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The Two-Phase Newsvendor Problem with Optimally-Timed Additional Replenishment

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Abstract

Recent advancements in Information Technology have provided an opportunity to significantly improve the effectiveness of inventory systems. The use of in-cycle demand information enables faster reaction to demand fluctuations. In particular, for single-period systems such as the newsvendor system, one way to exploit the newly available data is to perform an additional review (AR) of inventory at an endogenously determined, a priori set time during the sales period, and perform an additional replenishment if necessary. We jointly find the optimal quantity to order before the start of sales, the optimal AR timing, and the optimal quantity to replenish at the time of the AR. Through a computational experiment, we show that our algorithm quickly finds the optimal solution. Using the same computational experiment, we gain additional insight into the problem.
1 Introduction and Literature Review

Two major changes have affected supply chain management (SCM) in recent years. The first is the increased product variety accompanied by a shortening of product lifecycles. This resulted in short-term, highly uncertain demand profiles, making single-period inventory models particularly relevant and widely used.

The second change is the recent advancements in Information Technology, such as EDI (Electronic Data Interchange) systems and RFID (Radio Frequency Identification) tags, which provide decision makers in the supply chain (SC) with extensive, accurate, and often real-time, data. Wisely used, this newly available data can assist in improving the performance of the SC. For single-period systems such as the newsvendor system, one way to leverage this new information is to perform an additional review (AR) of inventory at an endogenously determined, a priori set time during the sales period, essentially creating two sub-periods. Our goal is to utilize this in-cycle sales information to react to and address demand fluctuations. We simultaneously optimize the initial order quantity, the timing of the AR, and the quantity to replenish at the time of the AR. Whenever the words “replenishment” or “lotsizing” are used, they can equally stand for production or procurement.

To clarify the problem setting, consider the following motivating examples.

A typical bakery bakes its bread in the early morning and sells it throughout the day. At some a priori set point during the day, after observing partial demand, the bakery may decide to bake an additional quantity to match the supply with the expected demand for the rest of the day. Due to the nature of demand, missed sales are lost, and at the end of the day, the surplus is salvaged or disposed of. The cost of holding the inventory during the sales period is effectively zero. In an attempt to avoid overproduction, some bakeries routinely practice this double-baking.

A printing house of books and periodicals decides, before the sales period begins, that at a certain moment during the period, after obtaining partial sales information, it may print and distribute another batch of the same product to meet the expected demand during the rest of the period. Due to the nature of these products, the print industry fits very well into the newsvendor (NV) framework. Unmet demand is lost, surplus inventory is scrapped at the end of the period or is redirected to a secondary sales channel, and there is no holding cost during the period. In case of a double-printing, the most substantial part of the fixed cost, i.e., preparing the printing plates, is not incurred for the second production since these are saved from the initial production. Because the printing process is considerably faster than the demand rate, the lead time is considered negligible.
and additional review and production are easily implementable. A similar argument can be made for high fashion and technology-related products.

A representative example of the print industry is Yedioth Group, the largest media group in Israel. Our model, after modification due to logistical constraints, was implemented at Yedioth; see Smirnov (2016) for details.

The AR policy addressed in the current work is static, in that the AR timing is set a priori. The reason for investigating this type of policy is that in many industries, in particular, in those mentioned above, production facilities and resources are shared by multiple products. In this way, while the timing of the AR and production is potentially flexible, one must reserve production capacity in advance. Otherwise, the resources are likely to be occupied by other products at the desired time of the additional production.

A correctly selected AR timing can improve the system performance compared to an arbitrarily selected timing. McGavin et al. (1993), who address a different but related model, have motivated our work by showing, via a numerical example, that the AR timing affects performance. The effect of varying the reorder intervals in a similar system with an infinite planning horizon and periodic review was numerically examined by van der Heijden (1999).

Our work is related to two lines of research. The first is the two-phase NV problem (e.g., Bulinskaya 1964) and the second is the problem of determining the optimal review timing in a stochastic inventory system. We use the term “period” if a holding cost is present, and the term “sub-period” if a single time unit is divided into two parts and a disposal cost is present only at the end of the period.

The two-phase NV problem has been addressed in several contexts. The first is information updating in inventory management of seasonal goods, such as apparel, or goods with short lifecycles, such as electronics (Bradford and Sugrue 1990, Fisher and Raman 1996, Choi et al. 2003, Bensoussan et al. 2004, Cachon and Swinney 2009, 2011). After obtaining partial or full sales data, demand forecasts are updated and additional replenishment and pricing decisions can be made, sometimes before the selling season begins and sometimes during its course. A more general setting includes multiple phases with a replenishment opportunity before each phase begins, and is not limited to a single location. The approach usually applied to this kind of problems is stochastic dynamic programming (Murray and Silver 1966, Crowston et al. 1973, Hartung 1973, Eppen and Iyer 1997, Avrahami et al. 2014), sometimes combined with Bayesian information updating (Murray and Silver 1966, Eppen and Iyer 1997). While the above mentioned studies perform forecast updating, an issue
that is not addressed in the current work, they do not consider timing decisions. Unlike these studies, we do optimize the AR timing.

The decision taken before the second phase of the sales period is not limited to replenishment. For instance, in a real-data-driven case study, Mostard and Teunter (2006) considered the possibility of returns during the sales period and reselling the returned items in a NV setting. Song and Zipkin (2012) considered a case of long lead times, where replenishment decisions based on early sales information include shrinking the batch size by canceling previously reserved production capacity and/or selling through intermediate channels at a penalty cost.

