke as driver). If Janneke and Rick carpool, the costs are €14 per day (with Janneke as driver). If Tim and Rick carpool, the costs are €16 per day (with Tim as driver). If all employees carpool, the costs are €20 per day (with Tim as driver). So, if they all cooperate, they can save €14. This situation can be modeled by a game (N, v) with player set $N = \{1, 2, 3\}$ with 1 representing Janneke, 2 representing Tim, and 3 representing Rick, and a characteristic cost saving function

$$v(M) = \begin{cases} 0 & \text{if} \quad M \in \{\{1\}, \{2\}, \{3\}\} \\ 4 & \text{if} \quad M = \{2, 3\} \\ 8 & \text{if} \quad M \in \{\{1, 2\}, \{1, 3\}\} \\ 14 & \text{if} \quad M = N. \end{cases}$$

This cooperative game is zero-normalized, monotonic and superadditive, but not convex. ◇

2.2.2 The core and balancedness

An *allocation* for a game (N, v) is an N-dimensional vector $x \in \mathbb{R}^N$ where player $i \in N$ receives x_i. An allocation is *efficient* if the value of the grand coalition N is divided completely amongst the players in N, i.e., if it holds that

$$\sum_{i \in N} x_i = v(N).$$

An allocation is *individual rational* if every player gets at least as much as what this player could obtain by staying alone, i.e., if $x_i \geq v(\{i\})$ for all $i \in N$ and *stable* if no group of players has an incentive to split off from the grand coalition N, i.e., if

$$\sum_{i \in M} x_i \geq v(M) \text{ for all } M \subseteq N.$$

The set of efficient and stable allocations of (N, v) is the *core* of (N, v) and denoted by

$$\mathscr{C}(N,v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = v(N), \;\; \sum_{i \in M} x_i \geq v(M) \;\text{ for all }\; M \subseteq N \right\}.$$

An efficient and stable allocation is called a *core allocation*. The core is a convex set and either empty or non-empty. In the latter case, it has exactly one or infinitely many core allocations. The following two examples illustrate the concept of the core.

Example 2.2. *Consider game (N, v) of example 2.1. Allocation $x = (13, 0, 1)$ is an efficient and individual rational allocation, but not stable as $x_2 + x_3 = 1 < 4 = v(\{2, 3\})$. It can be checked easily that allocation $x = (8, 2, 4)$ is efficient and stable. In figure 2.2 the set of all individual rational and efficient allocations is represented as a triangle in barycentric coordinates. In addition, the core is represented as well.* ◇

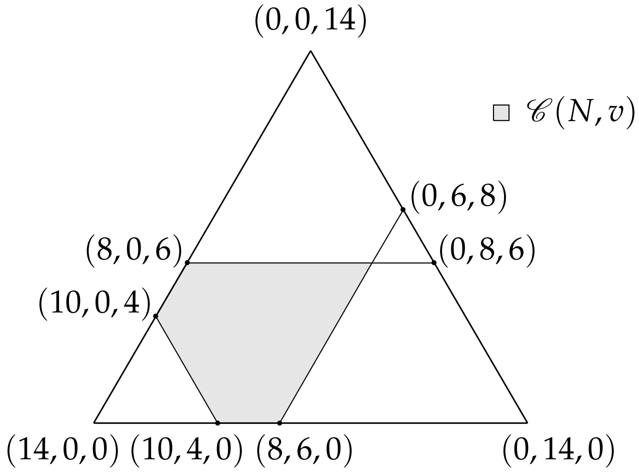

Figure 2.2: Graphical representation of the core of game (N, v).

Example 2.3. *Primary school Triangel wants to order a single bus (with 40 seats at a cost of €36) for a regional activity upcoming Thursday. As the total number of pupils is only 12, teacher Bert proposes to cooperate with other schools (that participate in the regional activity as well) to save money. He decides to call teacher Liesbeth of primary school Maasvlecht and teacher Johan of primary school Sint Michaël. It turns out that Liesbeth needs 26 seats and Johan needs 10 seats. So, if the three primary schools cooperate, they only need to order two buses and save*

€36. This situation can be modeled by a game (N, v) with player set $N = \{1, 2, 3\}$ with 1, 2, and 3 representing primary school Triangel, primary school Maasvlecht, and primary school Sint Michaël, respectively, and characteristic cost savings function

$$v(M) = \begin{cases} 0 & \text{if } M \in \{\{1\}, \{2\}, \{3\}\} \\ 36 & \text{otherwise.} \end{cases}$$

Now, suppose $x \in \mathscr{C}(N, v)$. By efficiency, it holds that $x_1 = 36 - (x_2 + x_3)$ and by stability that $x_2 + x_3 \geq 36$. Hence, $x_1 = 36 - (x_2 + x_3) \leq 36 - 36 = 0$. Similarly, we can obtain that $x_2 \leq 0$ and $x_3 \leq 0$. This conflicts with efficiency as $x_1 + x_2 + x_3 \leq 0 < 36 = v(\{1, 2, 3\})$ and thus we can conclude that the core of the corresponding game (N, v) is empty. ◇

There exist several sufficient and necessary conditions for a game to have a non-empty core. For instance, Bondareva [7] and Shapley [56] investigated sufficient and necessary conditions for games to have a non-empty core in terms of balanced maps. In order to describe these conditions, let N be a finite player set. We call a map $\kappa : 2^N \backslash \{\emptyset\} \to [0, 1]$ a *balanced map* for N if

$$\sum_{M \in 2^N :\, i \in M} \kappa(M) = 1 \quad \text{for all } i \in N.$$

So, a balanced map κ for N allocates weights (between zero and one) to coalitions in such a way that for each player $i \in N$ the sum of the weights of the coalitions to which this player belongs to equals one.

Example 2.4. *Consider game (N, v) with $N = \{1, 2, 3\}$ and map κ, given by*

$$\kappa(M) = \begin{cases} \frac{1}{4} & \text{if } \ M \in \{\{1\}, \{2, 3\}\} \\ \frac{3}{4} & \text{if } \ M \in \{\{3\}, \{1, 2\}\} \\ 0 & \quad \text{otherwise.} \end{cases}$$

For player 1 we have $\sum_{M \in 2^N \,:\, 1 \in M} \kappa(M) = \kappa(\{1\}) + \kappa(\{1, 2\}) = \frac{1}{4} + \frac{3}{4} = 1$. Similarly, this holds for (the equality related to) player 2 and player 3 as well and thus map κ is balanced. ◇

A collection $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ is called *balanced* if there exists a balanced map κ for which $\kappa(M) > 0$ for all $M \in \mathscr{B}$ and $\kappa(M) = 0$ otherwise. A collection $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ is called *minimal balanced* if there exists no proper subcollection of $\mathscr{B}$ that is also balanced.

Example 2.5. *Consider game (N, v) of example 2.4. Collection $\mathscr{B} = \{\{1\}, \{3\}, \{1, 2\}, \{2, 3\}\}$ is balanced as map κ, as presented in example 2.4, is balanced. However, this collection is not minimal balanced as there exists a proper subcollection, namely $\mathscr{B}' = \{\{1\}, \{2, 3\}\} \subset \mathscr{B}$, for which one can construct an associated balanced map κ' with $\kappa'(M) = 1$ for $M \in \{\{1\}, \{2, 3\}\}$ and $\kappa'(M) = 0$ otherwise, that is balanced as well.* ◇

An advantage of minimal balanced collections is that for every minimal balanced collection $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ there exists exactly one associated balanced map κ (Peleg and Sudhölter [41]). Moreover, a game (N, v) is called *balanced* if for every minimal balanced collection $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ with associated balanced map κ it holds that

$$\sum_{M \in \mathscr{B}} \kappa(M) \cdot v(M) \leq v(N). \tag{2.1}$$

Now, we are able to present a sufficient and necessary condition for core non-emptiness due to Bondareva [7] and Shapley [56].

Theorem 2.1. *A game (N, v) is balanced if and only if $\mathscr{C}(N, v) \neq \emptyset$.*

Example 2.6. *Consider game (N, v) of example 2.1. For every minimal balanced collection $\mathscr{B} \in \{\{\{1\}, \{2\}, \{3\}\}, \{\{1\}, \{2, 3\}\}, \{\{2\}, \{1, 3\}\}, \{\{3\}, \{1, 2\}\}\}$, the associated balanced map κ is given by $\kappa(M) = 1$ if $M \in \mathscr{B}$ and $\kappa(M) = 0$ otherwise. As $v(\{i\}) = 0$ for all $i \in N$ and $v(M) < v(N)$ for all $M \subseteq N$ with $|M| = 2$, equality (2.1) holds true. In addition to these minimal balanced collections, there exists only one other minimal balanced collection, namely $\mathscr{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ with associated balanced map κ, where $\kappa(M) = \frac{1}{2}$ for all $M \in \mathscr{B}$ and $\kappa(M) = 0$ otherwise. Observe that $\frac{1}{2} \cdot v(\{1, 2\}) + \frac{1}{2} \cdot v(\{1, 3\}) + \frac{1}{2} \cdot v(\{2, 3\}) = \frac{20}{2} < 14 = v(N)$. Hence, game (N, v) is balanced, i.e., has a non-empty core.* $\diamond$

A game (N, v) is called *totally balanced* if game (N, v) and all its subgames are balanced. It holds that any totally balanced game is superadditive and that any convex game is totally balanced. The reverse relationships do not hold true in general.

2.2.3 Allocation rules and allocation schemes

An *allocation rule (defined on games)* is a function f that assigns to any game (N, v) in a class of cooperative game a vector $f(N, v) \in \mathbb{R}^N$ satisfying $\sum_{i \in N} f_i(N, v) = v(N)$. A well known allocation rule defined on the set of all games is the *Shapley value*, proposed by Shapley [55]. This allocation rule can be described in several ways. One is to assign to each player a weighted average over all marginal contributions he can make to any possible coalition. Formally, for any game (N, v) the Shapley value can be defined by

$$\Phi_i(N, v) = \sum_{M \subseteq N \backslash \{i\}} \frac{|M|!(|N| - 1 - |M|)!}{|N|!} \left(v(M \cup \{i\}) - v(M)\right) \quad \text{for all } i \in N.$$

For any game (N, v) that is convex, it holds that $\Phi(N, v) \in \mathscr{C}(N, v)$. However, in the absence of convexity, $\Phi(N, v)$ may fall outside the core. Another well-known allocation rule defined on games is the nucleolus, which has, compared to the Shapley value, the advantage that it belongs to the core as long as the core is non-empty. We refer to Schmeidler [52] for more detailed information.

Finally, we introduce the notion of allocation schemes. For any game (N, v) an *allocation scheme* $y = (y_{i,M})_{M \subseteq N, i \in M}$ specifies an allocation for every coalition. A *population monotonic allocation scheme* (PMAS) for game (N, v), introduced by Sprumont [59], is an allocation scheme $(y_{i,M})_{M \subseteq N, i \in M}$ that is *efficient*, i.e., $\sum_{i \in M} y_{i,M} = v(M)$ for all $M \subseteq N$, and *population monotonic*, i.e., $y_{i,M} \leq y_{i,K}$ for all $M, K \subseteq N$ with $M \subseteq K$ and all $i \in M$. If a game (N, v) admits a PMAS y, then it is totally balanced and its allocation for the grand coalition, $(y_{i,N})_{i \in N}$, is a member of the core. In addition, for convex games the existence of a PMAS is guaranteed. Note that in section 4.4 and section 5.4 of this work, we also focus on allocation rules defined on situations (rather than on games).

Example 2.7. *Consider the cooperative game of example 2.1. The Shapley value is $\Phi(N, v) = (6, 4, 4)$ and it is a core allocation. In addition, a population monotonic allocation scheme, denoted by $y = (y_{i,M})_{M \subseteq N, i \in M}$, is presented in table 2.1. One can check that y is an allocation scheme as for all $M \subseteq N$ it holds that $\sum_{i \in M} y_{i,M} = v(M)$. In addition, one can check that y is population monotonic as $y_{i,M} \leq y_{i,K}$ for all $M, K \subseteq N$ with $M \subseteq K$ and all $i \in M$.* ◇

Table 2.1: a PMAS for cooperative game (N, v).

Coalition M	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$y_{1,M}$	0	$\star$	$\star$	6	6	$\star$	8
$y_{2,M}$	$\star$	0	$\star$	2	$\star$	2	3
$y_{3,M}$	$\star$	$\star$	0	$\star$	2	2	3

2.2.4 Overview of implications

In figure 2.3, an overview of the relationships among the various introduced properties is presented. Reverse relationships in figure 2.3 do not hold true in general.

$$
\begin{tikzcd}
& \Phi(N,v) \in \mathscr{C}(N,v) \arrow[r] & \mathscr{C}(N,v) \neq \emptyset \arrow[r, leftrightarrow] & (N,v) \text{ is balanced} \\
(N,v) \text{ is convex} \arrow[ur] \arrow[dr] & & & \\
& (N,v) \text{ admits a PMAS} \arrow[r] & (N,v) \text{ is totally balanced} \arrow[uu] \arrow[r] & (N,v) \text{ is superadditive}
\end{tikzcd}
$$

Figure 2.3: Implication between properties for any game (N, v).

2.2.5 Cooperative cost games

In the previous sections, we presented several examples in which players could save costs by coordinating their actions, e.g., by car pooling. For these situations, we formulated a cooperative game, with the value of each coalition representing a cost saving. Oppositely, the value of each coalition can also represent a cost. In that case, we call such a game a (cooperative) *cost game*. For cost games, the characteristic function is usually denoted by c instead of v where $c(M)$ represents the costs of coalition $M \subseteq N$. Now, we introduce some concepts for cost games, each corresponding to a concept for cooperative games. A cost game (N, c) is *subadditive* if $c(M) + c(K) \geq c(M \cup K)$ for all $M, K \subseteq N$ with $M \cap K = \emptyset$ and *concave* if $c(M \cup \{i\}) - c(M) \geq c(K \cup \{i\}) - c(K)$ for all $i \in N$ and all $M \subseteq K \subseteq N \backslash \{i\}$. An allocation $x \in \mathbb{R}^N$ describes the costs allocated to the players. In addition, an allocation is *efficient* if $\sum_{i \in N} x_i = c(N)$. An allocation is *individual rational* if $x_i \leq c(\{i\})$ for all $i \in N$. Moreover, an allocation is *stable* if $\sum_{i \in M} x_i \leq c(M)$ for all $M \subseteq N$. The set of efficient and stable allocation of (N, c) is the *core* of (N, c) and denoted by $\mathscr{C}(N, c) = \left\{x \in \mathbb{R}^N \middle| \sum_{i \in N} x_i = c(N), \sum_{i \in M} x_i \leq c(M) \text{ for all } M \subseteq N\right\}$. A cost game (N, c) is *balanced* if for every minimal balanced collection $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ with associated balanced map κ it holds that $\sum_{M \in \mathscr{B}} \kappa(M) \cdot c(M) \geq c(N)$. In line with theorem 2.1, a cost game (N, c) is balanced if and only if $\mathscr{C}(N, c) \neq \emptyset$. Finally, for each cost game (N, c) we can formulate a *cost savings game* as follows

$$v(M) = \sum_{i \in M} c(\{i\}) - c(M) \quad \text{for all } M \subseteq N.$$

Note that each cost savings game (N, v) is a cooperative game and zero-normalized by definition. There is an important relationship between cost games and cost savings games. Let (N, c) be a cost game and (N, v) be the associated cost savings game. Then $\mathscr{C}(N, c) \neq \emptyset$ if and only if $\mathscr{C}(N, v) \neq \emptyset$. We use this result in the upcoming example.

Example 2.8. *Consider the situation of example 2.3. This situation can also be modelled as a cost game (N, c) with player set $N = \{1, 2, 3\}$ and characteristic cost function*

$$c(M) = \begin{cases} 0 & \text{if } |M| = 0 \\ 36 & \text{if } |M| \in \{1, 2\} \\ 72 & \text{if } |M| = 3. \end{cases}$$

Suppose $x \in \mathscr{C}(N, c)$. By efficiency, it holds that $x_1 = 72 - (x_2 + x_3)$ and by stability that $x_2 + x_3 \leq 36$. Hence, $x_1 = 72 - (x_2 + x_3) \geq 72 - 36 = 36$. Similarly, we can obtain that $x_2 \geq 36$ and $x_3 \geq 36$. This conflicts with efficiency as $x_1 + x_2 + x_3 \geq 108 > 72 = c(\{1, 2, 3\})$ and thus $\mathscr{C}(N, c) = \emptyset$. We could have concluded this immediately as for cost savings game (N, v), which corresponds to the game in example 2.3, it holds that $\mathscr{C}(N, v) = \emptyset$. $\diamond$

2.3 Bipartite undirected graphs

In this section, we present some concepts of bipartite undirected graphs. An *undirected graph* is a pair $\mathscr{G} = (N, E)$ with N a non-empty, finite set and $E \subseteq \{\{i, j\} | i, j \in N, i \neq j\}$ a set of unordered pairs. The elements of N are called *nodes*, and the elements of E are called *edges*. An undirected graph (N, E) is called *bipartite* if there exists an $N_1, N_2 \subseteq N$ with $N_1 \cap N_2 = \emptyset$ and $N_1 \cup N_2 = N$ for which it holds that $E \subseteq \{\{i, j\} | i \in N_1, j \in N_2\}$.

Example 2.9. *Consider undirected graph $\mathscr{G} = (N, E)$ with nodes $N = \{1, 2, 3, 4, 5\}$ and edges $E = \{\{1, 4\}, \{2, 5\}, \{3, 5\}\}$. Take $N_1 = \{1, 2, 3\}$ and $N_2 = \{4, 5\}$. Then, it can be checked easily that $E \subset \{\{i, j\} | i \in N_1, j \in N_2\}$ and thus the undirected graph is bipartite.* ⋄

For notational convenience, we refer to a *bipartite undirected graph* as $\mathscr{G} = (N, L, E)$ with N and L representing two disjoint sets of nodes with $E \subseteq \{\{i, j\} | i \in N, j \in L\}$. A bipartite undirected graph $\mathscr{G} = (N, L, E)$ is called *isomorphic* to another bipartite undirected graph $\widetilde{\mathscr{G}} = (\widetilde{N}, \widetilde{L}, \widetilde{E})$ if there is a one-to-one correspondence between the nodes in N and $\widetilde{N}$ and a one-to-one correspondence between the nodes in L and $\widetilde{L}$ such that a node in N and a node in L are connected via an edge in E if and only if the corresponding nodes in $\widetilde{N}$ and $\widetilde{L}$ are connected via an edge in $\widetilde{E}$. Now, we present an example with two bipartite undirected graphs that turn out to be isomorphic.

Example 2.10. *Consider bipartite undirected graph $\mathscr{G}_1 = (N_1, L_1, E_1)$ with $N_1 = \{1, 2, 3\}$, $L_1 = \{4, 5, 6\}$, and $E_1 = \{\{1, 4\}, \{1, 6\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{3, 6\}\}$ and bipartite undirected graph $\mathscr{G}_2 = (N_2, L_2, E_2)$ with $N_2 = \{3, 2, 5\}$, $L_2 = \{4, 7, 1\}$, and $E_2 = \{\{3, 7\}, \{3, 4\}, \{5, 7\}, \{5, 1\}, \{2, 1\}, \{2, 4\}\}$. In figure 2.10, a graphical representation of each graph is presented.*

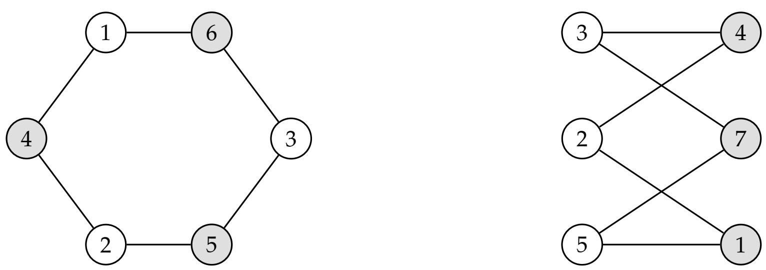

Figure 2.4: Two bipartite undirected graphs.

Bipartite undirected graph $\mathscr{G}_1$ and bipartite undirected graph $\mathscr{G}_2$ are isomorphic as node 1 of graph $\mathscr{G}_1$ corresponds to node 3 of graph $\mathscr{G}_2$ and nodes 2, 3, 4, 5, and 6 of graph $\mathscr{G}_1$ correspond to nodes 5, 2, 7, 1, and 4 of graph $\mathscr{G}_2$, respectively. Note, there are more options for correspondences. ⋄

2.4 Linear programming and integer linear programming

In this section, we present some basic concepts of linear programming problems and integer linear programming problems. Let $A \in \mathbb{R}^{w \times z}$, $b \in \mathbb{R}^w$ and $c \in \mathbb{R}^z$. Then, a *linear programming problem* is given by

$$\max\{c^T x \mid x \geq 0, Ax \leq b\}. \tag{2.2}$$

A solution $x \in \mathbb{R}^z$ is called *feasible* if $Ax \leq b$ and $x \geq 0$ simultaneously. A feasible solution $x \in \mathbb{R}^z$ is called *optimal* if $c^T x \geq c^T \widehat{x}$ for all feasible $\widehat{x} \in \mathbb{R}^z$. Sometimes, it turns out to be useful to deal with the *dual* of the original linear programming problem. The *dual* of a linear programming problem, as formulated in equation (2.2), is defined as

$$\min\{y^T b \mid y \geq 0,\ A^T y \geq c\}.$$

There is a well-known relationship between the original linear programming problem and its dual, also known as the *duality theorem* of linear programming.

Theorem 2.2. *Let $A \in \mathbb{R}^{w \times z}$, $b \in \mathbb{R}^w$, and $c \in \mathbb{R}^z$. Then*

$$\max\{c^T x \mid x \geq 0,\ Ax \leq b\} = \min\{y^T b \mid y \geq 0,\ A^T y \geq c\}$$

provided that both sets are non-empty.

Proof **:** See Schrijver [53, p. 90].

A *square submatrix* of a matrix $A \in \mathbb{R}^{w \times z}$ is a matrix $A' \in \mathbb{R}^{q \times q}$ formed by selecting q rows and q columns from the matrix A. Moreover, a matrix is *totally unimodular* if every square submatrix of A has determinant equal to $+1, -1$ or 0. If a linear programming problem has a totally unimodular matrix $A \in \mathbb{R}^{w \times z}$ and a vector $b \in \mathbb{Z}^w$ with only integer values, its related optimal solution is integer valued as well.

Theorem 2.3. *Let $A \in \mathbb{R}^{w \times z}$ be totally unimodular, $b \in \mathbb{Z}^w$, and $c \in \mathbb{R}^z$. Then the linear programming problem*

$$\max\{c^T x \mid x \geq 0,\ Ax \leq b\}$$

has integer optimal solutions, whenever it has a finite optimum.

Proof **:** See Wolsey [65, p. 40].

An *integer linear programming problem* is a linear programming problem with integrality constraints for all decision variables. In general, integer linear programming problems are much harder to solve than linear programming problems. However, some can be solved easily. For instance, this holds for integer linear programming problems from which one can derive an alternative linear programming problem where (i) the integrality constraint is relaxed, (ii) an *integer* solution is optimal, and (iii) the optimal value coincides with the optimal value of the original integer linear programming problem. Note, total unimodularity can play an important role here (see theorem 2.3).

2.5 Discrete time Markov decision processes

In this section, we present some basic concepts of (discrete time) Markov decision processes (MDP's). An MDP is a mathematical framework for modelling sequential decision problems under uncertainty. Consider a set $T = \mathbb{N} \cup \{0\}$ of decision epochs, a countable set $\mathscr{Y}$ of states, a finite set $\mathscr{A}(y)$ of actions for each $y \in \mathscr{Y}$, non-negative cost $C(y,a)$ for each $y \in \mathscr{Y}$ and all $a \in \mathscr{A}(y)$, and transition probabilities $p(y'|y,a)$ for all $y' \in \mathscr{Y}$, all $y \in \mathscr{Y}$, and all $a \in \mathscr{A}(y)$ with $\sum_{y' \in \mathscr{Y}} p(y'|y,a) = 1$ for all $y \in \mathscr{Y}$ and all $a \in \mathscr{A}(y)$. Tuple $(T, \mathscr{Y}, \mathscr{A}, C, p)$ with $\mathscr{A} = (\mathscr{A}(y))_{y \in \mathscr{Y}}$, $C = (C(y,a))_{y \in \mathscr{Y}, a \in \mathscr{A}(y)}$, and $p = (p(y'|y,a))_{y',y \in \mathscr{Y}, a \in \mathscr{A}(y)}$ is called a (discrete time) Markov decision process.

Let $t \in T$ be a time epoch. A *decision rule* $\omega_t = (\omega_t(y))_{y \in \mathscr{Y}}$ indicates for all states $y \in \mathscr{Y}$ which action to choose at time epoch t. In addition, a *policy* $\omega = (\omega_t)_{t \in T}$ is a sequence of decision rules for all time epochs. Let X_t with $t \in \mathbb{N} \cup \{0\}$ be a random variable that indicates the state at time t. Note that X_t depends on ω and X_0. If initially $X_0 = y \in \mathscr{Y}$, the *long-run average cost per time epoch* under policy ω are

$$J_\omega(y) = \limsup_{n \to \infty} \frac{1}{n} \mathbb{E}_\omega \left[\sum_{t=0}^{n-1} C(X_t, \omega(X_t)) | X_0 = y \right].$$

Let Ω be the set of all policies and $J^*(y) = \inf_{\omega \in \Omega} J_\omega(y)$ for all $y \in \mathscr{Y}$. There exists a class of MDP's for which there exists a constant J^* such that $J^* = J^*(y)$ for all $y \in \mathscr{Y}$. In that case, J^* is defined as the *minimal long-run average cost per time epoch*. A policy $\omega \in \Omega$ is *optimal* if $J_\omega(y) = J^*$ for all $y \in \mathscr{Y}$. A policy is *stationary* if there exists an f such that $\omega_t = f$ for all $t \in T$. We denote such a (stationary) policy by $f = (f(y))_{y \in \mathscr{Y}}$.

For an MDP, the *value function* $V_t(y)$ for all $y \in \mathscr{Y}$ and all $t \in T$ is defined by

$$V_{t+1}(y) = \min_{a \in \mathscr{A}(y)} \left\{ C(y,a) + \sum_{y' \in \mathscr{Y}} p(y'|y,a) \cdot V_t(y') \right\}, \tag{2.3}$$

with $V_0(y) = 0$ for all $y \in \mathscr{Y}$.

There exists an important result which states that, under two conditions, the minimal long-run average cost per time epoch exist, are attained under a stationary policy and moreover, coincide with the limit of the value function divided by the number of time epochs, when time goes to infinity. In these conditions, one refers to *irreducible* Markov chains and *positive recurrent* Markov chains. A Markov chain, which is attained under a given policy, is said to be *irreducible* if for all states $y \in \mathscr{Y}$ it holds that it can be reached from each other state $y' \in \mathscr{Y} \backslash \{y\}$. A Markov chain is said to be *positive recurrent* if for all states $y \in \mathscr{Y}$ it holds that the expected return time (to state y) is finite.

Theorem 2.4. *Let $(T, \mathscr{Y}, \mathscr{A}, C, p)$ be an MDP. If (i) there exists a stationary policy f inducing an irreducible and positive recurrent Markov chain on $\mathscr{Y}$, and satisfying $J_f(y) < \infty$ for all $y \in \mathscr{Y}$, and (ii) there exists an $\varepsilon > 0$ such that $\{y \in \mathscr{Y} | \exists a \in \mathscr{A}(y) : C(y,a) < J_f + \varepsilon\}$ is finite, then*

$$J^* = \lim_{t \to \infty} \frac{V_t(y)}{t} \text{ for all } y \in \mathscr{Y}, \tag{2.4}$$

and moreover, there exists an optimal stationary policy.

Proof : See proposition 4.3 of Sennott [54].

In this work, we restrict our attention to MDP's with finite state spaces. This implies that it becomes superfluous to check whether an irreducible Markov chain is positive recurrent as every irreducible Markov chain is positive recurrent in a finite state space (Modica and Poggiolini [34, Theorem 5.71 (ii)]). Moreover, the second condition of theorem 2.4 is always satisfied when $\mathscr{Y}$ is finite. We end this section with an example.

Example 2.11. *A well-motivated PhD-student is subject to the following behaviour; (i) after each hard working day, a sleepy day is needed to recover completely, (ii) after each sleepy day, the PhD-student can decide to work normal the next day, or to work hard again, (iii) after each normal day the student can plan to (a) continue working normal, but with probability $1/4$ he feels too jaded and instead of working normal a sleepy day is experienced, or (b) start working hard again, but with probability $7/8$ he feels too much stress and instead of working hard a sleepy day is experienced. During hard working days, a PhD student is productive 100% of the time, during normal days 50% of the time and during sleepy days 0% of the time. The criterium of the PhD-student is to minimize these non-productive moments in the long-run. It is unclear whether the PhD-student should (plan to) work hard or normal after a sleepy (normal) day in order to minimize the non-productive moments. For that reason, we set up an MDP.*

The state space is given $\mathscr{Y} = \{0, 1, 2\}$ where 0 corresponds to a sleepy day, 1 corresponds to a normal working day and 2 corresponds to a hard working day. The action space for the sleepy day is given by $\mathscr{A}(0) = \{1, 2\}$ where the actions reflect what he plans to do tomorrow, i.e., 1 *corresponds to a normal day and 2 corresponds to a hard working day. Similarly, the action space of the normal day is given by* $\mathscr{A}(1) = \{1, 2\}$ *and the action space of the hard working day by* $\mathscr{A}(2) = \{0\}$*, where 0 corresponds to a sleepy day. Moreover, for a sleepy day the transition probabilities are given by* $p(1|0,1) = p(2|0,2) = 1$ *and* $p(0|0,1) = p(2|0,1) = p(0|0,2) = p(1|0,2) = 0$*. For a normal day the transition probabilities are given by* $p(1|1,1) = \frac{3}{4}$*,* $p(0|1,1) = \frac{1}{4}$*,* $p(2|1,2) = \frac{1}{8}$*,* $p(0|1,2) = \frac{7}{8}$ *and* $p(2|1,1) = p(1|1,2) = 0$*. For a hard working day the transition probabilities are given by* $p(0|2,0) = 1$*,* $p(1|2,0) = 0$*, and* $p(2|2,0) = 0$*. Moreover, the costs obtained per state are* $C(0,1) = C(0,2) = 1$*,* $C(1,1) = C(1,2) = 0.5$ *and* $C(2,0) = 0$*. Note that these numbers represent the non-productivity per day. In figure 2.5 a graphical representation of the MDP of the PhD-student is presented.*

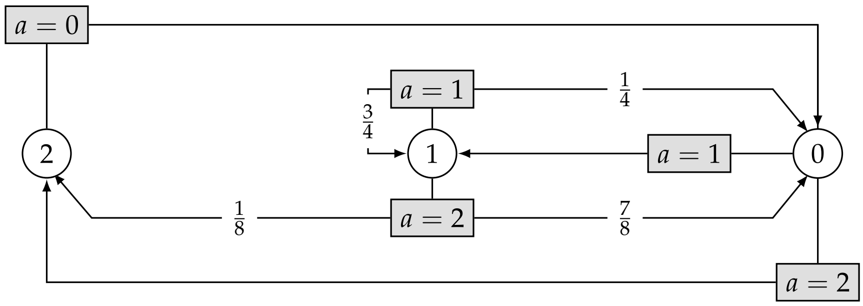

Figure 2.5: Graphical representation of MDP.

The value function is given by

$$
\begin{aligned}
V_{t+1}(0) &= 1 + \min\{V_t(1), V_t(2)\} \\
V_{t+1}(1) &= \frac{1}{2} + \min\left\{\frac{1}{8}V_t(2) + \frac{7}{8}V_t(0), \frac{3}{4}V_t(1) + \frac{1}{4}V_t(0)\right\} \\
V_{t+1}(2) &= V_t(0),
\end{aligned}
$$

for all $t \in \mathbb{N} \cup \{0\}$ *and* $V_0(0) = V_0(1) = V_0(2) = 0$*. Under stationary policy* f *with* $f(0) = 1$*,* $f(1) = 2$ *and* $f(2) = 0$ *it holds that* (i) *state* 0 *can be reached from state 1 and 2,* (ii) *state* 1 *can be reached by state 0 and 2, and* (iii) *state 2 can be reached by state 0 and 1, i.e., one obtains an irreducible, positive recurrent Markov chain, with corresponding costs that are bounded above by one (being non-productive always). Following theorem 2.4, there exists a stationary optimal policy and the minimal long-run average costs are given by* $\lim_{t\to\infty} V_t(0)/t$*. It can be checked easily that under stationary policy* f *with* $f(0) = 2$*,* $f(1) = 2$ *and* $f(2) = 0$ *the minimal long-run average costs are obtained and additionally that these costs equal 0.5. Note that it is optimal for the PhD-student to alternate between sleepy days and hard working days.* $\diamond$

Chapter 3

Literature review

3.1 Introduction

In the introduction of this monograph several types of operations research situations are introduced in which various parties can benefit by pooling their resources. The problem of how to distribute joint profit, costs, or cost savings amongst the parties can be addressed by using concepts of cooperative game theory. In literature, cooperative games arising from underlying operations research settings, in which one typically addresses an optimization problem per coalition, are often called OR games[2]. Borm et al. [8] give an overview of these OR games and divide them in five categories, namely connection, routing, scheduling, production, and inventory. In this chapter, we focus on the last category. In addition, we also focus on another interesting class of OR games (which is not included in the overview), namely location games. Finally, we position the resource pooling games we formulate in this monograph in the existing literature.

3.2 Inventory games

The class of inventory games can be divided into two subclasses, namely the subclass of deterministic inventory games (Fiestras-Janeiro et al. [16]) and the subclass of stochastic inventory games (Dror and Hartman [15]). In section 3.2.1, we discuss deterministic inventory games and in section 3.2.2, we discuss stochastic inventory games.

3.2.1 Deterministic inventory games

Well-known games within the subclass of deterministic inventory games are economic order quantity (EOQ) games (Meca et al. [33]) and economic lot sizing (ELS) games

[2]In this chapter, we introduce several OR games directly. Keep in mind that, although we describe these cooperative games directly, we implicitly describe the underlying operations research setting and from that point on describe the associated cooperative game.

(Van den Heuvel et al. [63]). In EOQ games, players face deterministic demand at a constant rate. In order to satisfy this demand, players place orders in a fixed frequency. Every order comes with a fixed ordering cost and holding inventory is costly as well. Players can collaborate by joint ordering and sharing the related (ordering) cost. Meca et al. [33] showed that the core of every EOQ game is non-empty. Several extensions as well as interesting properties of EOQ games have been studied by Meca et al. [32], Anily and Haviv [1], Dror and Hartman [14], and Zhang [69].

In ELS games, players face deterministic demand over a finite number of periods. Each player formulates a production plan that determines in which periods products are produced. There is a fixed cost for each production run and a holding cost (in each period) for products that are produced in preceding production runs. Players can cooperate by constructing and executing an optimal joint production plan. Van den Heuvel et al. [63] showed that ELS games have a non-empty core by using the balancedness conditions of Bondareva [7] and Shapley [56]. In Chen and Zhang [9], Zeng et al. [68], and Drechsel and Kimms [13] several extensions and properties regarding ELS games have been investigated.

3.2.2 Stochastic inventory games

Well-known inventory games within the subclass of stochastic inventory games are newsvendor games and various spare parts games. In newsvendor games, players face stochastic demand for a single item in a single period. Before realization of demand, players have to decide upon the number of items to buy. There is a cost for each item bought and a selling price for each item sold. Players can cooperate by joint ordering and pooling of their items. Müller et al. [37] and Slikker et al. [58] showed that newsvendor games have a non-empty core. We refer to Montrucchio et al. [35] for an overview of several results (and extensions) of newsvendor games.

For spare parts games, Wong et al. [66] are the pioneers. In their work, they investigated a situation in which players maintain a large number of machines, each containing a critical component that fails once in a while. Due to the uncertain nature of these failures, players keep spare parts in stock to prevent downtime of the machines. Players can cooperate by pooling their spare parts. It is assumed that pooling is facilitated by (expensive) lateral transshipments. Wong et al. [66] proposed four cost allocation rules and analysed them in a numerical experiment. Karsten et al. [24] investigated a quite similar spare parts game. However, they assumed that pooling is facilitated by free and instantaneous lateral transshipments. Under this assumption, they were able to provide sufficient and necessary conditions for core non-emptiness. Karsten and Basten [23] built upon the model of Karsten et al. [24] with the exception that backorders are allowed. Under this assumption, again sufficient and necessary conditions for core non-emptiness are derived. In addition, an allocation rule with appealing properties is presented. Guajardo and Rönnqvist [21] investigated a spare parts game with several players having different target service levels. When collaborating, new target service levels are set and full pooling is assumed. They provided a core element, provided that the core is non-empty. Recently, Karsten et al. [25] investigated a more general type of spare parts game, namely a resource pooling game, in which the resources may represent spare parts. In this type of game, players can collaborate by complete pooling of their resources and customer streams into a joint service system. Such a service system is modelled as an M/M/s queueing system that faces a fixed cost rate per server and homogeneous delay costs for waiting customers. Karsten et al. [25] provided several sufficient conditions for core non-emptiness of resource pooling games. This more general framework, in which resource pooling is modelled by queueing systems, has received much attention in literature (see, e.g., Anily and Haviv [3], Yu et al. [67] Garcia-Sanz et al. [17], Timmer and Scheinhardt [62], and González and Herrero [19]).

3.3 Location games

The class of location games can be divided into two subclasses, namely the subclass of games in which one has to position exactly one facility and the subclass of games in which one has to position multiple facilities. In section 3.3.1, we discuss location games with one facility and in section 3.3.2, we discuss location games with multiple facilities.

3.3.1 Single facility location games

In the subclass of location games with one facility, Granot [20] investigated a game in which one has to position a single facility with the objective to minimize the total costs, consisting of the costs of connecting players to the facility and the costs of opening this facility. Granot [20] investigated this type of game for the case in which potential facility locations as well as player locations are modelled as nodes, edges represent the distance between them, and the underlying graph structure is a tree. Puerto et al. [43] investigated the same type of game, except that potential facility locations as well as player locations are represented by points in a continuous space. In particular, Puerto et al. [43] provided several sufficient conditions for core non-emptiness for this type of game. Several extensions and variants of this type of game have been investigated. For instance, Mallozzi [31] included a regional dependent cost for positioning a facility and Puerto et al. [44] investigated a variant by assuming that the objective of the players is to minimize the maximal distance of a player to that facility.

3.3.2 Multiple facility location games

In the subclass of location games with multiple facilities, Kolen [27] investigated a game in which players deal with the problem of positioning several facilities so that the total cost of opening these facilities and costs of connecting the players to (at least) one facility is minimized. Under the assumption that the underlying graph structure is a tree, Kolen [27] showed that an optimal solution of the dual of a relaxation of the integer linear program (that solves the underlying problem) forms a core allocation. Goemans and Skutella [18] extended this result to the general case (without restrictions to the underlying graph structure). In addition, they extended the model by including that facilities can connect to a limited number of players only. They showed that the core is non-empty if and only if there is no integrality gap for a corresponding relaxation of the integer linear programming problem (that solves the underlying problem). Tamir [60] investigated a variant of this game in which facilities can only connect to players within a certain radius. Tamir [60] showed that the set of core allocations coincides with the set of optimal solutions of the dual of a relaxation of the integer linear programming problem (that solves the underlying problem) when the underlying graph structure is a tree. Deng et al. [12] extended this result to the situation with no restrictions on the underlying graph structure.

3.4 Contribution to the literature

Although several OR games have been investigated in the last decades, they do not provide a satisfactory answer to the research question of this monograph. In particular, for pooling of resources with characteristics as being critical, low-utilized and sometimes unavailable (see, e.g., section 1.1.1), no cooperative games have been formulated at all. Indeed, formulating a cooperative game corresponding to such underlying situation is new. In chapter 4, we formulate such a cooperative game and focus on several interesting properties they might satisfy. For pooling spare parts (see, e.g., section 1.1.2), quite some spare parts pooling games have been investigated. However, in all of these spare parts pooling games, full pooling is applied. This means that everyone can use spare parts from the pool whenever needed. For some spare parts pooling situations, e.g., the one in section 1.1.2, there may be reasons to apply another (and preferably smarter) pooling strategy. In this monograph, we formulate two types of spare parts pooling games in which different spare parts pooling strategies are applied. In chapter 5, we introduce the first spare parts pooling game in which players are assumed to apply a fixed critical level policy. We mainly focus on core non-emptiness of this game and the development of an allocation rule with appealing fairness properties. In chapter 6, we introduce the second spare parts pooling game in which players are assumed to apply an optimal pooling policy. This time, we only focus on core non-emptiness. Finally, for pooling resources that cover certain regions (see, e.g., section 1.1.3), results of (already studied) location games are not helpful as well. In particular, these types of location games all focus on the positioning of exactly one facility (which can be seen as a resource) or the positioning of multiple facilities to connect with (or cover) all locations. To the best of our knowledge, not any location game focusses on the situation in which a fixed number of facilities has to be positioned that, in total, not necessarily cover all the locations. In this monograph, we formulate a location game with such an assumption. Again, we mainly focus on core non-emptiness of this location game.

Chapter 4

Pooling of critical, low-utilization resources with unavailability

4.1 Introduction

In this chapter, which is based on Schlicher et al. [47], we investigate situations in which several independent service providers keep the same type of critical, low-utilization resources that are subject to unavailability. For instance, one can think of the Dutch railway setting of section 1.1.1 in which several contractors each own a critical, low-utilized tamping machine that is sometimes unavailable. One can also think of a setting with several maintenance companies, each having one repairman with specific knowledge for one and the same type of highly profitable machine. Repairmen are critical resources as they repair these machines. As machines break down only a few times per year and repair takes some hours only, utilization of repairmen is relatively low. However, due to, for instance, vacation or illness, repairmen may be unavailable for several weeks. In both examples, it is possible that demand occurs for an unavailable resource. For the railway setting, this leads to an inoperative part of the railway network and as a consequence to high social costs. For the specialized repairmen setting, this leads to downtime of the machine, which may be very costly as well. As the fraction of time resources are utilized is relative small, and so it is very unlikely that two or more service providers need a resource simultaneously, pooling of resources may be a natural option here. In particular, by resource pooling, a larger part of demand can be satisfied (directly) and so, additional profit can be realized in the long-run. This may encourage the service providers to pool their resources.

We examine such pooled situation by considering a stylized model of reality. In this model, we assume that (i) resources switch between being available and unavailable according to independent stochastic processes, (ii) it does not occur that two or more service providers need a resource simultaneously, (iii) transshipments of resources between service providers occur instantaneously at zero costs, and (iv) the profit of each service provider depends non-decreasingly on the long-term fraction of time that at least one resource is available. The first assumption is reasonable as there is no reason to assume that unavailability of a resource of a service provider would affect the unavailability of a resource of another service provider. The second assumption is a good approximation of reality when the fraction of time resources are utilized is relatively small, which implies that it is unlikely that two or more service providers need a resource simultaneously. The third assumption is realistic in settings in which service providers are located close (enough) to each other or in which substantial time is needed to prepare the environment for service. For the latter one, the Dutch railway setting is a good example as it already may take hours to create a safe work environment for tamping. Transshipment costs can be neglected as they are usually small relative to costs for unavailability. The last assumption is realistic in settings in which service providers operate for a sufficient long time. In this chapter, we analyse general non-decreasing profit functions as well as linear profit function, specifically. Non-linear profit functions fit well in situations in which service providers operate based on performance based contracts that reward service providers more when availability increases, e.g., the Dutch railway setting. Linear profit functions fit well in situations with highly profitable machines for which downtime costs increase proportional to the unavailability of the machines, e.g., the repairmen setting.

Based on the situation formulated above, we introduce an associated cooperative game. For this game, which we call an availability game, we show that the core is non-empty and present a population monotonic allocation scheme. In addition, we present several conditions for convexity of availability games. Moreover, we introduce four different allocation rules and investigate whether the allocations resulting from these allocation rules are increasing in the availability and in the profit function, i.e., satisfy monotonicity to profit and monotonicity to availability. We also investigate whether the allocations resulting from those allocation rules are the same for players who have the same profit function and availabilities, i.e., satisfy situation symmetry, and are the same for players who each bring in the same value for every possible coalition, i.e., satisfy game symmetry. Finally, we investigate whether the allocations resulting from the allocation rules are members of the core.

As players in a coalition of an availability game pool their resources completely, i.e., in an optimal way, we conclude that availability games belong to the class or OR games. However, there relationship with other OR games is limited. An exception is the class of cooperative reliability games (Bachrach et al. [4, 5, 6]). In cooperative reliability games, one considers a directed network with one sink and one source, where each link is controlled by a self-interested agent. Those links are subject to failures with some fixed probability. The agents can form coalitions to obtain connectivity from the sink to the target node. A fixed reward, which is equal to the probability of achieving connectivity for that coalition, should then be divided among the participating agents. Bachrach et al. [4] focused on how to approximate the Shapley value for large networks and Bachrach et al. [5, 6] focused on when cooperative reliability games are convex and balanced. Cooperative reliability games and cooperative availability games partially overlap as they focus both on situations in which players (or the resource of this player) can be unavailable according to some predefined probability. However, there is also a difference between these games. In cooperative reliability games, it is assumed that the reward obtained per coalition depends on a single societal profit function only, whereas our (availability) game assumes that the reward obtained per coalition depends on the sum of the profit functions of all players of that coalition. So, results of cooperative reliability games (Bachrach et al. [4, 5, 6]) cannot be applied to our setting.

The remainder of this chapter is as follows. In section 4.2, we introduce an availability situation and subsequently introduce the associated availability game. In section 4.3, we present general properties of availability games. Then, in section 4.4, we introduce four different allocation rules and investigate them on several properties. Finally, we describe our conclusions in section 4.5.

4.2 Model

In this section, we introduce availability situations and define the associated games, called availability games.

4.2.1 Availability situations

Consider an environment with $N \subseteq \mathbb{N}$ a finite set of service providers, each providing the same service with a single interchangeable resource (for example a tamping machine or a repairman). We assume those resources to switch between being available and unavailable according to underlying independent stochastic processes. In addition, it does not occur that two or more service providers need a resource simultaneously. Let $A_i \in (0,1)$ be the long term fraction of time that the resource of service provider $i \in N$ is available, i.e., the availability of service provider i, and let $1 - A_i$ be the long term fraction of time that the resource of service provider i is unavailable, i.e., the unavailability of service provider i. We assume that the profit function of each service provider depends on its availability. Let $P_i : [0,1] \to \mathbb{R}_+$ be this non-decreasing profit function. So, for availability A_i service provider $i \in N$ receives a profit of $P_i(A_i)$. We assume that there exists a $j \in N$ for which $P_j(1) - P_j(A_j) > 0$. So, there is at least one player who gains by improving availability. To analyse this setting, we define an *availability (AV) situation* as a tuple (N, A, P) with N, $A = (A_i)_{i \in N}$, and $P = (P_i)_{i \in N}$ as described above. We use θ to refer to an AV situation $\theta = (N, A, P)$ and θ' to refer to an(other) AV situation $\theta' = (N', A', P')$. Finally, we denote the set of AV situations by Θ.