Versions of the two-phase NV problem also emerge in the context of coordination in a SC of seasonal goods. Well-known coordination mechanisms (revenue sharing, buybacks, price protection, etc.) are modified to fit the case of two ordering opportunities with information updating between them (e.g., Donohue 2000, Lee et al. 2000, Taylor 2001, Linh and Hong 2009).

Determining the optimal review timing in a stochastic inventory system is a problem addressed in various settings, including multiple periods, echelons or locations (Ravindran 1972, Flynn and Garstka 1997, Chiang 2001, Rao 2003, Gaur et al. 2007, Wang and Tomlin 2009, Liu and Song 2012). Some of these studies involve lotsizing decisions in addition to review timing decisions. Ravindran (1972) proposed an algorithm for the best period length in a single-period problem and incorporated the resulting length in the lotsizing decision, but did not consider the option of an AR and replenishment during the sales period. Flynn and Garstka (1997) addressed a periodic review infinite horizon system when orders can arrive every period, but the reorder interval can consist of several periods and each order incurs a fixed cost. Only the problem of timing is considered, while the ordering policy is assumed known. Chiang (2001) proposed splitting an order into two deliveries and optimized the length of the review period and over the order-up-to level in a periodic-review infinite-horizon setting under a service level constraint. A similar setting is considered in Rao (2003), where optimization over the number of the periods between two successive reviews is performed, and the resulting number of periods is used in finding the optimal order-up-to level. The findings of Rao (2003) rely on joint convexity of the objective in the decision variables, a fact revisited by Liu and Song (2012) who provide optimization algorithms for the order-up-to level and the number of the periods between two successive reviews, when each decision variable can be either discrete or continuous. Gaur et al. (2007) derived the optimal order timing decision in a NV setting subject to price uncertainty using the real options methodology. Wang and Tomlin (2009) addressed a NV problem with a single ordering opportunity, lead time uncertainty and Markovian demand forecast.
updating. They analyzed the tradeoff between supply and demand uncertainty and determined the optimal order quantity and order timing. Other types of stochastic inventory systems with controllable review intervals include serial (Shang and Zhou 2009, 2010) and one-warehouse-multi-retailer (Shang et al. 2013, Wang 2013, Wang and Axsäter 2013). Wang (2013) performed a numerical optimization over the reorder interval lengths. Wang and Axsäter (2013) optimized the warehouse reorder interval length and combined this decision with lotsizing decisions under several simplifying assumptions, such as a base-stock policy at the warehouse and possibly negative allocation quantities to the retailers. In contrast to the reviewed studies, we consider both lotsizing and AR timing decisions in a single-period setting.

The remainder of this paper is organized as follows. In §2 we provide the problem description and introduce the notation. In §3 we analyze the problem and derive structural properties. In §4 we develop our optimization algorithm. In §5 we report on a computational experiment and discuss its results. We conclude the work in §6. Most of the non-trivial proofs appear in the appendix. Some technical proofs appear in the online supplement.

2 Problem Statement

We consider a finite planning horizon containing a single sales period, long enough to enable a meaningful division into two parts. We address the problem of a single location. Before the initial ordering decision, the stock level is zero. Units of product are discrete and demand follows a Poisson process. Lead time and fixed replenishment costs are negligible. The planning horizon is time-scaled to the interval \([0, 1]\), so that demand is represented by a homogeneous Poisson process with a constant arrival rate.

The lotsizing problem in the system described above is a standard NV problem. Its order of events is outlined here for completeness. At time 0, a lotsizing decision is made, variable ordering costs are incurred and items are received. Subsequently, the sales period starts and demand unfolds. At the end of the period, disposal costs are incurred for excess stock (if any) and penalty costs are incurred for lost sales (if any).

The system we address differs from the standard NV system in that after the sales period starts and before it ends, at an endogenously determined a priori set time, the stock level is reviewed and an additional replenishment is (possibly) performed. The AR can be performed at any moment throughout the sales period, and splits the period into two sub-periods. If there were lost sales
before the additional replenishment, a penalty cost is charged for the lost sales. No holding costs are incurred at the time of the AR. Figure 1 depicts the order of events in the proposed system.

![Figure 1: Order of events in the proposed system](image)

Throughout the study we use the following notation (additional notation used in a particular proof will be introduced as needed):

- $\mathbb{N}$: set of natural numbers, including zero
- $c$: unit ordering and replenishment cost
- $V$: unit selling price
- $h$: unit disposal cost at the end of the period
- $p$: unit penalty cost for lost sales, same for both sub-periods
- $N_t$: random variable with mean $\lambda t$ resulting from a Poisson counting process representing demand during a time interval of length $t$; by definition, $N_0 = N_{1-1} \equiv 0$
- $n$: realization of demand in the first sub-period
- $m$: realization of demand in the second sub-period
- $t$: AR timing, $t \in [0, 1]$
- $Q_0$: quantity ordered at time 0
- $Q(t, (Q_0 - N_t)^+)$: quantity replenished at time $t$, $t \in [0, 1]$, when the inventory level is $(Q_0 - N_t)^+$
- $S(t)$: order-up-to level at time $t$, $t \in [0, 1]$
- $x$: order-up-to level at the time of the AR, independent of the value of $t$
- $(x, Q_0)$: interval-initial quantity (IQ) pair
- $A(t)$: actual stock level at time $t$ after the AR
- $t^B_x$: breakpoint associated with the IQ pair $(x, Q_0)$
- $t^*_x(x, Q_0)$: stationary point on the time domain associated with the IQ pair $(x, Q_0)$
- $(Q_0, t)$: representation of a solution
- $(Q_0, t \in [t^B_{x+1}, t^B_x])$: representation of a solution when $t$ is within the interval $[t^B_{x+1}, t^B_x]$
- $Q^{U}_0(x) [Q^{L}_0(x)]$: upper [lower] bound on $Q^*_0$ when the AR timing lies between $t^B_{x+1}$ and $t^B_x$
- $\pi_t(Q_0, x, n, m)$: expected profit after the AR, when ordering $Q_0$ items at time 0, replenishing up-to $x$ at the time of the AR, demand realization is $n$ in the first sub-period and $m$ in the second sub-period
- $\pi_0(Q_0, t, S(t))$: expected profit for the period when ordering $Q_0$ units at time 0, performing the AR at time $t$ and replenishing up to $S(t)$ at time $t$
- $\pi_0(Q_0, t)$: expected profit for the period when ordering $Q_0$ units at time 0, performing the AR at time $t$ and replenishing optimally at time $t$

We define $z^+ \equiv \max\{z, 0\}$ and $z^- \equiv \max\{-z, 0\}$. We use $\Delta$ to denote a difference and $z^*$ to denote the optimal value of $z$. We use $\bar{z}$ ($\underline{z}$) to denote an upper (lower) bound on $z$. We define $E_Z$ as the expectation of the random variable $Z$. To avoid uninteresting situations, we make the standard assumptions $p + V > c$ and $c > -h$. 

6
Our problem is to select $Q_0$ and $t$ to maximize $\pi_0(Q_0,t)$. As we will see, the replenishment policy at the time of the AR is uniquely determined by $t$.

By definition, demand for the first sub-period ($[0,t]$) follows a Poisson distribution with parameter $\lambda t$, while demand for the second sub-period ($[t,1]$) follows a Poisson distribution with parameter $\lambda(1-t)$. The demands in the two sub-periods are independent.

3 Analysis

3.1 Preliminaries

At time 0, the system orders a quantity of $Q_0$ and incurs an initial variable ordering cost of $cQ_0$. During the first sub-period, $\min\{Q_0, N_t\}$ items are sold, hence by time $t$, the system receives a revenue of $V \min\{Q_0, N_t\}$. At time $t$, the system incurs lost sales costs for the first sub-period, $p(N_t - Q_0)^+$, and, if a replenishment of $Q(t, (Q_0 - N_t)^+)$ units is made, a cost of $cQ(t, (Q_0 - N_t)^+)$ as well. The second sub-period starts with an actual stock level of $(Q_0 - N_t)^+ + Q(t, (Q_0 - N_t)^+)$, comprising the surplus from the first sub-period and the quantity replenished at time $t$, each of which might be zero.

After the AR, sales continue and $\min\{(Q_0 - N_t)^+ + Q(t, (Q_0 - N_t)^+), N_{1-t}\}$ items are sold in the second sub-period, resulting in an additional revenue of $V \min\{(Q_0 - N_t)^+ + Q(t, (Q_0 - N_t)^+), N_{1-t}\}$. At time 1, the system incurs either disposal costs for surplus inventory or penalty costs for lost sales in the second sub-period, namely, $h((Q_0 - N_t)^+ + Q(t, (Q_0 - N_t)^+)) - N_{1-t}^+ + p(N_{1-t} - (Q_0 - N_t)^+ - Q(t, (Q_0 - N_t)^+))^+$. Given $n$ arrivals in the first sub-period, $A(t) = (Q_0 - n)^+ + Q(t, (Q_0 - n)^+)$, which is the actual inventory at time $t$, and we note that $A(t)$ only becomes known at time $t$.

Suppose for now that $t$ is fixed and consider the problem faced at time $t$.

Observation 1 Given $t \in [0,1]$, if $Q_0$ units are ordered at time 0 and there are $n$ arrivals before time $t$, the expected profit of the second sub-period, $[t,1]$, is the standard NV expected profit, i.e., $V \min\{N_{1-t}, A(t)\} - c(A(t) - (Q_0 - n)^+) - h\min\{(A(t) - N_{1-t})^+, -p\min\{N_{1-t}, (N_{1-t} - A(t))^+\}\}$.

This observation leads us to the following result.

Theorem 1 Given $t \in [0,1]$, the expected profit of the second sub-period is discrete concave in $S(t)$ and it holds that

$$S^*(t) = \min \left\{ S(t) \in \mathbb{N} : P(N_{1-t} \leq S(t)) \geq \frac{p + V - c}{p + V + h} \right\}. \tag{1}$$
Moreover, given $t \in [0,1]$, $S^*(t)$ is also the optimal order-up-to level at the time of the AR for our proposed system.

**Proof:** Observation 1 directly implies an order-up-to policy. Thus, the first part of Theorem 1 follows from Clark and Scarf (1960) and Taha (1997). The last part of Theorem 1 holds by noting that the expected profit in the first sub-period is independent of $S^*(t)$.

The proposed system clearly has an expected profit no less than a simple NV system solved at time 0. The system with an AR is more flexible and has more information, thus reducing demand uncertainty and enabling a better match between supply and demand. In fact, constraining $S(t) = 0$ for all $t$ reduces the problem to the simple NV system. Additionally, the proposed system with $t = 0$ or $t = 1$ is again the simple NV system. As a result, the AR timings $t = 0$ and $t = 1$ are not uniquely optimal for the proposed system. The following observation will be useful throughout the entire study.

**Observation 2** The probabilities $P(N_{1-t} \leq x)$ and $P(N_t \geq x)$ are increasing in $t$.

Combining Observation 2 and Theorem 1, we have:

**Observation 3** For all $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$, it holds that $S^*(t_1) \geq S^*(t_2)$.