4.2.2 Availability games

The service providers, from now on called players, can protect against unavailability by pooling their resources. We assume that transshipments of resources between two or multiple players occur instantaneously at zero costs. As no two (or more) players demand for a resource simultaneously, the availability of coalition $M \subseteq N$ becomes the long-term fraction of time that at least one resource is available. As resources get unavailable independently from each other, the availability of coalition M becomes

$$A^M = 1 - \prod_{j \in M} (1 - A_j). \tag{4.1}$$

As profit depends on this availability (only), the profit of player i as part of coalition M becomes $P_i(A^M)$ and thus the profit of coalition M becomes $\sum_{i \in M} P_i(A^M)$. Now, we proceed with associating an AV game to any AV situation.

Definition 4.1. *For any AV situation $\theta \in \Theta$, the game (N, v^θ) with*

$$v^\theta(M) = \sum_{i \in M} P_i \left(A^M \right) \tag{4.2}$$

for all $M \subseteq N$ with $M \neq \emptyset$ and $v^\theta(\emptyset) = 0$ is called the associated availability (AV) game.

Example 4.1. *Consider an AV situation $\theta \in \Theta$ with $N = \{1, 2, 3\}$, $A_1 = 0.6, A_2 = 0.9$, $A_3 = 0.5$, $P_1(x) = x, P_2(x) = 2x$, and $P_3(x) = 7x$. In table 4.1, the related availabilities and corresponding profits for (N, v^θ) are presented per coalition.* ◇

Table 4.1: Corresponding availabilities and profits.

M	$\emptyset$	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
A^M	-	0.60	0.90	0.50	0.96	0.80	0.95	0.98
$v^\theta(M)$	0	0.60	1.80	3.50	2.88	6.40	8.55	9.80

4.3 Properties of availability games

In this section, we present general properties of AV games. The following two lemmas will be used frequently. The first lemma states that the availability of a coalition is at least the availability of any of its subcoalitions, while the second lemma states that the profit of a coalition is at least the profit of any of its subcoalitions.

Lemma 4.1. *For every AV situation $\theta \in \Theta$ it holds for any $M, K \subseteq N$ with $M \subseteq K$ that*

$$\prod_{i \in M}(1 - A_i) \geq \prod_{i \in K}(1 - A_i).$$

Proof **:** Let $\theta \in \Theta$ be an AV situation and $M, K \subseteq N$ with $M \subseteq K$. We have $0 \leq 1 - A_i \leq 1$ for all $i \in N$ and consequently

$$\prod_{i \in M}(1 - A_i) \geq \prod_{i \in M}(1 - A_i) \cdot \prod_{i \in K \setminus M}(1 - A_i) = \prod_{i \in K}(1 - A_i),$$

where the inequality uses $0 \leq \prod_{i \in M}(1 - A_i) \leq 1$ for all $M \subseteq N$. □

Lemma 4.2. *For every AV situation $\theta \in \Theta$ it holds for any $M, K \subseteq N$ with $M \subseteq K$ and $i \in M$ that*

$$P_i\left(A^M\right) \leq P_i\left(A^K\right). \tag{4.3}$$

Proof **:** Let $\theta \in \Theta$ be an AV situation and $M, K \subseteq N$ with $M \subseteq K$. In addition, let $i \in M$. Then

$$P_i\left(A^M\right) = P_i\left(1 - \prod_{j \in M}(1 - A_j)\right) \leq P_i\left(1 - \prod_{j \in K}(1 - A_j)\right) = P_i\left(A^K\right),$$

where the inequality is a result of (*i*) lemma 4.1 and (*ii*) the non-decreasing property of P_i. The first and last equality follow from (4.1). □

As a result of lemma 4.2 we can now claim that AV games are monotonic.

Proposition 4.1. *Every AV game (N, v^θ) is monotonic.*

Proof **:** Let $\theta \in \Theta$ be an AV situation and (N, v^θ) be the associated AV game. Now, let $M, K \subseteq N$ with $M \subseteq K$. Then

$$v^\theta(M) = \sum_{i \in M} P_i(A^M) \leq \sum_{i \in M} P_i\left(A^K\right) \leq \sum_{i \in K} P_i\left(A^K\right) = v^\theta(K).$$

The first and last equality hold by definition. The first inequality holds by lemma 4.2 and the second one holds as $P_i(x) \in \mathbb{R}_+$ for all $x \in [0, 1]$. □

In addition, we are able to show that every AV game (N, v^θ) is superadditive.

Proposition 4.2. *Every AV game is superadditive.*

Proof **:** Let $\theta \in \Theta$ be an AV situation and (N, v^θ) be the associated AV game. Let $M, K \subseteq N$ with $M \cap K = \emptyset$. Then

$$\begin{aligned} v^\theta(M) + v^\theta(K) = \sum_{i \in M} P_i\left(A^M\right) + \sum_{i \in K} P_i\left(A^K\right) &\leq \sum_{i \in M} P_i\left(A^{M \cup K}\right) + \sum_{i \in K} P_i\left(A^{M \cup K}\right) \\ &= \sum_{i \in M \cup K} P_i\left(A^{M \cup K}\right) = v^\theta(M \cup K). \end{aligned}$$

where the inequality holds by lemma 4.2. □

Superadditivity does not suffice to conclude that the core is non-empty. Following the implications in figure 2.3, convexity of a game is a sufficient condition for core non-emptiness. The next example shows that AV games are not convex in general.

Example 4.2. *Consider the situation of example 4.1. Observe that* $v^\theta(\{1,2,3\}) - v^\theta(\{2,3\}) = 9.80 - 8.55 = 1.25 < 2.90 = 6.40 - 3.50 = v^\theta(\{1,3\}) - v^\theta(\{3\})$ *and we can conclude that the game is not convex.* ◇

Despite the fact that AV games are not convex in general, non-emptiness of the core can still be proven.

Theorem 4.1. *Every AV game has a non-empty core.*

Proof **:** Let $\theta \in \Theta$ be an AV situation and (N, v^θ) be the associated AV game. Let $(x_i)_{i \in N}$ be the allocation with

$$x_i = P_i\left(A^N\right) \text{ for all } i \in N.$$

First, observe that

$$\sum_{i \in N} x_i = \sum_{i \in N} P_i\left(A^N\right) = v^\theta(N),$$

and thus the allocation is efficient. Secondly, observe that for any $M \subseteq N$

$$\sum_{i \in M} x_i = \sum_{i \in M} P_i\left(A^N\right) \geq \sum_{i \in M} P_i\left(A^M\right) = v^\theta(M),$$

where the inequality holds by lemma 4.2. Given that $\sum_{i \in M} x_i \geq v^\theta(M)$ for all $M \subseteq N$, the allocation is stable as well. Hence, $(x_i)_{i \in N}$ is an efficient and stable allocation and thus always a member of the core. We conclude that the core is non-empty. □

We can also claim that every AV game has a PMAS.

Theorem 4.2. *For every AV situation $\theta \in \Theta$ allocation scheme $(a_{i,M})_{i\in M, M\subseteq N}$, given by*

$$a_{i,M} = P_i\left(A^M\right) \text{ for all } i \in M \text{ and all } M \subseteq N$$

is a PMAS for (N, v^θ).

Proof **:** Let $\theta \in \Theta$ be an AV situation. Then, observe that

$$\sum_{i\in M} a_{i,M} = \sum_{i\in M} P_i\left(A^M\right) = v^\theta(M)$$

for all $M \subseteq N$. Secondly, observe that for any $M, K \subseteq N$ with $M \subseteq K$ and $i \in M$ we have

$$a_{i,M} = P_i\left(A^M\right) \leq P_i\left(A^K\right) = a_{i,K}$$

and so $(a_{i,M})_{i\in M, M\subseteq N}$ is a PMAS. □

Following the implications in figure 2.3, every game with a PMAS is superadditive, has a non-empty core, and is totally balanced. Since every AV game has a PMAS, proposition 4.2 and theorem 4.1 can be seen as corollaries of theorem 4.2. Moreover, we can conclude (from the implications in figure 2.3) that every AV game is totally balanced as well.

Corollary 4.1. *Every AV game is totally balanced.*

In example 4.2, it is illustrated that AV games are not convex in general. However, it is of interest to investigate if there exist necessary and sufficient conditions for a class of AV situations for which convexity can be ensured. We investigate the class of AV situations with linear profit functions, i.e., the class of AV situations for which for every player $i \in N$, there exists a $p_i \in \mathbb{R}_+$ such that $P_i(x) = p_i x$ for all $x \in [0, 1]$. These situations will be called linear AV situations.[3] The set of linear AV situations will be denoted by Θ^L.

Definition 4.2. *Let $\theta \in \Theta^L$ be a linear AV situation. Function $\mathscr{I}_{ij}(\theta)$ is defined by*

$$\mathscr{I}_{ij}(\theta) = \sum_{k\in N} p_k A_i A_j - p_i A_j - p_j A_i \quad \text{for all } i, j \in N \text{ with } i \neq j.$$

Theorem 4.3. *For every linear AV situation $\theta \in \Theta^L$ with $|N| \geq 2$ the associated AV game (N, v^θ) is convex if and only if*

$$\mathscr{I}_{ij}(\theta) \leq 0 \quad \text{for all } i, j \in N \text{ with } i \neq j.$$

[3] AV games originating from linear AV situations with $p_j > 0$ for some $j \in N$ and $p_i = 0$ for all $i \in N \backslash \{j\}$ may be recognised as Big Boss games (Muto et al. [38]). As a consequence, AV games originating from linear AV situations may be recognized as linear combinations of Big Boss games. However, this does not hold for AV games in general.

***Proof* :** Let $\theta \in \Theta^L$ be a linear AV situation with $|N| \geq 2$ and (N, v^θ) be the associated AV game. We show that the associated AV game is convex if and only if $\mathscr{J}_{ij}(\theta) \leq 0$ for all $i, j \in N$ with $i \neq j$.

($\Rightarrow$) Suppose the AV game is convex. Hence,

$$v^\theta(M \cup \{i,j\}) - v^\theta(M \cup \{j\}) - (v^\theta(M \cup \{i\}) - v^\theta(M)) \geq 0 \tag{4.4}$$

for all $i, j \in N$ with $i \neq j$ and all $M \subseteq N \backslash \{i,j\}$. Let $i, j \in N$ with $i \neq j$ and $M \subseteq N \backslash \{i,j\}$. Based on (4.4) it holds that

$$\begin{aligned}
0 &\leq v^\theta(M \cup \{i,j\}) - v^\theta(M \cup \{j\}) - (v^\theta(M \cup \{i\}) - v^\theta(M)) \\
&= \sum_{k \in M \cup \{i,j\}} p_k \left(1 - \prod_{l \in M \cup \{i,j\}} (1 - A_l)\right) - \sum_{k \in M \cup \{j\}} p_k \left(1 - \prod_{l \in M \cup \{j\}} (1 - A_l)\right) \\
&\quad - \sum_{k \in M \cup \{i\}} p_k \left(1 - \prod_{l \in M \cup \{i\}} (1 - A_l)\right) + \sum_{k \in M} p_k \left(1 - \prod_{l \in M} (1 - A_l)\right) \\
&= \sum_{k \in M \cup \{j\}} p_k \left(A_i \prod_{l \in M \cup \{j\}} (1 - A_l)\right) + p_i \left(1 - \prod_{l \in M \cup \{i,j\}} (1 - A_l)\right) \\
&\quad - \sum_{k \in M} p_k \left(A_i \prod_{l \in M} (1 - A_l)\right) - p_i \left(1 - \prod_{l \in M \cup \{i\}} (1 - A_l)\right) \\
&= \prod_{l \in M \cup \{j\}} (1 - A_l) \left(\sum_{k \in M \cup \{j\}} p_k A_i - p_i (1 - A_i)\right) \\
&\quad - \prod_{l \in M} (1 - A_l) \left(\sum_{k \in M} p_k A_i - p_i (1 - A_i)\right) \\
&= \prod_{l \in M} (1 - A_l) \left((1 - A_j) \left(\sum_{k \in M \cup \{i,j\}} p_k A_i - p_i\right) - \left(\sum_{k \in M \cup \{i\}} p_k A_i - p_i\right)\right),
\end{aligned}$$

where the first equality follows by definition. The second equality follows by combining all terms $k \in M \cup \{j\}$ from the first and second summation into one summation and combining all terms $k \in M$ from the third and fourth summation into one summation. In the two new summations, we combine the product terms and use that $A_i = 1 - (1 - A_i)$. Finally, we write down the terms that are left from the original summations. In the third equality the product term $\prod_{l \in M \cup j} (1 - A_l)$ is taken out of the first and second term and the product term $\prod_{l \in M} (1 - A_l)$ is taken out of the third and fourth term. Moreover, $p_i \cdot 1$ and $-p_i \cdot 1$ cancel each other out. In the fourth equality, the product term $\prod_{l \in M} (1 - A_l)$ is taken out of the whole equality and $-p_i(1 - A_i)$ is written as $p_i A_i - p_i$, where $p_i A_i$ is included in the summation finally.

As $A_i \in (0,1)$ for all $i \in N$, it holds that $\prod_{l\in M}(1-A_l) > 0$. If the last expression is divided by $\prod_{l\in M}(1-A_l)$, we obtain

$$\begin{aligned} 0 &\le (1-A_j)\left(\sum_{k\in M\cup\{i,j\}} p_k A_i - p_i\right) - \left(\sum_{k\in M\cup\{i\}} p_k A_i - p_i\right) \\ &= p_j A_i - A_j\left(\sum_{k\in M\cup\{i,j\}} p_k A_i - p_i\right). \end{aligned}$$

This is equivalent to

$$\sum_{k\in M\cup\{i,j\}} p_k A_i A_j - p_i A_j - p_j A_i \le 0. \tag{4.5}$$

As $i,j \in N$ with $i \neq j$ and $M \subseteq N\backslash\{i,j\}$ were chosen arbitrarily, (4.5) holds for any $i,j \in N$ with $i \neq j$ and all $M \subseteq N\backslash\{i,j\}$. In particular, (4.5) holds for any $i,j \in N$ with $i \neq j$ and $M = N\backslash\{i,j\}$. For $M = N\backslash\{i,j\}$ the left side of (4.5) coincides with $\mathscr{I}_{ij}(\theta)$ and thus $\mathscr{I}_{ij}(\theta) \le 0$ for all $i,j \in N$ with $i \neq j$.

($\Leftarrow$) Now, we assume that

$$\mathscr{I}_{ij}(\theta) \le 0$$

for all $i,j \in N$ with $i \neq j$. Then, for a given $i,j \in N$ with $i \neq j$ it holds that

$$A_j\left(\sum_{k\in N} p_k A_i - p_i\right) - p_j A_i \le 0.$$

Now, let $M \subseteq N\backslash\{i,j\}$. As $\sum_{k\in M\cup\{i,j\}} p_k A_i \le \sum_{k\in N} p_k A_i$, we can conclude that

$$A_j\left(\sum_{k\in M\cup\{i,j\}} p_k A_i - p_i\right) - p_j A_i \le A_j\left(\sum_{k\in N} p_k A_i - p_i\right) - p_j A_i \le 0.$$

This implies that

$$\begin{aligned} 0 &\le -A_j\left(\sum_{k\in M\cup\{i,j\}} p_k A_i - p_i\right) + p_j A_i \\ &= -A_j\left(\sum_{k\in M\cup\{i,j\}} p_k A_i - p_i\right) + \left(\sum_{k\in M\cup\{i,j\}} p_k A_i - p_i\right) - \left(\sum_{k\in M\cup\{i\}} p_k A_i - p_i\right) \\ &= (1-A_j)\left(\sum_{k\in M\cup\{i,j\}} p_k A_i - p_i\right) - \left(\sum_{k\in M\cup\{i\}} p_k A_i - p_i\right) \end{aligned}$$

Multiplying the last expression by $\prod_{l\in M}(1-A_l) > 0$ results into

$$\prod_{l\in M}(1-A_l)\left((1-A_j)\left(\sum_{k\in M\cup\{i,j\}} p_k A_i - p_i\right) - \left(\sum_{k\in M\cup\{i\}} p_k A_i - p_i\right)\right) \geq 0.$$

From proof ($\Rightarrow$) we know that this inequality coincides with

$$v(M\cup\{i,j\}) - v(M\cup\{j\}) - (v(M\cup\{i\}) - v(M)) \geq 0. \tag{4.6}$$

As $i,j \in N$ with $i \neq j$ and $M \subseteq N\backslash\{i,j\}$ were chosen arbitrarily, we can conclude that (4.6) holds for any $i,j \in N$ with $i \neq j$ and all $M \subseteq N\backslash\{i,j\}$. Using recursive arguments it can be seen that (4.6) is sufficient to show convexity (as defined on page 10). □

Example 4.3. *Consider the (linear) AV situation of example 4.1. Note that* $p_1 = 1$, $p_2 = 2$ *and* $p_3 = 7$. *Then,* $\mathscr{I}_{12}(\theta)$ *is given by*

$$\mathscr{I}_{12}(\theta) = (1+2+7)\cdot 0.6\cdot 0.9 - 1\cdot 0.9 - 2\cdot 0.6 = 3.3 > 0.$$

As derived directly in example 4.2, the game is indeed not convex. ◇

For linear AV situations $\theta \in \Theta^L$ with $p_i = \overline{p} \in \mathbb{R}_+$ for all $i \in N$, theorem 4.3 reduces to an easier result as described in proposition 4.3.

Proposition 4.3. *For every linear AV situation* $\theta \in \Theta^L$ *with* $N = \{1,2,\ldots,n\}$ *and* $n \geq 2$, $p_i = \overline{p} \in \mathbb{R}_+$ *for all* $i \in N$, *and* $A_1 \geq A_2 \geq \ldots \geq A_n$, *the associated AV game* (N, v^θ) *is convex if and only if*

$$|N|A_1A_2 - A_1 - A_2 \leq 0.$$

Proof **:** Let $\theta \in \Theta^L$ be a linear AV situation with $N = \{1,2,\ldots,n\}$ and $n \geq 2$, $p_i = \overline{p} \in \mathbb{R}_+$ for all $i \in N$, and $A_1 \geq A_2 \geq \ldots \geq A_n$. Let (N, v^θ) be the associated AV game. We show that the associated AV game is convex if and only if $|N|A_1A_2 - A_1 - A_2 \leq 0$.

($\Rightarrow$) Suppose the associated AV game is convex. Then, by theorem 4.3, $\mathscr{I}_{ij}(\theta) \leq 0$ for all $i,j \in N$ with $i \neq j$ and so

$$\mathscr{I}_{12}(\theta) = |N|\overline{p}A_1A_2 - \overline{p}A_2 - \overline{p}A_1 \leq 0.$$

As $\overline{p} \in \mathbb{R}_+$, we derive

$$|N|A_1A_2 - A_1 - A_2 \leq 0,$$

which concludes this part of the proof.

($\Leftarrow$) Suppose that $|N|A_1A_2 - A_1 - A_2 \leq 0$. Then, it holds that

$$A_1 \left(\frac{1}{2}|N|A_2 - 1 \right) + A_2 \left(\frac{1}{2}|N|A_1 - 1 \right) \leq 0. \tag{4.7}$$

As $0 \leq A_2 \leq A_1 < 1$, this implies that $\frac{1}{2}|N|A_2 - 1 \leq 0$ and $\frac{1}{2}|N|A_1 - 1 \leq 0$ or $\frac{1}{2}|N|A_2 - 1 \leq 0$ and $\frac{1}{2}|N|A_1 - 1 \geq 0$. We now investigate those different cases.

Case 1. $\frac{1}{2}|N|A_2 - 1 \leq 0$ and $\frac{1}{2}|N|A_1 - 1 \leq 0$.

As $\frac{1}{2}|N|A_1 - 1 \leq 0$, it holds that $\frac{1}{2}|N|A_j - 1 \leq \frac{1}{2}|N|A_1 - 1 \leq 0$ for all $j \in N$. As $A_i \in (0,1)$ for all $i \in N$, it holds that $A_i(\frac{1}{2}|N|A_j - 1) \leq 0$ for all $i,j \in N$. So, for all $i,j \in N$ with $i \neq j$, it holds that

$$A_i \left(\frac{1}{2}|N|A_j - 1 \right) + A_j \left(\frac{1}{2}|N|A_i - 1 \right) \leq 0.$$

Case 2. $\frac{1}{2}|N|A_2 - 1 \leq 0$ and $\frac{1}{2}|N|A_1 - 1 \geq 0$.

It holds that $A_1(\frac{1}{2}|N|A_j - 1) \leq A_1(\frac{1}{2}|N|A_2 - 1)$ for all $j \in N\backslash\{1\}$ and $A_j(\frac{1}{2}|N|A_1 - 1) \leq A_2(\frac{1}{2}|N|A_1 - 1)$ for all $j \in N\backslash\{1\}$. Therefore,

$$A_1(\frac{1}{2}|N|A_j - 1) + A_j(\frac{1}{2}|N|A_1 - 1) \leq A_1(\frac{1}{2}|N|A_2 - 1) + A_2(\frac{1}{2}|N|A_1 - 1) \leq 0$$

for all $j \in N\backslash\{1\}$.

Let $i \in N\backslash\{1\}$ and $j \in N$ with $j > i$. It holds that $\frac{1}{2}|N|A_j - 1 \leq \frac{1}{2}|N|A_i - 1 \leq \frac{1}{2}|N|A_2 - 1 \leq 0$. From this, we conclude that

$$A_i \left(\frac{1}{2}|N|A_j - 1 \right) + A_j \left(\frac{1}{2}|N|A_i - 1 \right) \leq 0. \tag{4.8}$$

Combining case 1 and case 2, we conclude that (4.8) holds for all $i \in N$ and all $j \in N$ with $j > i$. Since $\mathscr{J}_{ij}(\theta) = \mathscr{J}_{ji}(\theta)$ for all $i,j \in N$ with $i \neq j$, (4.8) holds for all $i,j \in N$ with $i \neq j$. As multiplying (4.8) with $\overline{p} \in \mathbb{R}_+$ will not affect the right hand side of the inequality, it holds that $\mathscr{J}_{ij}(\theta) \leq 0$ for all $i,j \in N$ with $i \neq j$. By theorem 4.3, the associated AV game is convex. □

Proposition 4.3 states that, under specific conditions, the associated AV game is convex. For example, AV games with only few players are more likely to be convex than games with many players (under the same highest and second highest availabilities). This may be due to the following effect. The additional profit player $i \in N$ generates when another player $j \in N\backslash\{i\}$ enters the coalition decreases by the size of the coalition player $i \in N$ belongs to. This effect may occur for linear AV situations $\theta \in \Theta^L$ where availabilities (and profits) are constant as well.

Proposition 4.4. *For every linear AV situation $\theta \in \Theta^L$ with $N = \{1, 2, \ldots, n\}$ and $n \geq 2$, $A_i = \overline{A}$ for all $i \in N$, and $p_1 \leq p_2 \leq \ldots \leq p_n$ the associated AV game (N, v^θ) is convex if and only if*

$$\overline{A} \leq \frac{p_1 + p_2}{\sum_{i \in N} p_i}.$$

Proof **:** Let $\theta \in \Theta^L$ be a linear AV situation with $N = \{1, 2, \ldots, n\}$ and $n \geq 2$, $A_i = \overline{A} \in (0, 1)$ for all $i \in N$, and $p_1 \leq p_2 \ldots \leq p_n$. Let (N, v^θ) be the associated AV game. We show that the associated AV game is convex if and only if $\overline{A} \leq \frac{p_1 + p_2}{\sum_{i \in N} p_i}$.

($\Rightarrow$) Suppose the associated AV game is convex. Then, by theorem 4.3, $\mathscr{I}_{ij}(\theta) \leq 0$ for all $i, j \in N$ with $i \neq j$ and so

$$\mathscr{I}_{12}(\theta) = \sum_{i \in N} p_i \overline{A}^2 - p_1 \overline{A} - p_2 \overline{A} \leq 0.$$

After some rewriting, we derive

$$\overline{A} \leq \frac{p_1 + p_2}{\sum_{i \in N} p_i},$$

which concludes this part of the proof.

($\Leftarrow$) Suppose that $0 < \overline{A} \leq \frac{p_1 + p_2}{\sum_{i \in N} p_i}$. After some rewriting, we derive

$$\overline{A} \left(\sum_{i \in N} p_i \overline{A} - p_1 \right) - p_2 \overline{A} \leq 0. \tag{4.9}$$

The left hand side of (4.9) coincides with $\mathscr{I}_{12}(\theta)$, and so $\mathscr{I}_{12}(\theta) \leq 0$. Now, observe that

$$\begin{aligned} 0 \geq \mathscr{I}_{12}(\theta) = \overline{A} \left(\sum_{k \in N} p_k \overline{A} - p_1 \right) - p_2 \overline{A} &= \overline{A}^2 \sum_{k \in N} p_k - \overline{A}(p_1 + p_2) \\ &\geq \overline{A}^2 \sum_{k \in N} p_k - \overline{A}(p_i + p_j) \\ &= \mathscr{I}_{ij}(\theta) \end{aligned}$$

for all $i, j \in N$ with $i \neq j$. This implies that $\mathscr{I}_{ij}(\theta) \leq 0$ for all $i, j \in N$ with $i \neq j$. By theorem 4.3, the associated AV game is convex. $\square$

Proposition 4.5. *For every linear AV situation $\theta \in \Theta^L$ with $N = \{1, 2, \ldots, n\}$ and $n \geq 2$, $p_i = \overline{p} \in \mathbb{R}_+$, and $A_i = \overline{A}$ for all $i \in N$ the associated AV game (N, v^θ) is convex if and only if*

$$\overline{A} \leq \frac{2}{|N|}.$$

Proof : Let $\theta \in \Theta^L$ be a linear AV situation with $N = \{1, 2, \ldots, n\}$ and $n \geq 2$, $A_i = \overline{A} \in (0,1)$ for all $i \in N$, and $p_i = \overline{p} \in \mathbb{R}_+$ for all $i \in N$. Let (N, v^θ) be the associated AV game. We show that the associated AV game is convex if and only if $\overline{A} \leq \frac{2}{|N|}$.

($\Rightarrow$) Suppose the associated AV game is convex. Then, by proposition 4.4, it holds that

$$\overline{A} \leq \frac{p_1 + p_2}{\sum_{k \in N} p_k} = \frac{2\overline{p}}{|N|\overline{p}} = \frac{2}{|N|},$$

which concludes the first part of the proof.

($\Leftarrow$) Suppose that $\overline{A} \leq \frac{2}{|N|}$. This implies that

$$\overline{A} \leq \frac{2}{|N|} = \frac{2\overline{p}}{|N|\overline{p}} = \frac{p_1 + p_2}{\sum_{k \in N} p_k},$$

and following proposition 4.4, the associated game is convex. □

4.4 Allocation rules

In the proof of theorem 4.1, an interesting allocation per AV situation, which can be seen as an allocation rule, is presented. Despite the fact that the allocation resulting from this allocation rule is a core member of every AV situation, it is not guaranteed that it satisfies other (appealing) properties. Additionally, there may exist other allocation rules that (*i*) allocate total profit based on other criteria, (*ii*) satisfy interesting properties and (*iii*) have an allocation that is a core member for every AV situation as well. For that reason, we introduce three other (interesting) allocation rules for AV situations. For the, in total, four allocation rules, we investigate if they satisfy monotonicity to availability, monotonicity to profit, situation symmetry, and game symmetry. Finally, we also investigate the core membership of the allocations resulting from the allocation rules.

4.4.1 Four allocation rules

First, we formally introduce an allocation rule defined on AV situations. Note that these allocation rules deviate (in definition) from the allocation rules defined in chapter 2. The allocation rules introduced in chapter 2 are defined on cooperative games, while in this chapter, we focus on allocation rules defined on situations.

Definition 4.3. *An allocation rule on AV situations is a mapping f that assigns to any AV situation $\theta \in \Theta$ a vector $f(\theta) \in \mathbb{R}^N$.*

We only pay attention to allocation rules that divide total profit, i.e., $\sum_{i \in N} f_i(\theta) = v^{\theta}(N)$ for any AV situation $\theta \in \Theta$. Recall that total profit only depends on (*i*) the availabilities and (*ii*) the profit functions of the different players. In what follows, we first introduce three intuitive allocation rules, each depending on the availabilities and profit functions of the different players of the corresponding AV situation. Then, we present the fourth allocation rule which is based on a well-known allocation rule for cooperative games, namely the Shapley value.

The first allocation rule (which is introduced in the proof of theorem 4.1 as an allocation for every AV situation) allocates to every player the profit that this player generates with its own profit function while being part of the grand coalition. It is based on the idea that a player who generates more profit than another player under the same availability should also be rewarded more. This allocation rule, which we call Own Profit (OP), is described for any AV situation $\theta \in \Theta$ by

$$OP_i(\theta) = P_i\left(A^N\right) \quad \text{for all } i \in N.$$

A possible drawback of the first allocation rule is that players are not rewarded directly for the impact of their own availability (on the profit functions of others). The second allocation rule overcomes this by allocating the total profit proportional to the availabilities of the players. The idea behind this allocation rule is that the more a player is available, the more it can help others and for this the player will be rewarded. Formally, for every AV situation $\theta \in \Theta$, this allocation rule, which we call Proportional to Availability (PA), is defined by

$$PA_i(\theta) = \frac{A_i}{\sum_{j \in N} A_j} v^{\theta}(N) \quad \text{for all } i \in N.$$

A possible drawback of the second allocation rule is that players are not rewarded directly for the profit generated with their own profit function while being part of the grand coalition. The third allocation rule will at least partly overcome this second drawback. In particular, this allocation rule first allocates the individual profit, i.e., the profit that every player would obtain in the individual situation, to every player. So, in this way, each player will be rewarded for their own availability and profit function. The remaining part of the total profit, i.e., the surplus, will be divided proportional to the players' profit loss due to unavailability. We measure the profit loss due to unavailability as the difference in profit a player can bring in with 100 % availability and with their (own) individual availability. The idea behind this part is that players are rewarded for their profit potential they can bring with 100% availability. Formally, for every AV situation $\theta \in \Theta$ this allocation rule, which we call Surplus Proportional to profit Potential (SPP), is defined by[4]

$$SPP_i(\theta) = v^{\theta}(\{i\}) + \frac{P_i(1) - P_i(A_i)}{\sum_{j \in N}\left[P_j(1) - P_j(A_j)\right]}\left(v^{\theta}(N) - \sum_{j \in N} v^{\theta}(\{j\})\right) \text{ for all } i \in N.$$

The last allocation rule that will be introduced is the Shapley value defined for AV situations. For every AV situation $\theta \in \Theta$, we define the Shapley value (SV) by

$$\mathrm{SV}_i(\theta) = \Phi_i(N, v^{\theta}) \quad \text{for all } i \in N.$$

4.4.2 Properties of allocation rules

In this section, we investigate whether the allocation rules satisfy intuitive properties as monotonicity to availability, monotonicity to profit, situation symmetry, and game symmetry. Finally, we also investigate whether the allocations resulting from the allocation rules are core members.

Monotonicity to availability

Suppose the availability of a player increases. Then, this specific player is able to generate more profit. Moreover, as the total availability increases, other players can generate more profit as well. Hence, it is natural to assume that players do not expect decreases in their allocations. We investigate whether the allocation rules allocate to all players not less when the availability of any player increases, i.e., satisfy monotonicity to availability.

Definition 4.4. *An allocation rule f satisfies monotonicity to availability on $D \subseteq \Theta$ if for any two AV situations $\theta, \theta' \in D$, where θ and θ' coincide except for the availability of player j with $A_j \leq A'_j$, it holds that*

$$f_i(\theta) \leq f_i(\theta') \quad \textit{for all } i \in N.$$

The following example shows that allocation rules PA, SPP, and SV do not satisfy monotonicity to availability on Θ.

Example 4.4. *Consider AV situation $\theta \in \Theta$ with $N = \{1,2,3\}$, $A_1 = 0.5$, $A_2 = 0.5$, $A_3 = 0.5$ and*

$$P_1(x) = P_2(x) = \begin{cases} x & \textit{if} \quad 0 \leq x \leq \frac{1}{2} \\ \frac{1}{2} & \textit{if} \quad \frac{1}{2} < x < 1 \\ 1 & \textit{if} \quad x = 1, \end{cases} \qquad P_3(x) = \begin{cases} x & \textit{if} \quad 0 \leq x \leq \frac{1}{2} \\ 1 & \textit{if} \quad \frac{1}{2} < x \leq 1. \end{cases}$$

[4] One can also be interested in the situation in which the profit loss due to unavailability is with respect to A^N (rather than 100%). Then, allocation rule SPP boils down to OP.

Moreover, consider situation $\theta' \in \Theta$, which coincides with θ except that $A'_3 = 0.75$. In table 4.2, the four allocations regarding those two situations θ and θ' are depicted for all three players.

Table 4.2: Allocations for AV game.

	i	OP_i	PA_i	SPP_i	SV_i		i	OP_i	PA_i	SPP_i	SV_i
	1	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{7}{12}$		1	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{1}{2}$	$\frac{1}{2}$
θ	2	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{7}{12}$	θ'	2	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{1}{2}$	$\frac{1}{2}$
	3	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{10}{12}$		3	1	$\frac{6}{7}$	1	1

Allocation rules PA, SPP, and SV do not satisfy monotonicity to availability, since

$$
\begin{aligned}
PA_2(\theta) &> PA_2(\theta'),\\
SPP_2(\theta) &> SPP_2(\theta'),\\
SV_2(\theta) &> SV_3(\theta').
\end{aligned}
\qquad \diamond
$$

We remark that a similar example can be constructed with continuous profit functions.

Proposition 4.6. *Allocation rule OP satisfies monotonicity to availability on Θ.*

Proof **:** Let $\theta \in \Theta$ be an AV situation and $\theta' \in \Theta$ be another AV situation that coincides with θ except for the availability of player j, i.e., $A_j \leq A'_j$. Then, it holds for any player $i \in N$ that

$$
\begin{aligned}
OP_i(\theta) &= P_i\left(1 - \prod_{k\in N}(1 - A_k)\right)\\
&= P_i\left(1 - \prod_{k\in N\setminus\{j\}}(1 - A_k)(1 - A_j)\right)\\
&= P'_i\left(1 - \prod_{k\in N\setminus\{j\}}(1 - A'_k)(1 - A_j)\right)\\
&\leq P'_i\left(1 - \prod_{k\in N\setminus\{j\}}(1 - A'_k)(1 - A'_j)\right)\\
&= OP_i(\theta').
\end{aligned}
$$

where the third equality results from $A_k = A'_k$ for all $k \neq j$ and $P_i = P'_i$ for all $i \in N$. The inequality results from $0 \leq A_j \leq A'_j \leq 1$ with $0 \leq \prod_{k\in N\setminus\{j\}}(1 - A'_k) \leq 1$, and the fact that P'_i is non-decreasing. □

For linear AV situations, we obtain different results regarding monotonicity to availability. The following example will be used to show that allocation rules PA and SV do not satisfy monotonicity to availability on Θ^L.

Example 4.5. *Consider the (linear) AV situation $\theta \in \Theta^L$ of example 4.1. Moreover, consider situation $\theta' \in \Theta^L$, which coincides with θ except that $A_1' = 0.8$ now. In table 4.3, the four allocations regarding those two situations θ and θ' are depicted for all three players. All numbers are rounded to two decimals.*

Table 4.3: Allocations for AV game.

	i	OP_i	PA_i	SPP_i	SV_i		i	OP_i	PA_i	SPP_i	SV_i
	1	0.98	2.94	0.98	1.28		1	0.99	3.60	0.99	1.52
θ	2	1.96	4.41	1.99	2.95	θ'	2	1.98	4.05	2.00	2.70
	3	6.86	2.45	6.83	5.57		3	6.93	2.25	6.91	5.68

Allocation rules PA and SV do not satisfy monotonicity to availability, since

$$\begin{aligned} PA_2(\theta) &> PA_2(\theta'),\\ SV_2(\theta) &> SV_2(\theta'). \end{aligned} \qquad \diamond$$

Proposition 4.7. *Allocation rules OP and SPP satisfy monotonicity to availability on Θ^L.*

Proof **:** *(i) OP*. Follows directly from proposition 4.6 as $\Theta^L \subseteq \Theta$.

(ii) SPP. Let $\theta \in \Theta^L$ be a linear AV situation and $\theta' \in \Theta^L$ be another linear AV situation that only deviates in the availability of player $j \in N$ with $A_j \leq A_j'$. We claim that the derivative of $SPP_i(\theta)$ for any player $i \in N$ is non-negative with respect to availability A_j. Note that $P_i(1) - P_i(A_i) = p_i - p_iA_i = p_i(1 - A_i)$ for all $i \in N$. Let $i \in N$. Then, allocation $SPP_i(\theta)$ can be rewritten as

$$\begin{aligned} SPP_i(\theta) &= p_iA_i + \frac{p_i(1-A_i)}{\sum_{k\in N} p_k(1-A_k)} \left(\sum_{t\in N} p_t \left(1 - \prod_{k\in N}(1-A_k) \right) - \sum_{l\in N} p_lA_l \right) \\ &= p_iA_i + \left(1 - \frac{\sum_{l\in N\setminus\{i\}} p_l(1-A_l)}{\sum_{l\in N} p_l(1-A_l)} \right) \left(\sum_{t\in N} p_t \left(1 - A_t - \prod_{k\in N}(1-A_k) \right) \right) \\ &= p_iA_i + \sum_{t\in N} p_t \left(1 - A_t - \prod_{k\in N}(1-A_k) \right) - \sum_{l\in N\setminus\{i\}} p_l(1-A_l) \\ &\qquad + \frac{\sum_{l\in N\setminus\{i\}} p_l(1-A_l)}{\sum_{l\in N} p_l(1-A_l)} \sum_{t\in N} p_t \prod_{k\in N}(1-A_k) \\ &= p_iA_i + p_i(1-A_i) - \sum_{t\in N} p_t \prod_{k\in N}(1-A_k) \\ &\qquad + \frac{\sum_{l\in N\setminus\{i\}} p_l(1-A_l)}{\sum_{l\in N} p_l(1-A_l)} \sum_{t\in N} p_t \prod_{k\in N}(1-A_k) \end{aligned}$$

$$
\begin{aligned}
&= p_i - \sum_{t\in N} p_t \prod_{k\in N}(1-A_k)\left(1 - \frac{\sum_{l\in N\setminus\{i\}} p_l(1-A_l)}{\sum_{l\in N} p_l(1-A_l)}\right)\\
&= p_i - \sum_{t\in N} p_t \prod_{k\in N}(1-A_k)\left(\frac{p_i(1-A_i)}{\sum_{l\in N} p_l(1-A_l)}\right).
\end{aligned}
$$

As at least for one player $k \in N$, $p_k(1) - p_k(A_k) > 0$, function $SPP_i(\theta)$ is continuous and differentiable in A_j. Let $j \in N$. The derivative of $SPP_j(\theta)$ to A_j is given by

$$
\begin{aligned}
\frac{d}{dA_j} SPP_j(\theta) &= -\frac{\sum_{l\in N} p_l(1-A_l)\cdot(-p_j) - p_j(1-A_j)\cdot(-p_j)}{\left(\sum_{l\in N} p_l(1-A_l)\right)^2} \sum_{t\in N} p_t \prod_{k\in N}(1-A_k)\\
&\quad - \frac{p_j(1-A_j)}{\sum_{l\in N} p_l(1-A_l)} \sum_{t\in N} p_t \prod_{k\in N\setminus\{j\}}(1-A_k)\cdot(-1)\\
&= \frac{p_j}{\left(\sum_{l\in N} p_l(1-A_l)\right)^2}\Bigg(\sum_{l\in N\setminus\{j\}} p_l(1-A_l)\sum_{t\in N} p_t \prod_{k\in N}(1-A_k)\\
&\qquad + \sum_{l\in N} p_l(1-A_l)\sum_{t\in N} p_t \prod_{k\in N}(1-A_k)\Bigg)\\
&= \frac{p_j \sum_{t\in N} p_t \prod_{k\in N}(1-A_k)}{\left(\sum_{l\in N} p_l(1-A_l)\right)^2}\left(\sum_{l\in N\setminus\{j\}} p_l(1-A_l) + \sum_{l\in N} p_l(1-A_l)\right) \geq 0.
\end{aligned}
$$

Note that all terms are non-negative and thus the derivative is non-negative as well. Hence, $SPP_j(\theta)$ is non-decreasing in A_j. This implies that $SPP_j(\theta) \leq SPP_j(\theta')$. Let $j \in N$ and $i \in N\setminus\{j\}$. Then, the derivative of $SPP_i(\theta)$ to A_j is given by

$$
\begin{aligned}
&\frac{d}{dA_j} SPP_i(\theta)\\
&= -\frac{0 - (p_i(1-A_i)\cdot(-p_j))}{\left(\sum_{l\in N} p_l(1-A_l)\right)^2}\sum_{t\in N} p_t \prod_{k\in N}(1-A_k)\\
&\quad - \frac{p_i(1-A_i)}{\sum_{l\in N} p_l(1-A_l)}\sum_{t\in N} p_t \prod_{k\in N\setminus\{j\}}(1-A_k)\cdot(-1)\\
&= \frac{p_i(1-A_i)}{\left(\sum_{l\in N} p_l(1-A_l)\right)^2}\left(-p_j\sum_{t\in N} p_t \prod_{k\in N}(1-A_k) + \sum_{t\in N} p_t \prod_{k\in N\setminus\{j\}}(1-A_k)\sum_{l\in N} p_l(1-A_l)\right)\\
&= \frac{p_i(1-A_i)}{\left(\sum_{l\in N} p_l(1-A_l)\right)^2}\left(\sum_{t\in N} p_t \prod_{k\in N\setminus\{j\}}(1-A_k)\right)\left(-p_j(1-A_j) + \sum_{l\in N} p_l(1-A_l)\right)\\
&= \frac{p_i(1-A_i)}{\left(\sum_{l\in N} p_l(1-A_l)\right)^2}\left(\sum_{t\in N} p_t \prod_{k\in N\setminus\{j\}}(1-A_k)\right)\left(\sum_{l\in N\setminus\{j\}} p_l(1-A_l)\right) \geq 0.
\end{aligned}
$$

Note that all terms are non-negative and thus the derivative is non-negative as well.

Hence, $SPP_i(\theta)$ is non-decreasing in A_j for all $i \in N \backslash \{j\}$. From this, we conclude that $SPP_i(\theta) \leq SPP_i(\theta')$ for all $i \in N$. □

Monotonicity to profit

Suppose the profit function of a player changes such that the difference between the outcomes of the old and the new profit function is non-decreasing in the argument, i.e. in (total) availability. Then, this specific player is able to generate more profit. Despite that the other players do not generate more profit themselves, they are responsible (in terms of availability) for the (extra) profit of the specific player as well. Hence, it is natural to assume that players do not expect decreases in their allocations. We investigate whether the allocation rules allocate to all players not less when the difference between the outcome of the new and old profit function of a specific player is non-decreasing in (total) availability, i.e., satisfy monotonicity to profit.

Definition 4.5. *An allocation rule f satisfies monotonicity to profit on $D \subseteq \Theta$ if for any two AV situations $\theta, \theta' \in D$, where θ and θ' coincide except for the profit of player j with $P'_j(x) - P_j(x)$ non-decreasing in x, it holds that*

$$f_i(\theta) \leq f_i(\theta') \text{ for all } i \in N.$$

The following example shows that allocation rule SPP does not satisfy monotonicity to profit on Θ^L.

Example 4.6. *Consider (linear) AV situation $\theta \in \Theta^L$ with $N = \{1, 2, 3\}$, $A_1 = 0.6$, $A_2 = 0.7$, $A_3 = 0.5$, $p_1 = 1$, $p_2 = 3$, and $p_3 = 9$. Moreover, consider (linear) AV situation $\theta' \in \Theta^L$, which coincides with θ except that $p'_1 = 10$ now. In table 4.4, the four allocations regarding those two situations θ and θ' are depicted for all three players. Note that all numbers are rounded to two decimals.*

Table 4.4: Allocations for AV game.

	i	OP_i	PA_i	SPP_i	SV_i		i	OP_i	PA_i	SPP_i	SV_i
	1	0.94	4.07	0.95	1.69		1	9.40	6.89	9.44	8.83
θ	2	2.82	4.75	2.88	3.54	θ'	2	2.82	8.04	2.87	4.38
	3	8.46	3.39	8.39	6.98		3	8.46	5.74	8.37	7.46

Allocation rule SPP does not satisfy monotonicity to profit, since

$$SPP_3(\theta) > SPP_3(\theta'). \qquad \diamond$$

Proposition 4.8. *Allocation rules OP, PA, and SV satisfy monotonicity to profit on Θ.*

Proof **:** Let $\theta, \theta' \in \Theta$ be two AV situations where θ and θ' coincide except for the profit of player j with $P'_j(x) - P_j(x)$ non-decreasing in x. As $P'_k(x) - P_k(x) = 0$ for all $k \in N\backslash\{j\}$, it holds that $P'_i(x) \geq P_i(x)$ for all $i \in N$. Hence, it holds for all $i \in N$ that

$$OP_i(\theta) = P_i\left(1 - \prod_{k\in N}(1 - A_k)\right) \leq P'_i\left(1 - \prod_{k\in N}(1 - A'_k)\right) = OP_i(\theta')$$

given that $A_k = A'_k$ for all $k \in N$.

In the same line, it holds that

$$\begin{aligned} PA_i(\theta) = \frac{A_i}{\sum_{k\in N} A_k}\sum_{k\in N} P_k\left(1 - \prod_{h\in N}(1 - A_h)\right) &\leq \frac{A'_i}{\sum_{k\in N} A'_k}\sum_{k\in N} P'_k\left(1 - \prod_{h\in N}(1 - A'_h)\right) \\ &= PA_i(\theta'), \end{aligned}$$

given that $A_k = A'_k$ for all $k \in N$.

Finally, let $i \in N$ and $M \subseteq N\backslash\{i\}$. Then

$$\begin{aligned} &v^{\theta'}(M \cup \{i\}) - v^{\theta'}(M) \\ =& \sum_{k\in M\cup\{i\}} P'_k\left(1 - \prod_{l\in M\cup\{i\}}(1 - A_l)\right) - \sum_{k\in M} P'_k\left(1 - \prod_{l\in M}(1 - A_l)\right) \\ =& \sum_{k\in M}\left(P'_k\left(1 - \prod_{l\in M\cup\{i\}}(1 - A_l)\right) - P'_k\left(1 - \prod_{l\in M}(1 - A_l)\right)\right) + P'_i\left(1 - \prod_{l\in M\cup\{i\}}(1 - A_l)\right) \\ \geq& \sum_{k\in M}\left(P_k\left(1 - \prod_{l\in M\cup\{i\}}(1 - A_l)\right) - P_k\left(1 - \prod_{l\in M}(1 - A_l)\right)\right) + P_i\left(1 - \prod_{l\in M\cup\{i\}}(1 - A_l)\right) \\ =& v^{\theta}(M \cup \{i\}) - v^{\theta}(M). \end{aligned}$$

where the inequality holds, as

i) if $j \in M$ (and thus $j \neq i$) then $P'_j(y) - P'_j(x) \geq P_j(y) - P_j(x)$ for $y \geq x$ and $P'_l = P_l$ for all $l \in M\backslash\{j\}$ and $P'_i = P_i$.

ii) if $j \notin M$ and $i = j$ then $P'_l = P_l$ for all $l \in M$ and $P'_i \geq P_i$.

iii) if $j \notin M$ and $i \neq j$ then $P'_l = P_l$ for all $l \in M$ and $P'_i = P_i$ and so the inequality becomes equality.