### 3.2 Objective Function

Recall that our objective function is to maximize the expected profit for the period. In its most general form, the expected profit, $\pi_0$, is a function of three decision variables, namely the quantity to order at time 0 ($Q_0$), the AR timing ($t$) and the order-up-to level at the time of the AR ($S(t)$).

Conditioning on the number of arrivals in the first sub-period and on the number of arrivals in the second sub-period, and removing the dependence between the AR timing and the order-up-to level at the time of the AR, the expected profit for the period is:

$$
\pi_0(Q_0, t, x) = V \sum_{n=0}^{\infty} \min\{Q_0, n\} P(N_t = n) - cQ_0 - p \sum_{n=0}^{\infty} (n - Q_0)^+ P(N_t = n)
$$

$$
+ \sum_{n=0}^{\infty} \left[ \sum_{m=0}^{\infty} \pi_t(Q_0, x, n, m) P(N_{1-t} = m) \right] P(N_t = n).
$$

(2)

The expression in square brackets in Eq. (2) reflects the profit-to-go, where:

$$
\pi_t(Q_0, x, n, m) = V \min\{\max\{x, (Q_0 - n)^+\}, m\} - c (\max\{x, (Q_0 - n)^+\} - (Q_0 - n)^+)
$$

$$
- h (\max\{x, (Q_0 - n)^+\} - m)^+ - p (m - \max\{x, (Q_0 - n)^+\})^+.\quad (3)
$$
To gain insight into the problem, let us examine Figure 2, which depicts $\pi_0$ as a function of $t$ for a fixed $Q_0 = 16$ for various order-up-to levels at the time of the AR for a particular problem instance. Each curve corresponds to an order-up-to level $x$ $(x \in \{0, \ldots, S^*(0)\})$. For example, the left-most curve in Figure 2 represents the function $\pi_0(16, t, 16)$, the next curve to the right represents the function $\pi_0(16, t, 15)$, and so on. The horizontal line in Figure 2 represents the function $\pi_0(16, t, 0)$.

![Image](image_url)

**Figure 2:** Expected profit as a function of $t$ for a fixed $Q_0 = 16$ and various values of $x$

Theorem 1 established that the optimal order-up-to level at time $t$ depends only on $t$, thus the dimension of the problem can be reduced from three variables to two variables by defining $\pi_0(Q_0, t) \equiv \pi_0(Q_0, t, S^*(t))$. Graphically, this means that in Figure 2, we select the top curve for every possible AR timing, thus obtaining a “pseudo-cloud-shaped” curve (see, e.g., Liu and Song 2012, where they discuss a cloud-shaped curve); see Figure 3.

In the problem we investigate, however, $Q_0$ is not fixed. We now formulate our objective function as a stochastic planning problem with recourse:

$$
\max_{Q_0, t} \pi_0(Q_0, t) = \max_{Q_0, t} \left\{ \mathbb{E}_{N_t} \left[ V \cdot \min\{Q_0, N_t\} - cQ_0 - p(N_t - Q_0)^+ \\
+ \mathbb{E}_{N_{1-t}} \left[ V \cdot \min\{A^*(t), N_{1-t}\} - c\left(A^*(t) - (Q_0 - N_t)^+\right) \\
- h(A^*(t) - N_{1-t})^+ - p(N_{1-t} - A^*(t))^+ \right]\right]\right\},
$$

where $Q_0 \in \mathbb{N}$ and $t \in [0, 1]$ are the first-stage decision variables (note that having the timing of the recourse as a decision variable is non-standard in stochastic dynamic programming), and $A^*(t) = \max\{S^*(t), (Q_0 - N_t)^+\}$ is the recourse action. The expected profit function is thus a collection of
Using algebra and the relation $z = z^+ - z^-$, the expected profit for the period (Eq. 4) can be rewritten as:

$$
\max_{Q_0, t} \pi_0(Q_0, t) = (p + V + h) \cdot \max_{Q_0, t} \left\{ \frac{(V - c) \lambda}{p + V + h} - E_{N_t} \left[ E_{N_{1-t}} [(A^*(t) - N_{1-t})^+] \right] \right\} + \frac{p + V - c}{p + V + h} E_{N_t} \left[ E_{N_{1-t}} [A^*(t) - N_{1-t}] - (N_t - Q_0)^+ \right].
$$

Examine Eq. (5), we get the following result.

**Theorem 2** The cost parameters $(V, c, h$ and $p)$ influence the optimal solution $(Q_0^*, t^*)$ only through the standard NV critical ratio, $(p + V - c)/(p + V + h)$.

**Proof:** The first term inside the maximum in Eq. (5) is constant, and thus does not affect the optimal solution. Note that $A^*(t)$ itself depends on the cost parameters only through the critical ratio; see Eq. (1). Thus, the second and third terms inside the maximum depend on the cost parameters.
parameters only through the critical ratio, \((p + V - c)/(p + V + h)\). Since the constant \((p + V + h)\) multiplies all the terms in the right-hand-side of Eq. (5), it does not affect the optimal solution. □

Theorem 2 is used in our experimental design to reduce the number of parameters (see §5). Note that this result is similar to the result regarding the optimal order quantity in the standard NV problem.