As $v^{\theta'}(M \cup \{i\}) - v^{\theta'}(M) \geq v^{\theta}(M \cup \{i\}) - v^{\theta}(M)$ for any $i \in N$ and $M \subseteq N\{i\}$, and given that the Shapley value of a cooperative game is a weighted average over those marginal contributions, it follows that

$$SV_i(\theta) = \Phi_i(N, v^{\theta}) \leq \Phi_i(N, v^{\theta'}) = SV_i(\theta') \text{ for all } i \in N,$$

which concludes the proof. □

Situation Symmetry

Suppose that two players have the same profit function and availability. Then, those players both generate the same profit and both help other players (in terms of availability) in the same way. Hence, it is natural to assume that those players expect the same allocation. We investigate whether the allocation rules indeed allocate the same to both players, i.e., satisfy situation symmetry. For this we introduce some new definitions.

Definition 4.6. *For any AV situation $\theta \in \Theta$, players $i, j \in N$ with $i \neq j$ are called situation symmetric if*

$$P_i(x) = P_j(x) \text{ for all } x \in [0,1] \text{ and } A_i = A_j.$$

Definition 4.7. *An allocation rule f satisfies situation symmetry on $D \subseteq \Theta$ if for all $\theta \in D$ and all situation symmetric players $i, j \in N$ with $i \neq j$ it holds that*

$$f_i(\theta) = f_j(\theta).$$

Now, we show that allocation rules OP, PA, SPP, and SV satisfy situation symmetry on Θ. Although the proof for the first three allocation rules is almost straightforward, we present them for completeness.

Proposition 4.9. *Allocation rules OP, PA, SPP, and SV satisfy situation symmetry on Θ.*

Proof **:** Let $\theta \in \Theta$ be an AV situation and $i, j \in N$ with $i \neq j$ two players who are situation symmetric. For allocation rule OP it holds that

$$OP_i(\theta) = P_i\left(1 - \prod_{k \in N}(1 - A_k)\right) = P_j\left(1 - \prod_{k \in N}(1 - A_k)\right) = OP_j(\theta),$$

as $P_i(x) = P_j(x)$ for all $x \in [0,1]$. For allocation rule PA, it holds that

$$PA_i(\theta) = \frac{A_i}{\sum_{k \in N} A_k} v^{\theta}(N) = \frac{A_j}{\sum_{k \in N} A_k} v^{\theta}(N) = PA_j(\theta),$$

as $A_i = A_j$. For allocation rule SPP, it holds that

$$\begin{aligned}
SPP_i(\theta) &= v^\theta(\{i\}) + \frac{P_i(1) - P_i(A_i)}{\sum_{k \in N} [P_k(1) - P_k(A_k)]} \left(v^\theta(N) - \sum_{l \in N} v^\theta(\{l\}) \right) \\
&= P_i(A_i) + \frac{P_i(1) - P_i(A_i)}{\sum_{k \in N} [P_k(1) - P_k(A_k)]} \left(v^\theta(N) - \sum_{l \in N} v^\theta(\{l\}) \right) \\
&= P_j(A_j) + \frac{P_j(1) - P_j(A_j)}{\sum_{k \in N} [P_k(1) - P_k(A_k)]} \left(v^\theta(N) - \sum_{l \in N} v^\theta(\{l\}) \right) \\
&= v^\theta(\{j\}) + \frac{P_j(1) - P_j(A_j)}{\sum_{k \in N} [P_k(1) - P_k(A_k)]} \left(v^\theta(N) - \sum_{l \in N} v^\theta(\{l\}) \right) \\
&= SPP_j(\theta),
\end{aligned}$$

As $P_i(x) = P_j(x)$ for all $x \in [0,1]$ and $A_i = A_j$. Finally, for allocation rule SV, it holds that $P_i(x) = P_j(x)$ for all $x \in [0,1]$ and $A_i = A_j$. This implies that $v^\theta(M \cup \{i\}) = v^\theta(M \cup \{j\})$ for all $M \subseteq N \backslash \{i,j\}$. Now, we make use of two definitions and one proposition that will be proven later on. Based on definition 4.8, player i and j are game symmetric. Based on proposition 4.10 and definition 4.9, we conclude that $SV_i(\theta) = SV_j(\theta)$. □

Game Symmetry

Suppose that player i and player j have the same individual profit, but not necessarily the same profit functions and availabilities. Now, assume that the total profit of any coalition including player i equals the total profit of that same coalition, except that player i is now replaced by player j. So, both players are symmetric, but now in terms of the associated AV game. In this case, it is natural to assume that both players expect the same allocation. Now, we investigate whether the allocation rules allocate the same to both players, i.e., satisfy game symmetry. In doing so, we first introduce the definition of symmetric players in terms of AV games.

Definition 4.8. *For any AV situation $\theta \in \Theta$ players $i, j \in N$ with $i \neq j$ are called game symmetric if for the associated AV game (N, v^θ) it holds that*

$$v^\theta(M \cup \{i\}) = v^\theta(M \cup \{j\}) \text{ for all } M \subseteq N \backslash \{i,j\}.$$

Definition 4.9. *An allocation rule f satisfies game symmetry on $D \subseteq \Theta$ if for all $\theta \in D$ and all game symmetric players $i, j \in N$ with $i \neq j$ it holds that*

$$f_i(\theta) = f_j(\theta).$$

The following example shows that allocation rules OP, PA, and SPP do not satisfy game symmetry on Θ^L.

Example 4.7. *Consider (linear) AV situation $\theta \in \Theta^L$ with $N = \{1,2,3\}, A_1 = 0.7$, $A_2 = 0.8$, $A_3 = 0.9, p_1 = 9, p_2 = 40$, and $p_3 = 7$. Then $v^\theta(\{1\}) = 6.3 = v^\theta(\{3\})$ and $v^\theta(\{1,2\}) = 40.06 = v^\theta(\{2,3\})$. From this, we can conclude that player 1 and 3 are game symmetric. The corresponding allocations are presented in table 4.5. All numbers are rounded to two decimals.*

Table 4.5: Allocations for AV game.

	i	OP_i	PA_i	SPP_i	SV_i
	1	8.95	16.24	8.92	9.18
θ	2	39.76	18.55	39.76	37.30
	3	6.96	20.87	6.98	9.18

The allocation rules OP, PA, and SPP do not satisfy game symmetry, since

$$\begin{aligned} OP_1(\theta) &> OP_3(\theta), \\ PA_1(\theta) &< PA_3(\theta), \\ SPP_1(\theta) &> SPP_3(\theta). \end{aligned}$$

◇

Proposition 4.10. *Allocation rule SV satisfies game symmetry on Θ.*

Proof **:** Let $\theta \in \Theta$ be an AV situation. Moreover, let $i, j \in N$ with $i \neq j$ be two game symmetric players in (N, v^θ). Following Shapley [55], it holds that $\Phi_i(N, v^\theta) = \Phi_j(N, v^\theta)$. As a consequence, $SV_i(\theta) = \Phi_i(N, v^\theta) = \Phi_j(N, v^\theta) = SV_j(\theta)$. From this, we conclude that SV satisfies game symmetry. □

The core

In section 4.3, we showed core non-emptiness of AV games. This result was based on finding an allocation that always belongs to the core. Now, we investigate whether the allocations resulting from the four different allocation rules are always members of the core as well.

The following example shows that there exists an AV situation $\theta \in \Theta$ for which allocations $PA(\theta)$, $SPP(\theta)$ and $SV(\theta)$ are not core elements.

Example 4.8. *Consider the first AV situation of example 4.4. Then, the associated game (N, v^θ) is given by $v^\theta(\{1\}) = v^\theta(\{2\}) = v^\theta(\{3\}) = \frac{1}{2}$, $v^\theta(\{1,3\}) = v^\theta(\{2,3\}) = 1\frac{1}{2}$, $v^\theta(\{1,2\}) = 1$, and $v^\theta(\{1,2,3\}) = 2$. The allocations resulting from allocation rules PA, SPP and SV (see* *table 4.2) are not elements of the core, since*

$$\begin{aligned}
PA_1(\theta) + PA_3(\theta) &= \frac{2}{3} + \frac{2}{3} < 1\frac{1}{2} = v^{\theta}(\{1,3\}), \\
SPP_1(\theta) + SPP_3(\theta) &= \frac{2}{3} + \frac{2}{3} < 1\frac{1}{2} = v^{\theta}(\{1,3\}), \\
SV_1(\theta) + SV_3(\theta) &= \frac{7}{12} + \frac{10}{12} = 1\frac{5}{12} < 1\frac{1}{2} = v^{\theta}(\{1,3\}).
\end{aligned}$$

◇

Theorem 4.1 states that the core is non-empty. We proved this by showing that each allocation of OP can be recognized as a core allocation. So, the next result is immediate.

Corollary 4.2. *For every AV situation $\theta \in \Theta$ it holds that*

$$OP(\theta) \in \mathscr{C}(N, v^{\theta}).$$

The following example shows that there exists a linear AV situation $\theta \in \Theta^L$ for which allocation $PA(\theta)$ and $SV(\theta)$ are not core elements.

Example 4.9. *Consider the (linear) AV situation $\theta \in \Theta^L$ of example 4.1. The related allocations are presented in table 4.6. All values are rounded to two decimal places.*

Table 4.6: Allocations for AV game.

	i	OP_i	PA_i	SPP_i	SV_i
	1	0.98	2.94	0.98	1.28
θ	2	1.96	4.41	1.99	2.95
	3	6.86	2.45	6.83	5.57

The allocations resulting from allocation rules PA and SV are not elements of the core, since

$$\begin{aligned}
PA_2(\theta) + PA_3(\theta) &< 4.42 + 2.46 = 6.88 < v^{\theta}(\{2,3\}), \\
SV_2(\theta) + SV_3(\theta) &< 2.96 + 5.58 = 8.54 < v^{\theta}(\{2,3\}).
\end{aligned}$$

◇

Based on corollary 4.2 and example 4.9, it remains to investigate whether allocations resulting from SPP are core members. We make use of the following lemma.

Lemma 4.3. *Let $\theta \in \Theta^L$ be a linear AV situation with $x_i = 1 - A_i$ for all $i \in N$. Then for all $M \subseteq N$ it holds that*

$$\sum_{i \in M} p_i \left(\prod_{j \in M} x_j \right) \geq \frac{\sum_{i \in M} p_i x_i}{\sum_{j \in N} p_j x_j} \sum_{i \in N} p_i \left(\prod_{j \in N} x_j \right).$$

Proof : Let $\theta \in \Theta^L$ be a linear AV situation and $x_i = 1 - A_i$ for all $i \in N$. In addition, let $M \subseteq N$. Then, it holds that

$$\sum_{i \in M} p_i x_i \sum_{j \in M} p_j \left(1 - \prod_{k \in N \setminus M} x_k\right) \geq 0.$$

Moreover, it holds that

$$\sum_{i \in N \setminus M} p_i x_i \sum_{k \in M} p_k - \sum_{i \in N \setminus M} p_i \prod_{j \in N \setminus M} x_j \sum_{k \in M} p_k x_k \geq 0.$$

Now, when both parts are summed, we obtain

$$\begin{aligned}
0 &\leq \sum_{i \in M} p_i x_i \sum_{j \in M} p_j \left(1 - \prod_{k \in N \setminus M} x_k\right) + \sum_{i \in N \setminus M} p_i x_i \sum_{k \in M} p_k - \sum_{i \in N \setminus M} p_i \prod_{j \in N \setminus M} x_j \sum_{k \in M} p_k x_k \\
&= \sum_{i \in M} p_i x_i \sum_{j \in M} p_j - \sum_{i \in M} p_i x_i \sum_{j \in M} p_j \prod_{k \in N \setminus M} x_k + \sum_{i \in N \setminus M} p_i x_i \sum_{k \in M} p_k \\
&\quad - \sum_{i \in N \setminus M} p_i \prod_{j \in N \setminus M} x_j \sum_{k \in M} p_k x_k \\
&= \sum_{i \in N} p_i x_i \sum_{j \in M} p_j - \sum_{i \in M} p_i x_i \sum_{j \in M} p_j \prod_{k \in N \setminus M} x_k - \sum_{i \in N \setminus M} p_i \prod_{j \in N \setminus M} x_j \sum_{k \in M} p_k x_k \\
&= \sum_{i \in N} p_i x_i \sum_{j \in M} p_j - \sum_{i \in M} p_i x_i \sum_{j \in M} p_j \prod_{k \in N \setminus M} x_k - \sum_{k \in M} p_k x_k \sum_{i \in N \setminus M} p_i \left(\prod_{j \in N \setminus M} x_j\right) \\
&= \sum_{i \in N} p_i x_i \sum_{j \in M} p_j - \sum_{i \in M} p_i x_i \sum_{j \in N} p_j \left(\prod_{k \in N \setminus M} x_k\right),
\end{aligned}$$

where the equalities hold by rewriting. From the last expression, we derive

$$\sum_{i \in N} p_i x_i \sum_{j \in M} p_j \geq \sum_{i \in M} p_i x_i \sum_{j \in N} p_j \left(\prod_{k \in N \setminus M} x_k\right).$$

Multiplying both sides by $\prod_{j \in M} x_j (\geq 0)$ and subsequently dividing both sides by $\sum_{j \in N} p_j x_j (> 0)$ gives

$$\sum_{i \in M} p_i \prod_{j \in M} x_i \geq \frac{\sum_{i \in M} p_i x_i}{\sum_{j \in N} p_j x_j} \sum_{i \in N} p_i \prod_{j \in N} x_i,$$

which concludes the proof. □

Based on lemma 4.3 we can show that for each linear AV situation $\theta \in \Theta^L$ it holds that allocation $SPP(\theta)$ is a core member.

Proposition 4.11. *For every AV situation $\theta \in \Theta^L$ it holds that*

$$SPP(\theta) \in \mathscr{C}(N, v^\theta).$$

Proof **:** Let $\theta \in \Theta^L$ be a linear AV situation. Note that $P_i(1) - P_i(A_i) = p_i - p_i A_i = p_i(1 - A_i)$ for all $i \in N$. It holds that

$$\begin{aligned}
\sum_{i \in N} SPP_i(\theta) &= \sum_{i \in N} \left(v^\theta(\{i\}) + \frac{p_i(1-A_i)}{\sum_{j \in N} p_j(1-A_j)} \left(v^\theta(N) - \sum_{k \in N} v^\theta(\{k\}) \right) \right) \\
&= \sum_{i \in N} v^\theta(\{i\}) + v^\theta(N) - \sum_{k \in N} v^\theta(\{k\}) \\
&= v^\theta(N),
\end{aligned}$$

and thus $SPP(\theta)$ is efficient. Secondly, let $M \subseteq N$, then

$$\begin{aligned}
v^\theta(M) &= \sum_{i \in M} p_i \left(1 - \prod_{j \in M} (1 - A_j) \right) \\
&= \sum_{i \in M} p_i A_i + \sum_{i \in M} p_i(1 - A_i) - \sum_{i \in M} p_i \prod_{j \in M} (1 - A_j) \\
&\leq \sum_{i \in M} v^\theta(\{i\}) + \sum_{i \in M} p_i(1-A_i) - \frac{\sum_{i \in M} p_i(1-A_i)}{\sum_{k \in N} p_k(1-A_k)} \sum_{i \in N} p_i \prod_{j \in N} (1 - A_j) \\
&= \sum_{i \in M} v^\theta(\{i\}) + \frac{\sum_{i \in M} p_i(1-A_i)}{\sum_{k \in N} p_k(1-A_k)} \left(\sum_{k \in N} p_k(1-A_k) - \sum_{i \in N} p_i \prod_{j \in N} (1 - A_j) \right) \\
&= \sum_{i \in M} v^\theta(\{i\}) + \frac{\sum_{i \in M} p_i(1-A_i)}{\sum_{k \in N} p_k(1-A_k)} \left(\sum_{k \in N} p_k \left(1 - \prod_{j \in N} (1 - A_j) \right) - \sum_{k \in N} p_k A_k \right) \\
&= \sum_{i \in M} \left(v^\theta(\{i\}) + \frac{p_i(1-A_i)}{\sum_{k \in N} p_k(1-A_k)} \left(v^\theta(N) - \sum_{k \in N} v^\theta(\{k\}) \right) \right) \\
&= \sum_{i \in M} SPP_i(\theta),
\end{aligned}$$

where the inequality is a result of lemma 4.3 with $S = M$ and $x_j = 1 - A_j$ for all $j \in N$. Hence, $SPP(\theta)$ is also stable and thus a member of the core. □

Following the implications in figure 2.3, the Shapley value is a member of the core if the associated game is convex. In theorem 4.3 necessary and sufficient conditions are given for convexity of games with underlying linear AV situations. Combining them leads to the last result of this chapter immediately.

Corollary 4.3. *For any linear AV situation $\theta \in \Theta^L$ with $\mathscr{I}_{ij}(\theta) \leq 0$ for all $i, j \in N$ with $i \neq j$, it holds that $SV(\theta) \in \mathscr{C}(N, v^\theta)$.*

4.5 Conclusions

We formulated a stylized model of reality in which several independent service providers can collaborate by pooling their critical, low-utilization resources that are subject to unavailability. We examined the allocation of the joint profit for such a pooled situation by studying an associated cooperative game. For this game, which we called an availability game, we showed core non-emptiness and provided conditions under which convexity is ensured. In addition, we showed the existence of a PMAS for availability games. Moreover, we discussed four allocation rules and investigated whether they satisfy monotonicity to availability, monotonicity to profit, situation symmetry, and game symmetry. In addition, we investigated whether the allocations resulting for those allocation rules are core members. In table 4.7 and table 4.8 all results are summarized.

Table 4.7: Results for AV situations.

Properties	OP	PA	SPP	SV
Monotonicity to availability	✓	×	×	×
Monotonicity to profit	✓	✓	×	✓
Situation symmetry	✓	✓	✓	✓
Game symmetry	×	×	×	✓
Member of the core	✓	×	×	×

✓: satisfies property

× : does not (always) satisfy property

Table 4.8: Results for linear AV situations.

Properties	OP	PA	SPP	SV
Monotonicity to availability	✓	×	✓	×
Monotonicity to profit	✓	✓	×	✓
Situation symmetry	✓	✓	✓	✓
Game symmetry	×	×	×	✓
Member of the core	✓	×	✓	$\times^{\star}$

$\star$: satisfies property if conditions of corollary 4.3 hold.

For further research, it would be interesting to study the following extensions. First, one could look at an extended model with the property that two or more service providers may demand for a resource simultaneously. In this case, it is not always possible (anymore) that another resource can take over demand. Moreover, we can extend the model by including that (*i*) the individual resource availability depends on the number of parties that participate in the pool and (ii) exchange costs are included for transporting resources to other parties in case resources are pooled. Finally, one can try to completely characterize one (or several) allocation rule(s) introduced in this chapter by various fairness properties.

Chapter 5

Spare parts pooling games under a critical level policy

5.1 Introduction

In the last decades, spare parts pooling has shown its potential in several industries, including the airline industry (Kilpi and Vepsäläinen [26]) and the electricity market (Kukreja et al. [28]). In these industries, one typically applies full pooling. This means that all parties are allowed to take spare parts from the pool whenever needed. Although such form of pooling may reduce total costs significantly, it does not always perform well. For instance, when the impact of an out-of-stock situation differs per party, full pooling may even increase costs. As an example, one can think of the railway setting of section 1.1.2 in which contractors face different penalty costs for their out-of-stock situations. When those contractors decide to pool their inventories, it may be of interest to reserve some spare parts for the one with the higher penalty costs. Indeed, for such kind of situations, it may be of interest to apply another pooling strategy.

In this chapter, based on Schlicher et al. [49], we investigate a situation with several service providers, each keeping a single spare part in stock to protect for downtime of its technical systems, e.g., a complete railway segment. The cost related to the downtime of a single technical system may differ per service provider, while the cost for holding a spare part is equal for everyone. The intensity by which failures occur is the same for all service providers. This assumption is reasonable in situations in which service providers have similar types of technical systems that are used frequently. Moreover, spare parts can be transshipped instantaneously at negligible costs. This assumption is reasonable for situations in which (*i*) removing a defect component of a technical system may already take hours, which makes that transhipment will not affect the duration of the replacement and (*ii*) transshipment costs are negligible compared to downtime costs. Finally, spare parts are ordered at a supplier with ample (production) capacity.

In order to reduce total downtime costs, service providers can cooperate by pooling their spare parts according to a critical level policy. Under such a policy, the number of available spare parts left determines which service providers are allowed to satisfy demand. In this chapter, we apply an intuitive critical level policy. If one spare part is left, the service provider with the highest downtime costs is allowed to satisfy demand only. When two spare parts are left, the service providers with the highest downtime costs and second highest downtime costs are allowed to satisfy demand only, and so on. As one-by-one players are added to the group of players that are allowed to satisfy demand, we refer to this policy as the one-by-one critical level policy and refer to this form of pooling as one-by-one pooling. This form of pooling has the appealing property that, under full cooperation, the long-term average costs are at most equal to the sum of the long-term average costs when the service providers operate individually.

For the situation as described above, we formulate an associated cooperative game. For this cooperative game, which we call a one-by-one pooling game, we present necessary and sufficient conditions for convexity and show that the coalitional values may be recognized as a convex combination of values of Böhm-Bawerk horse market games. As a consequence, we can show our main contribution: one-by-one pooling games have a non-empty core. In addition, we present a class of allocation rules for which the resulting allocations are core members. Finally, we show that there exists an intuitive and simple allocation rule in this class of allocation rules. The resulting allocations of this allocation rule coincide with the average of so-called extreme Böhm-Bawerk horse market allocations. An extreme Böhm-Bawerk horse market allocation is defined as an allocation resulting from an allocation rule out of this class of allocation rules, where based on an arbitrary order of the player set, a player selects core allocations in its own best interest of Böhm-Bawerk horse market games in which he is active, under the restriction that the players before him in the corresponding order recursively selected their related core allocations. Although our focus is spare parts oriented, our results may be applicable to other types of resources as well. For instance, it may be applied to a region with several individual taxi drivers with different contractual agreements for similar-sized companies. These drivers may pool their taxis to increase total profit.

One-by-one pooling games have a strong connection with spare parts games formulated by Karsten et al. [24]. In these games, they consider the same underlying spare parts setting (as we do), except that spare parts are pooled according to a full pooling strategy and the number of spare parts in stock per player is general (instead of exactly one). Clearly, we deviate from Karsten et al. [24] as we apply another pooling strategy that performs well in situations in which downtime costs differ per player.

The remainder of this chapter is as follows. We start in section 5.2 with preliminaries on Böhm-Bawerk horse market games. In section 5.3, simple spare parts situations and the associated one-by-one pooling games are introduced. In section 5.4, we analyse one-by-one pooling games. Finally, we draw conclusions in section 5.5.

5.2 Preliminaries on Böhm-Bawerk horse market games

A Böhm-Bawerk horse market situation is a two-sided market with homogenous goods, e.g., horses. In this market, there are sellers that each have one good for sale and buyers that each want to buy one good. The finite set of sellers and buyers together is denoted by $N \subseteq \mathbb{N}$. The set of sellers is denoted by $S \subseteq N$ and the set of buyers is denoted by $B = N \backslash S$. Each seller (or buyer) $i \in N$ has a valuation w_i for its good. Without loss of generality, we assume that $w_i \geq w_j$ if $i, j \in N$ with $i < j$. To analyse this setting, we define a *Böhm-Bawerk horse market (BBHM) situation* as a tuple (N, S, w) with N, S, and $w = (w_i)_{i \in N}$ as described above. For short, we use γ to refer to such a BBHM situation $\gamma = (N, S, w)$ and Γ for the set of BBHM situations. For every $\gamma \in \Gamma$, it holds for any buyer $j \in B$ and any seller $i \in S$ that if $w_j < w_i$, no good will be traded between buyer j and seller i, and if $w_j \geq w_i$, buyer j and seller i can trade a good with a joint profit of $w_j - w_i \geq 0$. Now, let $A_{S,B}$ be defined as $A_{S,B} = (a_{ij})_{i \in S, j \in B}$, where $a_{ij} = \max\{w_j - w_i, 0\}$. A matching between set $S_1 \subseteq S$ and set $B_1 \subseteq B$ is a subset of pairs $\zeta \subseteq S_1 \times B_1$, where each seller (of S_1) as well as each buyer (of B_1) belongs to at most one pair in ζ. Let $\mathscr{M}(S_1, B_1)$ be the set of all such possible matchings. A matching $\zeta \in \mathscr{M}(B_1, S_1)$ is called *optimal* on (B_1, S_1) if for all $\zeta' \in \mathscr{M}(B_1, S_1)$ it holds that

$$\sum_{(i,j) \in \zeta} a_{ij} \geq \sum_{(i,j) \in \zeta'} a_{ij}.$$

Note that an optimal matching on (B_1, S_1) always exists as the total number of possible matchings is finite. The joint profit of any coalition $M \subseteq N$ is defined as the sum of the joint profits of all pairs of an optimal matching $\zeta \in \mathscr{M}(S \cap M, B \cap M)$. Note that if M exists of buyers only or sellers only, then this joint profit equals zero. Now, we associate a BBHM game to any BBHM situation.

Definition 5.1. *For any $\gamma \in \Gamma$, the associated BBHM game (N, v^γ) is given by*

$$v^\gamma(M) = \max_{\zeta \in \mathscr{M}(S \cap M, B \cap M)} \left\{ \sum_{(i,j) \in \zeta} a_{ij} \right\} \quad \textit{for all } M \subseteq N.$$

BBHM games are well-studied (see, e.g., Shapley and Shubik [57], Moulin [36], and Núñez and Rafels [40]). In particular, Shapley and Shubik [57] showed that the core of a BBHM game consists of a line segment. In order to define this segment, we introduce some additional notation. For any $M \subseteq N$, we define bijection $\sigma_M : M \to \{1,2,\dots,|M|\}$ with $\sigma_M(i) = |\{1,2,\dots,i\} \cap M|$ for any $i \in M$, i.e., with $\sigma_M(i)$ the position of player $i \in M$ in coalition M. Next, let function σ_M^{-1} be defined as the inverse of σ_M. Let $z = \min\{|S|,|B|\}$ and $l = \max\{i \in \{1,2,...,z\} \mid w_{\sigma_B^{-1}(i)} - w_{\sigma_S^{-1}(|S|+1-i)} \geq 0\}$, i.e., l be the variable indicating the number of sellers (or buyers) that trade a good. Those sellers (buyers) are called *active* sellers (buyers) and the remaining sellers (buyers) are called *non-active* sellers (buyers). For any BBHM situation $\gamma \in \Gamma$ with $S \neq \emptyset, N$, i.e., with at least one and at most $|N|-1$ sellers, we introduce

$$\underline{\delta} = \max\left\{w_{\sigma_S^{-1}(|S|-l+1)}\ ,\ w_{\sigma_B^{-1}(l+1)}\right\}, \quad \overline{\delta}\ = \min\left\{w_{\sigma_S^{-1}(|S|-l)}\ ,\ w_{\sigma_B^{-1}(l)}\right\},$$

with $w_{\sigma_S^{-1}(0)} = \infty$ and $w_{\sigma_B^{-1}(|B|+1)} = -\infty$. For any BBHM situation $\gamma \in \Gamma$ with $S = \emptyset$, let $\underline{\delta} = \overline{\delta} = w_{\sigma_N^{-1}(1)}$ and for any BBHM situation $\gamma \in \Gamma$ with $S = N$, let $\underline{\delta} = \overline{\delta} = w_{\sigma_N^{-1}(|N|)}$. We refer to $\underline{\delta}$ and $\overline{\delta}$ as the *extreme prices* to trade a good. Now, for any BBHM situation $\gamma \in \Gamma$, vector $\mathscr{U}(\gamma,\alpha)$ defined by

$$\mathscr{U}_i(\gamma,\alpha) = \begin{cases} \max\{\underline{\delta} + \alpha(\overline{\delta} - \underline{\delta}) - w_i, 0\} & \text{if } i \in S \\ \\ \max\left\{w_i - (\underline{\delta} + \alpha(\overline{\delta} - \underline{\delta})), 0\right\} & \text{if } i \in B \end{cases}$$

with $\alpha \in [0,1]$ for all $i \in N$, is a core element of the associated BBHM game. Varying between $\alpha = 0$, i.e., the *buyer optimal* core allocation, and $\alpha = 1$, i.e., the *seller optimal* core allocation, describes the whole core segment. Now, we formulate the result of Shapley and Shubik [57], stating that the core of a BBHM game consists of a line segment.

Theorem 5.1. *For every $\gamma \in \Gamma$ the core of the associated BBHM game (N, v^γ) is non-empty. In particular, the core is given by*

$$\mathscr{C}(N, v^\gamma) = \left\{\mathscr{U}(\gamma,\alpha) \in \mathbb{R}_+^N \mid \alpha \in [0,1]\right\}.$$

One can check easily that for any *non-active* player $i \in N$, i.e., any player who will not trade a good, $\mathscr{U}_i(\gamma,\alpha) = 0$. We end this section with an example.

Example 5.1. *Consider the following BBHM situation $\gamma \in \Gamma$ with $N = \{1,2,3\}$, $S = \{2\}$, and $w = (5,2,1)$. Then, $\underline{\delta} = \max\{2,1\} = 2$ and $\overline{\delta} = \min\{\infty,5\} = 5$. As a consequence, the core is given by $\mathscr{C}(N,v^\gamma) = \{(3 \cdot (1-\alpha), 3 \cdot \alpha, 0) \mid \alpha \in [0,1]\}$.* ◇

5.3 Model

In this section, we introduce simple spare parts situations and define the associated games, called one-by-one pooling games.

5.3.1 Simple spare parts situations

Consider a service provider i who keeps spare parts in stock to prevent costly downtime of its technical systems. We limit ourselves to one critical component, i.e., one stock-keeping unit, which is subject to failures. A failure of a component immediately leads to a demand for a spare part. This occurs according to a Poisson process with stationary rate $\lambda \in \mathbb{R}_+$. We assume that service provider i starts with one spare part in stock only. If a spare part is on hand when demand occurs, this demand is always satisfied and a replenishment order is placed immediately at a supplier with ample replenishment capacity. The lead times of such orders are independent and identically distributed according to a general distribution function with mean $\mu^{-1} \in \mathbb{R}_+$. If no spare part is available when demand occurs, an emergency delivery is initiated. The expected costs associated with the extra idleness of the technical system (until the spare part arrives), emergency shipment, and so on, shortly called downtime costs, are $d_i \in \mathbb{R}_+$ for service provider i. The holding costs per spare part are $h \in \mathbb{R}_+$ per time unit and are paid over inventory on-hand and ordered spare parts at the supplier. Note, the sum of the inventory on-hand and ordered spare parts is a constant, namely one. Finally, we assume that the service provider is interested in its long-term average costs per time unit.

Now, consider an environment with several service providers. To analyse such setting, we define a *simple spare parts (SSP) situation* as a tuple (N, μ, λ, d, h) with $N \subseteq \mathbb{N}$ the finite set of service providers, $\mu \in \mathbb{R}_+$ the inverse of the mean of a general distribution function generating lead times of replenishment orders for all players, $\lambda \in \mathbb{R}_+$ the demand rate of the Poisson process (which is the same) for each player, $d = (d_i)_{i \in N}$ the vector of downtime costs with $d_i \in \mathbb{R}_+$ the downtime costs of player i and $d_i \geq d_j$ if $i, j \in N$ and $i < j$, and $h \in \mathbb{R}_+$ the holding cost per spare part per unit time (which is the same) for each player. We use θ to refer to an SSP situation $\theta = (N, \mu, \lambda, d, h)$ and Θ for the set of all SSP situations. As the names of players can be relabeled, the non-increasing property of the vector of downtime costs is without loss of generality.

5.3.2 One-by-one pooling games

The service providers, from now on called players, can cooperate by pooling their spare parts. For any coalition $M \subseteq N$ we assume that (*i*) transshipments (of spare parts) occur instantaneously at negligible costs and (*ii*) one-by-one pooling is applied. Under this policy, demand of player $i \in M$ is filled as long as inventory on-hand is at least equal to critical level $\sigma_M(i)$. So, the player with the highest downtime costs can satisfy demand as long as inventory on-hand is at least one, the player with the second highest downtime costs can satisfy demand as long as inventory on-hand is at least two, and so on. In addition, after any satisfied demand, a replenishment order is placed immediately at a joint supplier with (i) ample replenishment capacity and (ii) production times that are generally distributed with mean μ. If player $i \in M$ faces demand, while inventory on-hand is below critical level $\sigma_M(i)$, an emergency delivery is initiated and related downtime costs of d_i are incurred. As players are interested in the long-term average costs per time unit, we determine the steady state probabilities of coalition M with $m = |M|$ players and $i \in \{0, 1, 2, \ldots, m\}$ spare part(s) in stock. Let $\pi(m, i)$ be defined as the steady state probability of coalition M with m players and i spare part(s) in stock. We now present a closed-form description of these probabilities.

Lemma 5.1. *For every SSP situation $\theta \in \Theta$ the steady state probabilities are given for all $M \subseteq N$ and all $i \in \{0, 1, \ldots, m\}$ by*

$$\pi(m,i) = \binom{m}{i} \cdot \left(\frac{\mu}{\lambda+\mu}\right)^{i} \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^{m-i}.$$

Proof **:** See, for instance, Van Houtum and Kranenburg [64, p.74, eq. 4.1].

For coalition $M \subseteq N$ the expected holding costs per time unit are $m \cdot h$ as the sum of the inventory on-hand and ordered spare parts equals m. The expected downtime costs per time unit in state i are $\pi(m, i) \cdot \lambda \cdot \sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)}$. Summing over all $m + 1$ states results in the expected downtime costs per time unit in steady state for coalition M.

Definition 5.2. *For every SSP situation $\theta \in \Theta$ the expected costs per time unit in steady state for any coalition $M \subseteq N$ are denoted by*

$$c^{\theta}(M) = \sum_{i=0}^{m} \left[\pi(m,i) \cdot \lambda \cdot \sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)} \right] + m \cdot h. \tag{5.1}$$

Example 5.2. *Consider SSP situation $\theta = (N, \mu, \lambda, d, h) \in \Theta$ with $N = \{1,2,3\}$, $d_1 = 5$, $d_2 = 2, d_3 = 1$, $\lambda = 1, \mu = 2$, and $h = \frac{1}{3}$. In table 5.1, the steady state probabilities for every possible cardinality of coalitions are presented.*

Table 5.1: Steady state probabilities.

i	0	1	2	3
$\pi(1,i)$	$\frac{1}{3}$	$\frac{2}{3}$		
$\pi(2,i)$	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{4}{9}$	
$\pi(3,i)$	$\frac{1}{27}$	$\frac{6}{27}$	$\frac{12}{27}$	$\frac{8}{27}$

For example, for coalition $M = \{1,2\}$ we obtain

$$c^{\theta}(\{1,2\}) = \frac{1}{9} \cdot (1 \cdot (5+2)) + \frac{4}{9} \cdot (1 \cdot 2) + \frac{4}{9} \cdot (1 \cdot 0) + 2 \cdot \frac{1}{3} = 2\frac{1}{3}.$$

Similarly, the costs of the other coalitions can be determined. In table 5.2, the corresponding costs for all coalitions are depicted. ◇

Table 5.2: Corresponding costs per coalition.

M	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$c^{\theta}(M)$	0	2	1	$\frac{2}{3}$	$2\frac{1}{3}$	$1\frac{7}{9}$	$1\frac{4}{9}$	$2\frac{11}{27}$

We proceed with associating a (cost savings) game to any SSP situation. We refer to such a game as a one-by-one pooling game.

Definition 5.3. *For any SSP situation $\theta \in \Theta$, the game (N, v^{θ}) with*

$$v^{\theta}(M) = \sum_{i \in M} c^{\theta}(\{i\}) - c^{\theta}(M) \tag{5.2}$$

for all coalitions $M \subseteq N$ is called the associated one-by-one pooling (OBOP) game.

It is easily seen that the cost savings do not depend on the holding costs. So, OBOP games are completely determined by the savings in downtime costs. The same result holds if holding costs are paid over inventory on-hand only (Schlicher et al. [49, p.12]).

Lemma 5.2. *For every SSP situation $\theta \in \Theta$, it holds for all $M \subseteq N$ that*

$$v^{\theta}(M) = \lambda \cdot \sum_{i=1}^{m-1} \left[\pi(m,i) \cdot \left(\frac{m-i}{m} \cdot \sum_{j=1}^{i} d_{\sigma_M^{-1}(j)} - \frac{i}{m} \cdot \sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)} \right) \right]. \tag{5.3}$$

Proof **:** Let $\theta \in \Theta$ be an SSP situation and $M \subseteq N$. First, we show that

$$\pi(1,0) = \left[\sum_{i=0}^{m} \frac{\pi(m,i)}{m} \cdot (m-i) \right]. \tag{5.4}$$

Observe that

$$\begin{aligned}
\pi(1,0) = 1 - \pi(1,1) = 1 - \frac{\mu}{\mu+\lambda} & \\
&= \frac{1}{m} \cdot m - \frac{1}{m} \cdot m \cdot \frac{\mu}{\mu+\lambda} \\
&= \frac{1}{m} \cdot m - \frac{1}{m} \cdot \sum_{i=1}^{m} \pi(m,i) \cdot i \\
&= \frac{1}{m} \left[m - \sum_{i=1}^{m} \pi(m,i) \cdot i \right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{m}\left[\sum_{i=0}^{m} \pi(m,i)\cdot m - \sum_{i=1}^{m} \pi(m,i)\cdot i\right] \\
&= \frac{1}{m}\left[\sum_{i=0}^{m} \pi(m,i)\cdot m - \sum_{i=0}^{m} \pi(m,i)\cdot i\right] \\
&= \left[\sum_{i=0}^{m} \frac{\pi(m,i)}{m}\cdot (m-i)\right].
\end{aligned}
$$

The first equality holds as $\pi(1,0)+\pi(1,1)=1$. The second equality holds by lemma 5.1. The third equality holds by multiplying by one. The fourth equality holds as the term $m\cdot \mu/(\mu+\lambda)$ can be recognized as the expectation of the binomial distribution formulated in lemma 5.1. The fifth equality holds by some rewriting. The sixth equality holds as $\sum_{i=0}^{m}\pi(m,i)=1$. The seventh equality holds by adding $\pi(m,0)\cdot 0=0$. The last equality holds by combining summations and some rewriting.

Secondly, observe that

$$
\begin{aligned}
\sum_{i\in M} c^{\theta}(\{i\}) &= \sum_{i\in M}\left[\pi(1,0)\cdot\lambda\cdot d_{\sigma^{-1}_{\{i\}}(1)}\right] + m\cdot h \\
&= \lambda\cdot\pi(1,0)\cdot\sum_{j=1}^{m} d_{\sigma^{-1}_{M}(j)} + m\cdot h \\
&= \lambda\cdot\left[\sum_{i=0}^{m}\frac{\pi(m,i)}{m}\cdot(m-i)\right]\cdot\sum_{j=1}^{m} d_{\sigma^{-1}_{M}(j)} + m\cdot h,
\end{aligned}
$$

where the first equality holds by definition. The second equality holds as $\pi(1,0)\cdot\lambda$ is independent of i and $\sum_{i\in M} d_{\sigma^{-1}_{\{i\}}(1)} = \sum_{j=1}^{m} d_{\sigma^{-1}_{M}(j)}$. The last equality holds by (5.4).

Finally, observe that

$$
\begin{aligned}
&\sum_{i\in M} c^{\theta}(\{i\}) - c^{\theta}(M) \\
=&\lambda\cdot\left[\sum_{i=0}^{m}\frac{\pi(m,i)}{m}\cdot(m-i)\right]\cdot\sum_{j=1}^{m} d_{\sigma^{-1}_{M}(j)} + m\cdot h - \left(\sum_{i=0}^{m}\left[\pi(m,i)\cdot\lambda\cdot\sum_{j=i+1}^{m} d_{\sigma^{-1}_{M}(j)}\right] + m\cdot h\right) \\
=&\lambda\cdot\left[\sum_{i=0}^{m}\frac{\pi(m,i)}{m}\cdot(m-i)\right]\cdot\sum_{j=1}^{m} d_{\sigma^{-1}_{M}(j)} - \lambda\cdot\sum_{i=0}^{m}\left[\pi(m,i)\cdot\sum_{j=i+1}^{m} d_{\sigma^{-1}_{M}(j)}\right] \\
=&\lambda\cdot\sum_{i=0}^{m}\left[\frac{\pi(m,i)}{m}\cdot(m-i)\cdot\left(\sum_{j=1}^{m} d_{\sigma^{-1}_{M}(j)} - \frac{m}{m-i}\cdot\sum_{j=i+1}^{m} d_{\sigma^{-1}_{M}(j)}\right)\right] \\
=&\lambda\cdot\sum_{i=0}^{m}\left[\frac{\pi(m,i)}{m}\cdot(m-i)\cdot\left(\sum_{j=1}^{m} d_{\sigma^{-1}_{M}(j)} - \frac{m+i-i}{m-i}\cdot\sum_{j=i+1}^{m} d_{\sigma^{-1}_{M}(j)}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&=\lambda \cdot \sum_{i=0}^{m} \left[\frac{\pi(m,i)}{m} \cdot (m-i) \cdot \left(\sum_{j=1}^{m} d_{\sigma_M^{-1}(j)} - \sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)} - \frac{i}{m-i} \cdot \sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)} \right) \right] \\
&=\lambda \cdot \sum_{i=0}^{m} \left[\frac{\pi(m,i)}{m} \cdot (m-i) \cdot \left(\sum_{j=1}^{i} d_{\sigma_M^{-1}(j)} - \frac{i}{m-i} \cdot \sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)} \right) \right] \\
&=\lambda \cdot \sum_{i=1}^{m-1} \left[\frac{\pi(m,i)}{m} \cdot (m-i) \cdot \left(\sum_{j=1}^{i} d_{\sigma_M^{-1}(j)} - \frac{i}{m-i} \cdot \sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)} \right) \right] \\
&=\lambda \cdot \sum_{i=1}^{m-1} \left[\pi(m,i) \cdot \left(\frac{m-i}{m} \cdot \sum_{j=1}^{i} d_{\sigma_M^{-1}(j)} - \frac{i}{m} \cdot \sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)} \right) \right].
\end{aligned}
$$

The first equality holds by definition. The second equality holds as $m \cdot h - m \cdot h = 0$. In the third equality, the term $\lambda \cdot \left[\sum_{i=0}^{m} \frac{\pi(m,i)}{m} \cdot (m-i) \right]$ is written in front of the second and fourth summation. In the fourth equality we add zero. In the fifth equality we split the fraction in front of the last summation into two parts. The sixth equality holds by combining the second and third summation. The seventh equality holds as for $i = 0$, $\sum_{j=1}^{i} d_{\sigma_M^{-1}(j)} - \frac{i}{m-i} \cdot \sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)} = 0$ and for $i = m$, $(m-i) = 0$. The last equality holds by some rewriting. This concludes the proof. □

Note that the term between the inner brackets of equation (5.3) of lemma 5.2 can be interpreted as the difference between the cost savings at the first i players and the extra costs at the remaining $m-i$ players.

Now, we show that players are at least as well off in a coalition rather than working individually. Equivalently, we show that under the one-by-one critical level policy, the long-term average costs are at most equal to the sum of the long-term average costs when service providers work individually.

Lemma 5.3. *For every SSP situation $\theta \in \Theta$ and all $M \subseteq N$ it holds that*

$$v^{\theta}(M) \geq 0.$$

Proof **:** Let Θ be an SSP situation and $M \subseteq N$. By lemma 5.2, the value of coalition M of the corresponding OBOP game is given by

$$v^{\theta}(M) = \lambda \cdot \sum_{i=1}^{m-1} \left[\pi(m,i) \cdot \left(\frac{m-i}{m} \cdot \sum_{j=1}^{i} d_{\sigma_M^{-1}(j)} - \frac{i}{m} \cdot \sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)} \right) \right].$$

As $\lambda, \mu > 0$, it is sufficient to check whether the last part of the summation is larger than or equal to zero. Now, take $i \in \{1, 2, ..., m-1\}$, then it holds that

$$\begin{aligned}
\frac{m-i}{m}\sum_{j=1}^{i} d_{\sigma_M^{-1}(j)} - \frac{i}{m}\sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)} &\geq \frac{m-i}{m}\sum_{j=1}^{i} d_{\sigma_M^{-1}(i)} - \frac{i}{m}\sum_{j=i+1}^{m} d_{\sigma_M^{-1}(i+1)} \\
&= \frac{m-i}{m}\cdot i \cdot d_{\sigma_M^{-1}(i)} - \frac{i}{m}\cdot (m-i)\cdot d_{\sigma_M^{-1}(i+1)} \\
&= \frac{m-i}{m}\cdot i \cdot \left(d_{\sigma_M^{-1}(i)} - d_{\sigma_M^{-1}(i+1)}\right) \\
&\geq 0.
\end{aligned}$$

where the first inequality holds as $d_{\sigma_M^{-1}(j)} \geq d_{\sigma_M^{-1}(i)}$ for all $j \in N$ with $j \leq i$ and $d_{\sigma_M^{-1}(j)} \leq d_{\sigma_M^{-1}(i+1)}$ for all $j \in N$ with $j \geq i+1$. The first equality holds as $d_{\sigma_M^{-1}(i)}$ and $d_{\sigma_M^{-1}(i+1)}$ are independent of j and thus the summations can be replaced by a factor i and $(m-i)$ upfront, respectively. The second equality holds by some rewriting. The last inequality holds as $((m-i)/m)\cdot i \geq 0$ and $d_{\sigma_M^{-1}(i)} \geq d_{\sigma_M^{-1}(i+1)}$ for all $i \in \{1, 2, \ldots, m-1\}$ and any $M \subseteq N$. This concludes the proof. □

We conclude this section with an example of an OBOP game.

Example 5.3. *Consider the situation of example 5.2. The cost savings for every coalition of the OBOP game can be calculated directly via lemma 5.2, or indirectly via definition 5.3 if the associated game is already determined. For example, the value of coalition $M = \{1, 2\}$ is*

$$v^{\theta}(\{1,2\}) = c^{\theta}(\{1\}) + c^{\theta}(\{2\}) - c^{\theta}(\{1,2\}) = 2 + 1 - 2\frac{1}{3} = \frac{2}{3}.$$

In a similar way, all other values can be determined. The cost savings for every coalition of the OBOP game are presented in table 5.3. ◇

Table 5.3: Corresponding costs savings per coalition.

M	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v^{\theta}(M)$	0	0	0	0	$\frac{2}{3}$	$\frac{8}{9}$	$\frac{2}{9}$	$1\frac{7}{27}$

5.4 Properties of one-by-one pooling games and a class of allocation rules

In this section, we first investigate whether OBOP games satisfy interesting properties. We start by considering convexity and then investigate whether OBOP games have a non-empty core. Then, we introduce and analyse a class of allocation rules. Finally, we study a simple and intuitive allocation rule within this class of allocation rules.

5.4.1 Convexity

The Shapley value is a core member when the corresponding game is convex (see, for instance, the implications in figure 2.3). As a consequence, the core of a convex game is non-empty. For that reason, we first investigate whether OBOP games are convex in general. The following example illustrates that this is not the case.