3.3 Breakpoints

Let us now consider the timing of the AR. If the AR is performed at the same time as the initial ordering, then it is optimal to replenish up to \(S^*(0)\) units. As the AR is performed later in the planning horizon, less inventory will be needed in the second sub-period (Observation 3). When the AR is performed very close to time 1, one would prefer not to replenish (i.e., order-up-to 0 units). Due to demand being integer, \(S^*(t)\) is a decreasing step function of \(t\) with a step height of one. For an illustrative example of the impact of \(t\) on \(S^*(t)\), see Figure 5.

A breakpoint is a point in the time dimension \((t \in [0, 1])\) at which one changes the optimal order-up-to level. By definition, we add two extreme breakpoints, \(t = 0\) and \(t = 1\). Thus, the breakpoints create \(S^*(0) + 1\) different intervals, each characterized by a different optimal order-up-to level, varying from \(S^*(0)\) down to 0. Note that these breakpoints are independent of \(Q_0\). Each “piece” of the pseudo-cloud-shaped function depicted in Figure 3 corresponds to the time interval between two consecutive breakpoints.

![Figure 5: Illustration of the impact of \(t\) on \(S^*(t)\) when \(S^*(0) = 8\)](image)

**Definition 1** For all \(x = 1, \ldots, S^*(0)\), the breakpoint associated with ordering up to \(x\) units at the
time of the AR is defined as

\[ t^B_x = \left\{ t \geq 0 : P(N_{1-t} < x) = \frac{p+V-c}{p+V+h} \right\} . \]  

(6)

The sequence of breakpoints is thus \( t^B_{S^*(0)+1} \equiv 0 \), \( t^B_{S^*(0)}, t^B_{S^*(0)-1}, \ldots, t^B_1, t^B_0 \equiv 1 \). We sometimes refer to the time interval \([t^B_x, t^B_x] \) as interval \( x \).

**Theorem 3** For all \( x = 1, \ldots, S^*(0) \), when \( t = t^B_x \), both \( x \) and \( x - 1 \) are optimal order-up-to levels.

### 3.4 Bounds on \( Q_0^* \)

We now establish bounds on the optimal quantity to order at time 0. We first establish an AR-timing-independent (i.e., independent of \( t \)) upper bound on \( Q_0^* \). Subsequently, we establish AR-timing-dependent upper and lower bounds on \( Q_0^* \). The following two lemmas will be useful in establishing these bounds.

**Lemma 1** Given \( t \in [0, 1] \), the expected finite difference in the expected profit from replenishing one additional unit at time 0 is

\[
\Delta \pi_0(Q_0, t) = (p + V - c)P(N_t > Q_0) 
+ \sum_{n=0}^{Q_0-S^*(0)+1} P(N_t = n) \left[ (p + V - c)P(N_{1-t} > Q_0 - n) - (c + h)P(N_{1-t} \leq Q_0 - n) \right].
\]

(7)

**Lemma 2** Given \( t \in [0, 1] \), \( \pi_0(Q_0, t) \) is discrete concave with respect to \( Q_0 \).

We are now ready to show that the optimal NV quantity at time 0 is an upper bound on \( Q_0^* \).

**Theorem 4** \( Q_0^* \leq S^*(0) \).

The intuition here is that since there is a chance to replenish at time \( t \), fewer units are needed at time zero. We will now establish interval-dependent bounds. For a given order-up-to level \( x \), we will establish bounds on \( Q_0^* \) for all \( t \) within interval \( x \).

**Theorem 5** Given \( t \in [t^B_{x+1}, t^B_x] \), it holds that \( Q_0^* \geq x \).

**Proof:** If we order less than \( x \) at time 0, when we arrive at time \( t \), obviously we will have to make an additional replenishment. So replenishing at least \( x \) units at time 0 cannot increase costs. Note that there is a positive probability that we will use the additional units to meet high demand in the first sub-period, thus reducing costs. □

A preliminary result subsequently used in several proofs is:
Lemma 3 Given a Poisson process with rate \( \lambda \), \( P(N_t = n) \) is unimodal in \( t \) on \([0, 1]\) and attains its maximum at \( t = \min\{\frac{n}{\lambda}, 1\} \).

Corollary 3.1 Given a Poisson process with rate \( \lambda \), \( P(N_t = n) \) attains its maximum over the interval \([t^B_{x+1}, t^B_x]\), \( K(x, n) \), as follows:

\[
K(x, n) = \begin{cases} 
P(N_{t_x} = n), & t_x < \frac{n}{\lambda} \\
P(N_{t_{x+1}} = n), & t_{x+1} > \frac{n}{\lambda} \\
P(N_x = n), & t_{x+1} \leq n/\lambda \leq t_x.
\end{cases}
\]

Our final bounds on \( Q^*_0 \) (see Theorem 6 below) are based on the following bounds on the finite difference of \( \pi_0(Q_0, t) \) (Eq. 7). Specifically, we have:

Lemma 4 Upper (\( \Delta \pi_0(Q_0, t \in [t^B_{x+1}, t^B_x]) \)) and lower (\( \Delta \pi_0(Q_0, t \in [t^B_{x+1}, t^B_x]) \)) bounds on \( \Delta \pi_0(Q_0, t) \) for all \( t \in [t^B_{x+1}, t^B_x] \) are, respectively,

\[
\Delta \pi_0(Q_0, t \in [t^B_{x+1}, t^B_x]) = (p + V - c)P(N_{t_x} > Q_0) + \sum_{n=0}^{(Q_0-x)^+} \left\{ \min\{P(N_{t_{x+1}} = n), P(N_{t_x} = n)\} \cdot \left[ (p + V - c)P(N_{1-t_{x+1}^B} > Q_0 - n) - (c + h)P(N_{1-t_{x+1}^B} \leq Q_0 - n) \right] \right\}, \tag{8}
\]

and

\[
\Delta \pi_0(Q_0, t \in [t^B_{x+1}, t^B_x]) = (p + V - c)P(N_{1-t_{x+1}^B} > Q_0) + \sum_{n=0}^{(Q_0-x)^+} K(x, n) \cdot \left[ (p + V - c)P(N_{1-t_{x+1}^B} > Q_0 - n) - (c + h)P(N_{1-t_{x+1}^B} \leq Q_0 - n) \right]. \tag{9}
\]