Example 5.4. *Consider the situation of example 5.3. Observe that* $v^\theta(\{1,2,3\}) - v^\theta(\{1,3\}) = 1\frac{7}{27} - \frac{8}{9} = \frac{10}{27} < \frac{18}{27} = \frac{2}{3} - 0 = v^\theta(\{1,2\}) - v^\theta(\{1\})$ *and we can conclude that the OBOP game is not convex.* ◇

Despite that OBOP games are not convex in general, we can identify necessary and sufficient conditions for which convexity can be ensured.

Theorem 5.2. *For every SSP situation* $\theta \in \Theta$ *with* $|N| \geq 3$ *the associated OBOP game is convex if and only if*

$$d_i - 2d_j + d_k = 0 \quad \forall i,j,k \in N \text{ with } i < j < k. \tag{5.5}$$

Proof **:** We distinguish between the situations with exactly three players and the situations with more than three players. Let $\theta \in \Theta$ be an SSP situation with $|N| = 3$. Without loss of generality, assume that $N = \{1,2,3\}$.

($\Rightarrow$) Suppose that the game is convex. We show that $d_1 - 2d_2 + d_3 = 0$. By convexity, it holds that

$$v^\theta(\{1,2,3\}) - v^\theta(\{1,3\}) - \Big(v^\theta(\{1,2\}) - v^\theta(\{1\})\Big) \geq 0.$$

By using lemma 5.2, we obtain

$$v^\theta(\{1,2,3\}) - v^\theta(\{1,3\}) - \Big(v^\theta(\{12\}) - v^\theta(\{1\})\Big) = \lambda\frac{\lambda\mu^2}{(\lambda+\mu)^3}(-d_1 + 2d_2 - d_3).$$

As $\lambda, \mu > 0$ it should (thus) hold that

$$d_1 - 2d_2 + d_3 \leq 0. \tag{5.6}$$

Moreover, it holds that

$$v^\theta(\{1,2,3\}) - v^\theta(\{2,3\}) - \Big(v^\theta(\{1,3\}) - v^\theta(\{3\})\Big) \geq 0.$$

By using lemma 5.2, we obtain

$$v^\theta(\{1,2,3\}) - v^\theta(\{2,3\}) - \Big(v^\theta(\{1,3\}) - v^\theta(\{3\})\Big) = \lambda\frac{\lambda^2\mu}{(\lambda+\mu)^3}(d_1 - 2d_2 + d_3).$$

As $\lambda, \mu > 0$ it should (thus) hold that

$$d_1 - 2d_2 + d_3 \geq 0. \tag{5.7}$$

As (5.6) and (5.7) should hold both, we conclude that

$$d_1 - 2d_2 + d_3 = 0.$$

($\Leftarrow$) Suppose that $d_1 - 2d_2 + d_3 = 0$. Now, we show that the inequality $(v^\theta(K \cup \{i\}) - v^\theta(K))- (v^\theta(M \cup \{i\}) - v^\theta(M)) \geq 0$ holds for all $M, K \subseteq \{1,2,3\} \backslash \{i\}$ and all $i \in \{1,2,3\}$. From the only-if-part we know that

$$v^\theta(\{1,2,3\}) - v^\theta(\{1,3\}) - \Big(v^\theta(\{1,2\}) - v^\theta(\{1\})\Big) = 0,$$

$$v^\theta(\{1,2,3\}) - v^\theta(\{2,3\}) - \Big(v^\theta(\{1,3\}) - v^\theta(\{3\})\Big) = 0.$$

By using lemma 5.2, we obtain

$$\begin{aligned}&v^\theta(\{1,2,3\}) - v^\theta(\{2,3\}) - \Big(v^\theta(\{1,2\}) - v^\theta(\{2\})\Big)\\ &= \lambda \frac{\lambda\mu}{(\lambda+\mu)^3} \cdot ((d_1 - d_2)\lambda + (d_2 - d_3)\mu) \geq 0,\end{aligned}$$

where the inequality holds as $\lambda, \mu > 0$ and $d_1 \geq d_2 \geq d_3$. By lemma 5.2, we obtain

$$\begin{aligned}&v^\theta(\{1,2,3\}) - v^\theta(\{1,2\}) - \Big(v^\theta(\{2,3\}) - v^\theta(\{2\})\Big)\\ &= \lambda \frac{\lambda\mu}{(\lambda+\mu)^3} \cdot ((d_1 - d_2)\lambda + (d_2 - d_3)\mu) \geq 0,\end{aligned}$$

where the inequality holds as $\lambda, \mu > 0$ and $d_1 \geq d_2 \geq d_3$. By lemma 5.2, we obtain

$$\begin{aligned}&v^\theta(\{1,2,3\}) - v^\theta(\{1,2\}) - \Big(v^\theta(\{1,3\}) - v^\theta(\{1\})\Big)\\ &= -\lambda \frac{\lambda\mu^2}{(\lambda+\mu)^3}(d_1 - 2d_2 + d_3) = 0,\end{aligned}$$

where the equality holds as $d_1 - 2d_2 + d_3 = 0$. By lemma 5.2, we obtain

$$\begin{aligned}&v^\theta(\{1,2,3\}) - v^\theta(\{1,3\}) - \Big(v^\theta(\{2,3\}) - v^\theta(\{3\})\Big)\\ &= \lambda \frac{\lambda^2\mu}{(\lambda+\mu)^3}(d_1 - 2d_2 + d_3) = 0,\end{aligned}$$

where the inequality holds as $d_1 - 2d_2 + d_3 = 0$. Finally, let $i, j \in N$ with $i \neq j$. Then

$$v^\theta(\{i,j\}) - v^\theta(\{i\}) - (v^\theta(\{j\}) - v^\theta(\emptyset)) = v(\{i,j\}) \geq 0,$$

where the equality holds as $v^\theta(\{i\}) = v^\theta(\{j\}) = v^\theta(\emptyset) = 0$. The inequality holds by lemma 5.3. Hence, $(v^\theta(K \cup \{i\}) - v^\theta(K)) - (v^\theta(M \cup \{i\}) - v^\theta(M)) \geq 0$ for all $M, K \subseteq \{1,2,3\} \backslash \{i\}$ and all $i \in \{1,2,3\}$. Using recursive arguments, convexity follows.

Now, let $\theta \in \Theta$ be an SSP situation with $|N| \geq 4$.

($\Rightarrow$) Suppose that the game is convex. As a consequence, any subgame with player set N' where $N' \subset N$ and $|N'| = 3$ is convex as well. From this, we conclude by the first part of this proof that

$$d_i - 2d_j + d_k = 0 \quad \forall i, j, k \in N \text{ with } i < j < k.$$

($\Leftarrow$) Suppose that $d_i - 2d_j - d_k = 0$ for all $i, j, k \in N$ with $i < j < k$. Given that $d_{\sigma_N^{-1}(1)} - 2d_{\sigma_N^{-1}(2)} - d_{\sigma_N^{-1}(k-1)} = 0$ and $d_{\sigma_N^{-1}(1)} - 2d_{\sigma_N^{-1}(2)} + d_{\sigma_N^{-1}(k)} = 0$ for all $k \in \{4, 5, \ldots, |N|\}$, we can immediately conclude that $d_{\sigma_N^{-1}(3)} = d_{\sigma_N^{-1}(4)} = \ldots = d_{\sigma_N^{-1}(|N|)}$. From $d_{\sigma_N^{-1}(1)} - 2d_{\sigma_N^{-1}(3)} + d_{\sigma_N^{-1}(4)} = 0$ and $d_{\sigma_N^{-1}(2)} - 2d_{\sigma_N^{-1}(3)} + d_{\sigma_N^{-1}(4)} = 0$ we can conclude that $d_{\sigma_N^{-1}(1)} = d_{\sigma_N^{-1}(2)}$ and from $d_{\sigma_N^{-1}(1)} - 2d_{\sigma_N^{-1}(2)} + d_{\sigma_N^{-1}(4)} = 0$ and $d_{\sigma_N^{-1}(1)} - 2d_{\sigma_N^{-1}(3)} + d_{\sigma_N^{-1}(4)} = 0$ we can conclude that $d_{\sigma_N^{-1}(2)} = d_{\sigma_N^{-1}(3)}$. Hence, for every possible solution, it should hold that $d_{\sigma_N^{-1}(1)} = d_{\sigma_N^{-1}(2)} = \ldots = d_{\sigma_N^{-1}(|N|)}$. Now, we show that $(d_i)_{i \in N}$ with $d_i = z \in \mathbb{R}_+$ for all $i \in N$ is a feasible solution. Let $i, j, k \in N$ with $i < j < k$. Then, $d_{\sigma_N^{-1}(i)} - 2d_{\sigma_N^{-1}(j)} + d_{\sigma_N^{-1}(k)} = z - 2z + z = 0$. Hence, $d_i = z \in \mathbb{R}_+$ is indeed feasible. For this feasible solution, it holds that all downtime costs are similar and thus no cost savings are obtained in the corresponding game, i.e., $v^\theta(M) = 0$ for all $M \subseteq N$. We conclude that the game is additive and as a consequence the game is convex. $\square$

Note that for SSP situations with three players only, the associated OBOP game is convex if and only if $d_{\sigma_N^{-1}(1)} - 2d_{\sigma_N^{-1}(2)} + d_{\sigma_N^{-1}(3)} = 0$. For SSP situations with more than three players, the game is convex if and only if all downtime costs are equal implying that the game is additive.

Corollary 5.1. *For every SSP situation $\theta \in \Theta$ with $|N| \geq 4$ the OBOP game is convex if and only if there exists a $z \in \mathbb{R}_+$ such that $d_i = z \in \mathbb{R}_+$ for all $i \in N$.*

5.4.2 The core

The results of section 5.4.1 provide a necessary and sufficient condition for convexity of OBOP games. These results leave the issue of core non-emptiness unanswered for a large class of OBOP games. This issue is the main topic of this section. The following example illustrates a phenomenon which turns out to be useful in here.

Example 5.5. *Consider an SSP situation $\theta \in \Theta$ with $N = \{1, 2, 3\}$. Suppose that player 1 is out-of-stock, while player 2 and player 3 do have a spare part available. Then, in the grand coalition, player 1 has the right to use the spare part of player 3 if necessary, based on the one-by-one critical level policy, while player 3 is not allowed to use its own spare part. Player 1 can* *be recognized as a buyer with value $\lambda \cdot d_1$ per time unit for a spare part and player 3 can be recognized as a seller with value $\lambda \cdot d_3$ per time unit for a spare part. So, an additional value of $\lambda(d_1 - d_3) > 0$ per time unit can be realized. By lemma 5.2, this situation has steady state probability $\frac{\lambda\mu^2}{(\lambda+\mu)^3}$. In a similar way, one can analyse the situations with player 2 out-of-stock, player 3 out-of-stock, player 1 and player 2 out-of-stock, and so on. By combining all these additional values and corresponding probabilities, we obtain $v^{\theta}(N)$.* ◇

Example 5.5 illustrates that the value of the grand coalition of a 3-person OBOP game can in fact be recognized as a convex combination of values of BBHM games. We show this result in general. For this, we need to introduce some new notation and formalize some aspects. Let n be the cardinality of player set N, i.e., $n = |N|$. For every SSP situation $\theta \in \Theta$ we introduce 2^n different BBHM situations, where every BBHM situation has a unique set of sellers. Let $\theta \in \Theta$ be an SSP situation. Then, for every $S \subseteq N$ the related BBHM situation will be denoted by $\gamma_S^{\theta} = (N, S, w)$, where $w = (\lambda \cdot d_i)_{i \in N}$.

Example 5.6. *Consider the situation of example 5.4. The games associated with all corresponding BBHM situations are depicted in table 5.4. Note that in table 5.4 we use that $i \in \{1,2,3\}$.*

Table 5.4: Corresponding values for BBHM games.

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v^{\gamma_S^{\theta}}(\{1,2,3\})$	0	0	$5-2$	$5-1$	0	$2-1$	$5-1$	0
$v^{\gamma_S^{\theta}}(\{1,2\})$	0	0	$5-2$	0	0	0	$5-2$	0
$v^{\gamma_S^{\theta}}(\{1,3\})$	0	0	0	$5-1$	0	0	$5-1$	0
$v^{\gamma_S^{\theta}}(\{2,3\})$	0	0	0	$2-1$	0	$2-1$	0	0
$v^{\gamma_S^{\theta}}(\{i\})$	0	0	0	0	0	0	0	0
$v^{\gamma_S^{\theta}}(\emptyset)$	0	0	0	0	0	0	0	0

For example, for coalition $M = \{1,3\}$, in BBHM situation $\gamma_{\{3\}}^{\theta}$, we have $\mathscr{M}(A_{S\cap M, B\cap M}) = \{\emptyset, \{(3,1)\}\}$. Hence, the value of coalition M is

$$v^{\gamma_{\{3\}}^{\theta}}(\{1,3\}) = \max_{\mu \in \{\emptyset, \{(3,1)\}\}} \left\{ \sum_{(i,j)\in\mu} a_{ij} \right\} = a_{31} = \max\{\lambda \cdot (d_3 - d_1), 0\} = 4.$$ ◇

Recall that for any BBHM situation γ_S^{θ} with $S \subseteq N$ a buyer (or seller) that buys (or sells) a spare part is called an active buyer (active seller), whether a buyer (seller) that buys (sells) no spare part is called a non-active buyer (non-active seller). Recall that $d_i \geq d_j$ for all $i, j \in N$ with $i < j$ and so $\lambda \cdot d_i \geq \lambda \cdot d_j$ for all $i, j \in N$ with $i < j$. This implies the following result. Note that (in this result) we use $s = |S|$ for every $S \subseteq N$.

Lemma 5.4. *For every SSP situation $\theta \in \Theta$ and every seller set $S \subseteq N$ there exists an optimal matching in γ_S^θ for which all players $i \in N$ with $1 \leq \sigma_N^{-1}(i) \leq s$ have (the right to use) a spare part after that specific optimal matching.*

In the remainder of this chapter, we assume that the specific matching of lemma 5.4 is applied. Based on this assumption, it is straightforward to determine which buyers and sellers are the active and non-active ones. If player $i \in N$ is a seller, it is active if and only if $s+1 \leq \sigma_N^{-1}(i) \leq n$ and if player $i \in N$ is a buyer, it is active if and only if $1 \leq \sigma_N^{-1}(i) \leq s$.

Now, it is easy to determine (*i*) the coalitions in which a player is active (non-active) and subsequently (*ii*) the total joint profit under a given SSP situation $\theta \in \Theta$ and a fixed number of sellers.

Lemma 5.5. *For every $\theta \in \Theta$ it holds for all $k \in \{1, 2, \ldots, n-1\}$ that*

$$\sum_{S \subseteq N : |S| = k} v^{\gamma_S^\theta}(N) = \lambda \cdot \binom{n}{k} \left(\sum_{j=1}^{k} d_{\sigma_N^{-1}(j)} - \frac{k}{n} \sum_{j=1}^{n} d_{\sigma_N^{-1}(j)} \right). \tag{5.8}$$

Proof : Let $\theta \in \Theta$ be an SSP situation and $k \in \{1, 2, \ldots, n-1\}$. Based on lemma 5.4 it holds that player $j \in N$ is an active buyer in SSP situation γ_S^θ with $|S| = k$ if and only if $1 \leq \sigma_N^{-1}(j) \leq k$. Let player $j \in N$ be a buyer with $1 \leq \sigma_N^{-1}(j) \leq k$. It is easily seen that, from the $\binom{n}{k}$ BBHM games with k sellers, $\binom{n-1}{k}$ times player j is an active buyer. This results in a total value of $\binom{n-1}{k} \cdot \lambda \cdot d_{\sigma_N^{-1}(j)}$. Similarly, player $j \in N$ is an active seller in SSP situation γ_S^θ with $|S| = k$ if and only if $k+1 \leq \sigma_N^{-1}(j) \leq n$. Let player $j \in N$ be a seller and $k+1 \leq \sigma_N^{-1}(j) \leq n$. It is easily seen that, from the $\binom{n}{k}$ BBHM games, $\binom{n-1}{k-1}$ times player j is an active seller. This results in total costs of $\binom{n-1}{k-1} \cdot \lambda \cdot d_{\sigma_N^{-1}(j)}$. The total joint profit of all BBHM games with k sellers is

$$\begin{aligned}
\sum_{S \subseteq N : |S| = k} v^{\gamma_S^\theta}(N) &= \binom{n-1}{k} \sum_{j=1}^{k} \lambda \cdot d_{\sigma_N^{-1}(j)} - \binom{n-1}{k-1} \sum_{j=k+1}^{n} \lambda \cdot d_{\sigma_N^{-1}(j)} \\
&= \lambda \frac{(n-1)!}{(k-1)!(n-1-k)!} \left(\frac{1}{k} \sum_{j=1}^{k} d_{\sigma_N^{-1}(j)} - \frac{1}{n-k} \sum_{j=k+1}^{n} d_{\sigma_N^{-1}(j)} \right) \\
&= \lambda \frac{(n-1)!}{(k-1)!(n-1-k)!} \left(\left(\frac{1}{k} + \frac{1}{n-k} \right) \sum_{j=1}^{k} d_{\sigma_N^{-1}(j)} - \frac{1}{n-k} \sum_{j=1}^{n} d_{\sigma_N^{-1}(j)} \right) \\
&= \lambda \frac{(n-1)!}{(k-1)!(n-1-k)!} \left(\frac{n}{k(n-k)} \sum_{j=1}^{k} d_{\sigma_N^{-1}(j)} - \frac{1}{n-k} \sum_{j=1}^{n} d_{\sigma_N^{-1}(j)} \right) \\
&= \lambda \binom{n}{k} \left(\sum_{j=1}^{k} d_{\sigma_N^{-1}(j)} - \frac{k}{n} \sum_{j=1}^{n} d_{\sigma_N^{-1}(j)} \right),
\end{aligned}$$

where the first equality follows from the explanation above. The second equality holds by writing λ and part of the binomial coefficient upfront. The third equality holds by adding zero. The fourth and fifth equality hold by some rewriting. $\square$

Lemma 5.5 states that the total joint profit under a fixed number of sellers consists of the product of (i) the total number of possible sets with k sellers and (ii) the total value for the first k players to have the right to use spare parts minus the initial value of any player before interchanging, which occurs with chance k/n.

In order to state and show our main result, we need another definition and lemma.

Definition 5.4. *For every SSP situation $\theta \in \Theta$ and all $S \subseteq N$, $p(\gamma_S^\theta)$ is defined as*

$$p(\gamma_S^\theta) = \frac{\mu^s \lambda^{n-s}}{(\lambda+\mu)^n}.$$

Lemma 5.6. *For any pair of SSP situations $\theta, \theta' \in \Theta$ with $N \subseteq N'$, $\lambda = \lambda'$, $\mu = \mu'$, $d_i = d_i'$ for all $i \in N$ and $h = h'$, it holds that*

$$v^\theta(M) = v^{\theta'}(M) \qquad \text{for all } M \subseteq N,$$
$$v^{\gamma_S^\theta}(M) = v^{\gamma_S^{\theta'}}(M) \qquad \text{for all } M \subseteq N \text{ and all } S \subseteq N.$$

Note that the result of lemma 5.6 follows directly, as the value of any coalition $M \subseteq N$ depends on the parameters of the players in coalition M only.

Theorem 5.3. *For every SSP situation $\theta \in \Theta$ and associated OBOP game (N, v^θ), it holds that*

$$v^\theta = \sum_{S \subseteq N} p(\gamma_S^\theta) \cdot v^{\gamma_S^\theta}. \tag{5.9}$$

***Proof* :** Let $\theta \in \Theta$ be an SSP situation. Then

$$\begin{aligned}
&\sum_{S \subseteq N} p(\gamma_S^\theta) \cdot v^{\gamma_S^\theta}(N) \\
=&\sum_{k=1}^{n-1} \left[\sum_{S \subseteq N: |S|=k} \left(\frac{\mu}{\lambda+\mu}\right)^k \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^{n-k} \cdot v^{\gamma_S^\theta}(N) \right] \\
=&\sum_{k=1}^{n-1} \left[\left(\frac{\mu}{\lambda+\mu}\right)^k \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^{n-k} \cdot \sum_{S \subseteq N: |S|=k} v^{\gamma_S^\theta}(N) \right] \\
=&\sum_{k=1}^{n-1} \left[\left(\frac{\mu}{\lambda+\mu}\right)^k \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^{n-k} \cdot \lambda \cdot \binom{n}{k} \cdot \left(\sum_{j=1}^k d_{\sigma_N^{-1}(j)} - \frac{k}{n} \sum_{j=1}^n d_{\sigma_N^{-1}(j)} \right) \right] \\
=&\lambda \cdot \sum_{k=1}^{n-1} \left[\pi(n,k) \cdot \left(\sum_{j=1}^k d_{\sigma_N^{-1}(j)} - \frac{k}{n} \sum_{j=1}^k d_{\sigma_N^{-1}(j)} - \frac{k}{n} \sum_{j=k+1}^n d_{\sigma_N^{-1}(j)} \right) \right]
\end{aligned}$$

$$
\begin{aligned}
&= \lambda \cdot \sum_{k=1}^{n-1} \left[\pi(n,k) \cdot \left(\frac{n-k}{n} \sum_{j=1}^{k} d_{\sigma_N^{-1}(j)} - \frac{k}{n} \sum_{j=1}^{n} d_{\sigma_N^{-1}(j)} \right) \right] \\
&= v^{\theta}(N),
\end{aligned}
$$

where the first equality holds by (*i*) $v^{\gamma_N^{\theta}}(N) = v^{\gamma_{\emptyset}^{\theta}}(N) = 0$, i.e., the values of the BBHM games with sellers only (or buyers only) are zero, (*ii*) definition 5.4, and (*iii*) conditioning on the cardinality of S. The second equality holds as the binomial coefficient depends on k only. The third equality holds by lemma 5.5. The fourth equality holds by lemma 5.1 and splitting the last sum. The fifth equality holds by some rewriting. The last equality holds by lemma 5.2.

Now, let $M \subset N$. In addition, let $\widetilde{\theta} = (\widetilde{N}, \widetilde{\mu}, \widetilde{\lambda}, \widetilde{d}, \widetilde{h})$ be a new SSP situation with $\widetilde{N} = M$, $\widetilde{\lambda} = \lambda$, $\widetilde{\mu} = \mu$, $\widetilde{h} = h$ and $\widetilde{d} = (d_i)_{i \in M}$. Then

$$
\begin{aligned}
\sum_{S \subseteq N} p(\gamma_S^{\theta}) \cdot v^{\gamma_S^{\theta}}(M) &= \sum_{S_1 \subseteq M} \sum_{S_2 \subseteq N \backslash M} p(\gamma_{S_1 \cup S_2}^{\theta}) \cdot v^{\gamma_{S_1 \cup S_2}^{\theta}}(M) \\
&= \sum_{S_1 \subseteq M} \sum_{S_2 \subseteq N \backslash M} p(\gamma_{S_1 \cup S_2}^{\theta}) \cdot v^{\gamma_{S_1}^{\theta}}(M) \\
&= \sum_{S_1 \subseteq M} v^{\gamma_{S_1}^{\theta}}(M) \sum_{S_2 \subseteq N \backslash M} p(\gamma_{S_1 \cup S_2}^{\theta}) \\
&= \sum_{S_1 \subseteq M} v^{\gamma_{S_1}^{\theta}}(M) \sum_{S_2 \subseteq N \backslash M} \frac{\mu^{s_1+s_2} \lambda^{n-s_1-s_2}}{(\lambda+\mu)^n} \\
&= \sum_{S_1 \subseteq M} v^{\gamma_{S_1}^{\theta}}(M) \cdot \frac{\mu^{s_1} \lambda^{m-s_1}}{(\lambda+\mu)^m} \sum_{S_2 \subseteq N \backslash M} \frac{\mu^{s_2} \lambda^{n-m-s_2}}{(\lambda+\mu)^{n-m}} \\
&= \sum_{S_1 \subseteq M} v^{\gamma_{S_1}^{\theta}}(M) \cdot \frac{\mu^{s_1} \lambda^{m-s_1}}{(\lambda+\mu)^m} \sum_{k=0}^{n-m} \binom{n-m}{k} \frac{\mu^{k} \lambda^{n-m-k}}{(\lambda+\mu)^{n-m}} \\
&= \sum_{S_1 \subseteq M} v^{\gamma_{S_1}^{\theta}}(M) \cdot \frac{\mu^{s_1} \lambda^{m-s_1}}{(\lambda+\mu)^m} \frac{(\lambda+\mu)^{n-m}}{(\lambda+\mu)^{n-m}} \\
&= \sum_{S_1 \subseteq M} \frac{\mu^{s_1} \lambda^{m-s_1}}{(\lambda+\mu)^m} \cdot v^{\gamma_{S_1}^{\theta}}(M) \\
&= \sum_{S_1 \subseteq M} p(\gamma_{S_1}^{\widetilde{\theta}}) \cdot v^{\gamma_{S_1}^{\widetilde{\theta}}}(M) = v^{\widetilde{\theta}}(M) = v^{\theta}(M),
\end{aligned}
$$

where the first equality follows by rewriting the summation as two summations. The second equality holds as sellers in $S_2 \subseteq N \backslash M$ do not influence $v^{\gamma_{S_1 \cup S_2}^{\theta}}(M)$. The third equality holds as $v^{\gamma_{S_1}^{\theta}}(M)$ does not depend on S_2. The fourth equality holds by definition 5.4. The fifth equality holds as $\frac{\mu^{s_1} \lambda^{m-s_1}}{(\lambda+\mu)^m}$ does not depend on S_2. The sixth equality holds as the argument in the second summation only depends on the cardinality of S_2.

The seventh equality holds by the binomium of Newton. The eighth equality holds as $\frac{(\lambda+\mu)^{n-m}}{(\lambda+\mu)^{n-m}} = 1$. The ninth equality holds as $p(\gamma_{S_1}^{\widetilde{\theta}}) = \frac{\mu^{s_1}\lambda^{m-s_1}}{(\lambda+\mu)^m}$ and $v^{\gamma_{S_1}^{\theta}}(M) = v^{\gamma_{S_1}^{\widetilde{\theta}}}(M)$ by lemma 5.6. The tenth equality holds by the first part of the proof (with a game with player set M). The last equality holds by lemma 5.6. □

We illustrate the result of theorem 5.3 in the following example.

Example 5.7. *Consider the situation of example 5.6. Then*

$$p(\gamma_S^{\theta}) = \begin{cases} \frac{8}{27} & \textit{if} \quad |S| = 3 \\ \frac{4}{27} & \textit{if} \quad |S| = 2 \\ \frac{2}{27} & \textit{if} \quad |S| = 1 \\ \frac{1}{27} & \textit{if} \quad |S| = 0. \end{cases}$$

For example, for coalition $M = \{2,3\}$, we obtain, using table 5.4 and theorem 5.3

$$v^{\theta}(\{2,3\}) = \frac{1}{27} \cdot 0 + \frac{2}{27} \cdot (0 + 0 + (2-1)) + \frac{4}{27} \cdot (0 + (2-1) + 0) + \frac{8}{27} \cdot 0 = \frac{2}{9}.$$

It is easily checked that all other values can be determined via table 5.4 as well. ◇

The next lemma formally expresses that a positive linear combination of several games with a non-empty core constitutes a game with a non-empty core as well.

Lemma 5.7. *Let $(v^k)_{k=1}^l$ be a set of games with the same player set N, and $x^k \in \mathscr{C}(N, v_k)$ for $k = 1, 2, \ldots, l$. Then, it holds for all $(\alpha_k)_{k=1}^l \in \mathbb{R}_+^l$ that*

$$y = \sum_{k=1}^{l} \alpha_k \cdot x^k \in \mathscr{C}(N, z),$$

where $z(M) = \sum_{k=1}^{l} \alpha_k \cdot v^k(M)$ for all $M \subseteq N$.

Proof **:** Let $(\alpha_k)_{k=1}^l \in \mathbb{R}_+^l$ and $M \subseteq N$. Then

$$\sum_{i \in M} y_i = \sum_{i \in M} \sum_{k=1}^{l} \alpha_k x_i^k = \sum_{k=1}^{l} \alpha_k \sum_{i \in M} x_i^k \geq \sum_{k=1}^{l} \alpha_k v^k(M) = z(M),$$

where the inequality becomes an equality for $M = N$. □

Using lemma 5.7, we conclude that every OBOP game has a non-empty core.

Theorem 5.4. *Every OBOP game has a non-empty core.*

Proof : Let $\theta \in \Theta$ be an SSP situation and (N, v^θ) be the associated OBOP game. Moreover, for every BBHM situation γ_S^θ with $S \subseteq N$ it holds, based on theorem 5.1, that the associated BBHM game $(N, v^{\gamma_S^\theta})$ has a non-empty core. Then, by lemma 5.7 and theorem 5.3, the associated OBOP game (N, v^θ) has a non-empty core. □

A next and natural step is to investigate wether OBOP games are totally balanced.

Theorem 5.5. *Every OBOP game is totally balanced.*

Proof : Let $\theta \in \Theta$ be an SSP situation and (N, v^θ) be the associated OBOP game. Let $M \subseteq N$ and consider subgame (M, v_M^θ) with $v_M^\theta(K) = v^\theta(K)$ for all $K \subseteq M$. Using lemma 5.6 there exists a $\theta' \in \Theta$ with $N' = M$, $\lambda' = \lambda$, $\mu' = \mu$, and $d_i' = d_i$ for all $i \in N'$ for which holds that $v^\theta(K) = v^{\theta'}(K)$ for all $K \subseteq N'$. Hence, $v_M^\theta(K) = v^{\theta'}(K)$ for all $K \subseteq M$. Using theorem 5.4, $(N', v^{\theta'})$ is balanced and so (M, v_M^θ) is balanced as well. We conclude that OBOP games are totally balanced. □

It is well-known (see, e.g., the implications in figure 2.3) that totally balanced games are superadditive. So, OBOP games are superadditive. In addition, as OBOP games are zero-normalized, OBOP games are monotonic as well.

Corollary 5.2. *Every OBOP game is superadditive and monotonic.*

5.4.3 A class of allocation rules

In this section, we first introduce (a class of) allocation rules defined on SSP situations. Formally, an allocation rule on SSP situations is defined as a mapping f that assigns to any SSP situation $\theta \in \Theta$ a vector $f(\theta) \in \mathbb{R}^N$. In section 5.4.2, we learned that OBOP games can be recognized as convex combinations of BBHM games. Now, we construct a class of allocation rules that exploits this. Recall from section 5.2 that any core element of a BBHM situation $\gamma \in \Gamma$ can be described as $\mathscr{U}(\gamma, \alpha)$ for some $\alpha \in [0,1]$. Now, for any $\theta \in \Theta$ and all related BBHM situations, i.e., all possible sets of sellers, we set $\alpha \in [0,1]$, i.e., we split the (trading) profit between the active buyers and active sellers in such a way that it is a core member of the associated BBHM game. Then, we multiply these outcomes with the probability that these BBHM situations occur and finally we add all terms. As for every possible set of sellers $S \subseteq N$ we have to select a parameter, we can formulate a whole class of allocation rules.

Definition 5.5. *For all $\widehat{\alpha} \in [0,1]^{2^N}$ allocation rule $\mathscr{F}^{\widehat{\alpha}}$ assigns to any SSP situation $\theta \in \Theta$ allocation*

$$\mathscr{F}^{\widehat{\alpha}}(\theta) = \sum_{S \subseteq N} p(\gamma_S^\theta) \cdot \mathscr{U}(\gamma_S^\theta, \widehat{\alpha}_S). \tag{5.10}$$

In the upcoming example, we illustrate such an allocation rule.

Example 5.8. *Consider the situation of example 5.7. In table 5.5, the extreme prices $\underline{\delta}$ and $\overline{\delta}$ are presented for any γ_S^θ with $S \subseteq N$. Note that the extreme prices for $S = \{2\}$ are presented in example 5.1 as well.*

Table 5.5: Corresponding extreme buyer (seller) prices.

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\overline{\delta}$	0	0	5	5	0	2	2	0
$\underline{\delta}$	0	0	2	2	0	1	1	0

Let $\widehat{\alpha}_S = \frac{1}{2}$ for all $S \subseteq N$. This fixes a core element for all $(N, v^{\gamma_S^\theta})$ with $S \subseteq N$. In table 5.6, all these core elements are presented.

Table 5.6: Corresponding core elements for BBHM situations.

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\mathscr{U}_1(\gamma_S^\theta, \frac{1}{2})$	0	0	$1\frac{1}{2}$	$1\frac{1}{2}$	0	0	$3\frac{1}{2}$	0
$\mathscr{U}_2(\gamma_S^\theta, \frac{1}{2})$	0	0	$1\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
$\mathscr{U}_3(\gamma_S^\theta, \frac{1}{2})$	0	0	0	$2\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0

For example, given that $S = \{2\}$, the payoff of player 1 is

$$\mathscr{U}_1\left(\gamma_{\{2\}}^\theta, \frac{1}{2}\right) = \max\{5 - (2 + \frac{1}{2} \cdot (5-2)), 0\} = 1\frac{1}{2}.$$

Now, by combining terms, we obtain

$$\mathscr{F}_1^{\widehat{\alpha}}(\theta) = \frac{1}{27} \cdot 0 + \frac{2}{27} \cdot \left(0 + 1\frac{1}{2} + 1\frac{1}{2}\right) + \frac{4}{27} \cdot \left(0 + 0 + 3\frac{1}{2}\right) + \frac{8}{27} \cdot 0 = \frac{20}{27}$$
$$\mathscr{F}_2^{\widehat{\alpha}}(\theta) = \frac{1}{27} \cdot 0 + \frac{2}{27} \cdot \left(0 + 1\frac{1}{2} + 0\right) + \frac{4}{27} \cdot \left(0 + \frac{1}{2} + 0\right) + \frac{8}{27} \cdot 0 = \frac{5}{27}$$
$$\mathscr{F}_3^{\widehat{\alpha}}(\theta) = \frac{1}{27} \cdot 0 + \frac{2}{27} \cdot \left(0 + 0 + 2\frac{1}{2}\right) + \frac{4}{27} \cdot \left(0 + \frac{1}{2} + \frac{1}{2}\right) + \frac{8}{27} \cdot 0 = \frac{9}{27}.$$

It is easily checked that $(\frac{20}{37}, \frac{5}{27}, \frac{9}{27}) \in \mathscr{C}(N, v^\theta)$. ◇

The next result follows directly from theorem 5.1, lemma 5.7, and theorem 5.3.

Corollary 5.3. *For every SSP situation $\theta \in \Theta$ it holds for any $\widehat{\alpha} \in [0,1]^{2^N}$ that $\mathscr{F}^{\widehat{\alpha}}(\theta) \in \mathscr{C}(N, v^\theta)$.*

One may wonder whether every element of the core of an OBOP game results from an allocation rule $\mathscr{F}^{\widehat{\alpha}}$ for some $\widehat{\alpha} \in [0,1]^{2^N}$, i.e., if $\left\{ \mathscr{F}^{\widehat{\alpha}} \mid \widehat{\alpha} \in [0,1]^{2^N} \right\} = \mathscr{C}(N, v^{\theta})$. The following example shows that this is not the case in general.

Example 5.9. *Consider the situation of example 5.8 and* $x = (1\frac{1}{27}, \frac{6}{27}, 0) \in \mathscr{C}(N, v^{\theta})$. *Let* $\widehat{\alpha} \in [0,1]^{2^N}$, *then*

$$\mathscr{U}_3\left(\gamma^{\theta}_{\{3\}}, \widehat{\alpha}_{\{3\}}\right) = \underline{\delta} + \widehat{\alpha}_{\{3\}}(\overline{\delta} - \underline{\delta}) - \lambda \cdot d_3 = 1 + 3 \cdot \widehat{\alpha}_{\{3\}} > 0,$$

as $\widehat{\alpha}_{\{3\}} \in [0,1]$. *In addition,* $p\left(\gamma^{\theta}_{\{3\}}\right) = \frac{2}{27} > 0$ *and thus* $\mathscr{F}^{\widehat{\alpha}}_3(\theta) > 0$.

We conclude that $x \neq (\mathscr{F}^{\widehat{\alpha}}_i(\theta))_{i \in N}$ *for all* $\widehat{\alpha} \in [0,1]^{2^N}$. *In addition, a graphical representation of* $\left\{ \mathscr{F}^{\widehat{\alpha}} \mid \widehat{\alpha} \in [0,1]^{2^N} \right\}$ *and the core is represented in figure 5.1.* ◇

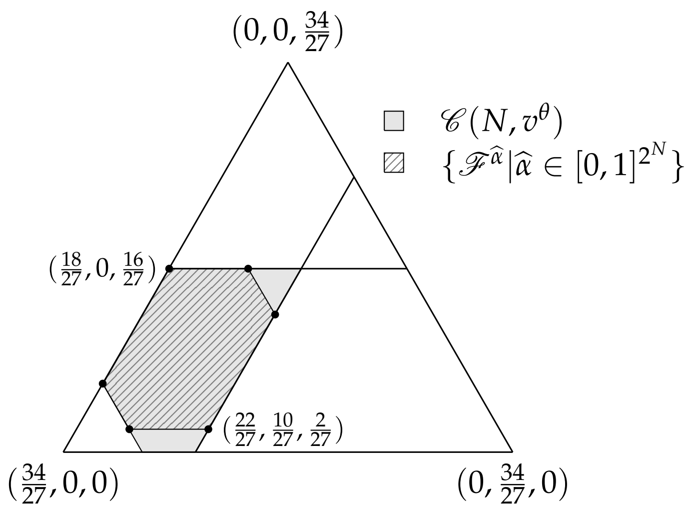

Figure 5.1: $\mathscr{C}(N, v^{\theta})$ and $\{\mathscr{F}^{\widehat{\alpha}} | \widehat{\alpha} \in [0,1]^{2^N}\}$ of game (N, v^{θ}).

Figure 5.1 shows that $\left\{ \mathscr{F}^{\widehat{\alpha}} \mid \widehat{\alpha} \in [0,1]^{2^N} \right\}$ is a convex set[5] and is spanned by six vectors only. These vectors turn out to be special allocations. For example, vector $(\frac{18}{27}, 0, \frac{16}{27})$ is the allocation where, based on order (3,1,2), a player selects core allocations in its own best interest of BBHM games in which he is active under the restriction that the players before him in the corresponding order recursively selected their related core allocations. For example, for order (3,1,2), first player 3 selects core allocations of BBHM games $(N, v^{\gamma^{\theta}_{\{3\}}})$, $(N, v^{\gamma^{\theta}_{\{1,3\}}})$, and $(N, v^{\gamma^{\theta}_{\{2,3\}}})$. As player 3 is an active seller in all of these BBHM games, player 3 selects three times the seller optimal core allocation. Subsequently player 1 has the right to select the core allocation of the remaining BBHM game $(N, v^{\gamma^{\theta}_{\{2\}}})$. As

[5]One can check easily that this set is convex by the linearity of $\mathscr{F}^{\widehat{\alpha}}$ in $\widehat{\alpha}$. See also the first part of the proof of theorem 5.7.

player 1 is an active buyer in this BBHM game, player 1 selects the buyer optimal core allocation. In a similar way, the other vectors can be obtained by considering the different orders of the player set. Note that for BBHM games without active players, and so no trading profit, there is no need to select a core allocation.

Now, we formalize these vectors. An order on N is a bijection $\omega : \{1, 2, \ldots, |N|\} \to N$, where $\omega(j)$ indicates which player is in position j and $\omega^{-1}(i)$ indicates the position of player i. The set of all orders is denoted by Ω^N. For any $S \subseteq N$ let l_S be the number of active sellers (buyers) that trade a good in γ_S^θ. For any $S \subseteq N$ with at least one active seller in γ_S^θ let $N^S = \left\{\sigma_B^{-1}(i) \mid 1 \leq i \leq l_S\right\} \cup \left\{\sigma_S^{-1}(i) \mid s - l_S + 1 \leq i \leq s\right\}$ be the subset of N with active sellers and buyers, and let $i_S^* \in N^S$ such that $\omega^{-1}(i_S^*) \leq \omega^{-1}(i)$ for all $i \in N^S$, i.e., i_S^* the first active player according to ω. Then, for any $S \subseteq N$ with $l_S \geq 1$ we define

$$\alpha_S^\omega = \begin{cases} 1 & \text{if } i_S^* \in S \\ \\ 0 & \text{if } i_S^* \in B. \end{cases}$$

Hence, if the first active player is a seller, it selects the seller optimal core allocation and if the first active player is a buyer, it selects the buyer optimal core allocation. For any $S \subseteq N$ for which $l_S = 0$, and thus $\mathscr{U}_i(\gamma_S^\theta, \alpha_S) = 0$ for all $i \in N$ and any $\alpha_S \in [0,1]$, we set (arbitrarily) $\alpha_S^\omega = \frac{1}{2}$. For any $\omega \in \Omega$ this fixes vector $\alpha^\omega = (\alpha_S^\omega)_{S \subseteq N}$. We refer to such vectors as *extreme BBHM allocations*.

Example 5.10. *Consider the situation of example 5.9. For order* $(2, 1, 3)$, *the first active player in BBHM game* $(N, v^{\gamma_{\{2\}}^\theta})$ *is a seller, namely player 2, and as a consequence* $\alpha_{\{2\}}^{(2,1,3)} = 1$. *In table 5.7 all extreme BBHM allocations are presented.* ◇

Table 5.7: All extreme BBHM allocations.

ω	$\alpha_\emptyset^\omega$	$\alpha_{\{1\}}^\omega$	$\alpha_{\{2\}}^\omega$	$\alpha_{\{3\}}^\omega$	$\alpha_{\{1,2\}}^\omega$	$\alpha_{\{1,3\}}^\omega$	$\alpha_{\{2,3\}}^\omega$	$\alpha_{\{1,2,3\}}^\omega$
(1,2,3)	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$
(1,3,2)	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$
(2,1,3)	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$
(2,3,1)	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$
(3,1,2)	$\frac{1}{2}$	$\frac{1}{2}$	0	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$
(3,2,1)	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$

It is a natural choice to consider the average over all extreme BBHM allocations, which is in line with the Shapley value and the Alexia value. The Alexia value (Tijs et al. [61]) results from averaging the lexicographical optimal points of the core.

Example 5.11. *Consider the situation of example 5.10. In table 5.8 $\mathscr{F}^{\alpha^\omega}$ is represented for all $\omega \in \Omega$ as well as the average of these vectors. In addition, the Shapley value is given by $\left(\frac{49}{81}, \frac{22}{81}, \frac{31}{81}\right)$ and the Alexia value by $\left(\frac{19}{27}, \frac{6}{27}, \frac{9}{27}\right)$.* $\diamond$

Table 5.8: $\mathscr{F}^{\alpha^\omega}$ for all $\omega \in \Omega$ and average.

ω	(1,2,3)	(1,3,2)	(2,1,3)	(2,3,1)	(3,1,2)	(3,2,1)	average
$\mathscr{F}_1^{\widehat{\alpha}^\omega}(\theta)$	$\frac{28}{27}$	$\frac{28}{27}$	$\frac{22}{27}$	$\frac{12}{27}$	$\frac{18}{27}$	$\frac{12}{27}$	$\frac{20}{27}$
$\mathscr{F}_2^{\widehat{\alpha}^\omega}(\theta)$	$\frac{4}{27}$	0	$\frac{10}{27}$	$\frac{10}{27}$	0	$\frac{6}{27}$	$\frac{5}{27}$
$\mathscr{F}_3^{\widehat{\alpha}^\omega}(\theta)$	$\frac{2}{27}$	$\frac{6}{27}$	$\frac{2}{27}$	$\frac{12}{27}$	$\frac{16}{27}$	$\frac{16}{27}$	$\frac{9}{27}$

Note that the average of $\mathscr{F}^{\alpha^\omega}$ deviates from the Shapley value and the Alexia value. However, it does not deviate from the allocation of (the simple) allocation rule $\mathscr{F}^{\widehat{\alpha}}(\theta)$ with $\widehat{\alpha}_S = \frac{1}{2}$ for all $S \subseteq N$. This is no coincidence.

Theorem 5.6. *Let $\theta \in \Theta$ and $\widehat{\alpha}_S = \frac{1}{2}$ for all $S \subseteq N$. Then*

$$\mathscr{F}^{\widehat{\alpha}}(\theta) = \frac{1}{n!} \sum_{\omega \in \Omega} \mathscr{F}^{\alpha^\omega}(\theta).$$

Proof **:** First, observe that for every $S \subseteq N$ with $l_S \geq 1$, i.e., with at least one trading couple, the number of orders $\omega \in \Omega$ for which an active seller selects $\alpha_S^\omega = 1$ equals the number of orders $\omega \in \Omega$ for which an active buyer selects $\alpha_S^\omega = 0$ as we consider all possible orders of the player set. As $|\Omega| = n!$, the total number of times an active seller (buyer) selects is $n!/2$. Secondly, observe that for every $S \subseteq N$ with $l_S = 0$, i.e., with no trading couple, $\mathscr{U}_i(\gamma_S^\theta, \widehat{\alpha}_S) = 0$ for any $\widehat{\alpha}_s \in [0,1]$ and all $i \in N$. Now, observe that

$$\begin{aligned}
\mathscr{F}^{\widehat{\alpha}}(\theta) &= \sum_{S \subseteq N} p(\gamma_S^\theta) \cdot \mathscr{U}\left(\gamma_S^\theta, \frac{1}{2}\right) \\
&= \sum_{S \subseteq N} p(\gamma_S^\theta) \cdot \left(\frac{1}{2} \cdot \mathscr{U}(\gamma_S^\theta, 0) + \frac{1}{2} \cdot \mathscr{U}(\gamma_S^\theta, 1)\right) \\
&= \frac{1}{n!} \sum_{S \subseteq N} p(\gamma_S^\theta) \cdot \left(\frac{n!}{2} \cdot \mathscr{U}(\gamma_S^\theta, 0) + \frac{n!}{2} \cdot \mathscr{U}(\gamma_S^\theta, 1)\right) \\
&= \frac{1}{n!} \sum_{S \subseteq N} p(\gamma_S^\theta) \sum_{\omega \in \Omega} \mathscr{U}(\gamma_S^\theta, \alpha_S^\omega) \\
&= \frac{1}{n!} \sum_{\omega \in \Omega} \sum_{S \subseteq N} p(\gamma_S^\theta) \cdot \mathscr{U}(\gamma_S^\theta, \alpha_S^\omega) \\
&= \frac{1}{n!} \sum_{\omega \in \Omega} \mathscr{F}^{\alpha^\omega}(\theta).
\end{aligned}$$

The first and last equality hold by definition. The second equality holds as $\mathscr{U}(\gamma_S^\theta, \frac{1}{2}) = \frac{1}{2} \cdot \mathscr{U}(\gamma_S^\theta, 0) + \frac{1}{2} \cdot \mathscr{U}(\gamma_S^\theta, 1)$ for any $S \subseteq N$. The third and fifth equality hold by some rewriting. The fourth equality is a result of the description given at the start of the proof. This concludes the proof. $\square$

Recall that figure 5.1 showed us that $\{\mathscr{F}^{\widehat{\alpha}} | \widehat{\alpha} \in [0,1]^{2^N}\}$ is spanned by the extreme BBHM allocations only. This does not hold in general.