Proof: Lemma 4 follows by bounding the various probabilities in Eq. (7) over the interval \([t^B_{x+1}, t^B_x]\) either from above or below, depending on the sign preceding each term. We use Observation 2, Corollary 3.1 and the fact that \( c + h \) and \( p + V - c \) are positive. \( \square \)

We are now ready to establish the final bounds on the optimal quantity to order at time 0.

Theorem 6 Upper and lower bounds on \( Q^*_0 \) for all \( t \in [t^B_{x+1}, t^B_x] \) are, respectively,

\[
Q^U_0(x) = \min\{S^*(0), \min\{Q_0 \in N : \Delta \pi_0(Q_0, t \in [t^B_{x+1}, t^B_x]) \leq 0\}\},
\]

and

\[
Q^L_0(x) = \max\{x, \min\{Q_0 \in N : \Delta \pi_0(Q_0, t \in [t^B_{x+1}, t^B_x]) \leq 0\}\}.
\]
3.5 Analysis with Respect to the Additional Review Timing

In this section we provide an analysis of the expected profit as a function of the AR timing \( (t) \) for given \( Q_0 \) and \( x \). The results of this analysis are used later in our solution algorithm (§4).

3.5.1 Properties of the Objective Function

The following result is useful for identifying structural properties of the problem:

**Lemma 5** Given \( Q_0 \) and \( x \), the derivative of the expected profit for the period with respect to \( t \) is

\[
\frac{\partial \pi_0(Q_0, t, x)}{\partial t} = -\lambda(p + V - c)P(N_t \geq Q_0) \\
-\lambda P(N_t \geq Q_0 - x) \left[ (c + h)P(N_{1-t} < x) - (p + V - c)P(N_{1-t} \geq x) \right].
\] (10)

Intuitively, Eq. (10) can be understood by noting that increasing \( t \) by a small amount implies a reduction in the expected profit of \( p + V - c \) for each additional arrival beyond the quantity available for the first sub-period \( (Q_0) \). If stock is replenished up to \( x \) at the time of the AR (with probability \( P(N_t \geq Q_0 - x) \)), then increasing \( t \) affects the expected profit by the NV derivative associated with the second sub-period (the expression in square brackets in Eq. 10). If no replenishment occurs at the time of the AR, then the derivative is zero.

Returning to the pseudo-cloud-shaped function (see Figure 3), we note that for AR timings other than the breakpoints, it holds that \( \partial \pi_0(Q_0, t) / \partial t = \partial \pi_0(Q_0, t, x) / \partial t \). Additionally, we note that \( \partial \pi_0(Q_0, t) / \partial t \) is undefined at the breakpoints because two different order-up-to levels are used to the left and right of the breakpoint. To calculate the left and right derivatives, respectively, at a breakpoint \( t_B^x \), we take the limit of Eq. (10) evaluated at order-up-to levels \( x \) and \( x - 1 \), respectively. The following result allows us to eliminate the breakpoints from further consideration.

**Theorem 7** No breakpoint \( t_B^x \) \( (x = 1, ..., S^*(0)) \) is optimal.

To explore concavity, it is interesting to look at the second derivative of the expected profit with respect to \( t \). Exploring concavity is necessary for our solution algorithm (§4).

**Lemma 6** Given \( Q_0 \) and \( x \), the second derivative of the expected profit for the period with respect
to \( t \) is

\[
\frac{\partial^2 \pi_0(Q_0,t,x)}{\partial t^2} = -\lambda^2(p + V - c)P(N_t = Q_0 - 1) \\
-\lambda^2(p + V + h)P(N_t \geq Q_0 - x)P(N_{1-t} = x - 1) \\
-\lambda^2P(N_t = Q_0 - x - 1) [(c + h)P(N_{1-t} < x) - (p + V - c)P(N_{1-t} \geq x)].
\]

(11)

Despite the fact that \( \pi_0(Q_0,t) \) appears, for most instances, to be piecewise concave, we have found examples for which this is not the case. However, we have the following weaker result.

**Theorem 8** Given \( Q_0 \) and \( x, Q_0 \geq x, \pi_0(Q_0,t,x) \) is strictly unimodal in \( t \) for \( x > 0 \) and constant for \( x = 0 \).

Theorem 8 implies that \( \pi_0(Q_0,t) \) is piecewise-strictly-pseudo-concave in \( t \) for \( t < t_1^B \) and is constant for \( t > t_1^B \). As a result, the expected profit can be optimized interval-wise with respect to \( t \) (for all \( x \) and for all \( Q_0 \)) and the global maximum can be selected by comparison of values from a finite set. Our final algorithm (§4) uses this fact along with Theorem 9. Theorem 8 also allows us to eliminate intervals from our search space according to the following observation.

**Observation 4** For a given \( Q_0 \), if \( \lim_{t \to t_{x+1}^B} \partial \pi_0(Q_0,t)/\partial t < 0 \) or \( \lim_{t \to t_x^B} \partial \pi_0(Q_0,t)/\partial t > 0 \) \((x \in \{0, \ldots, S^*(0)\})\), the global maximizer of \( \pi_0(Q_0,t) \) is not found in the interval \([t_x^B, t_{x+1}^B] \).

### 3.5.2 Bounds on the Expected Profit Between Breakpoints

Given \( Q_0 \) and \( x \), we derive an upper bound on the expected profit in the interval \([t_{x+1}^B, t_x^B] \). This bound is utilized in our solution algorithm (§4) to eliminate intervals from consideration. The bound on the expected profit uses bounds on the first and second derivatives with respect to the time domain, as reflected in the following lemmas.

**Lemma 7** Given \( Q_0 \) and \( x \), an upper bound on the second derivative of \( \pi_0(Q_0,t) \) with respect to \( t \) over \([t_{x+1}^B, t_x^B] \) is

\[
\frac{\partial^2 \pi_0(Q_0,t \in [t_{x+1}^B, t_x^B])/\partial t^2}{\partial t^2} \equiv -\lambda^2(p + V - c) \min \left\{ P(N_{t_{x+1}^B} = Q_0 - 1), P(N_{t_x^B} = Q_0 - 1) \right\} \\
-\lambda^2(p + V + h)P(N_{t_{x+1}^B} \geq Q_0 - x) \min \left\{ P(N_{1-t_{x+1}^B} = x - 1), P(N_{1-t_x^B} = x - 1) \right\} \\
-\lambda^2 \cdot \mathcal{K}(x, Q_0 - x - 1) \cdot [(c + h)P(N_{1-t_{x+1}^B} < x) - (p + V - c)P(N_{1-t_{x+1}^B} \geq x)].
\]

(12)
Theorem 9: Given \( Q_0, t \in [t_{x+1}^B, t_x^B] \) is concave over \( [t_{x+1}^B, t_x^B] \). Similar to the bound on the second derivative, we derive upper and lower bounds on the derivative of the expected profit with respect to \( t \).

**Proof:** If the expected profit is concave over interval \( t \), the derivative is decreasing in \( t \). Hence, its highest (lowest, respectively) value is found at the left (right, respectively) boundary. Otherwise, the bounds are derived from Eq. (10) similarly to the bounds on the finite difference (Eqs. 8 and 9). □

Clearly, if \( \partial^2 \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B])/\partial t^2 < 0 \), then \( \pi_0(Q_0, t) \) is concave over \( [t_{x+1}^B, t_x^B] \). Given \( \pi_0(Q_0, t) \) and \( \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B]) \), we derive upper and lower bounds on the derivative of the expected profit with respect to \( t \).

**Lemma 8** Given \( Q_0 \) and \( x \), an upper and a lower bound on the derivative of \( \pi_0(Q_0, t) \) with respect to \( t \) over \( [t_{x+1}^B, t_x^B] \) are, respectively,

\[
\partial \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B])/\partial t \equiv \begin{cases} \\
\partial \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B])/\partial t |_{t=t_{x+1}^B}, & \text{if } \partial^2 \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B])/\partial t^2 < 0 \\
-\lambda(p + V - c)P(N_{t_{x+1}^B} \geq Q_0) & -\lambda P(N_{t_{x+1}^B} \geq Q_0 - x)(c + h)P(N_{1-t_{x+1}^B} < x) \text{ otherwise,} \\
-(p + V - c)P(N_{1-t_{x+1}^B} \geq x), \end{cases}
\]

and

\[
\partial \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B])/\partial t \equiv \begin{cases} \\
\partial \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B])/\partial t |_{t=t_x^B}, & \text{if } \partial^2 \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B])/\partial t^2 < 0 \\
-\lambda(p + V - c)P(N_{t_x^B} \geq Q_0) & -\lambda P(N_{t_x^B} \geq Q_0 - x)(c + h)P(N_{1-t_x^B} < x) \text{ otherwise,} \\
-(p + V - c)P(N_{1-t_x^B} \geq x), \end{cases}
\]

**Proof:** If the expected profit is concave over interval \( x \), the derivative is decreasing in \( t \). Hence, its highest (lowest, respectively) value is found at the left (right, respectively) boundary. Otherwise, the bounds are derived from Eq. (10) similarly to the bounds on the finite difference (Eqs. 8 and 9). □

We are now ready to derive an upper bound on the expected profit in an interval.

**Theorem 9** Given \( Q_0 \) and \( x \), an upper bound on the expected profit over \( [t_{x+1}^B, t_x^B] \) is

\[
\pi_0(Q_0, t \in [t_{x+1}^B, t_x^B]) = \pi_0(Q_0, t_{x+1}^B, x) + \partial \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B])/\partial t |_{t=t_{x+1}^B},
\]

where

\[
t_{x+1}^B = \frac{\partial \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B])/\partial t \cdot t_{x+1}^B - \partial \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B])/\partial t \cdot t_x^B + \partial \pi_0(Q_0, t_x^B, x) - \pi_0(Q_0, t_{x+1}^B, x)}{\partial \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B])/\partial t - \partial \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B])/\partial t}.
\]
The upper bound on the expected profit is found at the intersection of two linear functions that bound the expected profit—one starting at $t^B_{x+1}$ and having a slope corresponding to the upper bound on the slope, and the other ending at $t^B_x$ and having a slope corresponding to the lower bound on the slope. An upper bound on the objective value corresponds to the intersection of these two lines. Theorem 9 is illustrated in Figure 6.

Figure 6: Illustration of the upper bound on the expected profit from Theorem 9.