Theorem 5.7. *For any $\theta \in \Theta$ it holds that*

$$convexhull \left\{ \mathscr{F}^{\widehat{\alpha}^\omega} \mid \omega \in \Omega \right\} \subseteq \left\{ \mathscr{F}^{\widehat{\alpha}} | \widehat{\alpha} \in [0,1]^{2^N} \right\}$$

and there exists a $\theta \in \Theta$ for which this relation is strict.

Proof **:** First we show that $convexhull \left\{ \mathscr{F}^{\widehat{\alpha}^\omega} \mid \omega \in \Omega \right\} \subseteq \left\{ \mathscr{F}^{\widehat{\alpha}} \mid \widehat{\alpha} \in [0,1]^{2^N} \right\}$ for all $\theta \in \Theta$ and subsequently that there exists a $\theta \in \Theta$ for which $\left\{ \mathscr{F}^{\widehat{\alpha}} \mid \widehat{\alpha} \in [0,1]^{2^N} \right\} \not\subseteq convexhull \left\{ \mathscr{F}^{\widehat{\alpha}^\omega} \mid \omega \in \Omega \right\}$.

Let $\theta \in \Theta$, $\widehat{\alpha}, \widehat{\alpha}' \in [0,1]^{2^N}$ and $\beta \in [0,1]$. Then observe that

$$\begin{aligned} \beta \mathscr{F}^{\widehat{\alpha}} + (1-\beta)\mathscr{F}^{\widehat{\alpha}'} &= \beta \sum_{S \subseteq N} p(\gamma_S^\theta) \cdot \mathscr{U}(\gamma_S^\theta, \widehat{\alpha}_S) + (1-\beta) \sum_{S \subseteq N} p(\gamma_S^\theta) \cdot \mathscr{U}(\gamma_S^\theta, \widehat{\alpha}'_S) \\ &= \sum_{S \subseteq N} p(\gamma_S^\theta) \cdot \mathscr{U}(\gamma_S^\theta, \beta\widehat{\alpha}_S + (1-\beta)\widehat{\alpha}'_S) \\ &= \mathscr{F}^{\beta\widehat{\alpha} + (1-\beta)\widehat{\alpha}'}. \end{aligned}$$

The second equality holds as $\mathscr{U}(\gamma_S^\theta, \alpha)$ is linear in α for all $S \subseteq N$. We conclude that $\left\{ \mathscr{F}^{\widehat{\alpha}} \mid \widehat{\alpha} \in [0,1]^{2^N} \right\}$ is a convex set. Moreover, observe that for all $\omega \in \Omega$ there exists an $\widehat{\alpha} \in [0,1]^{2^N}$ such that $\alpha^\omega = \widehat{\alpha}$. So, $convexhull \left\{ \mathscr{F}^{\widehat{\alpha}^\omega} \mid \omega \in \Omega \right\} \subseteq \left\{ \mathscr{F}^{\widehat{\alpha}} \mid \widehat{\alpha} \in [0,1]^{2^N} \right\}$.

Consider $\theta \in \Theta$ with $N = \{1,2,3,4,5\}$ and $d_1 > d_2 > d_3 > d_4 > d_5$. This implies that for all $S \subseteq N$ it holds that $\mathscr{U}_i(\gamma_S^\theta, 0) < \mathscr{U}_i(\gamma_S^\theta, 1)$ if $i \in S$ and active and $\mathscr{U}_i(\gamma_S^\theta, 0) > \mathscr{U}_i(\gamma_S^\theta, 1)$ if $i \in B$ and active. For all $S \subseteq N$ let $\widehat{\alpha}_S^* = 1$ if $3 \in B$ and $\widehat{\alpha}_S^* = 0$ if $3 \in S$. First, observe that

$$\mathscr{U}_3(\gamma_S^\theta, \widehat{\alpha}_S^*) \leq \mathscr{U}_3(\gamma_S^\theta, \alpha) \text{ for all } \alpha \in [0,1] \text{ and all } S \subseteq N.$$

Note that in combination with (5.10) it holds for all $\omega \in \Omega$ that $\mathscr{F}_3^{\widehat{\alpha}^*} < \mathscr{F}_3^{\alpha^\omega}$ if there exists an $S \subseteq N$ for which $\mathscr{U}_3\left(\gamma_S^\theta, \widehat{\alpha}_S^*\right) < \mathscr{U}_3\left(\gamma_S^\theta, \alpha_S^\omega\right)$.

Now, we show that for all $\omega \in \Omega$ there exists an $S \subseteq N$ for which holds that $\mathscr{U}_3\left(\gamma_S^\theta, \widehat{\alpha}_S^*\right) < \mathscr{U}_3\left(\gamma_S^\theta, \alpha_S^\omega\right)$. Let $\omega \in \Omega$. We condition on $\omega(1)$.

Case 1. $\omega(1) = 5$.

For $S = \{3,5\}$ player 3 and player 5 are active sellers. Hence, $\alpha^{\omega}_{\{3,5\}} = 1$ while $\widehat{\alpha}^{*}_{\{3,5\}} = 0$ implying $\mathscr{U}_3\left(\gamma^{\theta}_{\{3,5\}}, \widehat{\alpha}^{*}_{\{3,5\}}\right) < \mathscr{U}_3\left(\gamma^{\theta}_{\{3,5\}}, \alpha^{\omega}_{\{3,5\}}\right)$.

Case 2. $\omega(1) = 4$.

Similar argument as in case 1 with $S = \{3,4\}$.

Case 3. $\omega(1) = 3$.

Same argument as case 1.

Case 4. $\omega(1) = 2$.

For $S = \{1,4,5\}$ player 2 and player 3 are active buyers. Hence, $\alpha^{\omega}_{\{1,4,5\}} = 0$ while $\widehat{\alpha}^{*}_{\{1,4,5\}} = 1$ implying that $\mathscr{U}_3\left(\gamma^{\theta}_{\{1,4,5\}}, \widehat{\alpha}^{*}_{\{1,4,5\}}\right) < \mathscr{U}_3\left(\gamma^{\theta}_{\{1,4,5\}}, \alpha^{\omega}_{\{1,4,5\}}\right)$.

Case 5. $\omega(1) = 1$.

Similar argument as in case 4 with $S = \{2,4,5\}$.

So, for all $\omega \in \Omega$ there exists an $S \subseteq N$ for which $\mathscr{U}_3\left(\gamma^{\theta}_{S}, \widehat{\alpha}^{*}_{S}\right) < \mathscr{U}_3\left(\gamma^{\theta}_{S}, \alpha^{\omega}_{S}\right)$ and thus $\mathscr{F}^{\widehat{\alpha}^*}_3(\theta) < \mathscr{F}^{\alpha^{\omega}}_3(\theta)$. Hence, $\left\{ \mathscr{F}^{\widehat{\alpha}} \mid \widehat{\alpha} \in [0,1]^{2^N} \right\} \not\subseteq$ *convexhull* $\left\{ \mathscr{F}^{\widehat{\alpha}^{\omega}} \mid \omega \in \Omega \right\}$. □

We conclude this section by showing that the average of the allocations resulting from all allocation rules with $\alpha \in \{0,1\}^{2^N}$ coincides with the average of the extreme BBHM allocations.

Theorem 5.8. *Let $\theta \in \Theta$ and $\widehat{\alpha}_S = \frac{1}{2}$ for all $S \subseteq N$. Then*

$$\mathscr{F}^{\widehat{\alpha}}(\theta) = \frac{1}{2^{|2^N|}} \sum_{\tau \in \{0,1\}^{2^N}} \mathscr{F}^{\tau}(\theta).$$

Proof : Consider set $\{0,1\}^{2^N}$ and let $S \subseteq N$. The number of $\tau \in \{0,1\}^{2^N}$ for which $\tau_S = 1$ coincides with the number of $\tau \in \{0,1\}^{2^N}$ for which $\tau_S = 0$. As the total number of unique $\tau \in \{0,1\}^{2^N}$ is given by $2^{|2^N|}$, we conclude that the total number of $\tau \in \{0,1\}^{2^N}$ for which $\tau_S = 1$ ($\tau_S = 0$) equals $2^{|2^N|}/2$. Now, observe that

$$\begin{aligned}
\mathscr{F}^{\widehat{\alpha}}(\theta) &= \sum_{S \subseteq N} p(\gamma^{\theta}_S) \cdot \mathscr{U}\left(\gamma^{\theta}_S, \frac{1}{2}\right) \\
&= \sum_{S \subseteq N} p(\gamma^{\theta}_S) \cdot \left(\frac{1}{2} \cdot \mathscr{U}(\gamma^{\theta}_S, 0) + \frac{1}{2} \cdot \mathscr{U}(\gamma^{\theta}_S, 1)\right) \\
&= \frac{1}{2^{|2^N|}} \sum_{S \subseteq N} p(\gamma^{\theta}_S) \cdot \left(\frac{2^{|2^N|}}{2} \cdot \mathscr{U}(\gamma^{\theta}_S, 0) + \frac{2^{|2^N|}}{2} \cdot \mathscr{U}(\gamma^{\theta}_S, 1)\right) \\
&= \frac{1}{2^{|2^N|}} \sum_{S \subseteq N} p(\gamma^{\theta}_S) \sum_{\tau \in \{0,1\}^{2^N}} \mathscr{U}(\gamma^{\theta}_S, \tau_S)
\end{aligned}$$

$$= \frac{1}{2^{|2^N|}} \sum_{\tau \in \{0,1\}^{2^N}} \sum_{S \subseteq N} p(\gamma_S^\theta) \cdot \mathscr{U}(\gamma_S^\theta, \tau_S)$$
$$= \frac{1}{2^{|2^N|}} \sum_{\tau \in \{0,1\}^{2^N}} \mathscr{F}^\tau(\theta).$$

The first and last equality hold by definition. The second equality holds as $\mathscr{U}(\gamma_S^\theta, \frac{1}{2}) = \frac{1}{2} \cdot \mathscr{U}(\gamma_S^\theta, 0) + \frac{1}{2} \cdot \mathscr{U}(\gamma_S^\theta, 1)$ for all $S \subseteq N$. The third and fifth equality hold by rewriting. The fourth equality is a result of the description given at the start of the proof. $\square$

5.5 Conclusions

In this chapter, we considered an environment in which several independent service providers with possibly different downtime costs could cooperate by pooling their (single) spare parts according to the one-by-one critical level policy. For the associated one-by-one pooling game, we presented necessary and sufficient conditions to ensure convexity. Thereafter, we showed that values of one-by-one pooling games could be recognized as convex combinations of values of Böhm-Bahwerk horse market games and, as a consequence, we could prove that one-by-one pooling games are (totally) balanced. In addition, we presented a class of allocation rules for which the allocations are core members. Finally, we showed that within this class of allocation rules, there exists a quite intuitive and simple allocation rule for which the related allocation coincides with the average of the extreme Böhm-Bahwerk horse market allocations. The main managerial insight of this chapter is that cooperation is beneficial in our spare parts setting and can be supported by an efficient and stable allocation.

The work in this chapter can be extended in several ways. For instance, one can focus on situations with (i) players having an arbitrary number of spare parts in stock (rather than one), (ii) arrival intensities that may differ per player, (iii) non-zero lateral transshipment costs or (iv) another (and maybe even smarter) pooling strategy. For all these extensions, it is unlikely that the proof technique of formulating a game as a convex combination of Böhm-Bahwerk horse market games can be used again. This proof technique heavily depends on the fact that the steady state probabilities of being in a certain state remains the same for the situation with and without collaboration. By including different spare parts levels, arrival rates, lateral transshipment costs or even another critical level policy, these steady state probabilities will not (necessarily) remain the same for the situation with and without collaboration and thus would not allow for a similar proof technique.

In chapter 6, we make a first attempt to the last extension. In particular, we consider a spare parts situation that has quite some similarities with the one in this chapter and assume that an optimal spare parts pooling strategy is used. Finally, we want to point out that chapter 5 reveals the concept of convex combinations of Böhm-Bahwerk horse market games, which to the best of our knowledge, has not received any attention in literature so far. Most of the results obtained in this chapter can be generalized to this class of (convex combinations of Böhm-Bahwerk horse market) games easily.

Chapter 6

Spare parts pooling under an optimal pooling policy

6.1 Introduction

In the previous chapter, we learned that smart spare parts pooling may pay off. First of all, it can decrease total cost significantly. Secondly, it can create incentives for all service providers to collaborate. In this chapter, which is based on Schlicher et al. [50], we go one step further. We investigate a situation in which several service providers can pool their spare parts according to an optimal pooling policy. In particular, we consider a situation that coincides with the one of chapter 5, except that service providers may now keep any number of spare parts in stock, demand rates may differ per service provider, and a joint supplier produces spare parts with a single server in which production rates are combined (instead of a joint supplier with multiple separate servers). It is well-known that the optimal policy for such a pooled situation is of the form of a critical level policy. So, there is some kind of a stratification that determines when players are allowed to make use of the pool. We refer to this as stratified pooling.

In line with previous chapters, we also formulate an associated cooperative game. For this game, which we call a stratified pooling game, we face the problem of a (possibly) changing (stratified pooling) strategy per coalition, which severely complicates the mathematical analysis. However, we are (still) able to show core non-emptiness of stratified pooling games. To this end, we use that the underlying resource pooling situation can be described by a Markov decision process and the optimal spare parts pooling strategy as a stationary decision rule in this Markov decision process. To the best of our knowledge, we are the first within the field of cooperative game theory who use this modelling technique to prove core non-emptiness. In addition, we show that when service providers would naively apply a full pooling strategy (rather than an optimal pooling strategy) core non-emptiness is not guaranteed any longer.

Clearly, stratified pooling games fit within the class of OR games. In particular, these games partially overlap with resource pooling games discussed in Anily and Haviv [2], Karsten et al. [24, 25], and chapter 5. For instance, Anily and Haviv [2] assume as well that service is provided on a single server only, and in Karsten et al. [24] and in chapter 5 it is also assumed that waiting is no option if demand occurs. However, these articles and chapter 5 also deviate from stratified pooling games in several ways. For instance, Karsten et al. [25] and Anily and Haviv [2] assume that waiting (in a queue) is allowed. Secondly, in Karsten et al. [24, 25] and in chapter 5 it is assumed that the supplier has ample capacity and so all outstanding replenishment orders (of spare parts), can be processed simultaneously, while in our situation spare parts orders are produced on a single server one-by-one. Finally, Karsten et al. [24, 25] and Anily and Haviv [2] assume that demand is accepted according to a full pooling strategy, in chapter 5, a one-by-one pooling strategy is applied, whereas we apply a stratified (and so, optimal) pooling strategy. We want to emphasize that this last modelling assumption is of particular interest. To the best of our knowledge, we are the first who investigate a situation in which optimal pooling is applied. In table 6.1 an overview of the relevant literature is represented, according to two modelling assumptions –the number of servers and whether waiting is allowed or not. We also denote the type of pooling strategy used.

Table 6.1: Classification on some cooperative resource pooling games.

		number of servers	
		Single	Multiple
Waiting	*Not allowed*	chapter 6 *(optimal pooling)*	Karsten et al. [24] *(full pooling)* chapter 5 *(one-by-one pooling)*
	Allowed	Anily and Haviv [2] *(full pooling)*	Karsten et al. [25] *(full pooling)*

The remainder of this chapter is organised as follows. In section 6.2, we introduce a spare parts situation and subsequently introduce the associated stratified pooling game. In section 6.3, we show that stratified pooling games have a non-empty core. In section 6.4, we describe our conclusions.

6.2 Model

In this section, we introduce spare parts situations and define the associated games, called stratified pooling games. Next, we discuss that an optimal spare parts pooling strategy can be described as a stationary decision rule in a Markov decision process.

6.2.1 Spare parts situations

As in chapter 5, we consider a service provider i who keeps spare parts in stock to prevent costly downtime of its technical systems, focus on one critical component that is subject to failures only, and assume that a failure of such component leads to a demand for a spare part immediately. However, this time, demand occurs according to a Poisson process with player specific stationary rate $\lambda_i \in \mathbb{R}_+$. Moreover, we assume that service provider i starts with $I_i \in \mathbb{N} \cup \{0\}$ spare part in stock. If a spare part is on hand when demand occurs, this demand is always satisfied and a replenishment order is placed immediately at a supplier that produces spare parts one-by-one. So, service provider i follows a basestock policy with base stock level I_i. The production times of an order are independent and identically distributed according to an exponential distribution with mean $\mu_i^{-1} \in \mathbb{R}_+$. If no spare part is available when demand occurs, an emergency delivery is initiated. The expected costs associated with the extra idleness of the technical system (until the spare part arrives), the emergency shipment, and so on, shortly called downtime costs, are $d_i \in \mathbb{R}_+$ for service provider i. We do not consider holding costs for ordered spare parts and inventory on-hand in this chapter. Such a cost term can be recognized as a fixed, additive term (see, e.g., chapter 5). Finally, we assume that the service provider is interested in its long-term average costs per time unit.

Now, consider an environment with several of these service providers. To analyse such a setting, we define a *spare parts (SP) situation* as a tuple (N, I, d, λ, μ) with $N \subseteq \mathbb{N}$ a finite set of service providers, $I = (I_i)_{i \in N}$ with $I_i \in \mathbb{N} \cup \{0\}$ the initial inventory on-hand of service provider $i \in N$, $d = (d_i)_{i \in N}$ with $d_i \in \mathbb{R}_+$ the downtime costs of service provider $i \in N$, $\lambda = (\lambda_i)_{i \in N}$ with $\lambda_i \in \mathbb{R}_+$ the demand rate of service provider $i \in N$, and $\mu = (\mu_i)_{i \in N}$ with $\mu_i \in \mathbb{R}_+$ the production rate of the supplier of service provider $i \in N$. Finally, we use Θ for the set of SP situations.

6.2.2 Stratified pooling games

Consider an SP situation $\theta \in \Theta$ and coalition $M \subseteq N$ with $M \neq \emptyset$. The service providers, from now on called players, in coalition M can collaborate by pooling their spare parts. In this pooled situation, the initial inventory on-hand of coalition M becomes $I_M = \sum_{i \in M} I_i$, transshipments (of spare parts) occur instantaneously at negligible costs, and the new (heterogeneous) demand rate of coalition M becomes $\lambda_M = \sum_{i \in M} \lambda_i$. We assume that the players in coalition M apply a policy for the acceptance (or rejection) of demand that minimizes the long-run average downtime costs per time unit. The acceptance of each demand leads to a replenishment order. This order is placed at a joint supplier that produces spare parts one-by-one. Similar to Anily and Haviv [2], the production times are exponentially distributed with mean $\mu_M^{-1} = (\sum_{i \in M} \mu_i)^{-1}$. If demand of player $i \in N$ is rejected, an emergency delivery is initiated and related downtime costs of d_i are incurred. Based on a classical result of Ha [22], the optimal policy of accepting (or rejecting) demand for this pooling situation can be described as a critical level policy. We refer to this as stratified pooling. We denote the minimal long-run average downtime costs per time unit for coalition M by $c^\theta(M)$ and we set $c^\theta(\emptyset) = 0$.

Now, we proceed with associating a stratified pooling game to any SP situation.

Definition 6.1. *For any SP situation $\theta \in \Theta$, the game (N, c^θ) with $c^\theta(M)$ representing the minimal long-run average downtime costs per time unit for all $M \subseteq N$ with $M \neq \emptyset$ and $c^\theta(\emptyset) = 0$ is called the associated stratified pooling (STP) game.*

Example 6.1. *Let $\theta \in \Theta$ be an SP situation with $N = \{1,2,3\}$, $I = (1,1,1)$, $d = (3,2,1)$, $\lambda = (1,1,1)$, and $\mu = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. For the grand coalition, the optimal policy turns out to be the one-by-one pooling strategy (as introduced in chapter 5). According to this policy, one can construct a corresponding Markov chain (see figure 6.1). In chapter 5, a similar Markov chain is obtained if production is exponentially distributed. Note, production rates would differ then.*

$$
\begin{tikzcd}
3 \arrow[r, bend left, "\lambda_{\{1,2,3\}}"] & 2 \arrow[l, bend left, "\mu_{\{1,2,3\}}"] \arrow[r, bend left, "\lambda_{\{1,2\}}"] & 1 \arrow[l, bend left, "\mu_{\{1,2,3\}}"] \arrow[r, bend left, "\lambda_{\{1\}}"] & 0 \arrow[l, bend left, "\mu_{\{1,2,3\}}"]
\end{tikzcd}
$$

Figure 6.1: Underlying Markov chain for grand coalition.

Based on this Markov chain it is easy to determine the steady state probabilities of state 0 $\left(\frac{16}{67}\right)$, state 1 $\left(\frac{24}{67}\right)$, state 2 $\left(\frac{18}{67}\right)$ and state 3 $\left(\frac{9}{67}\right)$. According to these steady state probabilities, one can determine the minimal long-run average costs per time unit as follows

$$c(N) = \frac{16}{67} \cdot (\lambda_1 \cdot d_1 + \lambda_2 \cdot d_2 + \lambda_3 \cdot d_3) + \frac{24}{67} \cdot (\lambda_2 \cdot d_2 + \lambda_3 \cdot d_3) + \frac{18}{67} \cdot (\lambda_3 \cdot d_3) = 2\frac{52}{67}.$$

Similarly, one can determine the costs of the other coalitions (see table 6.2 for these values). Note that $x = \left(1\frac{52}{67}, 1, 0\right) \in \mathscr{C}(N, c^\theta)$, i.e., the core of STP game (N, c^θ) is non-empty. ◇

Table 6.2: Corresponding costs per coalition.

M	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$c^\theta(M)$	0	2	$1\frac{1}{3}$	$\frac{2}{3}$	$2\frac{4}{5}$	2	$1\frac{3}{5}$	$2\frac{52}{67}$

Remark 6.1. *Core non-emptiness is not guaranteed for situations in which coalitions would naively apply a full pooling strategy. For instance, consider SP situation $\theta \in \Theta$ for which $N = \{1,2\}$, $I = (1,1)$, $d = (1,4)$, $\lambda = (5,1)$, and $\mu = (1,1)$. Under full pooling, we would have $c^{\theta}(\{1\}) + c^{\theta}(\{2\}) = 4\frac{1}{6} + 2 < 6\frac{3}{13} = c^{\theta}(\{1,2\})$, i.e., the core is empty. However, if stratified pooling is applied, $c^{\theta}(\{1,2\}) = 5\frac{2}{11}$ and so the core is non-empty again.*

6.2.3 Markov decision process formulation

In this section, we formulate the underlying spare parts (pooling) situation as a Markov decision process (MDP) and the optimal spare parts pooling strategy as a stationary decision rule in this MDP. Let $\theta \in \Theta$ be an SP situation and $M \subseteq N$ with $M \neq \emptyset$. Now, we present the corresponding MDP.

State and Action Spaces

We define the state space to be $\mathscr{Y}^M = \{0, 1, \ldots, I_M\}$ and the action space to be $\mathscr{A}^M = \prod_{y \in \mathscr{Y}^M, i \in M} \mathscr{A}_i^M(y)$, with $\mathscr{A}_i^M(y) = \{0,1\}$ for all $i \in M$ and all $y > 0$ and $\mathscr{A}_i^M(y) = \{0\}$ otherwise. In state $y \in \mathscr{Y}^M$, action 1 corresponds with the acceptance of a demand at a player, while action 0 corresponds with the rejection of such a demand.

Costs and transition probabilities

Let $\gamma = \sum_{i \in N} [\lambda_i + \mu_i]$. We use γ as a so-called uniformization rate, which is independent of M. In addition, let $\lambda_i^* = \lambda_i / \gamma$ and $\mu_i^* = \mu_i / \gamma$ for all $i \in N$. Now, $C^M(y, a)$ denotes the expected costs collected over a single uniformized time period, given that the system begins the period in state $y \in \mathscr{Y}^M$ and action $a = (a_i)_{i \in M} \in \mathscr{A}^M(y)$ is taken. For our situation, we have

$$C^M(y,a) = \sum_{i \in M} \lambda_i^* \cdot (1 - a_i) \cdot d_i \qquad \text{for all } y \in \mathscr{Y}^M \text{ and all } a \in \mathscr{A}^M(y).$$

Now, let $p^M(y'|y,a)$ denote the one-stage transition probability from state $y \in \mathscr{Y}^M$ to $y' \in \mathscr{Y}^M$ under action $a = (a_i)_{i \in M} \in \mathscr{A}^M(y)$. So, for all $y \in \mathscr{Y}^M$ and all $a \in \mathscr{A}^M(y)$

$$p^M(y'|y,a) = \begin{cases} \sum_{i \in M} \lambda_i^* \cdot a_i & \text{if} \quad y' = y - 1, y > 0 \\ \sum_{i \in M} \mu_i^* & \text{if} \quad y' = y + 1, y < I_M \\ 1 - \sum_{i \in M} \left[\lambda_i^* \cdot a_i + \mu_i^* \right] & \text{if} \quad y' = y < I_M \\ 1 - \sum_{i \in M} \left[\lambda_i^* \cdot a_i \right] & \text{if} \quad y' = y = I_M \\ 0 & \text{otherwise.} \end{cases}$$

Value function and equivalence

Now, we present the value function (as defined in (2.3)) in a form suitable for this chapter.

Lemma 6.1. *Let $\theta \in \Theta$ and $M \subseteq N$. Then, for all $y \in \mathscr{Y}^M$ and all $t \in \mathbb{N} \cup \{0\}$ it holds that*

$$V_{t+1}^M(y) = \sum_{i \in M} \left[\lambda_i^* \min_{l \in \{0, \min\{y,1\}\}} \left\{ V_t^M(y-l) + (1-l)d_i \right\} + \mu_i^* V_t^M(\min\{y+1, I_M\}) \right] + \left(1 - \sum_{i \in M} [\lambda_i^* + \mu_i^*] \right) \cdot V_t^M(y)$$

with $V_0^M(y) = 0$ for all $y \in \mathscr{Y}^M$.

Proof **:** We distinguish between the case $I_M = 0$ and $I_M > 0$.

Case 1. $I_M = 0$.

Let $j = 0 \in \mathscr{Y}^M$ and $t \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned}
V_{t+1}^M(j) &= \min_{a \in \mathscr{A}^M(j)} \left\{ C^M(j,a) + \sum_{y' \in \mathscr{Y}^M} p(y'|j,a) \cdot V_t^M(y') \right\} \\
&= C^M(j, 0^N) + \sum_{y' \in \mathscr{Y}^M} p(y'|j, 0^N) \cdot V_t^M(y') \\
&= \sum_{i \in M} \lambda_i^* \cdot d_i + 1 \cdot V_t^M(j) \\
&= \sum_{i \in M} \lambda_i^* \cdot d_i + \left(\sum_{i \in M} [\lambda_i^* + \mu_i^*] + 1 - \sum_{i \in M} [\lambda_i^* + \mu_i^*] \right) \cdot V_t^M(j) \\
&= \sum_{i \in M} \left[\lambda_i^* \cdot (d_i + V_t^M(j)) + \mu_i^* V_t^M(j) \right] + \left(1 - \sum_{i \in M} [\lambda_i^* + \mu_i^*] \right) \cdot V_t^M(j) \\
&= \sum_{i \in M} \left[\lambda_i^* \min_{l \in \{0, \min\{j,1\}\}} \left\{ V_t^M(j-l) + (1-l)d_i \right\} + \mu_i^* V_t^M(\min\{j+1, I_M\}) \right] \\
&\qquad + \left(1 - \sum_{i \in M} [\lambda_i^* + \mu_i^*] \right) \cdot V_t^M(j).
\end{aligned}$$

The first equality holds by definition. The second equality holds as action $a = 0^N$ is the only allowed action. The third equality holds by the definition of $C^M(j,a)$ and the fact that one can transit to the current state ($j = 0$) only. In the fourth equality, we add zero. The fifth equality holds by some rewriting. The last equality holds as $\min\{j,1\} = j = 0$ and $\min\{j+1, I_M\} = I_M = j = 0$ as well.

Case 2. $I_M > 0$.

We distinguish between the subcases $j = 0$, $0 < j < I_M$, and $j = I_M$.

Case 2.a $\quad j = 0$.

Let $t \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned}
V_{t+1}^M(j) &= \min_{a \in \mathscr{A}^M(j)} \left\{ C^M(j,a) + \sum_{y' \in \mathscr{Y}^M} p(y'|j,a) \cdot V_t^M(y') \right\} \\
&= C^M(j, 0^N) + \sum_{y' \in \mathscr{Y}^M} p(y'|j, 0^N) \cdot V_t^M(y') \\
&= \sum_{i \in M} \lambda_i^* \cdot d_i + \left(1 - \sum_{i \in M} \mu_i^*\right) \cdot V_t^M(j) + \sum_{i \in M} \mu_i^* \cdot V_t^M(j+1) \\
&= \sum_{i \in M} \lambda_i^* \cdot d_i + \left(\sum_{i \in M} \lambda_i^* + \sum_{i \in N \setminus M} \lambda_i^* + \sum_{i \in N \setminus M} \mu_i^* \right) \cdot V_t^M(j) + \sum_{i \in M} \mu_i^* \cdot V_t^M(j+1) \\
&= \sum_{i \in M} \lambda_i^* \cdot (d_i + V_t^M(j)) + \sum_{i \in M} \mu_i^* \cdot V_t^M(j+1) + \sum_{i \in N \setminus M} [\lambda_i^* + \mu_i^*] \cdot V_t^M(j) \\
&= \sum_{i \in M} \left[\lambda_i^* \min_{l \in \{0, \min\{j,1\}\}} \left\{ V_t^M(j-l) + (1-l) d_i \right\} + \mu_i^* V_t^M(\min\{j+1, I_M\}) \right] \\
&\qquad + \left(1 - \sum_{i \in M} [\lambda_i^* + \mu_i^*] \right) \cdot V_t^M(j).
\end{aligned}$$

The first equality holds by definition. The second equality holds as action $a = 0^N$ is the only allowed action. In the third equality, we use the definition of $C^M(j,a)$ and the fact that with probability $1 - \sum_{i \in M} \mu_i^*$ we remain in the same state and with probability $\sum_{i \in M} \mu_i^*$ we transit to state $j+1$. In the fourth equality, we use that $1 = \sum_{i \in N} \left[\lambda_i^* + \mu_i^*\right] = \sum_{i \in M} \lambda_i^* + \sum_{i \in N \setminus M} \lambda_i^* + \sum_{i \in N} \mu_i^*$. The fifth equality holds by some rewriting. The last equality holds as $\min\{j, 1\} = j = 0$ and $\min\{j+1, I_M\} = j+1 = 1$.

Case 2.b $\quad 0 < j < I_M$.

Let $j \in \mathscr{Y}^M \setminus \{0, I_M\}$ and $t \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned}
V_{t+1}^M(j) &= \min_{a \in \mathscr{A}^M(j)} \left\{ C^M(j,a) + \sum_{y' \in \mathscr{Y}^M} p(y'|j,a) \cdot V_t^M(y') \right\} \\
&= \min_{a \in \mathscr{A}^M(j)} \left\{ \sum_{i \in M} \lambda_i^* (1-a_i) d_i + \sum_{i \in M} \lambda_i^* a_i V_t^M(j-1) + \left(1 - \sum_{i \in M} [\lambda_i^* a_i + \mu_i^*]\right) V_t^M(j) \right. \\
&\qquad \left. + \sum_{i \in M} \mu_i^* V_t^M(j+1) \right\} \\
&= \min_{a \in \mathscr{A}^M(j)} \left\{ \sum_{i \in M} \left[\lambda_i^* \left((1-a_i)(d_i + V_t^M(j)) + a_i V_t^M(j-1) \right) + \mu_i^* V_t^M(j+1) \right] \right. \\
&\qquad \left. + \left(1 - \sum_{i \in M} [\lambda_i^* + \mu_i^*] \right) V_t^M(j) \right\}
\end{aligned}$$

$$= \sum_{i \in M} \left[\lambda_i^* \min_{l \in \{0, \min\{j,1\}\}} \left\{ V_t^M(j-l) + (1-l)d_i \right\} + \mu_i^* V_t^M(\min\{j+1, I_M\}) \right] + \left(1 - \sum_{i \in M} [\lambda_i^* + \mu_i^*]\right) \cdot V_t^M(j).$$

The first equality holds by definition. In the second equality, we use the definition of $C^M(j,a)$, the fact that with probability $\sum_{i \in M} \lambda_i^* a_i$ we transit to state $j-1$, with probability $1 - \sum_{i \in M} \left[\lambda_i^* a_i + \mu_i^*\right]$ we remain in the same state, and with probability $\sum_{i \in M} \mu_i^*$ we transit to state $j+1$. In the third equality, we did some rewriting and used $a_i = a_i - 1 + 1$. The last equality holds as $\min\{j,1\} = 1$, $\min\{j+1, I_M\} = j+1$, and the fact that a minimum of a sum of independent terms equals the sum of all these individual terms evaluated at their minimum.

Case 2.c $\quad j = I_M$.

Let $j = I_M \in \mathscr{Y}^M$ and $t \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned}
V_{t+1}^M(j) &= \min_{a \in \mathscr{A}^M(j)} \left\{ C^M(j,a) + \sum_{y' \in \mathscr{Y}^M} p(y'|j,a) \cdot V_t^M(y') \right\} \\
&= \min_{a \in \mathscr{A}^M(j)} \left\{ \sum_{i \in M} \lambda_i^*(1-a_i)d_i + \sum_{i \in M} \lambda_i^* a_i V_t^M(j-1) + \left(1 - \sum_{i \in M} \lambda_i^* a_i\right) V_t^M(j) \right\} \\
&= \min_{a \in \mathscr{A}^M(j)} \left\{ \sum_{i \in M} \lambda_i^*(1-a_i)d_i + \sum_{i \in M} \lambda_i^* a_i V_t^M(j-1) + \left(\sum_{i \in M} [\lambda_i^*(1-a_i) + \mu_i^*] \right.\right. \\
&\qquad \left.\left. + \sum_{i \in N \setminus M} [\lambda_i^* + \mu_i^*] \right) V_t^M(j) \right\} \\
&= \min_{a \in \mathscr{A}^M(j)} \left\{ \sum_{i \in M} \lambda_i^* \left[(1-a_i)(d_i + V_t^M(j)) + a_i V_t^M(j-1) + \mu_i^* V_t^M(j) \right] + \right. \\
&\qquad \left. \sum_{i \in N \setminus M} [\lambda_i^* + \mu_i^*] \, V_t^M(j) \right\} \\
&= \sum_{i \in M} \left[\lambda_i^* \min_{l \in \{0, \min\{j,1\}\}} \left\{ V_t^M(j-l) + (1-l)d_i \right\} + \mu_i^* V_t^M(\min\{j+1, I_M\}) \right] \\
&\qquad + \left(1 - \sum_{i \in M} [\lambda_i^* + \mu_i^*]\right) \cdot V_t^M(j).
\end{aligned}$$

The first equality holds by definition. In the second equality, we use the definition of $C^M(j,a)$, the fact that with probability $\sum_{i \in M} \lambda_i^* a_i$ we transit to state $j-1$, and with probability $1 - \sum_{i \in M} \lambda_i^* a_i$ we remain in the same state. In the third equality, we use that

$1 = \sum_{i \in N} \left[\lambda_i^* + \mu_i^*\right]$. The fourth equality holds by some rewriting. The last equality holds as $\min\{j, 1\} = 1$, $\min\{j+1, I_M\} = I_M$ and the fact that a minimum of a sum of independent terms equals the sum of all these individual terms evaluated at their minimum. This concludes the proof. $\square$

The formulation of the value function in lemma 6.1 can be interpreted in the following way. Let $M \subseteq N$, and $i \in M$. Then, with probability λ_i^* there is a demand arrival and, except for the state with no spare parts in stock, there is a possibility to (i) accept demand $(l = 1)$ or (ii) reject demand $(l = 0)$ and incur related costs d_i. Based on the the first decision $(l = 1)$ there is a transition to the state with one spare part less in stock and based on the second decision $(l = 0)$ there is a transition back to the same state. With probability μ_i^* there is, except for the state with no outstanding replenishment orders, a spare part replenishment, which leads to a transition to the state with one spare part more in stock. Finally, with probability $1 - \sum_{i \in M}(\lambda_i^* + \mu_i^*)(\geq 0)$ there is a dummy transition back to the same state, ensuring that the probabilities sum to one.

Example 6.2. *Consider the situation of example 6.1. Observe that $\gamma = 4\frac{1}{2}$, $\lambda_i^* = \frac{2}{9}$ for all $i \in N$, and $\mu_i^* = \frac{1}{9}$ for all $i \in N$. For instance, for coalition $M = \{1,2\}$ with $I_{\{1,2\}} = 2$, the value function for all $t \in \mathbb{N} \cup \{0\}$ is given by*

$$\begin{aligned}
V_{t+1}^{\{1,2\}}(0) &= \sum_{i=1}^{2} \left(\frac{2}{9} \left(V_t^{\{1,2\}}(0) + d_i \right) + \frac{1}{9} V_t^{\{1,2\}}(1) \right) + \frac{1}{3} V_t^{\{1,2\}}(0) \\
V_{t+1}^{\{1,2\}}(1) &= \sum_{i=1}^{2} \left(\frac{2}{9} \min_{l \in \{0,1\}} \left\{ V_t^{\{1,2\}}(1-l) + (1-l)d_i \right\} + \frac{1}{9} V_t^{\{1,2\}}(2) \right) + \frac{1}{3} V_t^{\{1,2\}}(1) \\
V_{t+1}^{\{1,2\}}(2) &= \sum_{i=1}^{2} \left(\frac{2}{9} \min_{l \in \{0,1\}} \left\{ V_t^{\{1,2\}}(2-l) + (1-l)d_i \right\} + \frac{1}{9} V_t^{\{1,2\}}(2) \right) + \frac{1}{3} V_t^{\{1,2\}}(2).
\end{aligned}$$

Note that for zero spare parts in stock, it is not possible to accept demand and for two spare parts in stock, a possible replenishment has no effect on the inventory on-hand. $\diamond$

Finally, we define g^M as the minimal long-run average costs per time epoch of the MDP. Now, we show that there is a direct relationship between g^M and the original minimal long-run average costs per time unit of coalition M.

Lemma 6.2. *Let $\theta \in \Theta$ and $M \subseteq N$ with $M \neq \emptyset$. Then*

$$c^\theta(M) = \gamma \cdot g^M = \gamma \cdot \lim_{t \to \infty} \frac{V_t^M(y)}{t} \quad \text{for all } y \in \mathscr{Y}^M.$$

Proof **:** The first equality holds by uniformization, which is allowed if transition rates are bounded and the MDP is unichain (Puterman [45, p.568]). We recall that interarrival times of demands as well as production times (of spare parts) are exponentially distributed with rates that are bounded above. In addition, for every stationary policy, the state without any outstanding replenishment orders is accessible from every state and so, the transition probability matrix consists of a single recurrent class. Hence, the MDP is unichain. With respect to the second equality, observe that state space $\mathscr{Y}^M$ and action space $\mathscr{A}^M$ of the MDP are finite. Under stationary policy $f = (f_i(y))_{y\in\mathscr{Y}^M, i\in M}$ with $f_i(y) = 1$ for all $i \in M$ and all $y > 0$ and $f_i(0) = 0$ for all $i \in M$, every state $y \in \mathscr{Y}^M$ is accessible from any state $y' \in \mathscr{Y}^M$ after (possibly) some arrivals and some (one-by-one) spare part replenishments. Hence, the related Markov chain is irreducible. An irreducible Markov chain with finite state space is positive recurrent (Modica and Poggiolini [34, theorem 5.71 (ii)]). Finally, observe that the long-run average costs per time epoch under policy f are bounded (naturally) by $\sum_{i\in M} \lambda_i^* \cdot d_i$ and as a result of theorem 2.4, the second equality follows. $\square$

Example 6.3. *Consider the situation of example 6.2. For coalition $M = \{1,2\}$ it holds that $\lim_{t\to\infty} V_t^{\{1,2\}}(0)/t = \frac{28}{45}$. This value can be interpreted as the minimal long-run average costs per time epoch of the corresponding MDP. Multiplying this value with the uniformization rate $\gamma(= 4\frac{1}{2})$ yields $\frac{126}{45}(= 2\frac{4}{5} = c^\theta(\{1,2\}))$, which can be recognised as the minimal long-run average costs per time unit of coalition M.* $\diamond$

6.3 The core

In this section, we focus on core non-emptiness of STP games in general. In section 2.2.2 (and section 2.2.5), we formulated sufficient and necessary condition for a (cost) game to have a non-empty core in terms of minimal balanced collections.

Let $N \subseteq \mathbb{N}$ be a finite player set. For every minimal balanced collection $\mathscr{B} \subseteq 2^N\backslash\{\emptyset\}$ there exists exactly one associated balanced map κ. For this balanced map it holds that $\kappa(M) \in \mathbb{Q}$ for all $M \in \mathscr{B}$ (Norde and Reijnierse [39, p. 325]). Let $\mathscr{B} \subseteq 2^N\backslash\{\emptyset\}$ be a minimal balanced collection. We define $\alpha \in \mathbb{N}$ as the smallest integer for which $\kappa(M) \cdot \alpha \in \mathbb{N}$ for all $M \in \mathscr{B}$ and use $b_M = \kappa(M) \cdot \alpha$ for all $M \in \mathscr{B}$ as a shorthand notation. In order to show balancedness of a game (N, c), it suffices to check if for each minimal balanced collection $\mathscr{B} \subseteq 2^N\backslash\{\emptyset\}$ it holds (see, e.g., section 2.2.5) that

$$\sum_{M\in\mathscr{B}} b_M \cdot c(M) \geq \alpha \cdot c(N). \tag{6.1}$$

We guide the reader through the steps of proving balancedness, i.e., showing that inequality (6.1) holds for any minimal balanced collection. The idea of this proof is threefold. First, we construct for each minimal balanced collection a number of copies for all its coalitions. Secondly, we set up an extensive MDP that can handle with the policies of all copied coalitions simultaneously. Thirdly, we show that in this extensive MDP, the optimal policy is a multiple of the optimal policy of the grand coalition. In the last two steps, the use of value functions turns out to be powerful.

Now, we introduce all remaining ingredients needed to prove balancedness. Let us introduce for each minimal balanced collection $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ another set $\mathscr{L}$ that contains for each $M \in \mathscr{B}$ exactly b_s labeled copies of coalition M.

Definition 6.2. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. Then, we define*

$$\mathscr{L} = \Big\{ \ (M,k) \ \mid \ M \in \mathscr{B}, k \in \{1,2,\dots,b_M\} \Big\}.$$

Example 6.4. *Let $\theta \in \Theta$ with $N = \{1,2,3,4\}$ and $\mathscr{B} = \{\{1\},\{2,3\},\{2,4\},\{3,4\}\}$ be a minimal balanced collection with $\kappa(\{1\}) = 1, \kappa(\{2,3\}) = \kappa(\{2,4\}) = \kappa(\{3,4\}) = \frac{1}{2}$. Hence, $\alpha = 2$, and so $\mathscr{L} = \{(\{1\},1),(\{1\},2),(\{2,3\},1),(\{2,4\},1),(\{3,4\},1)\}$.* $\diamond$

The labeled copies are called *labeled coalitions*. For each labeled coalition $(M,k) \in \mathscr{L}$ we denote the value function by $V^{M,k}(\cdot)$ and the inventory on-hand by $I_{M,k}$. In addition, we rewrite the labeled coalitional values (corresponding to a minimal balanced collection) as stated in lemma 6.2, i.e., as limits of value functions.

Lemma 6.3. *For every $\theta \in \Theta$ it holds for any minimal balanced collection $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ that*

$$\sum_{M \in \mathscr{B}} b_M \cdot c^\theta(M) = \gamma \cdot \lim_{t \to \infty} \frac{1}{t} \cdot \sum_{M \in \mathscr{B}} \sum_{k=1}^{b_M} V_t^{M,k}(I_{M,k}).$$

Proof : Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. It holds that

$$\sum_{M \in \mathscr{B}} b_M \cdot c^\theta(M) = \gamma \cdot \sum_{M \in \mathscr{B}} \sum_{k=1}^{b_M} \lim_{t \to \infty} \frac{V_t^{M,k}(I_{M,k})}{t} = \gamma \cdot \lim_{t \to \infty} \frac{1}{t} \cdot \sum_{M \in \mathscr{B}} \sum_{k=1}^{b_M} V_t^{M,k}(I_{M,k}).$$

The first equality holds by exploiting all labeled coalitions, lemma 6.2, and the fact that $I_{M,k} \in \mathscr{Y}^{M,k}$ for all $(M,k) \in \mathscr{L}$. The last equality holds as all limits are well-defined and all sums are finite. This concludes the proof. $\square$

As a next step, we show that for any minimal balanced collection $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ we can construct a value function (of some unspecified MDP) with a state space that keeps track of the inventory on-hand of every labeled coalition $(M,k) \in \mathscr{L}$, an action space that consists of all possible actions per labeled coalition $(M,k) \in \mathscr{L}$ given its inventory on-hand, and moreover, for which the related costs coincide with $\sum_{M \in \mathscr{B}} \sum_{k=1}^{b_M} V_t^{M,k}(I_{M,k})$ for all $t \in \mathbb{N} \cup \{0\}$. In order to do so, we first introduce a new state space.

Definition 6.3. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. Then, we define*

$$\mathscr{Y}^{\mathscr{B}} = \left\{ \ (r_z)_{z \in \mathscr{L}} \ \middle| \ r_z \in \{0, 1, \ldots, I_z\} \ \ \forall z (= (M,k)) \in \mathscr{L} \right\}.$$

Secondly, we introduce a new action space.

Definition 6.4. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. Then, for all $r \in \mathscr{Y}^{\mathscr{B}}$ and all $i \in N$ we define*

$$\mathscr{A}_i^{\mathscr{B}-}(r) = \left\{ \ (l_z)_{z \in \mathscr{L}} \ \middle| \ \begin{array}{cc} l_z \in \{0, \min\{1, r_z\}\} & \forall z \in \mathscr{L} : i \in M \\ l_z = 0 & \forall z \in \mathscr{L} : i \notin M \end{array} \right\}$$
$$\mathscr{A}_i^{\mathscr{B}+}(r) = \left\{ \ (l_z)_{z \in \mathscr{L}} \ \middle| \ \begin{array}{cc} l_z = \min\{1, I_z - r_z\} & \forall z \in \mathscr{L} : i \in M \\ l_z = 0 & \forall z \in \mathscr{L} : i \notin M \end{array} \right\}.$$

Subsequently, we introduce the new value function.

Definition 6.5. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. Then, for all $r \in \mathscr{Y}^{\mathscr{B}}$ and all $t \in \mathbb{N} \cup \{0\}$, we define the value function as*

$$V_{t+1}^{\mathscr{B}}(r) = \sum_{i \in N} \left[\lambda_i^* \min_{l \in \mathscr{A}_i^{\mathscr{B}-}(r)} \left\{ (\alpha - ||l||_1) d_i + V_t^{\mathscr{B}}(r - l) \right\} + \mu_i^* \min_{l \in \mathscr{A}_i^{\mathscr{B}+}(r)} \left\{ V_t^{\mathscr{B}}(r + l) \right\} \right]$$

with $V_0^{\mathscr{B}}(r) = 0$ for all $r \in \mathscr{Y}^{\mathscr{B}}$ and $|| \cdot ||_1$ the L^1 norm.

The new value function can be interpreted in the following way. Let $M \subseteq N$ and $i \in M$. With probability λ_i^* there is a demand for all labeled coalitions $(M,k) \in \mathscr{L}$ for which $i \in M$. Each such labeled coalition (M,k) has, except for the case with $r_{M,k} = 0$, the possibility to accept demand ($l_{M,k} = 1$), and always the possibility to reject demand ($l_{M,k} = 0$). For all (other) labeled coalitions $(M,k) \in \mathscr{L}$ for which $i \notin M$ it holds that there is no demand arrival and so $l_{M,k} = 0$. Based on these decisions, total costs equal $(\alpha - ||l||_1)d_i$ and one transits to $r - l$. With probability μ_i^* there is a spare part replenishment for each labeled coalition $(M,k) \in \mathscr{L}$ for which $i \in M$. For each labeled coalition $(M,k) \in \mathscr{L}$ with $i \in M$ and $r_{M,k} < I_{M,k}$ one accepts the spare part ($l_{M,k} = 1$). However, for each labeled coalition $(M,k) \in \mathscr{L}$ with $i \in M$ and $r_{M,k} = I_{M,k}$ the spare part is rejected ($l_{M,k} = 0$) as there are no outstanding replenishment orders[6]. For all (other) labeled coalitions $(M,k) \in \mathscr{L}$ for which $i \notin M$ it holds that there is no spare part replenishment and so $l_{M,k} = 0$. Based on the decisions made, one transits to state $r + l$.

[6]A possible interpretation could be to see this spare part replenishment in fact as a possible spare part replenishment where it is a spare part replenishment if $r_{M,k} < I_{M,k}$ and not if $r_{M,k} = I_{M,k}$.

Example 6.5. *Consider the situation of example 6.3 and $\mathscr{B} = \{\{1,2\},\{1,3\},\{2,3\}\}$. Observe that $\mathscr{B}$ is a minimal balanced collection with $\kappa(\{1,2\}) = \kappa(\{1,3\}) = \kappa(\{2,3\}) = \frac{1}{2}$. So, $\alpha = 2$ and $\mathscr{L} = \{(\{1,2\},1),(\{1,3\},1),(\{2,3\},1)\}$. Then, we have a state space*

$$\mathscr{Y}^{\mathscr{B}} = \{0,1,2\}^{\mathscr{L}}.$$

Moreover, for $r = (r_{(\{1,2\},1)}, r_{(\{1,3\},1)}, r_{(\{2,3\},1)}) = (1,2,0) \in \mathscr{Y}^{\mathscr{B}}$, we have an action space

$$\mathscr{A}_1^{\mathscr{B}-}((1,2,0)) = \{(0,0,0),(1,0,0),(0,1,0),(1,1,0)\}.$$

Note that the elements in this action space represent the possible actions that can be taken by all possible labeled coalitions $(\{1,2\},1)$, $(\{1,3\},1)$, and $(\{2,3\},1)$ whenever there is a demand for labeled coalition $(\{1,2\},1)$ and $(\{1,3\},1)$. As both labeled coalitions have at least one spare part in stock, they can both accept this demand, one of them can except, or both can reject. Similarly, for action space $(0,1,1) \in \mathscr{Y}^{\mathscr{B}}$, we have

$$\mathscr{A}_1^{\mathscr{B}-}((0,1,1)) = \{(0,0,0),(0,1,0)\}.$$

The action space illustrates that the spare part of labeled coalition $(\{2,3\},1)$ cannot be used by the labeled coalitions $(\{1,2\},1)$ and $(\{1,3\},1)$. $\diamond$

Now, we are able to show for all time moments the equivalence between the costs of the new value function and $\sum_{M\in\mathscr{B}}\sum_{k=1}^{b_M} V_t^{M,k}(r_{M,k})$.

Lemma 6.4. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N\backslash\{\emptyset\}$ be a minimal balanced collection. Then, for all $r \in \mathscr{Y}^{\mathscr{B}}$ and all $t \in \mathbb{N}\cup\{0\}$ it holds that*

$$\sum_{M\in\mathscr{B}}\sum_{k=1}^{b_M} V_t^{M,k}(r_{M,k}) = V_t^{\mathscr{B}}(r).$$

***Proof* :** This proof is by induction. By definition of the value functions $V_0^{M,k}(y) = 0$ for all $y \in \mathscr{Y}^{M,k}$, and all $M \in \mathscr{B}$ and all $k \in \{1,2,\ldots,b_M\}$. Similarly, $V_0^{\mathscr{B}}(r) = 0$ for all $r \in \mathscr{R}$ as well. Hence, $\sum_{M\in\mathscr{B}}\sum_{k=1}^{b_M} V_0^{M,k}(r_{M,k}) = V_0^{\mathscr{B}}(r)$ for all $r \in \mathscr{R}$. Let $t \in \mathbb{N}\cup\{0\}$ and assume that $\sum_{M\in\mathscr{B}}\sum_{k=1}^{b_M} V_t^{M,k}(r_{M,k}) = V_t^{\mathscr{B}}(r)$ for all $r \in \mathscr{R}$. Let $r \in \mathscr{R}$.

Now, observe that

$$\begin{aligned}
&\sum_{M\in\mathscr{B}}\sum_{k=1}^{b_M} V_{t+1}^{M,k}(r_{M,k}) \\
&= \sum_{M\in\mathscr{B}}\sum_{k=1}^{b_M}\left(\sum_{i\in M}\left[\lambda_i^* \min_{l\in\{0,\min\{1,r_{M,k}\}\}}\left\{V_t^{M,k}(r_{M,k}-l)+(1-l)d_i\right\}\right.\right. \\
&\qquad\left.\left. +\mu_i^* V_t^{M,k}(\min\{r_{M,k}+1, I_{M,k}\})\right] + \left(1-\sum_{i\in M}[\lambda_i^*+\mu_i^*]\right)V_t^{M,k}(y)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{M\in\mathscr{B}}\sum_{k=1}^{b_M}\Bigg(\sum_{i\in M}\Bigg[\lambda_i^* \min_{l\in\{0,\min\{1,r_{M,k}\}\}}\Big\{V_t^{M,k}(r_{M,k}-l)+(1-l)d_i\Big\} \\
&\qquad +\mu_i^* V_t^{M,k}(\min\{r_{M,k}+1,I_{M,k}\})\Bigg] + \sum_{i\in N\setminus M}\Big[\lambda_i^* V_t^{M,k}(r_{M,k})+\mu_i^* V_t^{M,k}(r_{M,k})\Big]\Bigg) \\
&= \sum_{i\in N}\Bigg[\lambda_i^*\Bigg(\sum_{\substack{M\in\mathscr{B}:\\ i\in M}}\sum_{k=1}^{b_M}\min_{l\in\{0,\min\{r_{M,k},1\}\}}\Big\{V_t^{M,k}(r_{M,k}-l)+(1-l)d_i\Big\} + \sum_{\substack{M\in\mathscr{B}:\\ i\notin M}}\sum_{k=1}^{b_M}V_t^{M,k}(r_{M,k})\Bigg) \\
&\qquad +\mu_i^*\Bigg(\sum_{\substack{M\in\mathscr{B}:\\ i\in M}}\sum_{k=1}^{b_M}V_t^{M,k}(\min\{r_{M,k}+1,I_{M,k}\}) + \sum_{\substack{M\in\mathscr{B}:\\ i\notin M}}\sum_{k=1}^{b_M}V_t^{M,k}(r_{M,k})\Bigg)\Bigg] \\
&= \sum_{i\in N}\Bigg[\lambda_i^*\Bigg(\min_{l\in\mathscr{A}_i^{\mathscr{B}-}(r)}\Bigg\{\sum_{M\in\mathscr{B}}\sum_{k=1}^{b_M}V_t^{M,k}(r_{M,k}-l_{M,k})+(\alpha-||l||_1)d_i\Bigg\}\Bigg)\Bigg] \\
&\qquad + \mu_i^*\Bigg(\min_{l\in\mathscr{A}_i^{\mathscr{B}+}(r)}\Big\{V_t^{\mathscr{B}}(r+l)\Big\}\Bigg)\Bigg] \\
&= \sum_{i\in N}\Bigg[\lambda_i^*\Bigg(\min_{l\in\mathscr{A}_i^{\mathscr{B}-}(r)}\Big\{V_t^{\mathscr{B}}(r-l)+(\alpha-||l||_1)d_i\Big\}\Bigg)+\mu_i^*\cdot\Bigg(\min_{l\in\mathscr{A}_i^{\mathscr{B}+}(r)}\Big\{V_t^{\mathscr{B}}(r+l)\Big\}\Bigg)\Bigg] \\
&= V_{t+1}^{\mathscr{B}}(r).
\end{aligned}
$$

The first equality holds by lemma 6.1. The second equality holds as $1-\sum_{i\in M}[\lambda_i^*+\mu_i^*]=\sum_{i\in N}[\lambda_i^*+\mu_i^*]-\sum_{i\in M}[\lambda_i^*+\mu_i^*]=\sum_{i\in N\setminus M}[\lambda_i^*+\mu_i^*]$. The third equality holds by conditioning on λ_i^* and μ_i^* for all $i\in N$. The fourth equality holds as the sum of minima can be rewritten as one minimum and $\mathscr{A}_i^{\mathscr{B}+}$ and $\mathscr{A}_i^{\mathscr{B}-}$ are defined such that the decisions made for all minima fit. Note that $l_{M,k}=0$ if $i\notin M$. The fifth equality holds by the induction hypothesis. The last equality holds by definition 6.5. $\square$

Observe that the action space in the new value function is quite restricted. For instance, upon a demand arrival of player $i\in M$, it is not possible to accept a demand for labeled coalition $(M,k)\in\mathscr{L}$ for which $i\in M$ and $r_{M,k}=0$, while other labeled coalitions $(M,k)\in\mathscr{L}$ for which $i\notin M$ may still be able to accept them. A similar reasoning holds for the replenishment of spare parts. It is not possible to accept a (replenished) spare part for any labeled coalition $(M,k)\in\mathscr{L}$ with $i\in M$ for which $r_{M,k}=I_{M,k}$, while other not fully replenished labeled coalitions are willing to accept (such replenished spare part). We introduce a value function that incorporates these extended possibilities. So, we introduce a value function (related to some unspecified MDP), that coincides with the value function of definition 6.5, except for an enlarged action space. In order to do so, we first need to introduce such an enlarged action space.

Definition 6.6. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. Then, for all $r \in \mathscr{Y}^{\mathscr{B}}$ and all $i \in N$ we define*

$$\mathscr{A}_i^{\mathscr{B}^*_-}(r) = \left\{ (l_z)_{z \in \mathscr{L}} \;\middle|\; l_z \in \mathbb{N} \cup \{0\} \;\; \forall z \in \mathscr{L}, \; \sum_{z \in \mathscr{L}} l_z \leq \alpha, \;\; r - l \in \mathscr{Y}^{\mathscr{B}} \right\}$$

$$\mathscr{A}_i^{\mathscr{B}^*_+}(r) = \left\{ (l_z)_{z \in \mathscr{L}} \;\middle|\; l_z \in \mathbb{N} \cup \{0\} \;\; \forall z \in \mathscr{L}, \; \sum_{z \in \mathscr{L}} l_z \leq \alpha, \;\; r + l \in \mathscr{Y}^{\mathscr{B}} \right\}.$$

The following result captures that these are indeed enlarged action spaces.

Lemma 6.5. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. Then, for all $r \in \mathscr{Y}^{\mathscr{B}}$ and all $i \in N$ it holds that $\mathscr{A}_i^{\mathscr{B}-}(r) \subseteq \mathscr{A}_i^{\mathscr{B}^*_-}(r)$ and $\mathscr{A}_i^{\mathscr{B}+}(r) \subseteq \mathscr{A}_i^{\mathscr{B}^*_+}(r)$.*

Example 6.6. *Consider the situation and minimal balanced collection of example 6.5. Then*

$$\mathscr{A}_1^{\mathscr{B}^*_-}((1,2,0)) = \{(0,0,0), (1,0,0), (0,1,0), (1,1,0), (0,2,0)\}.$$

Note that in comparison to action space $\mathscr{A}_1^{\mathscr{B}-}((1,2,0))$, we now have one additional action, namely $(0,2,0)$, which represents that demand for labeled coalitions $(\{1,2\},1)$ and $(\{1,3\},1)$ is taken from the stock of labeled coalition $(\{1,3\},1)$ only. Similarly, we have

$$\mathscr{A}_1^{\mathscr{B}^*_-}((0,1,1)) = \{(0,0,0), (0,1,0), (0,0,1), (0,1,1)\}.$$

Note that in comparison to action space $\mathscr{A}_1^{\mathscr{B}-}((0,1,1))$, we now have two additional actions. $\diamond$

Now, we present the value function with this enlarged action space.

Definition 6.7. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. For all $r \in \mathscr{Y}^{\mathscr{B}}$ and all $t \in \mathbb{N} \cup \{0\}$, we define*

$$V_{t+1}^{\mathscr{B}^*}(r) = \sum_{i \in N} \left[\lambda_i^* \min_{l \in \mathscr{A}_i^{\mathscr{B}^*_-}(r)} \left\{ (\alpha - ||l||_1) d_i + V_t^{\mathscr{B}^*}(r - l) \right\} + \mu_i^* \min_{l \in \mathscr{A}_i^{\mathscr{B}^*_+}(r)} \left\{ V_t^{\mathscr{B}^*}(r + l) \right\} \right]$$

with $V_0^{\mathscr{B}^*}(r) = 0$ for all $r \in \mathscr{Y}^{\mathscr{B}}$.

Incorporating an extended action space in the new value function leads to related costs that are smaller than or equal to the original costs of the value function. This result is formally described in the upcoming lemma.

Lemma 6.6. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. For all $r \in \mathscr{Y}^{\mathscr{B}}$ and all $t \in \mathbb{N} \cup \{0\}$ it holds that*

$$V_t^{\mathscr{B}}(r) \geq V_t^{\mathscr{B}^*}(r).$$

***Proof* :** This proof is by induction. By definition of the value functions $V_0^{\mathscr{B}}(r) = 0$ for all $r \in \mathscr{R}$. Similarly, $V_0^{\mathscr{B}^*}(r) = 0$ for all $r \in \mathscr{R}$. Hence, $V_0^{\mathscr{B}}(r) \geq V_0^{\mathscr{B}^*}(r)$ for all $r \in \mathscr{R}$. Let $t \in \mathbb{N} \cup \{0\}$ and assume that $V_t^{\mathscr{B}}(r) \geq V_t^{\mathscr{B}^*}(r)$ for all $r \in \mathscr{R}$. Let $r \in \mathscr{R}$.

Now, observe that

$$\begin{aligned}
V_{t+1}^{\mathscr{B}}(r) &= \sum_{i \in N} \left[\lambda_i^* \min_{l \in \mathscr{A}_i^{\mathscr{B}-}(r)} \left\{ (\alpha - ||l||_1) d_i + V_t^{\mathscr{B}}(r-l) \right\} + \mu_i^* \min_{l \in \mathscr{A}_i^{\mathscr{B}+}(r)} \left\{ V_t^{\mathscr{B}}(r+l) \right\} \right] \\
&\geq \sum_{i \in N} \left[\lambda_i^* \min_{l \in \mathscr{A}_i^{\mathscr{B}^*-}(r)} \left\{ (\alpha - ||l||_1) d_i + V_t^{\mathscr{B}^*}(r-l) \right\} + \mu_i^* \min_{l \in \mathscr{A}_i^{\mathscr{B}^*+}(r)} \left\{ V_t^{\mathscr{B}^*}(r+l) \right\} \right] \\
&= V_{t+1}^{\mathscr{B}^*}(r).
\end{aligned}$$

The first and last equality hold by definition. For the inequality, observe that $\mathscr{A}_i^{\mathscr{B}-}(r) \subseteq \mathscr{A}_i^{\mathscr{B}^*-}(r)$ for all $r \in \mathscr{R}$ and $\mathscr{A}_i^{\mathscr{B}+}(r) \subseteq \mathscr{A}_i^{\mathscr{B}^*+}(r)$ for all $r \in \mathscr{R}$ based on lemma 6.5. As adding terms to a set from which its minimum it selected does not increase the minimum and $V_t^{\mathscr{B}}(r) \geq V_t^{\mathscr{B}^*}(r)$ for all $r \in \mathscr{R}$ (by induction), the inequality follows. $\square$

Observe that the costs in the new value function depend on the total number of accepted (or rejected) spare parts only. In combination with the enlarged action space, this implies that for every minimal balanced collection, one can construct another value function with equal costs, in which (i) the state space only depends on the total inventory on-hand and (ii) the action space only depends on the total number of spare parts to accept (or reject) upon demand arrival and the number of spare parts to replenish. We introduce this value function (which is related to some MDP) and show cost-equivalence with the value function of definition 6.7.

Definition 6.8. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. Then, for all $j \in \{0, 1, \ldots, \alpha \cdot I_N\}$ and all $t \in \mathbb{N} \cup \{0\}$ we define*

$$V_{t+1}^{\alpha}(j) = \sum_{i \in N} \left[\lambda_i^* \min_{l \in \{0,..,\min\{\alpha, j\}\}} \left\{ (\alpha - l) d_i + V_t^{\alpha}(j-l) \right\} + \mu_i^* \min_{l \in \{0,..,\min\{\alpha, \alpha \cdot I_N - j\}\}} V_t^{\alpha}(j+l) \right]$$

with $V_0^{\alpha}(j) = 0$ for all $j \in \{0, 1, \ldots, \alpha \cdot I_N\}$.

Lemma 6.7. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. For all $r \in \mathscr{Y}^{\mathscr{B}}$ it holds that*

$$V_t^{\mathscr{B}^*}(r) = V_t^{\alpha}(||r||_1) \qquad \textit{for all } t \in \mathbb{N} \cup \{0\}.$$

***Proof* :** This proof is by induction. By definition of the value functions $V_0^{\mathscr{B}^*}(r) = 0$ for all $r \in \mathscr{R}$ and $V_0^{\alpha}(||r||_1) = 0$ for all $r \in \mathscr{R}$. Hence, $V_0^{\mathscr{B}^*}(r) = V_0^{\alpha}(||r||_1)$ for all $r \in \mathscr{R}$. Let $t \in \mathbb{N} \cup \{0\}$ and assume that $V_t^{\mathscr{B}^*}(r) = V_t^{\alpha}(||r||_1)$ for all $r \in \mathscr{R}$. Let $r \in \mathscr{R}$.

Now, observe that

$$\begin{aligned}
V_{t+1}^{\mathscr{B}^*}(r) &= \sum_{i\in N}\lambda_i^* \min_{l\in\mathscr{A}_i^{\mathscr{B}^*-}(r)}\left\{(\alpha-||l||_1)d_i+V_t^{\mathscr{B}^*}(r-l)\right\}+\sum_{i\in N}\mu_i^* \min_{l\in\mathscr{A}_i^{\mathscr{B}^*+}(r)} V_t^{\mathscr{B}^*}(r+l)\\
&= \sum_{i\in N}\lambda_i^* \min_{z\in\{0,1,\ldots,\min\{\alpha,||r||_1\}\}}\left\{\min_{l\in\mathscr{A}_i^{\mathscr{B}^*-}(r):||l||_1=z}\left\{(\alpha-z)d_i+V_t^{\mathscr{B}^*}(r-l)\right\}\right\}\\
&\quad+\sum_{i\in N}\mu_i^* \min_{z\in\{0,1,\ldots,\min\{\alpha,\alpha\cdot I_N-||r||_1\}\}}\left\{\min_{l\in\mathscr{A}_i^{\mathscr{B}^*+}(r):||l||_1=z} V_t^{\mathscr{B}^*}(r+l)\right\}\\
&= \sum_{i\in N}\lambda_i^* \min_{z\in\{0,1,\ldots,\min\{\alpha,||r||_1\}\}}\{(\alpha-z)d_i+V_t^{\alpha}(||r||_1-z)\}\\
&\quad+\sum_{i\in N}\mu_i^* \min_{z\in\{0,1,\ldots,\min\{\alpha,\alpha\cdot I_N-||r||_1\}\}}\left\{V_t^{\alpha}(||r||_1+z)\right\}\\
&=V_{t+1}^{\alpha}(||r||_1).
\end{aligned}$$

The first equality holds by definition 6.7. The second equality holds by rewriting the minimum as a two-step minimization. The third equality holds by the induction hypothesis. The last equality holds by definition 6.8. This concludes the proof. □

As a next step we identify some interesting properties of the new value function.

Lemma 6.8. *Let $\theta\in\Theta$ and $\mathscr{B}\subseteq 2^N\backslash\{\emptyset\}$ be a minimal balanced collection. For all $t\in\mathbb{N}\cup\{0\}$ it holds that*

(i) $V_t^{\alpha}(j)\geq V_t^{\alpha}(j+1)$ *for all* $j\in\{0,1,\ldots,\alpha\cdot I_N-1\}$;

(ii) $V_t^{\alpha}(j)+V_t^{\alpha}(j+2)\geq 2\cdot V_t^{\alpha}(j+1)$ *for all* $j\in\{0,1,\ldots,\alpha\cdot I_N-2\}$;

(iii) $V_t^{\alpha}(k+j)+V_t^{\alpha}(k+j+2)=2\cdot V_t^{\alpha}(k+j+1)$ *for all* $j\in\{0,1,\ldots,\alpha-2\}$ *and all* $k\in\{0,\alpha,2\cdot\alpha,\ldots,(I_N-1)\cdot\alpha\}$.

Proof **:** This proof is by induction. (i) By definition of the value functions $V_0^{\alpha}(j)=0$ for all $j\in\{0,1,\ldots,\alpha\cdot I_N\}$. Hence, $V_0^{\alpha}(j)\geq V_0^{\alpha}(j+1)$ for all $j\in\{0,1,\ldots,\alpha\cdot I_N-1\}$. Let $t\in\mathbb{N}\cup\{0\}$ and assume that $V_t^{\alpha}(j)\geq V_t^{\alpha}(j+1)$ for all $j\in\{0,1,\ldots,\alpha\cdot I_N-1\}$. Let $j\in\{0,1,\ldots,\alpha\cdot I_N-1\}$. Now, observe that

$$\begin{aligned}
V_{t+1}^{\alpha}(j) &= \sum_{i\in N}\lambda_i^* \min_{l\in\{0,1,\ldots,\min\{\alpha,j\}\}}\left\{(\alpha-l)d_i+V_t^{\alpha}(j-l)\right\}\\
&\quad+\sum_{i\in N}\mu_i^* \min_{l\in\{0,1,\ldots,\min\{\alpha,\alpha\cdot I_N-j\}\}}\left\{V_t^{\alpha}(j+l)\right\}\\
&\geq \sum_{i\in N}\lambda_i^* \min_{l\in\{0,1,\ldots,\min\{\alpha,j+1\}\}}\left\{(\alpha-l)d_i+V_t^{\alpha}(j+1-l)\right\}\\
&\quad+\sum_{i\in N}\mu_i^* \min_{l\in\{0,1,\ldots,\min\{\alpha,\alpha\cdot I_N-j\}\}}\left\{V_t^{\alpha}(j+l)\right\}
\end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{i\in N} \lambda_i^* \min_{l\in\{0,1,\ldots,\min\{\alpha,j+1\}\}} \left\{(\alpha-l)d_i + V_t^\alpha(j+1-l)\right\} \\
&\quad + \sum_{i\in N} \mu_i^* V_t^\alpha(\min\{j+\alpha+1, \alpha\cdot I_N\}) \\
&\geq \sum_{i\in N} \lambda_i^* \min_{l\in\{0,1,\ldots,\min\{\alpha,j+1\}\}} \left\{(\alpha-l)d_i + V_t^\alpha(j+1-l)\right\} \\
&\quad + \sum_{i\in N} \mu_i^* \min_{l\in\{0,1,\ldots,\min\{\alpha,\alpha\cdot I_N-(j+1)\}\}} \left\{V_t^\alpha(j+1+l)\right\}. \\
&= V_{t+1}^\alpha(j+1).
\end{aligned}
$$

The first inequality holds as $V_t^\alpha(j-l) \geq V_t^\alpha(j+1-l)$ for all $l \in \{0,1,\ldots,\min\{\alpha,j\}\}$ (by the induction hypothesis) and the fact that adding a (possible) term to a set from which its minimum is selected will not increase the minimum. In addition, the second inequality holds as $\min_{l\in\{0,1,\ldots,\min\{\alpha,\alpha\cdot I_N-j\}\}}\{V_t^\alpha(j+l)\} \geq V_t^\alpha(\min\{j+\alpha,\alpha\cdot I_N\})$ $\geq V_t^\alpha(\min\{j+1+\alpha,\alpha\cdot I_N\})$. Note that the inequalities are a direct consequence of the induction hypothesis, namely $V_t^\alpha(j) \geq V_t^\alpha(j+1)$ for all $j \in \{0,1,\ldots,\alpha\cdot I_N - 1\}$. The third inequality holds as adding (possible) terms to a set from which its minimum is selected will not increase the minimum.

(ii) First, the value function will be rewritten. From (i) of lemma 6.8 it follows for all $j \in \{0,1,\ldots,\alpha\cdot I_N\}$ and all $t \in \mathbb{N}\cup\{0\}$ that

$$
\begin{aligned}
V_{t+1}^\alpha(j) &= \sum_{i\in N} \lambda_i^* \min_{l\in\{0,1,\ldots,\min\{\alpha,j\}\}} \left\{(\alpha-l)d_i + V_t^\alpha(j-l)\right\} + \sum_{i\in N} \mu_i^* V_t^\alpha(\min\{j+\alpha,\alpha\cdot I_N\}) \\
&= \sum_{i\in N} \lambda_i^* \left[\min_{l\in\{0,1,\ldots,\min\{\alpha,j\}\}} \left\{(j-l)d_i + V_t^\alpha(j-l)\right\} + (\alpha-j)d_i\right] \\
&\quad + \sum_{i\in N} \mu_i^* V_t^\alpha(\min\{j+\alpha,\alpha\cdot I_N\}) \\
&= \sum_{i\in N} \lambda_i^* \left[\min_{l\in\{\max\{0,j-\alpha\},\ldots,j\}} \left\{ld_i + V_t^\alpha(l)\right\} + (\alpha-j)d_i\right] \\
&\quad + \sum_{i\in N} \mu_i^* V_t^\alpha(\min\{j+\alpha,\alpha\cdot I_N\}),
\end{aligned}
$$

where the second equality holds as $(a-l)d_i = (\alpha-j)d_i + (j-l)d_i$. The third equality holds by substituting $j-l$ into a single variable.

In addition, we define for all $j \in \{0,1,\ldots,\alpha\cdot I_N\}$ and all $t \in \mathbb{N}\cup\{0\}$

$$
\begin{aligned}
V_t^{\alpha_1}(j) &= \sum_{i\in N} \lambda_i^* \left[\min_{l\in\{\max\{0,j-\alpha\},\ldots,j\}} \left\{ld_i + V_t^\alpha(l)\right\} + (\alpha-j)d_i\right] \\
V_t^{\alpha_2}(j) &= \sum_{i\in N} \mu_i^* V_t^\alpha(\min\{j+\alpha,\alpha\cdot I_N\}).
\end{aligned}
$$

Note that $V_t^{\alpha}(j) = V_t^{\alpha_1}(j) + V_t^{\alpha_2}(j)$ for all $j \in \{0, 1, \ldots, \alpha \cdot I_N\}$ and all $t \in \mathbb{N} \cup \{0\}$.

Now, we prove that $V_0^{\alpha}(j) + V_0^{\alpha}(j+2) \geq 2 \cdot V_0^{\alpha}(j+1)$ for all $j \in \{0, 1, \ldots, \alpha \cdot I_N - 2\}$ and all $t \in \mathbb{N} \cup \{0\}$. By definition of the value functions $V_0^{\alpha}(j) = 0$ for all $j \in \{0, 1, \ldots, \alpha \cdot I_N\}$. Hence, $V_0^{\alpha}(j) + V_0^{\alpha}(j+2) \geq 2 \cdot V_0^{\alpha}(j+1)$ for all $j \in \{0, 1, \ldots, \alpha \cdot I_N - 2\}$. Let $t \in \mathbb{N} \cup \{0\}$ and assume that $V_t^{\alpha}(j) + V_t^{\alpha}(j+2) \geq 2 \cdot V_t^{\alpha}(j+1)$ for all $j \in \{0, 1, \ldots, \alpha \cdot I_N - 2\}$.

First, we focus on $V_t^{\alpha_2}(j)$ and thereafter on $V_t^{\alpha_1}(j)$. Let $j \in \{0, 1, \ldots, \alpha \cdot (I_N - 1) - 2\}$. Now, observe that

$$\begin{aligned}
V_{t+1}^{\alpha_2}(j) + V_{t+1}^{\alpha_2}(j+2) &= \sum_{i \in N} \mu_i^* V_t^{\alpha}(\min\{j + \alpha, \alpha \cdot I_N\}) \\
&\quad + \sum_{i \in N} \mu_i^* V_t^{\alpha}(\min\{j + 2 + \alpha, \alpha \cdot I_N\}) \\
&= \sum_{i \in N} \mu_i^* V_t^{\alpha}(j + \alpha) + \sum_{i \in N} \mu_i^* V_t^{\alpha}(j + 2 + \alpha) \\
&= \sum_{i \in N} \mu_i^* \left(V_t^{\alpha}(j + \alpha) + V_t^{\alpha}(j + 2 + \alpha)\right) \\
&\geq 2 \sum_{i \in N} \mu_i^* V_t^{\alpha}(j + 1 + \alpha) \\
&= 2 \sum_{i \in N} \mu_i^* V_t^{\alpha}(\min\{j + 1 + \alpha, \alpha \cdot I_N\}) \\
&= 2V_{t+1}^{\alpha_2}(j+1),
\end{aligned}$$

where the inequality holds by the induction hypothesis.

Let $j \in \{\alpha \cdot (I_N - 1) - 1, \alpha \cdot (I_N - 1), \ldots, \alpha \cdot I_N - 2\}$. Now, observe that

$$\begin{aligned}
V_{t+1}^{\alpha_2}(j) + V_{t+1}^{\alpha_2}(j+2) &= \sum_{i \in N} \mu_i^* V_t^{\alpha}(\min\{j + \alpha, \alpha \cdot I_N\}) \\
&\quad + \sum_{i \in N} \mu_i^* V_t^{\alpha}(\min\{j + 2 + \alpha, \alpha \cdot I_N\}) \\
&\geq \sum_{i \in N} \mu_i^* V_t^{\alpha}(\alpha \cdot I_N) + \sum_{i \in N} \mu_i^* V_t^{\alpha}(\alpha \cdot I_N) \\
&= 2 \sum_{i \in N} \mu_i^* V_t^{\alpha}(\alpha \cdot I_N) \\
&= 2 \sum_{i \in N} \mu_i^* V_t^{\alpha}(\min\{j + 1 + \alpha, \alpha \cdot I_N\}) \\
&= 2V_{t+1}^{\alpha_2}(j+1),
\end{aligned}$$

where the inequality holds by (i). Hence, for all $j \in \{0, 1, \ldots, \alpha \cdot I_N - 2\}$ it holds that $V_{t+1}^{\alpha_2}(j) + V_{t+1}^{\alpha_2}(j+2) \geq 2V_{t+1}^{\alpha_2}(j+1)$.

Let $j \in \{\alpha, \alpha+1, \ldots, \alpha \cdot I_N - 2\}$. Now, observe that

$$
\begin{aligned}
&V_{t+1}^{\alpha_1}(j) + V_{t+1}^{\alpha_1}(j+2) \\
&= \sum_{i \in N} \lambda_i^* \min_{l \in \{\max\{0, j-\alpha\}, \ldots, j\}} \left\{ l d_i + V_t^{\alpha}(l) \right\} + \sum_{i \in N} \lambda_i^* \min_{l \in \{\max\{0, j+2-\alpha\}, \ldots, j+2\}} \left\{ l d_i + V_t^{\alpha}(l) \right\} \\
&= \sum_{i \in N} \lambda_i^* \min_{\substack{l_1 \in \{j-\alpha, \ldots, j\} \\ l_2 \in \{j+2-\alpha, \ldots, j+2\}}} \left\{ (l_1 + l_2) d_i + V_t^{\alpha}(l_1) + V_t^{\alpha}(l_2) \right\} \\
&\geq \sum_{i \in N} \lambda_i^* \min \left[\left\{ (2l_3 + 1) d_i + V_t^{\alpha}(l_3) + V_t^{\alpha}(l_3 + 1) | l_3 = j + 1 - \alpha, \ldots, j \right\} \right. \\
&\qquad\qquad \left. \cup \left\{ 2 \left(l_3 d_i + V_t^{\alpha}(l_3) \right) | l_3 = j + 1 - \alpha, \ldots, j + 1 \right\} \right] \\
&= 2 \sum_{i \in N} \lambda_i^* \min_{l_3 \in \{j+1-\alpha, j+2-\alpha, \ldots, j+1\}} \left\{ V_t^{\alpha}(l_3) + l_3 d_i \right\} \Big\} \\
&= 2 \sum_{i \in N} \lambda_i^* \min_{l_3 \in \{\max\{0, j+1-\alpha\}, \ldots, j+1\}} \left\{ V_t^{\alpha}(l_3) + l_3 d_i \right\} \Big\} \\
&= 2 V_{t+1}^{\alpha_1}(j+1).
\end{aligned}
$$

The inequality holds as for any $l_1, l_2 \in \{0, 1, \ldots, \alpha \cdot I_N - 2\}$ with $l_1 \leq l_2$ it holds that $V_t^{\alpha}(l_1) + V_t^{\alpha}(l_2) \geq V_t^{\alpha}(\lfloor (l_1 + l_2)/2 \rfloor) + V_t^{\alpha}(\lceil (l_1 + l_2)/2 \rceil)$ based on the induction hypothesis. This implies that for any $l_1, l_2 \in \{0, 1, \ldots, \alpha \cdot I_N - 2\}$ with $l_1 + l_2$ odd, it follows that $(l_1 + l_2) d_i + V_t^{\alpha}(l_1) + V_t^{\alpha}(l_2) \geq (2l_3 + 1) d_i + V_t^{\alpha}(l_3) + V_t^{\alpha}(l_3 + 1)$ where $l_3 = \lfloor (l_1 + l_2)/2 \rfloor$. For any $l_1, l_2 \in \{0, 1, \ldots, \alpha \cdot I_N - 2\}$ with $l_1 + l_2$ even, it follows that $(l_1 + l_2) d_i + V_t^{\alpha}(l_1) + V_t^{\alpha}(l_2) \geq 2 l_3 d_i + 2 V_t^{\alpha}(l_3)$ with $l_3 = (l_1 + l_2)/2$. The third equality holds as $\min\{2a, a + b, 2b\} = \min\{2a, 2b\}$ for any $a, b \in \mathbb{R}$. The second last equality holds as $j \geq \alpha$.

Let $j \in \{0, 1, \ldots, \alpha - 1\}$. Now, observe that

$$
\begin{aligned}
&V_{t+1}^{\alpha_1}(j) + V_{t+1}^{\alpha_1}(j+2) \\
&= \sum_{i \in N} \lambda_i^* \min_{l \in \{\max\{0, j-\alpha\}, \ldots, j\}} \left\{ l d_i + V_t^{\alpha}(l) \right\} + \sum_{i \in N} \lambda_i^* \min_{l \in \{\max\{0, j+2-\alpha\}, \ldots, j+2\}} \left\{ l d_i + V_t^{\alpha}(l) \right\} \\
&\geq \sum_{i \in N} \lambda_i^* \min_{\substack{l_1 \in \{0, \ldots, j\} \\ l_2 \in \{0, \ldots, j+2\}}} \left\{ (l_1 + l_2) d_i + V_t^{\alpha}(l_1) + V_t^{\alpha}(l_2) \right\} \\
&\geq \sum_{i \in N} \lambda_i^* \min \left[\left\{ (2l_3 + 1) d_i + V_t^{\alpha}(l_3) + V_t^{\alpha}(l_3 + 1) | l_3 = 0, \ldots, j \right\} \right. \\
&\qquad\qquad \left. \cup \left\{ 2 \left(l_3 d_i + V_t^{\alpha}(l_3) \right) | l_3 = 0, \ldots, j + 1 \right\} \right]
\end{aligned}
$$

$$
\begin{aligned}
&= 2\sum_{i\in N}\lambda_i^* \min_{l_3\in\{0,1,\dots,j+1\}}\{V_t^{\alpha}(l_3)+l_3 d_i\}\Big\}\\
&= 2\sum_{i\in N}\lambda_i^* \min_{l_3\in\{\max\{0,j+1-\alpha\},\dots,j+1\}}\{V_t^{\alpha}(l_3)+l_3 d_i\}\Big\}\\
&= 2V_{t+1}^{\alpha_1}(j+1).
\end{aligned}
$$

The first inequality holds as adding a (possible) term to a set from which its minimum is selected will not increase the minimum. The arguments of the other (in)equalities are similar to the ones of case $j\in\{\alpha,\alpha+1,\dots,\alpha\cdot I_N-2\}$. So, for all $j\in\{0,1,\dots,\alpha\cdot I_N-2\}$ it holds that $V_{t+1}^{\alpha_1}(j)+V_{t+1}^{\alpha_1}(j+2)\geq 2V_{t+1}^{\alpha_1}(j+1)$.

We conclude that for all $j\in\{0,1,\dots,\alpha\cdot I_N-2\}$, it holds that

$$
\begin{aligned}
V_{t+1}^{\alpha}(j)+V_{t+1}^{\alpha}(j+2) &= V_{t+1}^{\alpha_1}(j)+V_{t+1}^{\alpha_2}(j)+V_{t+1}^{\alpha_1}(j+2)+V_{t+1}^{\alpha_2}(j+2)\\
&\geq 2V_{t+1}^{\alpha_1}(j+1)+2V_{t+1}^{\alpha_2}(j+1)\\
&= 2V_{t+1}^{\alpha}(j+1).
\end{aligned}
$$

(iii) By definition of the value functions $V_0^{\alpha}(j)=0$ for all $j\in\{0,1,\dots,\alpha\cdot I_N\}$. Hence, $V_0^{\alpha}(k+j)+V_0^{\alpha}(k+j+2)=2\cdot V_0^{\alpha}(k+j+1)$ for all $j\in\{0,1,\dots,\alpha-2\}$ and all $k\in\{0,\alpha,2\cdot\alpha,\dots,(I_N-1)\cdot\alpha\}$. Let $t\in\mathbb{N}\cup\{0\}$ and assume that $V_t^{\alpha}(k+j)+V_t^{\alpha}(k+j+2)=2\cdot V_t^{\alpha}(k+j+1)$ for all $j\in\{0,1,\dots,\alpha-2\}$ and all $k\in\{0,\alpha,2\cdot\alpha,\dots,(I_N-1)\cdot\alpha\}$.

First, observe that function $V_t^{\alpha}(j)+j\cdot d_i$ is convex in j for all $i\in N$ as $V_t^{\alpha}(\cdot)$ is convex by (ii) and $j\cdot d_i$ is linear. By our induction hypothesis, $V_t^{\alpha}(k+j)+V_t^{\alpha}(k+j+2)=2\cdot V_t^{\alpha}(k+j+1)$ for all $j\in\{0,1,\dots,\alpha-2\}$ and all $k\in\{0,\alpha,2\cdot\alpha\dots,(I_N-1)\cdot\alpha\}$. So, $V_t^{\alpha}(k+j)+(k+j)d_i+V_t^{\alpha}(k+j+2)+(k+j+2)d_i=2\cdot V_t^{\alpha}(k+j+1)+2\cdot(k+j+1)d_i$ for all $j\in\{0,1,\dots,\alpha-2\}$, all $k\in\{0,\alpha,2\cdot\alpha\dots,(I_N-1)\cdot\alpha\}$ and all $i\in N$. For this, we conclude that there exists an $h\in\{0,\alpha,2\alpha,\dots,\dots,(I_N-1)\cdot\alpha\}$ for which it holds that $V_t^{\alpha}(h)+h\cdot d_i\leq V_t^{\alpha}(j)+j\cdot d_i$ for all $j\in\{0,1,\dots,\alpha\cdot I_N\}$ and all $i\in N$. Let $h\in\{0,\alpha,\dots,I_N\cdot\alpha\}$. Then, for all $k\in\{0,\alpha,\dots,h-\alpha\}$ and all $j\in\{0,1,\dots,\alpha\}$ it holds that

$$
\begin{aligned}
&\sum_{i\in N}\lambda_i^*\left[\min_{l\in\{\min\{0,k+j-\alpha\},\dots,k+j\}}\left\{ld_i+V_t^{\alpha}(l)\right\}+(\alpha-(k+j))d_i\right]\\
&=\sum_{i\in N}\lambda_i^*\left[(k+j)d_i+V_t^{\alpha}(k+j)+(\alpha-(k+j))d_i\right] \qquad (6.2)\\
&=\sum_{i\in N}\lambda_i^*\left[V_t^{\alpha}(k+j)+\alpha d_i\right].
\end{aligned}
$$

For $k = h$ and all $j \in \{0, 1, \ldots, \alpha\}$ it holds that

$$\begin{aligned}
&\sum_{i \in N} \lambda_i^* \left[\min_{l \in \{\min\{0, k+j-\alpha\}, \ldots, k+j\}} \left\{ l d_i + V_t^\alpha(l) \right\} + (\alpha - (k+j)) d_i \right] \\
&= \sum_{i \in N} \lambda_i^* \left[k \cdot d_i + V_t^\alpha(k) + (\alpha - (k+j)) d_i \right] \\
&= \sum_{i \in N} \lambda_i^* \left[V_t^\alpha(k) + (\alpha - j) d_i \right].
\end{aligned} \tag{6.3}$$

For all $k \in \{h + \alpha, h + 2 \cdot \alpha, \ldots, (I_N - 1) \cdot \alpha\}$ and all $j \in \{0, 1, \ldots, \alpha\}$ it holds that

$$\begin{aligned}
&\sum_{i \in N} \lambda_i^* \left[\min_{l \in \{\min\{0, k+j-\alpha\}, \ldots, k+j\}} \left\{ l d_i + V_t^\alpha(l) \right\} + (\alpha - (k+j)) d_i \right] \\
&= \sum_{i \in N} \lambda_i^* \left[(k + j - \alpha) d_i + V_t^\alpha(k + j - \alpha) + (\alpha - (k+j)) d_i \right] \\
&= \sum_{i \in N} \lambda_i^* \left[V_t^\alpha(k + j - \alpha) \right].
\end{aligned} \tag{6.4}$$

In addition, for all $k \in \{0, \alpha, \ldots (I_N - 2) \cdot \alpha\}$ and all $j \in \{0, 1, \ldots, \alpha - 2\}$ it holds that

$$\begin{aligned}
&\sum_{i \in N} \mu_i^* V_t^\alpha(\min\{k + j + \alpha, \alpha \cdot I_N\}) + \sum_{i \in N} \mu_i^* V_t^\alpha(\min\{k + j + 2 + \alpha, \alpha \cdot I_N\}) \\
&= \sum_{i \in N} \mu_i^* V_t^\alpha(k + j + \alpha) + \sum_{i \in N} \mu_i^* V_t^\alpha(k + j + 2 + \alpha) \\
&= 2 \sum_{i \in N} \mu_i^* V_t^\alpha(k + j + 1 + \alpha) \\
&= 2 \sum_{i \in N} \mu_i^* V_t^\alpha(\min\{k + j + 1 + \alpha, \alpha \cdot I_N\}).
\end{aligned}$$

Moreover, for $k = (I_N - 1) \cdot \alpha$ and all $j \in \{0, 1, \ldots, \alpha - 2\}$ it holds that

$$\begin{aligned}
&\sum_{i \in N} \mu_i^* V_t^\alpha(\min\{k + j + \alpha, \alpha \cdot I_N\}) + \sum_{i \in N} \mu_i^* V_t^\alpha(\min\{k + j + 2 + \alpha, \alpha \cdot I_N\}) \\
&= \sum_{i \in N} \mu_i^* V_t^\alpha(\alpha \cdot I_N) + \sum_{i \in N} \mu_i^* V_t^\alpha(\alpha \cdot I_N) \\
&= 2 \sum_{i \in N} \mu_i^* V_t^\alpha(\alpha \cdot I_N) \\
&= 2 \sum_{i \in N} \mu_i^* V_t^\alpha(\min\{k + j + 1, \alpha \cdot I_N\}).
\end{aligned}$$

Hence, for all $k \in \{0, \alpha, \ldots, (I_N - 1) \cdot \alpha\}$ and all $j \in \{0, 1, \ldots, \alpha - 2\}$ it holds that

$$\begin{aligned}
&\sum_{i \in N} \mu_i^* V_t^\alpha(\min\{k + j + \alpha, \alpha \cdot I_N\}) + \sum_{i \in N} \mu_i^* V_t^\alpha(\min\{k + j + 2 + \alpha, \alpha \cdot I_N\}) \\
&= 2 \sum_{i \in N} \mu_i^* V_t^\alpha(\min\{k + j + 1, \alpha \cdot I_N\}).
\end{aligned} \tag{6.5}$$

For all $k \in \{0, 1, \ldots, h - \alpha\}$ and all $j \in \{0, 1, \ldots, \alpha - 2\}$ it holds that

$$
\begin{aligned}
&V^{\alpha}_{t+1}(k+j) + V^{\alpha}_{t+1}(k+j+2)\\
=&\sum_{i\in N} \lambda^*_i \left[V^{\alpha}_t(k+j) + \alpha \cdot d_i \right] + \sum_{i\in N} \mu^*_i V^{\alpha}_t(\min\{k+j+\alpha, \alpha \cdot I_N\})\\
&+ \sum_{i\in N} \lambda^*_i \left[V^{\alpha}_t(k+j+2) + \alpha \cdot d_i \right] + \sum_{i\in N} \mu^*_i V^{\alpha}_t(\min\{k+j+2+\alpha, \alpha \cdot I_N\})\Big)\\
=&2\sum_{i\in N} \lambda^*_i \left[V^{\alpha}_t(k+j+1) + \alpha \cdot d_i \right] + \sum_{i\in N} 2\mu^*_i V^{\alpha}_t(\min\{k+j+1+\alpha, \alpha \cdot I_N\})\\
=&V^{\alpha}_{t+1}(k+j+1).
\end{aligned}
$$

The first equality holds by (6.2). The second equality holds by the induction hypothesis and (6.5).

For $k = h$ and all $j \in \{0, 1, \ldots, \alpha - 2\}$ it holds that

$$
\begin{aligned}
&V^{\alpha}_{t+1}(k+j) + V^{\alpha}_{t+1}(k+j+2)\\
=&\sum_{i\in N} \lambda^*_i \left[V^{\alpha}_t(k) + (\alpha - j) \cdot d_i \right] + \sum_{i\in N} \mu^*_i V^{\alpha}_t(\min\{k+j+\alpha, \alpha \cdot I_N\})\\
&+ \sum_{i\in N} \lambda^*_i \left[V^{\alpha}_t(k) + (\alpha - (j+2)) \cdot d_i \right] + \sum_{i\in N} \mu^*_i V^{\alpha}_t(\min\{k+j+2+\alpha, \alpha \cdot I_N\})\\
=&2\sum_{i\in N} \lambda^*_i \left[V^{\alpha}_t(k) + (\alpha - (j+1)) \cdot d_i \right] + \sum_{i\in N} 2\mu^*_i V^{\alpha}_t(\min\{k+j+1+\alpha, \alpha \cdot I_N\})\\
=&V^{\alpha}_{t+1}(k+j+1).
\end{aligned}
$$

The first equality holds by (6.3). The second equality holds by the induction hypothesis and (6.5).

For all $k \in \{h + \alpha, h + 2 \cdot \alpha, \ldots, (I_N - 1) \cdot \alpha\}$ and all $j \in \{0, 1, \ldots, \alpha - 2\}$ it holds that

$$
\begin{aligned}
&V^{\alpha}_{t+1}(k+j) + V^{\alpha}_{t+1}(k+j+2)\\
=&\sum_{i\in N} \lambda^*_i \left[V^{\alpha}_t(k+j-\alpha) \right] + \sum_{i\in N} \mu^*_i V^{\alpha}_t(\min\{k+j+\alpha, \alpha \cdot I_N\})\\
&+ \sum_{i\in N} \lambda^*_i \left[V^{\alpha}_t(k+j+2-\alpha) \right] + \sum_{i\in N} \mu^*_i V^{\alpha}_t(\min\{k+j+2+\alpha, \alpha \cdot I_N\})\\
=&2\sum_{i\in N} \lambda^*_i \left[V^{\alpha}_t(k+j+1-\alpha) \right] + \sum_{i\in N} 2\mu^*_i V^{\alpha}_t(\min\{k+j+1+\alpha, \alpha \cdot I_N\})\\
=&V^{\alpha}_{t+1}(k+j+1).
\end{aligned}
$$

The first equality holds by (6.4). The second equality holds by the induction hypothesis and (6.5). This concludes the proof. □

The first property of lemma 6.8 states that the costs decrease in the total inventory on-hand. As a direct consequence, it is optimal for the spare part replenishment part of the value function to choose an action that increases the state most. The second property states that the marginal decrease in costs will not increase in the inventory on-hand. By exploiting the first two properties, we can show that the third property holds true. The third property states that it is optimal for all states that are multiples of α to choose the action that accepts all demand, i.e., α spare parts, or nothing upon demand arrival. In particular, this implies that the value function of the states that are multiples of α depend on the value function of states that are multiples of α only.

Lemma 6.9. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. Then, for all $j \in \{0, \alpha, \ldots, I_N \cdot \alpha\}$ and all $t \in \mathbb{N} \cup \{0\}$ it holds that*

$$V_t^\alpha(j) = \alpha \cdot V_t^N\left(\frac{j}{\alpha}\right).$$

Proof **:** Based on (i) and (iii) of lemma 6.8, it can be seen easily that for all $j \in \{0, \alpha, \ldots, \alpha \cdot I_N\}$ and all $t \in \mathbb{N} \cup \{0\}$ it holds that

$$V_{t+1}^\alpha(j) = \sum_{i \in N}\left[\lambda_i^* \min_{l \in \{0, \min\{j, \alpha\}\}}\{V_t^\alpha(j-l) + (\alpha - l)d_i\}\right] + \sum_{i \in N}\mu_i^* V_t^\alpha(\min\{j+\alpha, \alpha \cdot I_N\}).$$

By definition of the value functions $V_0^\alpha(j) = 0$ for all $j \in \{0, \alpha, \ldots, I_N \cdot \alpha\}$ and $V_0^N(j) = 0$ for all $j \in \{0, 1, \ldots, I_N\}$. Hence, $V_0^\alpha(j) = \alpha \cdot V_0^N\left(\frac{j}{\alpha}\right)$ for all $j \in \{0, \alpha, \ldots, I_N \cdot \alpha\}$. Let $t \in \mathbb{N} \cup \{0\}$ and assume that $V_t^\alpha(j) = \alpha \cdot V_t^N\left(\frac{j}{\alpha}\right)$ for all $j \in \{0, \alpha, \ldots, I_N \cdot \alpha\}$.

Let $j \in \{0, \alpha, \ldots, I_N \cdot \alpha\}$. Then, observe that

$$\begin{aligned}
V_{t+1}^\alpha(j) &= \left(\sum_{i \in N}\left[\lambda_i^* \min_{l \in \{0, \min\{j, \alpha\}\}}\{V_t^\alpha(j-l) + (\alpha - l)d_i\}\right] + \sum_{i \in N}\mu_i^* V_t^\alpha(\min\{j+\alpha, \alpha \cdot I_N\}\right) \\
&= \left(\sum_{i \in N}\left[\lambda_i^* \min_{l \in \{0, \min\{j, \alpha\}\}}\left\{\alpha \cdot V_t^N\left(\frac{j-l}{\alpha}\right) + (\alpha - l)d_i\right\}\right]\right. \\
&\qquad \left. + \sum_{i \in N}\mu_i^* V_t^\alpha(\min\{j+\alpha, \alpha \cdot I_N\})\right) \\
&= \left(\sum_{i \in N}\left[\lambda_i^* \min_{z \in \{0, \min\{\frac{j}{\alpha}, 1\}\}}\left\{\alpha \cdot V_t^N\left(\frac{j}{\alpha} - z\right) + \alpha \cdot (1-z)d_i\right\}\right]\right. \\
&\qquad \left. + \sum_{i \in N}\mu_i^* V_t^\alpha(\min\{j+\alpha, \alpha \cdot I_N\}\right)
\end{aligned}$$

$$= \left(\sum_{i \in N} \left[\lambda_i^* \min_{z \in \{0, \min\{\frac{j}{\alpha}, 1\}\}} \left\{ \alpha \cdot V_t^N \left(\frac{j}{\alpha} - z \right) + \alpha \cdot (1-z) d_i \right\} \right] \right.$$
$$\left. + \sum_{i \in N} \alpha \cdot \mu_i^* V_t^N (\min\{ \frac{j}{\alpha} + 1, I_N \}) \right)$$
$$= \alpha \cdot \left(\sum_{i \in N} \left[\lambda_i^* \min_{z \in \{0, \min\{\frac{j}{\alpha}, 1\}\}} \left\{ V_t^N \left(\frac{j}{\alpha} - z \right) + (1-z) d_i \right\} \right] \right.$$
$$\left. + \sum_{i \in N} \mu_i^* V_t^N (\min\{ \frac{j}{\alpha} + 1, I_N \}) \right)$$
$$= \alpha \cdot V_{t+1}^N \left(\frac{j}{\alpha} \right).$$

The first equality holds by definition. The second equality holds by the induction hypothesis. The third equality holds by introducing a new variable $z = l/\alpha$. The fourth equality holds by the induction hypothesis (again). The fifth equality holds as α can be taken outside the summations. The last equality holds by lemma 6.1. □

Using lemma 6.9, we conclude that for every minimal balanced collection $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ the related costs $\sum_{M \in \mathscr{B}} \sum_{k=1}^{b_M} V_t^{M,k}(I_{M,k})$ are at least at large as α-times the costs of $V_t^N(I_N)$ for all time moments. All ingredients are present to prove balancedness of STP games!

Theorem 6.1. *Every STP game is balanced.*

Proof **:** Let $\theta \in \Theta$ and (N, c^θ) be the associated STP game and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. In addition, let $r^* = (I_{M,k})_{(M,k) \in \mathscr{L}}$. Then, observe that

$$\sum_{M \in \mathscr{B}} b_M \cdot c^\theta(M) = \gamma \cdot \lim_{t \to \infty} \frac{1}{t} \sum_{M \in \mathscr{B}} \sum_{k=1}^{b_M} V_t^{M,k}(I_{M,k}) \geq \gamma \cdot \lim_{t \to \infty} \frac{V_t^{\mathscr{B}^*}(r^*)}{t} = \gamma \cdot \lim_{t \to \infty} \frac{V_t^\alpha(\alpha \cdot I_N)}{t}$$
$$= \gamma \cdot \lim_{t \to \infty} \alpha \cdot \frac{V_t^N(I_N)}{t}$$
$$= \alpha \cdot c^\theta(N).$$

The first equality holds by lemma 6.3. The inequality holds by lemma 6.4 and lemma 6.6. The second equality holds by lemma 6.7 and the fact that $\sum_{(M,k) \in \mathscr{L}} I_{M,k} = \alpha \cdot I_N$. The third equality holds by lemma 6.9. The last equality holds by taking $\alpha \in \mathbb{N}$ outside the limit (which is allowed as it is a constant) and subsequently applying lemma 6.2. We conclude that inequality (6.1) is satisfied and so STP games are balanced. □

Based on theorem 2.1, which states that a game has a non-empty core if and only if it is balanced, the next result follows immediately.

Corollary 6.1. *Every STP game has a non-empty core.*

6.4 Conclusions

In this chapter, we investigated a situation in which several independent service providers with (possibly) different downtime costs can cooperate by pooling their spare parts according to an optimal pooling strategy, i.e., by applying stratified pooling. We assumed that service providers can order (new) spare parts at a single supplier that produces spare parts one-by-one. For the associated stratified pooling game, we showed that the underlying resource pooling situation can be described by a Markov decision process and the optimal spare parts pooling strategy as a stationary decision rule in this Markov decision process. This modelling technique turned out to be crucial in proving core non-emptiness. In addition, we showed that when service providers would naively apply a full pooling strategy, core non-emptiness is not guaranteed.

For further research, the following extensions are of interest. First, one can extend the model, in line with chapter 5, by assuming that the joint supplier has multiple parallel servers (instead of one single server in which production rates are combined). It is not straightforward to apply our proof technique to that situation as the rate by which spare parts are replenished is no longer the sum of the rates of the players who collaborate, which is essential in our proof. Another possible extension is the one in which players optimize the number of spare parts to stock as well. In that case, players face a trade off between holding costs and downtime costs under an optimal pooling strategy. It is also not straightforward to apply our proof technique to this situation, as the number of spare parts to stock for the grand coalition is no longer the sum of the number of spare parts of the players who collaborate, which also is essential in our proof.

Chapter 7

Maximal covering location games

7.1 Introduction

In this chapter, which is based on Schlicher et al. [51], we focus on the well-known maximal covering location problem (Church and ReVelle [10]). In this location model, a single decision maker has to position a predetermined number of resources in order to maximize profit of the covered demand points, where a demand point is covered if a resource is positioned within a certain radius. The maximal covering location problem has proven to be useful in many settings, e.g., for positioning of emergency vehicles (Li et al. [30]), cell towers (Lee and Murray [29]), and retail stores (Plastria and Vanhaverbeke [42]). Another interesting setting is the one with several small-sized regions, e.g., villages, municipalities, or a (part of a) region of a railway contractor as described in section 1.1.3, that each may or may not own a single resource to cover their region completely. If those regions pool their resources a maximal covering location problem arises. Typically, additional coverage, and so additional profit, can be realized and a joint profit allocation issue arises amongst the collaborating regions.

We investigate this allocation aspect by introducing a maximal covering location situation in which regions, i.e., service providers, are represented by single demand points that may or may not keep a single resource. For such a situation, an associated maximal covering location game is introduced. For this cooperative game, we provide several types of sufficient conditions for core non-emptiness. These sufficient conditions are in terms of the number of players, the type of graph, the number of resources, and an underlying integer linear program. For each condition we provide an example showing that when the condition is not satisfied, core non-emptiness is not guaranteed.

Maximal covering location games clearly belong to the class of OR games and in particular to the class of location games. In line with the classification made in chapter 3 of this monograph, maximal covering location games belong to the subclass of games in which one has to position (possibly) multiple facilities. To the best of our knowledge, we are the first who investigate a cooperative location game with the property that not all players are necessarily connected to a facility.

The outline of this chapter is as follows. In section 7.2, we introduce maximal covering location situations and subsequently introduce the associated maximal covering location games. Then, in section 7.3, we present our results regarding these games. Finally, we draw conclusions in section 7.4. Some (extensive) proofs are presented in appendix 7.A.

7.2 Model

In this section, we introduce maximal covering location situations and subsequently define the associated games, called maximal covering location games.

7.2.1 Maximal covering location situations

We consider an environment with a finite set $N \subseteq \mathbb{N}$ of regions, i.e., service providers, and a finite set $L \subseteq \mathbb{N}$ of possible resource locations. The distance between region $i \in N$ and resource location $j \in L$ is denoted by $d_{ij} \in \mathbb{R}_+$. The maximal cover distance (for a resource) is denoted by $\mathscr{D} \in \mathbb{R}_+$. For any region $i \in N$, we introduce $r_i \in \{0,1\}$, where $r_i = 1$ indicates that region $i \in N$ owns a resource and $r_i = 0$ indicates that region $i \in N$ does not own a resource. Every region $i \in N$ with $r_i = 1$ positions its resource at any resource location $j \in L$ and if $d_{ij} \leq \mathscr{D}$, i.e., if the region is covered by the resource, a profit of $p_i \in \mathbb{R}_+$ is obtained. To analyse this setting, we define a *maximal covering location (MCL) situation* as a tuple $(N, L, p, r, d, \mathscr{D})$ with N, L, $p = (p_i)_{i \in N}$, $r = (r_i)_{i \in N}$, $d = (d_{ij})_{i \in N, j \in L}$, and $\mathscr{D}$ as described above. For short, we use θ to refer to such an MCL situation $\theta = (N, L, p, r, d, \mathscr{D})$ and Θ for the set of MCL situations. In addition, for all $\theta \in \Theta$, we define $N_j = \{i \in N | d_{ij} \leq \mathscr{D}\}$ for all $j \in L$, $L_i = \{j \in L | d_{ij} \leq \mathscr{D}\}$ for all $i \in N$, and construct a corresponding (bipartite undirected) graph $\mathscr{G} = (N, L, E)$ with N and L the sets of nodes and $E = \{\{i, j\}_{i \in N, j \in L_i}\}$ the set of edges. An edge between region $i \in N$ and resource location $j \in L$ indicates that the distance between these nodes is no more than $\mathscr{D}$, implying that region i is covered when a resource is positioned at location j.

7.2.2 Maximal covering location games

As some regions, from now on called players, may not own a resource, additional profit can be realized when resources are pooled amongst the players. In line with Church and ReVelle [10], we assume for any coalition $M \subseteq N$ that coverage of any player $i \in M$ by one (or possibly multiple) resource(s) results into a profit of p_i. As a consequence, any

M faces the joint problem of where to position the resources such that the sum of the individual profits (of coalition M) is maximized. For every MCL situation $\theta \in \Theta$ and all $M \subseteq N$ this corresponding MCL problem can be formulated as

$$\begin{aligned} MCL^{\theta}(M): \quad \max \quad & \sum_{i \in M} p_i \cdot y_i \\ \text{s.t.} \quad & y_i - \sum_{j \in L_i} x_j \leq 0 && \forall i \in M \\ & \sum_{j \in L} x_j \leq \sum_{i \in M} r_i \\ & x_j \in \{0,1\} && \forall j \in L \\ & y_i \in \{0,1\} && \forall i \in M. \end{aligned}$$

The first constraint ensures that the profit of player $i \in M$ is obtained only if at least one resource of coalition M is positioned within distance $\mathscr{D}$. The second constraint ensures that the total number of resources used does not exceed the number of available resources of coalition M. The third and fourth constraint enforce integrality of the variables. Note that a solution of the MCL problem indicates at which resource locations a resource is positioned and which players obtain a profit. In particular, if a resource is positioned at resource location $j \in L$, then $x_j = 1$ and otherwise $x_j = 0$. Similarly, if player $i \in M$ obtains profit p_i, then $y_i = 1$, otherwise $y_i = 0$.

In the remainder of this chapter, we denote for every MCL situation $\theta \in \Theta$ and all $M \subseteq N$ the optimal value of $MCL^{\theta}(M)$ by $\mathbf{opt}(MCL^{\theta}(M))$.

Example 7.1. *Let $\theta \in \Theta$ be an MCL situation with $N = \{1,2,3\}, L = \{4,5\}$, $p = (1,2,3)$, $r = (1,0,0)$, $d_{14} = d_{24} = d_{25} = d_{35} = 1$, $d_{15} = d_{34} = 2$, and $\mathscr{D} = 1$. Observe that $L_1 = \{4\}, L_2 = \{4,5\}$, and $L_3 = \{5\}$. The corresponding graph $\mathscr{G} = (N, L, E)$ with $E = \{\{1,4\},\{2,4\},\{2,5\},\{3,5\}\}$ is represented in figure 7.1.*

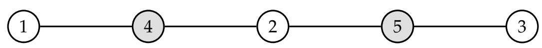

Figure 7.1: Graph corresponding to MCL situation.

For coalition $M = \{1,3\}$, the maximization problem boils down to a trade off between a profit of 1, when the resource is positioned at location 4 and a profit of 3 when the resource is positioned at location 5. Hence, $\mathbf{opt}(MCL^{\theta}(\{1,3\})) = 3$. ◇

We proceed with associating an MCL game to any MCL situation.

Definition 7.1. *For any MCL situation $\theta \in \Theta$, the game (N, v^θ) with*

$$v^\theta(M) = \mathbf{opt}(MCL^\theta(M))$$

for all $M \subseteq N$ with $M \neq \emptyset$ and $v^\theta(\emptyset) = 0$ is called the associated maximal covering location (MCL) game.

Now, we present an example of an MCL game.

Example 7.2. *Consider the situation of example 7.1. The associated MCL game is presented in table 7.1.* $\diamond$

Table 7.1: Corresponding profit per coalition of MCL game.

M	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v^\theta(M)$	0	1	0	0	3	3	0	5

7.3 Properties of maximal covering location games

In this section, we present general properties of MCL games. We start by showing that the value of the union of two disjoint coalitions is larger than or equal to the sum of the values of the disjoint subcoalitions and that the value of every coalition is at least the value of any of its subcoalitions.

Proposition 7.1. *Every MCL game is superadditive and monotonic.*

Proof **:** First we show that every MCL game is superadditive. Let $\theta \in \Theta$ be an MCL situation and (N, v^θ) be the associated MCL game. In addition, let $M, K \subseteq N$ with $M \cap K = \emptyset$. Moreover, let $((x_j^M)_{j\in L}, (y_i^M)_{i\in M})$ be an optimal solution of coalition M and $((x_j^K)_{j\in L}, (y_i^K)_{i\in K})$ be an optimal solution of coalition K. Now, we construct a solution $((x_j)_{j\in L}, (y_i)_{i\in M\cup K})$, given by

$$x_j = \max\{x_j^M, x_j^K\} \quad \forall j \in L, \quad y_i = \begin{cases} y_i^M & \text{if } i \in M \\ y_i^K & \text{if } i \in K. \end{cases}$$

We claim that $((x_j)_{j\in L}, (y_i)_{i\in M\cup K})$ is feasible. Let $i \in M$, then

$$y_i - \sum_{j\in L_i} x_j = y_i^M - \sum_{j\in L_i} \max\{x_j^M, x_j^K\} \leq y_i^M - \sum_{j\in L_i} x_j^M \leq 0,$$

where the equality holds by definition. The first inequality holds as $x_j^M \leq \max\{x_j^M, x_j^K\}$ for all $j \in L_i$, and the second inequality holds by the feasibility of $((x_j^M)_{j\in L}, (y_i^M)_{i\in M})$. In a similar way, this holds for any $i \in K$. Now, observe that

$$\begin{aligned}\sum_{j\in L} x_j - \sum_{i\in M\cup K} r_i &= \sum_{j\in L} \max\{x_j^M, x_j^K\} - \sum_{i\in M\cup K} r_i \\ &\leq \sum_{j\in L} x_j^M + \sum_{j\in L} x_j^K - \sum_{i\in M\cup K} r_i \\ &= \sum_{j\in L} x_j^M - \sum_{i\in K} r_i + \sum_{j\in L} x_j^K - \sum_{i\in M} r_i \leq 0,\end{aligned}$$

where the first and second equality hold by definition. The first inequality holds as $\max\{x_j^M, x_j^K\} \leq x_j^M + x_j^K$ for all $j \in L$ and the second inequality holds by the feasibility of $((x_j^M)_{j\in L}, (y_i^M)_{i\in M})$ and $((x_j^K)_{j\in L}, (y_i^K)_{i\in K})$. Finally, observe that $x_j \in \{0,1\}$ for all $j \in L$, and $y_i \in \{0,1\}$ for all $i \in M \cup K$. Hence, $((x_j)_{j\in L}, (y_i)_{i\in M\cup K})$ is a feasible solution. From this, we conclude that

$$v^\theta(M\cup K) \geq \sum_{i\in M\cup K} p_i \cdot y_i = \sum_{i\in M} p_i \cdot y_i^M + \sum_{i\in K} p_i \cdot y_i^K = v^\theta(M) + v^\theta(K),$$

where the first equality holds as $((x_j)_{j\in L}, (y_i)_{i\in M\cup K})$ is a feasible solution, with associated profit that is at most as much as the profit under an optimal solution of coalition $M \cup K$. The first and second equality hold by definition.

Now, we show that every MCL game is monotonic. Let $\theta \in \Theta$ be an MCL situation and (N, v^θ) be the associated MCL game. In addition, let $M, K \subseteq N$ with $M \subseteq K$. First, observe that $v(K\backslash M) \geq 0$ as $\sum_{i\in K\backslash M} p_i \cdot y_i = 0$ under feasible solution $x_j = 0$ for all $j \in L$ and $y_i = 0$ for all $i \in K\backslash M$. Then, observe that

$$v^\theta(M) \leq v^\theta(M) + v^\theta(K\backslash M) \leq v^\theta(K),$$

where the first equality holds as $v^\theta(K\backslash M) \geq 0$. The second equality holds as MCL games are superadditive. This concludes the proof. □

A natural next step is to investigate whether MCL games have a non-empty core. The following example illustrates that this is not the case in general.

Example 7.3. *Let $\theta \in \Theta$ be an MCL situation with $N = \{1,2,3,4\}, L = \{5,6,7,8\}$, $p = (1,1,1,1)$, $r = (1,1,0,0)$, $d_{15} = d_{17} = d_{25} = d_{26} = d_{36} = d_{37} = d_{48} = 1$, $d_{16} = d_{18} = d_{27} = d_{28} = d_{35} = d_{38} = d_{45} = d_{46} = d_{47} = 2$, and $\mathscr{D} = 1$. Then, $L_1 = \{5,7\}$, $L_2 = \{5,6\}$, $L_3 = \{6,7\}$, and $L_4 = \{8\}$. The corresponding graph $\mathscr{G} = (N, L, E)$ with $E = \{\{1,5\},\{1,7\},\{2,5\},\{2,6\},\{3,6\},\{3,7\},\{4,8\}\}$ is represented in figure 7.2.*

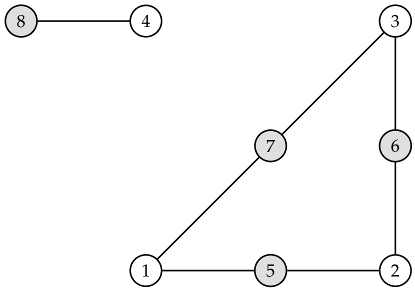

Figure 7.2: Graph corresponding to MCL situation.

Observe that $v^\theta(N) = 3$, $v^\theta(N\backslash\{i\}) = 2$ for $i \in \{1,2\}$, and $v^\theta(N\backslash\{i\}) = 3$ for $i \in \{3,4\}$. Now, suppose the core is non-empty. Let $x \in \mathscr{C}(N, v^\theta)$. As $x_i = \sum_{j\in N} x_j - \sum_{j\in N\backslash\{i\}} x_j = v^\theta(N) - \sum_{j\in N\backslash\{i\}} x_j \leq v^\theta(N) - v^\theta(N\backslash\{i\})$ for all $i \in N$, we obtain $x_1 \leq 1, x_2 \leq 1, x_3 \leq 0$, and $x_4 \leq 0$. This conflicts with efficiency, i.e., $\sum_{i\in N} x_i \leq 2 < 3 = v^\theta(N)$. Hence, we conclude that the core is empty. $\diamond$

The graph corresponding to the MCL situation of example 7.3 contains a cycle. In some other cooperative games related to problems in combinatorial optimization non-emptiness of the core is guaranteed when cycles are not present in the corresponding graph (see, e.g., Deng et al. [11] or Tamir [60]). One may wonder whether this holds for MCL games as well. The following 6-person MCL situation with a corresponding graph without cycles illustrates that this is not the case in general.

Example 7.4. *Let $\theta \in \Theta$ be an MCL situation with $N = \{1,2,3,4,5,6\}$, $L = \{7,8,9,10\}$, $p = (1,1,1,1,1,1)$, $r = (1,0,0,0,0,1)$, $d_{i7} = 1$ for $i \in \{1,2\}$, $d_{i7} = 2$ for $i \in \{3,4,5,6\}$, $d_{i8} = 1$ for $i \in \{2,3,5\}$, $d_{i8} = 2$ for $i \in \{1,4,6\}$, $d_{i9} = 1$ for $i \in \{3,4\}$, $d_{i9} = 2$ for $i \in \{1,2,5,6\}$, $d_{i10} = 1$ for $i \in \{5,6\}$, $d_{i10} = 2$ for $i \in \{1,2,3,4\}$, and $\mathscr{D} = 1$. Then, $L_1 = \{7\}, L_2 = \{7,8\}$, $L_3 = \{8,9\}$, $L_4 = \{9\}$, $L_5 = \{8,10\}$, and $L_6 = \{10\}$. The corresponding graph $\mathscr{G} = (N, L, E)$ with $E = \{\{1,7\},\{2,7\},\{2,8\},\{3,8\},\{3,9\},\{4,9\},\{5,8\},\{5,10\}, \{6,10\}\}$ is represented in figure 7.3.*

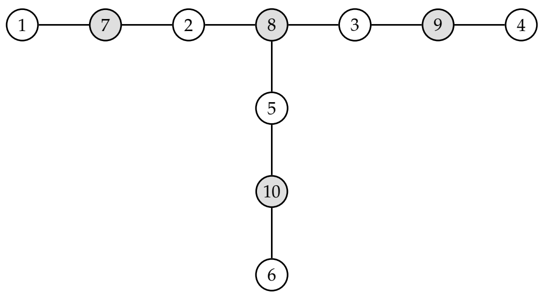

Figure 7.3: Graph corresponding to MCL situation.

Observe that $v^{\theta}(N) = 4$ *by positioning the two resources at any two resource locations. In addition,* $v^{\theta}(N\backslash\{i\}) = 4$ *for* $i \in \{2,3,4,5\}$*, and* $v^{\theta}(N\backslash\{i\}) = 3$ *for* $i \in \{1,6\}$*. Now, suppose the core is non-empty. Let* $x \in \mathscr{C}(N, v^{\theta})$*. As* $x_i \leq v^{\theta}(N) - v^{\theta}(N\backslash\{i\})$ *for all* $i \in N$*, we obtain* $x_1 \leq 1, x_6 \leq 1$ *and* $x_i \leq 0$ *for* $i \in N\backslash\{1,6\}$*. This conflicts with efficiency, i.e.,* $\sum_{i\in N} x_i \leq 2 < 4 = v^{\theta}(N)$*. Hence, the core is empty.* $\diamond$

Example 7.3 demonstrates that the core may be empty from 4-person games on. In addition, example 7.4 shows that under the assumption that the corresponding graph contains no cycles the core may be empty from 6-person games on. With all this in mind we address the following two questions in the remainder of this chapter; (i) is the core non-empty up to 3-person games in general and (ii) is the core non-empty up to 5-person games when the corresponding graph (of the MCL situation) contains no cycles? By addressing those issues, some other interesting results come along.

Proposition 7.2. *For every MCL situation* $\theta \in \Theta$ *and associated MCL game* (N, v^{θ})*, the core is non-empty if k players with* $k \in \{0, 1, |N| - 1, |N|\}$ *own a resource.*

Proof **:** We consider the four possibilities of k separately.

Case 1. $k = 0$.

Let $\theta_0 \in \Theta$ be an MCL situation where no player owns a resource, i.e., $k = 0$, and (N, v^{θ_0}) be the associated MCL game. As no player owns a resource, it holds that $v^{\theta_0}(M) = 0$ for all $M \subseteq N$. So, the game is additive and thus we can conclude that $\mathscr{C}(N, v^{\theta_0}) \neq \emptyset$.

Case 2. $k = 1$.

Let $\theta_1 \in \Theta$ be an MCL situation where exactly one player owns a resource, i.e., $k = 1$, and (N, v^{θ_1}) be the associated MCL game. Assume that player $i^* \in N$ owns a resource. Now, let $x_{i^*} = v^{\theta_1}(N)$ and $x_i = 0$ for all $i \in N\backslash\{i^*\}$. Then, for any $M \subseteq N$ with $i^* \notin M$, it holds that $\sum_{i\in M} x_i = 0 = v^{\theta_1}(M)$. For any $M \subseteq N$ with $i^* \in M$, it holds that $\sum_{i\in M} x_i = x_{i^*} = v^{\theta_1}(N) \geq v^{\theta_1}(M)$. The last inequality holds as MCL games are monotonic. Finally, observe that $\sum_{i\in N} x_i = v^{\theta_1}(N)$ and thus $x \in \mathscr{C}(N, v^{\theta_1})$.

Case 3. $k = |N| - 1$.

Let $\theta_{|N|-1} \in \Theta$ be an MCL situation where everyone owns a resource, except for one player, i.e., $k = |N| - 1$, and $(N, v^{\theta_{|N|-1}})$ be the associated MCL game. Let $i^* \in N$ be the player with $r_{i^*} = 0$. If $|N| \leq 2$, we end up in case 1 or 2. Hence, we can restrict attention to $|N| > 2$. We distinguish between two cases (and two subcases per case).

Case 3.a. For all $i \in N$ it holds that $L_i \neq \emptyset$.

Case 3.a.1. For all $i, i' \in N$ with $i \neq i'$ it holds that $L_i \cap L_{i'} = \emptyset$.

Observe that for every $i \in N$ there exists a $j \in L_i$ such that $j \notin L_{i'}$ for all $i' \in N \backslash \{i\}$. As $|N| - 1$ resources are available only, it follows that $v^{\theta_{|N|-1}}(N) = \sum_{i \in N} p_i - p_{\min(N)}$, where $p_{\min(M)} = \min_{i \in M} p_i$ for all $M \subseteq N$. Let $x_{i^*} = p_{i^*} - p_{\min(N)}$ and $x_i = p_i$ for all $i \in N \backslash \{i^*\}$. Note that $p_{\min(N)} \leq p_{\min(M)}$ for all $M \subseteq N$. Now, for any $M \subseteq N$ with $i^* \notin M$, it holds that $\sum_{i \in M} x_i = \sum_{i \in M} p_i = v^{\theta_{|N|-1}}(M)$, where the last equality holds as every $i \in M$ owns a resource. In addition, for any $M \subseteq N$ with $i^* \in M$, it holds that $\sum_{i \in M} x_i = \sum_{i \in M} p_i - p_{\min(N)} \geq \sum_{i \in M} p_i - p_{\min(M)} = v^{\theta_{|N|-1}}(M)$. Finally, observe that $\sum_{i \in N} x_i = \sum_{i \in N} p_i - p_{\min(N)} = v^{\theta_{|N|-1}}(N)$. We conclude that $x \in \mathscr{C}(N, v^{\theta_{|N|-1}})$.

Case 3.a.2. There exist $i, i' \in N$ with $i \neq i'$ for which $L_i \cap L_{i'} \neq \emptyset$.

Observe that $|L_i| \geq 1$ for all $i \in N$ and that there exists a $j \in L$ and $k, l \in N$ for which $j \in L_k \cap L_l$. Hence, $v^{\theta_{|N|-1}}(N) = \sum_{i \in N} p_i$. Now, let $x_i = p_i$ for all $i \in N$. Observe that for any $M \subseteq N$, it holds that $\sum_{i \in M} x_i = \sum_{i \in M} p_i \geq v^{\theta_{|N|-1}}(M)$ and $\sum_{i \in N} x_i = \sum_{i \in N} p_i = v^{\theta_{|N|-1}}(N)$. We conclude that $x \in \mathscr{C}(N, v^{\theta_{|N|-1}})$.

Case 3.b. There exists an $i \in N$ for which $L_i = \emptyset$.

Case 3.b.1. $L_{i^*} = \emptyset$.

As player i^* (also) has no resource, it can add nothing to any coalition. Hence, we can restrict ourselves to a subgame without player i^*, i.e., with player set $N \backslash \{i^*\}$ where each player owns a resource. Then, we end up in case 4.

Case 3.b.2. $L_{i^*} \neq \emptyset$.

Let $T = \{t \in N | L_t = \emptyset\}$. As $L_{i^*} \neq \emptyset$, it holds that $1 \leq |T| \leq |N| - 1$. Then, in the grand coalition, for any $b \in B = N \backslash T$ there is a resource available and so $v^{\theta_{|N|-1}}(N) = \sum_{i \in B} p_i$. Let $x_i = p_i$ for all $i \in B$. As for each player $t \in T$ it holds that they can never add profit p_t to any coalition, it holds for any $M \subseteq N$ that $\sum_{i \in M} x_i \geq \sum_{i \in M \cap B} x_i = v^{\theta_{|N|-1}}(M)$. Finally, $\sum_{i \in N} x_i = \sum_{i \in N \cap B} p_i = v^{\theta_{|N|-1}}(N)$. Hence, $x \in \mathscr{C}(N, v^{\theta_{|N|-1}})$.

Case 4. $k = |N|$.

Let $\theta_{|N|} \in \Theta$ be an MCL situation where every player owns a resource, i.e., $k = |N|$, and $(N, v^{\theta_{|N|}})$ be the associated MCL game. Let $H = \{h \in N | L_h \neq \emptyset\}$. As a consequence, $v^{\theta_{|N|}}(M) = \sum_{i \in M \cap H} p_i$ for all $M \subseteq N$. Now, let $x_i = p_i$ for each $i \in H$ and $x_i = 0$ for every $i \in N \backslash H$. Then, observe that $\sum_{i \in M} x_i = \sum_{i \in M \cap H} p_i = v^{\theta_{|N|}}(M)$ for each $M \subseteq N$. So, $x \in \mathscr{C}(N, v^{\theta_{|N|}})$, which concludes this proof. □

Remark 7.1. If the condition of proposition 7.2 is not satisfied, core non-emptiness is not guaranteed. In example 7.3 with four players and two resources the core is empty.

As a direct consequence of proposition 7.2, we can conclude that the core of every k-person MCL game with $k \in \{1,2,3\}$ is non-empty.

Theorem 7.1. *Every k-person MCL game with $k \in \{1,2,3\}$ has a non-empty core.*

We continue by addressing our second question of interest. For this, we introduce some definitions and present a proposition and two lemmas which are of interest by themselves as well. For every MCL situation $\theta \in \Theta$ and all $M \subseteq N$ we define $RMCL^{\theta}(M)$ as a relaxation of $MCL^{\theta}(M)$ where $x_j \geq 0$ for all $j \in L$ and $0 \leq y_i \leq 1$ for all $i \in M$. Note that $x_j \leq 1$ for all $j \in L$ is not taken into consideration. Based on this relaxation, we formulate a sufficient condition for non-emptiness of the core.

Proposition 7.3. *For every MCL situation $\theta \in \Theta$ with $\mathbf{opt}(RMCL^{\theta}(N)) = \mathbf{opt}(MCL^{\theta}(N))$, it holds that the core of the associated MCL game (N, v^{θ}) is non-empty.*

Proof **:** Let $\theta \in \Theta$ and assume that $\mathbf{opt}(MCL^{\theta}(N)) = \mathbf{opt}(RMCL^{\theta}(N))$. Let $M \subseteq N$. Note that $RMCL^{\theta}(M)$ is feasible (as $x_j = 0$ for all $j \in L$ and $y_i = 0$ for all $i \in M$ is a feasible solution) and bounded (by $\sum_{i \in M} p_i + 1$). Then, for the dual of $RMCL^{\theta}(M)$, which is given by

$$\begin{aligned} DRMCL^{\theta}(M): \ \min \quad & b \cdot \sum_{i \in M} r_i + \sum_{i \in M} c_i \\ \text{s.t.} \quad & a_i + c_i \geq p_i \quad \forall i \in M \\ & -\sum_{i \in N_j^M} a_i + b \geq 0 \quad \forall j \in L \\ & a_i, c_i \geq 0 \quad \forall i \in M \\ & b \geq 0, \end{aligned}$$

where $N_j^M = \{i \in M | d_{ij} \leq \mathscr{D}\}$, it holds based on the duality theorem of linear programming (Schrijver [53, p.90]) that

$$\mathbf{opt}(DRMCL^{\theta}(M)) = \mathbf{opt}(RMCL^{\theta}(M)). \tag{7.1}$$

Based on our assumption, it holds for $M = N$ that

$$\mathbf{opt}(MCL^{\theta}(N)) = \mathbf{opt}(DRMCL^{\theta}(N)). \tag{7.2}$$

Let $M \subseteq N$. We claim that the restriction of any feasible solution $((a_i)_{i \in N}, b, (c_i)_{i \in N})$ of $DRMCL^{\theta}(N)$ to M defined by $((a'_i)_{i \in M}, b', (c'_i)_{i \in M})$ with

$$\begin{aligned} & a'_i = a_i \text{ for all } i \in M \\ & b' = b \\ & c'_i = c_i \text{ for all } i \in M \end{aligned}$$

is a feasible solution of $DRMCL^\theta(M)$. Let $i \in M$. Then $p_i \leq a_i + c_i = a'_i + c'_i$. Again, let $i \in M$. Then $0 \leq -\sum_{i \in N_j^N} a_i + b \leq -\sum_{i \in N_j^M} a_i + b = -\sum_{i \in N_j^M} a'_i + b'$. Finally, as $a'_i = a_i \geq 0$ for all $i \in M$, $b' = b \geq 0$, and $c'_i = c_i \geq 0$ for all $i \in M$, we conclude that $((a'_i)_{i \in M}, b', (c'_i)_{i \in M})$ is a feasible solution of $DRMCL^\theta(M)$.

Now, let $((a^*_i)_{i \in N}, b^*, (c^*_i)_{i \in N})$ be an optimal solution of dual problem $DRMCL^\theta(N)$. Note that this solution exists by the duality theorem of linear programming (again). We construct an allocation $z = (z_i)_{i \in N}$ as follows

$$z_i = b^* \cdot r_i + c^*_i \text{ for all } i \in N.$$

Now, observe that

$$\begin{aligned} \sum_{i \in N} z_i = \sum_{i \in N} [b^* \cdot r_i + c^*_i] = b^* \sum_{i \in N} r_i + \sum_{i \in N} c^*_i &= \mathbf{opt}(DRMCL^\theta(N)) \\ &= \mathbf{opt}(MCL^\theta(N)) \\ &= v^\theta(N). \end{aligned}$$

where the fourth equality holds by equation (7.2). In addition, let $M \subseteq N$ and $((a'^*_i)_{i \in M}, b'^*, (c'^*_i)_{i \in M})$ be the restriction of $((a^*_i)_{i \in N}, b^*, (c^*_i)_{i \in N})$ to coalition M. Then, observe that

$$\begin{aligned} \sum_{i \in M} z_i = \sum_{i \in M} [b^* \cdot r_i + c^*_i] = b'^* \sum_{i \in M} r_i + \sum_{i \in M} c'^*_i \geq \mathbf{opt}(DRMCL^\theta(M)) &= \mathbf{opt}(RMCL^\theta(M)) \\ &\geq \mathbf{opt}(MCL^\theta(M)) \\ &= v^\theta(M), \end{aligned}$$

where the first inequality holds as the value of feasible solution $((a'^*_i)_{i \in M}, b'^*, (c'^*_i)_{i \in M})$ is more than or equal to the optimal value of $DRMCL^\theta(M)$. The third equality holds by equation (7.1). The second inequality holds as we restrict to integer solutions only. As there exists a $z = (z_i)_{i \in N}$ for which $\sum_{i \in N} z_i = v^\theta(N)$ and $\sum_{i \in M} z_i \geq v^\theta(M)$ for all $M \subseteq N$, we conclude that the core of (N, v^θ) is non-empty. □

Remark 7.2. Introducing a relaxation of an integer linear programming problem in order to find sufficient conditions for core non-emptiness (for different classes of situations) has been used by others as well (see, e.g., Deng et al. [11]). Moreover, it turns out that if the condition of proposition 7.3 is not satisfied, core non-emptiness cannot be guaranteed. For instance, in example 7.4 it holds that $\mathbf{opt}(MCL^\theta(N)) = 4$, $\mathbf{opt}(RMCL^\theta(N)) = 4.5$ (resulting from $x_j = \frac{1}{2}$ for all $j \in L$), and $\mathscr{C}(N, v^\theta) = \emptyset$. Finally, it turns out that the condition of proposition 7.3 is not necessary. For instance, in example 7.4 with $r_2 = r_3 = 1$ and $r_i = 0$ for all $i \in N \backslash \{2, 3\}$, it holds that $\mathscr{C}(N, v^\theta) = \{(0, 2, 2, 0, 0, 0, 0)\} \neq \emptyset$.

For all $\theta = (N, L, p, r, d, \mathscr{D}) \in \Theta$ and all $j \in L$, we define $\theta^{-j} = (N, L^{-j}, p, r, d^{-j}, \mathscr{D})$, where $L^{-j} = L \backslash \{j\}$ and $d^{-j} = (d_{ij})_{i \in N, j \in L \backslash \{j\}}$. In addition, for every $\theta \in \Theta$, resource location $j \in L$ is called *obsolete* if there exists a $k \in L \backslash \{j\}$ for which $N_j \subseteq N_k$.

Lemma 7.1. *For every MCL situation $\theta \in \Theta$ with obsolete resource location $j \in L$, it holds that*

$$v^{\theta}(M) = v^{\theta^{-j}}(M) \text{ for all } M \subseteq N.$$

Note that the result of lemma 7.1 follows directly, as (for every $M \subseteq N$) there exists a $k \in L \backslash \{j\}$ with $N_j \subseteq N_k$ which makes j superfluous. Now, we are able to affirmatively answer our second question of interest. We provide a sketch of the proof here. The complete (and quite extensive) proof is relegated to appendix 7.A.

As for every MCL situation with $0, 1, |N| - 1$, and $|N|$ resources the core of the associated MCL game is non-empty (proposition 7.2), it suffices to focus on the following MCL situations only; (i) $|N| = 4$ with two resources, (ii) $|N| = 5$ with two resources, and (iii) $|N| = 5$ with three resources. For all cases, it holds that if the number of resource locations is less than or equal to the number of resources, the core is non-empty. This holds as allocating p_i to all players $i \in N$ with $L_i \neq \emptyset$, i.e., for which there exists a resource location within distance $\mathscr{D}$, results into a core element. In addition, for every MCL situation with an obsolete resource location there exists another MCL situation without this obsolete resource location such that the coalitional values of the MCL games resulting from those MCL situations coincide (lemma 7.1). Hence, if the core is non-empty for the game corresponding to the MCL situation without the obsolete resource location, it is the case for the other game as well. So, it suffices to consider the three cases with the additional constraint that (i) $|L| \geq 3$ and (ii) the corresponding graphs are free of obsolete resource locations and free of cycles. All possible graphs for $|N| = 4$ with $|L| \geq 3$, obtained by conditioning on the degrees of the nodes, are presented below.

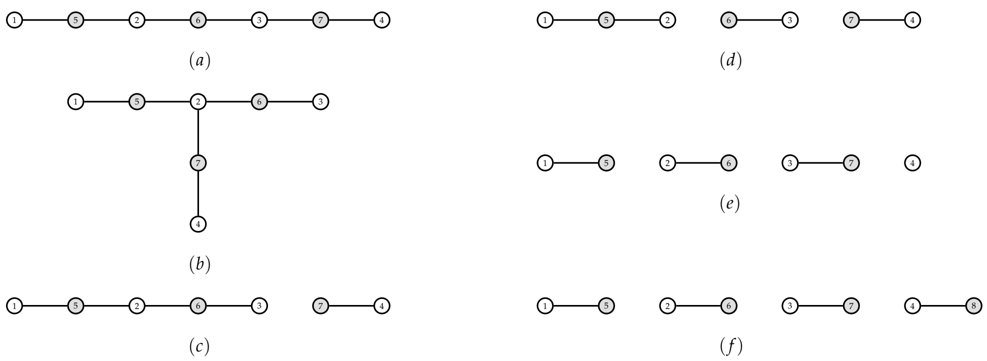

Figure 7.4: Possible corresponding graphs with $|N| = 4$ and $|L| \geq 3$.

Note that duplicates of the graphs of figure 7.4 due to relabeling of the player set and resource set are removed. Similarly, one can present the possible corresponding graphs (with no obsolete resource locations and no cycles) for $|N| = 5$ with $|L| \geq 3$. These graphs are presented in figure 7.5 below and are also obtained by conditioning on the degrees of the nodes. Note that duplicates of the graphs of figure 7.5 due to relabeling of the player set and resource set are removed (again).

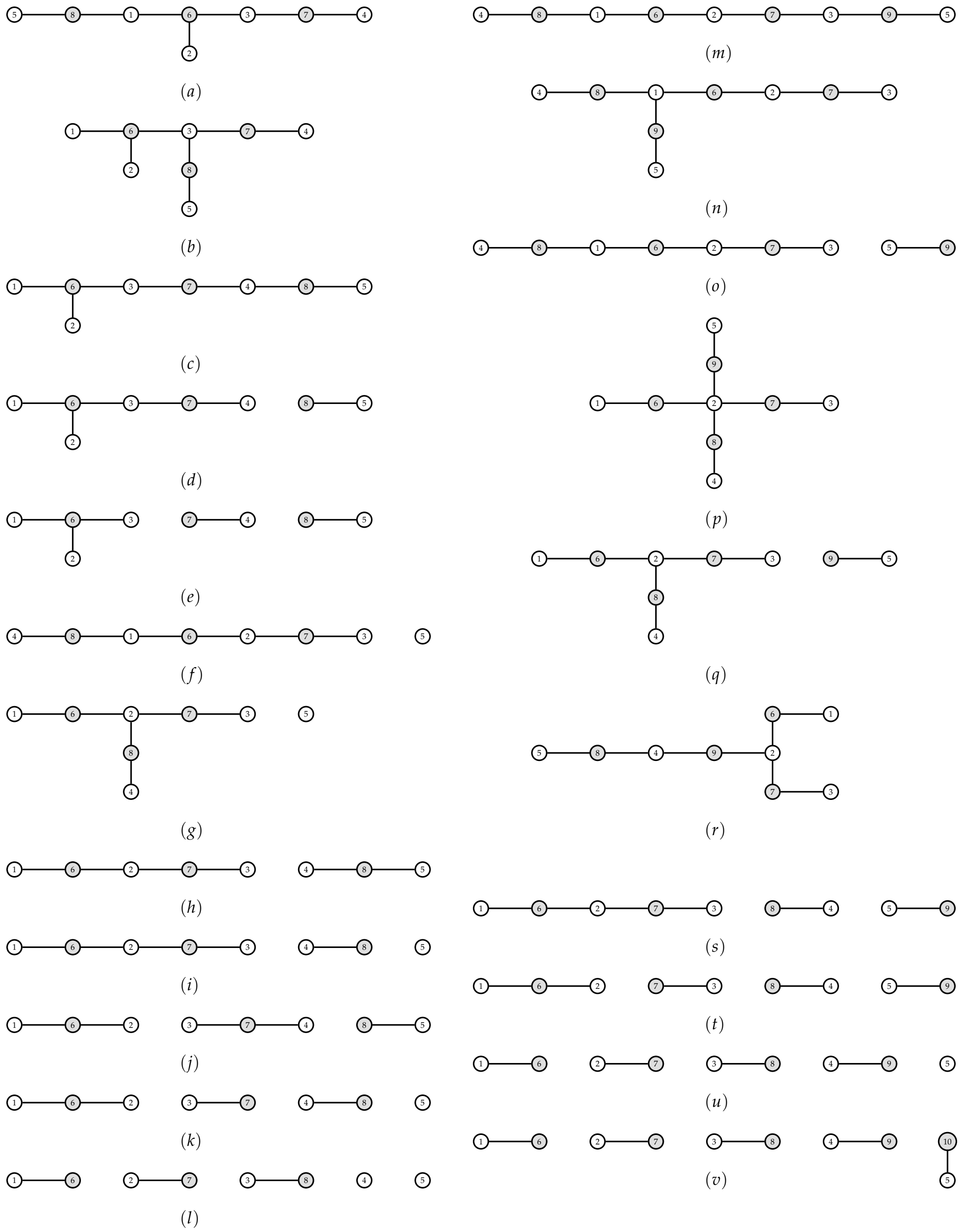

Figure 7.5: Possible corresponding graphs with $|N| = 5$ and $|L| \geq 3$.

For all remaining MCL situations, which all have a corresponding graph as presented in figure 7.4 or figure 7.5 (possibly after some relabeling of the player set and the resource location set), we can (re)formulate the relaxation of the MCL problem in standard LP-form, i.e., in matrix form $Ax \leq b$ with $x \geq 0$.

Example 7.5. *Let $\theta \in \Theta$ be an MCL situation with corresponding graph (e) of figure 7.4. Then, for the relaxation of the MCL problem, we obtain the following A and b.*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Note that vector x stands for $(y_1, y_2, y_3, y_4, x_5, x_6, x_7)$ where $x_5, x_6,$ and x_7 represent resource locations 5, 6, and 7, respectively. ◇

For all these MCL situations, we can show that matrix A (in standard LP-form) is totally unimodular. One can check, e.g., via standard software packages, that every submatrix of A (of all 28 graphs of figure 7.4 and figure 7.5) has determinant 1, 0 or -1. Moreover, vector b (of the standard LP-form) has all integer entries, $p_i \in \mathbb{R}_+$ for all $i \in N$, and the optimal value of the MCL problem is finite. So, we can conclude that the relaxation of the MCL problem has an integer optimal solution (theorem 2.3) that can straightforwardly be shown to be a binary optimal solution. Hence, the optimal value of the MCL problem coincides with the optimal value of the relaxation of the MCL problem and thus, the core is non-empty (proposition 7.3). This leads to our final theorem.

Theorem 7.2. *Every k-person MCL game with $k \in \{1, 2, 3, 4, 5\}$ and no cycles in the corresponding graph has a non-empty core.*

Remark 7.3. The more general case in which each player may own several resources to cover multiple demand points can be deducted from our situation easily by merging several players (with and without a resource) into one (super)player. All sufficient conditions can be translated to this situation easily, and will be in terms of settings with players who can be seen as forming such a superplayer.

7.4 Conclusions

In this chapter, we introduced and analysed maximal covering location games. As the core of a maximal covering location game may be empty, several sufficient conditions for core non-emptiness are presented. The first one is in terms of the number of players, the second one is in terms of the type of graph, the third one is in terms of the number of resources, and the last one is in terms of the underlying integer linear program. For each condition we provided an example showing that when the condition is not satisfied, core non-emptiness is not guaranteed.

For further research, it could be of interest to focus on other sufficient conditions for core non-emptiness or a sufficient and necessary condition for core non-emptiness. In addition, one could extend the underlying maximal covering location situations by including that (i) resources can cover a fixed number of regions only and (ii) regions can be covered partially by resources based on their distance from each other.

7.A Appendix

This appendix presents the complete proof of theorem 7.2 and two lemmas that turn out to be useful in proving this theorem.

Lemma 7.2. *Let $\theta \in \Theta$ with $|N| = 4$ and $|L| \geq 3$. If the corresponding graph $\mathscr{G} = (N, L, E)$ has no cycles nor obsolete resource locations, then graph $\mathscr{G} = (N, L, E)$ is isomorphic to one of the graphs represented in figure 7.4.*

Proof **:** If there exists a $j \in L$ for which $N_j = \emptyset$, then this resource location is obsolete. So, from now on, suppose $1 \leq |N_j| \leq 4$ for all $j \in L$. We distinguish between cases $|L| = 3, |L| = 4$, and $|L| \geq 5$. In each case, we force a contradiction (by showing that the graph has an obsolete resource location or a cycle) or present the possible graph(s).

Case 1. $|L| = 3$.

Let $L = \{5, 6, 7\}$ with $|N_5| \geq |N_6| \geq |N_7|$ without loss of generality. Now, we condition on the cardinality of N_5, N_6, and N_7.

Case 1.a. $|N_5| = 4$.

Note that $6 \in L$ is obsolete if $|N_6| = 1$. If $|N_6| \geq 2$, a cycle is formed.

Case 1.b. $|N_5| = 3, |N_6| = 3$.

We have $|N_5 \cap N_6| \geq 2$. A cycle is formed.

Case 1.c. $|N_5| = 3, |N_6| = 2$.

In order to avoid cycles, we have $|N_5 \cap N_6| = 1$. Let $N_5 = \{1,2,3\}$ and $N_6 = \{2,4\}$ without loss of generality. If $|N_7| = 1$, then $7 \in L$ is obsolete. If $|N_7| = 2$, a cycle is formed.

Case 1.d. $|N_5| = 3, |N_6| = 1$.

In order to avoid an obsolete resource location, we have $N_5 \cap N_6 = \emptyset$. Let $N_5 = \{1,2,3\}$ and $N_6 = \{4\}$ without loss of generality. If $|N_7| = 1$, then $7 \in L$ is obsolete.

Case 1.e. $|N_5| = 2, |N_6| = 2$.

In order to avoid cycles, we have $|N_5 \cap N_6| \leq 1$. Let (i) $N_5 = \{1,2\}$ and $N_6 = \{2,3\}$ or (ii) $N_5 = \{1,2\}$ and $N_6 = \{3,4\}$ without loss of generality.

(i) If $N_7 \in \{\{1,2\},\{1,3\},\{2,3\}\}$, a cycle is formed. For $N_7 \in \{\{1,4\},\{3,4\}\}$ we obtain a graph isomorphic to graph (a) (of figure 7.4). For $N_7 = \{2,4\}$ we obtain a graph isomorphic to graph (b). If $N_7 \in \{\{1\},\{2\},\{3\}\}$, we obtain an obsolete resource location. For $N_7 = \{4\}$ we obtain a graph isomorphic to graph (c).

(ii) If $N_7 \in \{\{1,2\},\{3,4\}\}$, a cycle is formed. For $N_7 \in \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$ we obtain a graph isomorphic to graph (a). If $|N_7| = 1$, then resource location $7 \in L$ is obsolete.

Case 1.f. $|N_5| = 2, |N_6| = 1, |N_7| = 1$.

In order to avoid obsolete resource locations, we have $N_5 \cap N_6 = N_5 \cap N_7 = N_6 \cap N_7 = \emptyset$. Let $N_5 = \{1,2\}, N_6 = \{3\}$, and $N_7 = \{4\}$ without loss of generality. Then, we obtain a graph isomorphic to graph (d).

Case 1.g. $|N_5| = 1, |N_6| = 1, |N_7| = 1$.

In order to avoid obsolete resource locations, we have $N_5 \cap N_6 = N_5 \cap N_7 = N_6 \cap N_7 = \emptyset$. Let $N_5 = \{1\}, N_6 = \{2\}$, and $N_7 = \{3\}$ without loss of generality. Then, we obtain a graph isomorphic to graph (e).

Case 2. $|L| = 4$.

Let $L = \{5,6,7,8\}$ with $|N_5| \geq |N_6| \geq |N_7| \geq |N_8|$ without loss of generality. We condition on the cardinality of N_5, N_6, N_7 and N_8. We make use of the results of case 1. For some cases this implies that we can conclude immediately that no graph exists (with no cycles nor obsolete resource locations). For the other cases, we can simply start with the obtained (sub)graphs and condition on N_8.

Case 2.a. $|N_5| \geq 3$.

A reasoning identical to cases 1.a, 1.b, 1.c, and 1.d. can be used.

Case 2.b. $|N_5| = 2, |N_6| = 2.$

By using the (same) reasoning of case 1.e we end up with subgraphs $(a), (b)$ and (c).

(a) $N_5 = \{1,2\}, N_6 = \{2,3\}$, and $N_7 = \{3,4\}$. If $|N_8| = 2$, a cycle is formed. If $|N_8| = 1$, resource location $8 \in L$ is obsolete.

(b) $N_5 = \{1,2\}, N_6 = \{2,3\}$, and $N_7 = \{2,4\}$. If $|N_8| = 2$, a cycle is formed. If $|N_8| = 1$, resource location $8 \in L$ is obsolete.

(c) $N_5 = \{1,2\}, N_6 = \{2,3\}$, and $N_7 = \{4\}$. If $|N_8| = 1$, resource location $8 \in L$ is obsolete.

Case 2.c. $|N_5| = 2, |N_6| = 1, |N_7| = 1.$

By using the (same) reasoning of case 1.f we end up with subgraph (d).

(d) $N_5 = \{1,2\}, N_6 = \{3\}$, and $N_7 = \{4\}$. If $|N_8| = 1$, we conclude that resource location $8 \in L$ is obsolete.

Case 2.d. $|N_5| = 1, |N_6| = 1, |N_7| = 1.$

By using the (same) reasoning of case 1.g we end up with subgraphs (e).

(e) $N_5 = \{1\}, N_6 = \{2\}$, and $N_7 = \{3\}$. If $N_8 \in \{\{1\},\{2\},\{3\}\}$, we obtain an obsolete resource location. For $N_8 = \{4\}$, we obtain a graph isomorphic to graph (f).

Case 3. $|L| \geq 5.$

Let $L = \{5, 6, \ldots, k\}$ with $k \geq 9$ where $|N_5| \geq |N_6| \geq \ldots \geq |N_k|$ without loss of generality. In line with our discussion at the start of case 2, we can simply start with the obtained (sub)graphs and condition on N_9.

Case 3.a. $|N_5| \geq 2.$

A reasoning identical to cases 2.a, 2.b, and 2.c. can be used.

Case 3.b. $|N_5| = 1, |N_6| = 1, |N_7| = 1.$

By using the (same) reasoning of case 2.d we end up with subgraphs (e).

(f) $N_5 = \{1\}, N_6 = \{2\}, N_7 = \{3\}$, and $N_8 = \{4\}$. If $|N_9| = 1$, then resource location $9 \in L$ is obsolete.

We conclude that the graph (with no cycles nor obsolete resource locations) is isomorphic to one of the graphs of figure 7.4. □

Lemma 7.3. *Let $\theta \in \Theta$ with $|N| = 5$ and $|L| \geq 3$. If the corresponding graph $\mathscr{G} = (N, L, E)$ has no cycles nor obsolete resource locations, then graph $\mathscr{G} = (N, L, E)$ is isomorphic to one of the graphs of figure 7.5.*

Proof **:** If there exists a $j \in L$ for which $N_j = \emptyset$, then this resource location is obsolete. So, from now on, suppose $1 \leq |N_j| \leq 5$ for all $j \in L$. We distinguish between cases $|L| = 3, |L| = 4$, $|L| = 5$ and $|L| \geq 6$. In each case, we force a contradiction (by showing that the graph has an obsolete resource location or a cycle) or present the possible graph(s).

Case 1. $|L| = 3$.

Let $L = \{6, 7, 8\}$ and assume without loss of generality that $|N_6| \geq |N_7| \geq |N_8|$. We condition on the cardinality of N_6, N_7 and N_8.

Case 1.a. $|N_6| = 5$.

Note that $7 \in L$ is obsolete if $|N_7| = 1$. If $|N_7| \geq 2$, a cycle is formed.

Case 1.b. $|N_6| = 4, |N_7| \geq 3$.

We have $|N_6 \cap N_7| \geq 2$. Hence, a cycle is formed.

Case 1.c. $|N_6| = 4, |N_7| = 2$.

In order to avoid cycles, we have $|N_6 \cap N_7| = 1$. Let $N_6 = \{1, 2, 3, 4\}$ and $N_7 = \{4, 5\}$ without loss of generality. If $|N_8| = 1$, resource location $8 \in L$ is obsolete. If $|N_8| = 2$, a cycle is formed.

Case 1.d. $|N_6| = 4, |N_7| = 1$.

In order to avoid an obsolete resource location, we have $N_6 \cap N_7 = \emptyset$. Let $N_6 = \{1, 2, 3, 4\}$ and $N_7 = \{5\}$ without loss of generality. If $|N_8| = 1$, then $8 \in L$ is obsolete.

Case 1.e. $|N_6| = 3, |N_7| = 3$.

In order to avoid cycles, we have $|N_6 \cap N_7| = 1$. Let $N_6 = \{1, 2, 3\}$ and $N_7 = \{3, 4, 5\}$ without loss of generality. If $2 \leq |N_8| \leq 3$, a cycle is formed. If $|N_8| = 1$, then $8 \in L$ is obsolete.

Case 1.f. $|N_6| = 3, |N_7| = 2, |N_8| = 2$.

In order to avoid an obsolete resource location, we have that $|N_6 \cap N_7| \leq 1$. Let (i) $N_6 = \{1, 2, 3\}$ and $N_7 = \{3, 4\}$ or (ii) $N_6 = \{1, 2, 3\}$ and $N_7 = \{4, 5\}$ without loss of generality.

(i) If $N_8 \in \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$, a cycle is formed. For $N_8 \in \{\{1, 5\}, \{2, 5\}\}$ we obtain a graph isomorphic to (a) (of figure 7.5). For $N_8 = \{3, 5\}$ we obtain a graph isomorphic to (b). For $N_8 = \{4, 5\}$ we obtain a graph isomorphic to graph (c).

(ii) If $N_8 \in \{\{1,2\},\{1,3\},\{2,3\},\{4,5\}\}$, a cycle is formed. For $N_8 \in \{\{1,4\},\{1,5\},\{2,4\},\{2,5\},\{3,4\},\{3,5\}\}$ we obtain a graph isomorphic to graph (c).

Case 1.g. $|N_6| = 3, |N_7| = 2, |N_8| = 1$.

In order to avoid an obsolete resource location, we have $|N_6 \cap N_7| \leq 1$. Now, let (i) $N_6 = \{1,2,3\}$ and $N_7 = \{3,4\}$ or (ii) $N_6 = \{1,2,3\}$ and $N_7 = \{4,5\}$ without loss of generality.

(i) If $N_8 \in \{\{1\},\{2\},\{3\},\{4\}\}$, the resource location is obsolete. For $N_8 = \{5\}$ we obtain a graph isomorphic to graph (d).

(ii) If $|N_8| = 1$, $8 \in L$ is obsolete.

Case 1.h. $|N_6| = 3, |N_7| = 1, |N_8| = 1$.

In order to avoid an obsolete resource location, we have $N_6 \cap N_7 = N_6 \cap N_8 = N_7 \cap N_8 = \emptyset$. Let $N_6 = \{1,2,3\}$, $N_7 = \{4\}$, and $N_8 = \{5\}$ without loss of generality. Then, we obtain a graph isomorphic to (e).

Case 1.i. $|N_6| = 2, |N_7| = 2, |N_8| = 2$.

In order to avoid an obsolete resource location, we have $|N_6 \cap N_7| \leq 1$. Let (i) $N_6 = \{1,2\}$ and $N_7 = \{2,3\}$ or (ii) $N_6 = \{1,2\}$ and $N_7 = \{3,4\}$ without loss of generality.

(i) If $N_8 \in \{\{1,2\},\{1,3\},\{2,3\}\}$ a cycle is formed. For $N_8 \in \{\{1,4\},\{1,5\},\{3,4\},\{3,5\}\}$ we obtain a graph isomorphic to (f). For $N_8 \in \{\{2,4\},\{2,5\}\}$ we obtain a graph isomorphic to (g). For $N_8 = \{4,5\}$ we obtain a graph isomorphic to (h).

(ii) If $N_8 \in \{\{1,2\},\{3,4\}\}$, a cycle is formed. For $N_8 \in \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$ we obtain a graph isomorphic to (f). For $N_8 \in \{\{1,5\},\{2,5\},\{3,5\},\{4,5\}\}$ we obtain a graph isomorphic to (h).

Case 1.j. $|N_6| = 2, |N_7| = 2, |N_8| = 1$.

In order to avoid an obsolete resource location, we have $|N_6 \cap N_7| \leq 1$. Let (i) $N_6 = \{1,2\}$ and $N_7 = \{2,3\}$ or (ii) $N_6 = \{1,2\}$ and $N_7 = \{3,4\}$ without loss of generality.

(i) If $N_8 \in \{\{1\},\{2\},\{3\}\}$ we obtain an obsolete resource location. For $N_6 \in \{\{4\},\{5\}\}$ we obtain a graph isomorphic to (i).

(ii) If $N_8 \in \{\{1\},\{2\},\{3\},\{4\}\}$ we obtain an obsolete resource location. For $N_8 = \{5\}$ we obtain a graph isomorphic to (j).

Case 1.k. $|N_6| = 2, |N_7| = 1, |N_8| = 1$.

In order to avoid an obsolete resource location, we have $N_6 \cap N_7 = N_6 \cap N_8 = N_7 \cap N_8 = \emptyset$. Let $N_6 = \{1,2\}$, $N_7 = \{3\}$, and $N_8 = \{4\}$ without loss of generality. Then, we obtain a graph isomorphic to (k).

Case 1.l. $|N_6| = 1, |N_7| = 1, |N_8| = 1.$

In order to avoid an obsolete resource location, we have $N_6 \cap N_7 = N_6 \cap N_8 = N_7 \cap N_8 = \emptyset$. Let $N_6 = \{1\}$, $N_7 = \{2\}$, and $N_8 = \{3\}$ without loss of generality. Then, we obtain a graph isomorphic to (l).

Case 2. $|L| = 4.$

Let $L = \{6,7,8,9\}$ with $|N_6| \geq |N_7| \geq |N_8| \geq |N_9|$ without loss of generality. In line with our discussion at the start of case 2 of the proof of lemma 7.2, we can simply start with the obtained (sub)graphs and condition on N_9.

Case 2.a. $|N_6| \geq 4.$

A reasoning identical to cases 1.a, 1.b, 1.c, and 1.d. can be used.

Case 2.b. $|N_6| = 3, |N_7| = 3.$

A reasoning identical to case 1.e. can be used.

Case 2.c. $|N_6| = 3, |N_7| = 2, |N_8| = 2.$

By using the (same) reasoning of case 1.f we end up with subgraphs (a), (b) and (c).

(a) $N_6 = \{1,2,3\}, N_7 = \{3,4\}, N_8 = \{1,5\}$. If $|N_9| = 2$ a cycle is formed. If $|N_9| = 1$, then $9 \in L$ is obsolete.

(b) $N_6 = \{1,2,3\}, N_7 = \{3,4\}, N_8 = \{3,5\}$. If $|N_9| = 2$ a cycle is formed. If $|N_9| = 1$, then $9 \in L$ is obsolete.

(c) $N_6 = \{1,2,3\}, N_7 = \{3,4\}, N_8 = \{4,5\}$. If $|N_9| = 2$ a cycle is formed. If $|N_9| = 1$, then $9 \in L$ is obsolete.

Case 2.d. $|N_6| = 3, |N_7| = 2, |N_8| = 1.$

By using the (same) reasoning of case 1.g we end up with subgraph (d).

(d) $N_6 = \{1,2,3\}, N_7 = \{3,4\}, N_8 = \{5\}$. If $|N_9| = 1$, then $9 \in L$ is obsolete.

Case 2.e. $|N_6| = 3, |N_7| = 1, |N_8| = 1.$

By using the (same) reasoning of case 1.h we end up with subgraph (e).

(e) $N_6 = \{1,2,3\}, N_7 = \{4\}, N_8 = \{5\}$. If $|N_9| = 1$, then $9 \in L$ is obsolete.

Case 2.f. $|N_6| = 2, |N_7| = 2, |N_8| = 2.$

By using the (same) reasoning of case 1.i we end up with subgraph (f), (g) and (h).

(f) $N_6 = \{1,2\}, N_7 = \{2,3\}, N_8 = \{1,4\}$. If $N_9 \in \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$ a cycle is formed. For $N_9 \in \{\{3,5\},\{4,5\}\}$ we obtain a graph isomorphic to

(m). For $N_9 \in \{\{1,5\},\{2,5\}\}$ we obtain a graph isomorphic to (n). If $N_9 \in \{\{1\},\{2\},\{3\},\{4\}\}$, resource location $9 \in L$ is obsolete. For $N_9 = \{5\}$ we obtain a graph isomorphic to (o).

(g) $N_6 = \{1,2\}, N_7 = \{2,3\}, N_8 = \{2,4\}$. If $N_9 \in \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$, a cycle is formed. For $N_9 \in \{\{1,5\},\{3,5\},\{4,5\}\}$ we obtain a graph isomorphic to (n). For $N_9 = \{2,5\}$ we obtain a graph isomorphic to (p). If $N_9 \in \{\{1\},\{2\},\{3\},\{4\}\}$, resource location $9 \in L$ is obsolete. For $N_9 = \{5\}$ we obtain a graph isomorphic to (q).

(h) $N_6 = \{1,2\}, N_7 = \{2,3\}, N_8 = \{4,5\}$. If $N_9 \in \{\{1,2\},\{1,3\},\{2,3\}, \{4,5\}\}$, a cycle is formed. If $N_9 \in \{\{1,4\},\{1,5\},\{3,4\},\{3,5\}\}$, we obtain a graph isomorphic to (m). If $N_9 \in \{\{2,4\},\{2,5\}\}$, we obtain a graph isomorphic to (r). If $|N_9| = 1$, then resource location $9 \in L$ is obsolete.

Case 2.g. $|N_6| = 2, |N_7| = 2, |N_8| = 1$.

By using the (same) reasoning of case 1.j we end up with subgraph (i) and (j).

(i) $N_6 = \{1,2\}, N_7 = \{2,3\}, N_8 = \{4\}$. If $N_9 \in \{\{1\},\{2\},\{3\},\{4\}\}$, then $9 \in L$ is an obsolete resource location. For $N_9 = \{5\}$ we obtain a graph isomorphic to (s).

(j) $N_6 = \{1,2\}, N_7 = \{3,4\}, N_8 = \{5\}$. If $|N_9| = 1$, then $9 \in L$ is obsolete.

Case 2.h. $|N_6| = 2, |N_7| = 1, |N_8| = 1$.

By using the (same) reasoning of case 1.k we end up with subgraph (k).

(k) $N_6 = \{1,2\}, N_7 = \{3\}, N_8 = \{4\}$. If $N_9 \in \{\{1\},\{2\},\{3\},\{4\}\}$, then resource location $9 \in L$ is obsolete. For $N_9 = \{5\}$ we obtain a graph isomorphic to (t).

Case 2.i. $|N_6| = 1, |N_7| = 1, |N_8| = 1$.

By using the (same) reasoning of case 1.l we end up with subgraph (l).

(l) $N_6 = \{1\}, N_7 = \{2\}, N_8 = \{3\}$. If $N_9 \in \{\{1\},\{2\},\{3\}\}$, then resource location $9 \in L$ is obsolete. For $N_9 \in \{\{4\},\{5\}\}$ we obtain a graph isomorphic to (u).

Case 3. $|L| = 5$.

Let $L = \{6,7,8,9,10\}$ with $|N_6| \geq |N_7| \geq |N_8| \geq |N_9| \geq |N_{10}|$ without loss of generality. In line with our discussion at the start of case 2 of the proof of lemma 7.2, we can simply start with the obtained (sub)graphs and condition on N_{10}.

Case 3.a. $|N_6| \geq 3$.

A reasoning identical to cases 2.a, 2.b, 2.c, 2.d, and 2.e. can be used.

Case 3.b. $|N_6| = 2, |N_7| = 2, |N_8| = 2$.

By using the (same) reasoning of case 2.f we end up with subgraph $(m), (n), (o), (p), (q)$, and (r).

(m) $N_6 = \{1,2\}, N_7 = \{2,3\}, N_8 = \{1,4\}$, $N_9 = \{3,5\}$. If $|N_{10}| = 2$, a cycle is formed. If $|N_{10}| = 1$, resource location $10 \in L$ is obsolete.

(n) $N_6 = \{1,2\}, N_7 = \{2,3\}, N_8 = \{1,4\}$, $N_9 = \{1,5\}$. If $|N_{10}| = 2$, a cycle is formed. If $|N_{10}| = 1$, resource location $10 \in L$ is obsolete.

(o) $N_6 = \{1,2\}, N_7 = \{2,3\}, N_8 = \{1,4\}$, $N_9 = \{5\}$. If $|N_{10}| = 1$, resource location $10 \in L$ is obsolete.

(p) $N_6 = \{1,2\}, N_7 = \{2,3\}, N_8 = \{2,4\}$, $N_9 = \{2,5\}$. If $|N_{10}| = 2$, a cycle is formed. If $|N_{10}| = 1$, resource location $10 \in L$ is obsolete.

(q) $N_6 = \{1,2\}, N_7 = \{2,3\}, N_8 = \{2,4\}$, $N_9 = \{5\}$. If $|N_{10}| = 1$, resource location $10 \in L$ is obsolete.

(r) $N_6 = \{1,2\}, N_7 = \{2,3\}, N_8 = \{4,5\}$, $N_9 = \{2,4\}$. If $|N_{10}| = 2$, a cycle is formed. If $|N_{10}| = 1$, resource location $10 \in L$ is obsolete.

Case 3.c. $|N_6| = 2, |N_7| = 2, |N_8| = 1$.

By using the (same) reasoning of case 2.g we end up with subgraph (s).

(s) $N_6 = \{1,2\}, N_7 = \{2,3\}, N_8 = \{4\}$, $N_9 = \{5\}$. If $|N_{10}| = 1$, resource location $10 \in L$ is obsolete.

Case 3.d. $|N_6| = 2, |N_7| = 1, |N_8| = 1$.

By using the (same) reasoning of case 2.h we end up with subgraph (t).

(t) $N_6 = \{1,2\}, N_7 = \{3\}, N_8 = \{4\}$, $N_9 = \{5\}$. If $|N_{10}| = 1$, resource location $10 \in L$ is obsolete.

Case 3.e. $|N_6| = 1, |N_7| = 1, |N_8| = 1$.

By using the (same) reasoning of case 2.i we end up with subgraph (u).

(u) $N_6 = \{1\}, N_7 = \{2\}, N_8 = \{3\}$, $N_9 = \{4\}$. If $N_{10} \in \{1,2,3,4\}$, resource location $10 \in L$ is obsolete. For $N_{10} = \{5\}$ we obtain a graph isomorphic to (v).

Case 4. $|J| \geq 6$.

Let $J = \{6,7,\ldots,k\}$ with $k \geq 12$ and $|N_6| \geq |N_7| \geq \ldots| \geq |N_k|$ without loss of generality. In line with our discussion at the start of case 2 of the proof of lemma 7.2, we can simply start with the obtained (sub)graphs and condition on N_{11}.

Case 4.a. $|N_6| \geq 2$.

A reasoning identical to cases 3.a, 3.b, 3.c, and 3.d. can be used.

Case 4.b. $|N_6| = 1, |N_7| = 1, |N_8| = 1$.

By using the (same) reasoning of case 3.e we end up with subgraph (v).

(v) $N_6 = \{1\}, N_7 = \{2\}, N_8 = \{3\}, N_9 = \{4\}, N_{10} = \{5\}$. Let $|N_{11}| = 1$. Then, we conclude that resource location $11 \in L$ is obsolete.

Finally, we conclude that the graph (with no cycles nor obsolete resource locations) is isomorphic to one of the graphs of figure 7.5. □

Theorem 7.2. *Every k-person MCL game with $k \in \{1,2,3,4,5\}$ and no cycles in the corresponding graph has a non-empty core.*

Proof **:** We distinguish in the number of players. For $k \in \{1,2,3\}$ the core is non-empty by theorem 7.1. Hence, we can restrict attention to $k = 4$ and $k = 5$.

Case 1. $k = 4$.

Based on proposition 7.2 and lemma 7.1, we can focus on MCL situations with two resources only and no obsolete resources in the corresponding graph. Let $\theta \in \Theta$ with $|N| = 4$, $\sum_{i \in N} r_i = 2$ and a corresponding graph $\mathscr{G} = (N, L, E)$ with no cycles nor obsolete resource locations. Now, we distinguish between two cases.

Case 1.a. $|L| \leq 2$.

Take $x_i = p_i$ for all $i \in N$ for which $L_i \neq \emptyset$ and $x_i = 0$ otherwise. Let $S \subseteq N$, then

$$v^\theta(S) \leq \sum_{i \in S : L_i \neq \emptyset} p_i = \sum_{i \in S : L_i \neq \emptyset} x_i = \sum_{i \in S : L_i \neq \emptyset} x_i + \sum_{i \in S : L_i = \emptyset} x_i = \sum_{i \in S} x_i.$$

The inequality is a natural upperbound. The first and third equality hold by definition. The second equality holds as $x_i = 0$ for all $i \in N$ for which $L_i = \emptyset$.

Case 1.b. $|L| \geq 3$.

For θ, i.e., the MCL situation under consideration, which has a corresponding graph $\mathscr{G} = (N, L, E)$ isomorphic to one of the graphs of figure 7.4 based on lemma 7.2, one can check easily, e.g., via standard software packages, that (i) matrix A of the corresponding linear programming problem $RMCL^\theta(N)$ in standard LP-form, i.e., in form $Ax \leq b$, is totally unimodular and (ii) vector b has all integer entries. In addition, $RMCL^\theta(N)$ has a finite optimum and so, by theorem 2.3, $RMCL^\theta(N)$ has integer optimal solutions. As $0 \leq y_i \leq 1$ for all $i \in N$, it follows directly that $0 \leq x_j \leq 1$ for all $j \in L$ for all optimal solutions of $RMCL^\theta(N)$. Hence, $RMCL^\theta(N)$ has an integer optimal solution with not only $y_i \in \{0,1\}$ for all $i \in N$, but also $x_j \in \{0,1\}$ for all $j \in L$. Based on proposition 7.3, the corresponding core of (N, v^θ) is non-empty.

Case 2. $k = 5$.

Based on proposition 7.2 and lemma 7.1, we can focus on MCL situations with two resources and no obsolete resources in the corresponding graph and MCL situations with three resources and no obsolete resources in the corresponding graph. First, we consider the situation with two resources.

Case 2.a. $\sum_{i \in N} r_i = 2$.

Let $\theta \in \Theta$ with $|N| = 5$, $\sum_{i \in N} r_i = 2$ and a corresponding graph $\mathscr{G} = (N, L, E)$ with no cycles nor obsolete resource locations. Now, we distinguish between two cases.

Case 2.a.1. $|L| \leq 2$.

We refer to case 1.a. for the proof.

Case 2.a.2. $|L| \geq 3$.

Similar to case 1.b. by taking into consideration the graphs of figure 7.5 instead of the graphs of figure 7.4.

Case 2.b. $\sum_{i \in N} r_i = 3$.

Let $\theta \in \Theta$ with $|N| = 5$, $\sum_{i \in N} r_i = 3$ and a corresponding graph $\mathscr{G} = (N, L, E)$ with no cycles nor obsolete resource locations. Now, we distinguish between two cases.

Case 2.b.1. $|L| \leq 3$.

We refer to case 1.a. for the proof.

Case 2.b.2 $|L| \geq 4$.

Similar to case 1.b. by taking into consideration the graphs of figure 7.5 instead of the graphs of figure 7.4.

We conclude that every k-person MCL game with $k \in \{1, 2, 3, 4, 5\}$ and no cycles in the corresponding graph has a non-empty core. $\square$

Chapter 8

Conclusions

8.1 Introduction

In the last chapter of this monograph, we summarize our results, present managerial insights, and identify interesting directions for future research.

8.2 Main results

In the introduction of this monograph, we introduced several types of railway oriented situations in which contractors could obtain cost savings by pooling their resources. We were interested in how to allocate these joint cost savings such that no individual contractor, nor any subgroup of contractors has reasons to split off from the collaboration. We tackled this cost allocation aspect by using concepts of cooperative game theory. In particular, we introduced several types of resource pooling situations in which service providers, e.g., contractors in the Dutch railway network, could pool their resources, and for each of them, we formulated an associated cooperative game. Some cooperative games are formulated from a cost savings perspective directly, while others are formulated (indirectly) from a profit or cost perspective. For all cooperative games, we mainly focussed on non-emptiness of the core. When complexity allowed, we also focussed on allocation rules and fairness properties they could satisfy.

In what follows, we shortly describe the various resource pooling situations and associated games and subsequently highlight the most important results.

In chapter 4, we introduced our first resource pooling situation in which several service providers could pool their critical, low-utilization resources with unavailability to increase their joint profit. For the associated availability game, we showed that the core is non-empty in general. In addition, we showed an even stronger result, namely the existence of a population monotonic allocation scheme. Moreover, we introduced four different allocation rules and investigated them on four fairness properties.

In chapter 5, we investigated an environment with several service providers who each kept a single spare part in stock to protect for downtime of their technical systems. The cost related to the downtime of these technical systems could differ per service provider. Service providers could reduce joint downtime costs by pooling their spare parts according to the one-by-one critical level policy, i.e., by one-by-one pooling. For the associated one-by-one pooling game, we showed an interesting relationship with Böhm-Bahwerk horse market games. As a consequence, we could show that our game has a non-empty core. In addition, we presented a class of allocation rules for which the resulting allocations are core members. Finally, we studied a simple allocation rule within this class of allocation rules that satisfied appealing fairness properties.

The second resource pooling situation showed that pooling of spare parts according to a smart pooling strategy could pay off. In particular, it created an incentive for service providers to collaborate. This stimulated us to investigate whether this holds for other types of resource pooling strategies as well. In chapter 6, we investigated this aspect for a resource pooling situation that has quite some similarities with the one of chapter 5. However, this time, we assumed that service providers could collaborate by pooling their spare parts according to an optimal pooling policy. We referred to this optimal form of pooling as stratified pooling. For the associated game, which we called a stratified pooling game, we showed that the underlying resource pooling situation could be described by a Markov decision process and the optimal spare parts pooling strategy as a stationary decision rule in this Markov decision process. In particular, we used this modelling technique to prove core non-emptiness of stratified pooling games.

Finally, in chapter 7, we studied a situation with several service providers who each may or may not own a single resource to cover their region completely. These service providers could increase total covering by pooling their resources. For the associated maximal covering location game, we showed that the core could be empty. However, we also provided several types of sufficient conditions (in terms of the number of players, the type of graph, the number of resources, and the underlying integer linear program) that ensured core non-emptiness. Finally, for each condition, we provided an example showing that when the condition is not satisfied, core non-emptiness is not guaranteed.

8.3 Managerial implications

As already discussed in the previous section, for three out of in total four cooperative resource pooling games, we were able to prove core non-emptiness in general. So, this implies that, from a theoretical perspective, there exists an allocation of the grand coalition's value that cannot be improved upon by any coalition. From a practical perspective, this means that investigating collaboration between several contractors in terms of pooling their tamping machines or spare parts (via one-by-one pooling or stratified pooling) is a worthwhile endeavour.

In the remainder of this section, we focus on the managerial insights for the developed allocation rules for availability games and one-by-one pooling games. In addition, we also focus on the managerial insights regarding the sufficient conditions we obtained for core non-emptiness of maximal covering location games.

For availability games, we introduced and discussed four allocation rules. It is likely that the intuitive and simple-to-implement allocation rules which divide profit proportionally to criteria such as availability and profit, may have some shortcomings. For instance, it is well possible that the allocations of these allocation rules may fall outside the core as well as that they may lack the monotonicity to availability property, which implies that an increase in the availability not necessarily implies an increase in the profit allocation. Oppositely, it seems that the allocation rule, that allocates the profit that a player generates with its own profit function while being part of the grand coalition, is a good starting point as it is easy to implement, core guaranteed, and dominates the allocation rules that divide profit proportionally to criteria such as availability and profit based on the investigated fairness properties.

For one-by-one pooling games, we discussed a complete class of allocation rules. An allocation rule within this class of allocation rules focusses on all possible situations with a unique combination of players keeping a spare part. For each of these situations, pooling of spare parts can be recognized as a trade of goods between the players, where the value of the good to trade depends on the downtime costs per contractor. Such allocation rule is a good starting point as it easy to implement and core guaranteed.

For maximal covering location games, we showed that the core may be empty. Indeed, one may face a problem by pooling the resources. However, in terms of maximal covering location games, there exist several sufficient conditions under which core non-emptiness can be guaranteed. For instance, if the relaxation of the integer linear programming problem has an optimal solution with integers only, core non-emptiness is guaranteed. Church and ReVelle [10] showed via a numerical experiment that maximal covering location problems with a relative dense network have a good chance of being solved to optimality via the relaxation of the integer linear programming problem. As the Dutch road network is quite dense and so railway segments can be reached via several directions, there is are good chance that pooling of the resources is of interest in the Dutch railway network as well. Based on the other two conditions, we indicate that including not too many service contractors, or having only a few (which sounds very undesirable for the Dutch railway network) or a relative high number of resources, may be of interest in terms of collaboration as well.

8.4 Future research

For all cooperative resource pooling games studied in this monograph, we already addressed several possible extensions for future research. In this last section, we point out another direction for future research which is of interest for all of them as well. So far, cooperative resource pooling games are approached from a strategic perspective. This means that we assumed that each player is willing to cooperate for an infinite time horizon and, in line with that, interested in the expected costs savings (per time unit). In reality, players collaborate for a finite period of time only. As a consequence, the realized cost savings under a cooperation for a couple of years will, most likely, deviate from the expected costs savings. From that perspective, it is of interest to investigate cooperative resource pooling games arising from finite horizon situations as well.

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Summary

Cooperative Resource Pooling Games

with applications to the railway sector

The Netherlands has one of the most intensively used railway networks of the world. Every day, more than one million passengers are transported on a compact, capacitated railway network of 6830 kilometers length. A highly reliable and available railway network is needed in order to guarantee such kind of service. However, there are disruptions too, which may affect availability of the railway network negatively. In that perspective, it is important that disruptions are repaired as soon as possible. This requires bringing the right service engineers, the right equipment, and the right spare parts to the disruption as quickly as possible. The current administrator of the Dutch railway network, i.e., ProRail, regulates this by outsourcing maintenance to several competitive contractors. Via performance based contracts, these contractors are held responsible for specific regions of the railway network. In particular, the contractors each hold an individual set of maintenance resources, dedicated for the execution of the maintenance in their own region. Looking from a national perspective, these dedicated maintenance resources can be used more efficiently. For instance, contractors can set up an arrangement in which a group of common maintenance resources are pooled and, as a consequence, can obtain some interesting cost savings. Although such cost savings can be considerably large, establishing a pooling arrangement between several independent, self-interested contractors is not easy! How can we, for instance, make sure that no individual contractor, nor any subgroup of contractors has reasons to split off from the collaboration? Such requirements are crucial for a sustainable cooperation and in that perspective, the construction of an allocation or rule that allocates the cost savings in such a way that no (group of) contractors want(s) to split off is a necessity.

In this monograph, we focus on this aspect for several types of situations in which service providers can pool their resources. These situations are all inspired by the Dutch railway sector. We address the cost savings allocation problem by making use of concepts of cooperative game theory. In particular, we formulate four resource pooling situations and for each of them we define an associated cooperative game. We formulate these

resource pooling games in terms of costs, (additional) profit, or cost savings directly. In any case, we are interested in how to allocate the total amount obtained under full collaboration. In particular, we are interested in the existence of allocations that makes no individual service provider, nor any subgroup of service providers worse off, or in terms of cooperative game theory, we are interested in core non-emptiness of the associated game. In addition, we focus on several allocation rules and appealing fairness properties they might satisfy.

Next, we describe the various resource pooling situations and associated cooperative games and subsequently highlight the most important results.

In the first resource pooling situation, we study an environment with several service providers who pool their critical, low-utilization resources with unavailability to increase joint profit. As an example, one can think of contractors who each own a single tamping machine. Such tamping machines are critical as tamping is required immediately, low-utilized as they are used a few times per year only, and sometimes unavailable, because they are subject to failures and repair leadtimes are long. For the associated availability game, we show that the core is non-empty in general. In addition, we show an even stronger result, namely the existence of an allocation of the joint profit for every possible coalition such that each player's payoff increases as the coalition to which the player belongs to grows larger. Moreover, we introduce four allocation rules and investigate them on several fairness properties. We investigate whether the allocations resulting from those allocation rules are increasing in the availability and in the profit function. Furthermore, we investigate whether the allocations resulting from those allocation rules are the same for players who are similar in the underlying setting or equivalent in terms of the associated availability game. Finally, we also investigate whether the allocations resulting from those allocation rules are members of the core.

In the second resource pooling situation, we consider an environment with several service providers, each keeping a single spare part in stock to protect against downtime of their technical systems. The costs related to the downtime of these technical systems are assumed to be different per service provider. As an example, one can think of contractors who each keep spare parts in stock for a specific railway segment with different penalty costs specified in their performance based contracts. We assume that service providers are able to reduce joint downtime costs by pooling the spare parts according to the one-by-one critical level policy. Under such a policy, players are added one-by-one to the group of players that are allowed to satisfy demand for an increasing number of spare parts in the on-hand stock. We refer to this as one-by-one pooling. For the associated one-by-one pooling game, we show an interesting relationship with Böhm-Bahwerk horse market games. As a consequence, we can show that our game has a non-empty core. In addition, we present a class of allocation rules for which the resulting allocations are core members. Last, we study a simple and intuitive allocation rule within this class of allocation rules that satisfies interesting fairness properties.

In the third resource pooling situation, we investigate an environment that has quite some similarities with the second resource pooling situation. However, this time, we assume that service providers can collaborate by pooling their spare parts according to an optimal pooling strategy, to which we refer to as stratified pooling. The assumption of optimal pooling makes the mathematical analysis quite challenging. Still, we are able to show that the core of the associated stratified pooling game is non-empty. To this end, we use that the underlying resource pooling situation can be described by a Markov decision process and the optimal spare parts pooling strategy as a stationary decision rule in this Markov decision process. In particular, we use this modelling technique to prove core non-emptiness of stratified pooling games.

In the last resource pooling situation, we study an environment with several service providers who each may or may not own a single resource to cover their region completely. The service providers can increase total covering by pooling their resources. As an example, these service providers can represent (a part of) a region of a contractor with or without a repair van. Covering a region (of a service provider) implies that one can avoid penalty costs set by performance based contracts. For the associated maximal covering location game, we show that the core may be empty. This implies that there exist situations for which the allocation of the joint profit can always be improved upon by at least one coalition. Although collaboration is not always beneficial, we provide several sufficient conditions that ensure core non-emptiness. These conditions are in terms of the number of players, the type of graph, the number of resources, and an underlying integer linear program. Finally, for each condition, we provide an example showing that when the condition is not satisfied, core non-emptiness is not guaranteed.

STELLINGEN

behorende bij het proefschrift

COOPERATIVE RESOURCE POOLING GAMES

WITH APPLICATIONS TO THE RAILWAY SECTOR

van

Loe Schlicher

i

Laat (N, c) een coöperatief spel zijn met $x \in \mathscr{C}(N, c) \neq \emptyset$. Voor een kosten besparingsspel (N, v) met als karakteristieke waardefunctie

$$v(S) = \sum_{i \in S} c(\{i\}) - c(S) \quad \text{voor alle } S \subseteq N,$$

geldt dat $y \in \mathscr{C}(N, v) \neq \emptyset$ met $y_i = c(\{i\}) - x_i$ voor alle $i \in N$.

[Hoofdstuk 2, Proefschrift]

ii

Het belonen van spelers met veelal onbeschikbare hulpmiddelen kan evenzeer lonend zijn daar hun potentieel in te brengen winst groot kan zijn.

[Hoofdstuk 4, Proefschrift]

iii

Laat (N, v^S) een coöperatief Böhm-Bawerk paardenmarkt spel zijn met S de verzameling van verkopers en $p_S \in [0, 1]$ voor alle $S \subseteq N$ zodanig dat $\sum_{S \subseteq N} p_S = 1$. Het coöperatieve spel (N, w) met als karakteristieke waardefunctie

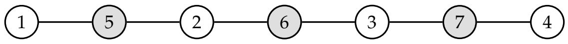
$$w = \sum_{S \subseteq N} p_S \cdot v^S$$

heeft een niet-lege core.

[Hoofdstuk 5, Proefschrift]

iv

Voor maximale dekking locatie spelen waarvoor spelers- en hulpmiddelenknopen elkaar steeds afwisselen in de onderliggende graaf (zie bijvoorbeeld figuur 1) geldt dat de core niet leeg is.



Figuur 1: Graaf met alternerende spelers- en hulpmiddelenknopen.

[Hoofdstuk 7, proefschrift]

v

Indien men samenwerking tussen aannemers in de Nederlandse spoorsector wil bevorderen, dient men het belang van de geografische inrichting van de contractgebieden in acht te nemen.

vi

Tijdelijke aanstellingen voor voltallig onderwijspersoneel zijn een schertsvertoning ten aanzien van kwaliteitsborging in het onderwijs.

vii

Het gebruik van Markov beslissingsprocessen voor het beschrijven van allerlei coöperatieve spelen, alsmede voor het aantonen van de niet-leegheid van de core, is een doortastende aanpak met veel potentie.

viii

In het hedendaagse wielrennen wordt het ondermaats presteren van de wereldkampioen vaak toegeschreven aan de vloek heersend op de regenboogtrui. Het zou beter zijn deze vloek te betitelen als het effect van regressie naar het gemiddelde.

ix

Het sequentieel neerzetten van dominostenen is gevaarlijk, aangezien het omvallen van de laatst gezette steen de complete rij kan verwoesten. Het inbouwen van buffers is hierbij essentieel. Voor de gestileerde situatie waarbij alleen de laatst neergezette steen kan omvallen, en wel met kans $p \in \left[0, \frac{1}{2}\right]$ naar links en met kans $p \in \left[0, \frac{1}{2}\right]$ naar rechts, is in figuur 2 een strategie getoond die de gemiddelde tijd voor het neerzetten van een vijftal dominostenen minimaliseert [1].

Stap 1 □ . . . .

Stap 2 □ □ . . .

Stap 3 □ □ . . □

Stap 4 □ □ . □ □

Stap 5 □ □ □ □ □

Figuur 2: Optimale volgorde om vijf dominostenen neer te zetten.

[1] J. Wijngaard. *Een analyse van een spelletje met dominostenen*,TH Eindhoven. THE/BDK/ORS, Vakgroep ORS : rapporten; Vol.7506, 1975.

x

Verscheidene (gepubliceerde) bronnen in de literatuur verwijzen naar stelling 8.4.5 afkomstig uit Puterman [2] om te claimen dat er een optimale strategie zou moeten bestaan welke *deterministisch* van aard is. Bij het naslaan van de betreffende stelling blijkt het verre van eenvoudig het woord deterministisch terug te vinden, noch het aantonen van het betreffende resultaat hiervan.

[2] M. Puterman. *Markov decision processes: discrete stochastic dynamic programming.* John Wiley & Sons, 2005.

xi

Het bewijzen dan wel ontkrachten van een vermoeden voelt als opgesloten zitten in een ruimte zonder sleutel.

xii

Blootstelling aan (trein)vertraging bevordert hieraan gerelateerd onderzoek.