4 Solution Algorithm

Our solution algorithm begins by finding an initial candidate solution. Subsequently, it checks combinations of intervals and initial quantities (a finite set). For each combination, the algorithm either finds a better solution (initial order quantity and AR timing) than the best so far, or demonstrates that the entire interval can be eliminated.

The main algorithm, Algorithm 1, finds the optimal solution. We rely on Theorem 6 in setting the start and end values of the inner loop. $(Q^*_0, t^*_0)$ is an optimal solution when the algorithm terminates, but while executing the algorithm, it is the best candidate solution found so far, and is not necessarily an optimal solution.

The purpose of Algorithm 2 (see below), which is executed for an IQ pair $(x, Q_0)$, is either to find a better candidate solution or to eliminate the IQ pair from consideration. The algorithm begins by
Algorithm 1: Find optimal \((Q_0^*, t^*)\)

**Data:** \(V, c, h, p\) and \(\lambda\)

**Result:** Optimal solution, \((Q_0^*, t^*)\), and optimal objective value, \(\pi_0(Q_0^*, t^*)\)

1: \((Q_0^*, t^*) \leftarrow \text{Initial candidate solution (see §4.1)}\)

For all possibly optimal interval–initial quantity pairs, examine the optimality of \((x, Q_0)\).

2: \textbf{for} \(x = 0, \ldots, S^*(0)\) \textbf{do}

3: \textbf{for} \(Q_0 = Q_0^L(x), \ldots, Q_0^U(x)\) \textbf{do}

4: Execute Algorithm 2 with arguments \(x\) and \(Q_0\) to find a better candidate solution, or eliminate the IQ pair \((x, Q_0)\) from consideration.

5: \textbf{end for}

6: \textbf{end for}

7: Return \((Q_0^*, t^*)\) and \(\pi_0(Q_0^*, t^*)\).

Applying various elimination tests, based on properties detailed and discussed in §3. The intervals that cannot be eliminated are optimized interval-wise and the best solution is selected from among them by comparison.

### 4.1 Finding an Initial Candidate Solution

Algorithm 1 begins by finding an initial candidate solution. The algorithm terminates with the optimal solution no matter what initial solution is used. The initial solution, however, does affect the execution time. Depending on the value of \(\lambda\), we apply two different heuristics.

When \(\lambda\) is small (\(\lambda < 16\)), the optimization algorithm is extremely fast, hence we do not invest significant time in finding the initial solution. In this case, we use a heuristic rule inspired by McGavin et al. (1993), who, although addressing a different problem, heuristically divided the period into two sub-periods of equal length. Thus, we set the initial AR timing to be 0.5 and optimize with respect to \(Q_0\) by enumerating all \(Q_0\)'s starting with \(S^*(0.5)\) until we find one for which Eq. (7) is non-positive (see Lemma 2). Alternatively, we could have performed a binary search with bounds of \(x\) and \(S^*(0)\).

For larger \(\lambda\), it becomes worthwhile to invest effort in finding a good initial solution. Although the optimal solution is not found at a breakpoint (Theorem 7), for \(\lambda \geq 16\), we find a two-dimensionally locally optimal breakpoint as the initial candidate solution. We use alternating optimization, i.e., starting with \(Q_0 = S^*(0)\), we alternately optimize over one of the decision vari-
ables ($Q_0$ and $t$) while holding the other constant. The algorithm converges since the objective value increases at each step and there is a finite number of candidates.

**Algorithm 2:** Find an interior candidate solution, $(Q^*_0, t^*)$, or eliminate the IQ pair $(x, Q_0)$

**Data:** Parameters: $V$, $c$, $h$, $p$, $\lambda$. Arguments: $x$, $Q_0$, $(Q^*_0, t^*)$ and $\pi_0(Q^*_0, t^*)$

**Result:** Interior optimal solution, $(Q^*_0, t^*)$, or a statement of elimination of the interval

1: if $\lim_{t \to t^*_x+} \partial \pi_0(Q_0, t)/\partial t < 0$ or $\lim_{t \to t^*_x-} \partial \pi_0(Q_0, t)/\partial t > 0$ then
2: $t^s_{(x,Q_0)} \leftarrow -1$ and eliminate all solutions $(Q_0, t \in (t^B_{x+1}, t^B_x))$
3: else
4: if $\pi_0(Q_0, t \in [t^B_{x+1}, t^B_x]) < \pi_0(Q^*_0, t^*)$ then
5: $t^s_{(x,Q_0)} \leftarrow -1$ and eliminate all solutions $(Q_0, t \in (t^B_{x+1}, t^B_x))$
6: else
7: $t^s_{(x,Q_0)} \leftarrow \{t \in [0,1] : \partial \pi_0(Q_0, t, x)/\partial t = 0\}$ (Newton’s method)
8: if $\pi_0(Q^*_0, t^*) > \pi_0(Q_0, t^s_{(x,Q_0)})$ then
9: Eliminate all solutions $(Q_0, t \in [t^B_{x+1}, t^B_x])$
10: else
11: $(Q^*_0, t^*) \leftarrow (Q_0, t^s_{(x,Q_0)})$ and $\pi_0(Q^*_0, t^*) \leftarrow \pi_0(Q_0, t^s_{(x,Q_0)})$
12: end if
13: end if
14: end if

**4.2 Algorithm 2**

For each IQ pair $(x, Q_0)$, we could simply apply the first-order condition for optimality. In order to shorten execution time, however, Algorithm 2 begins by a pair of elimination tests based on Observation 4 (Lines 1–2). The next test (Lines 3–5) is aimed at eliminating intervals whose expected profit is too low compared to the current best candidate solution. The upper bound on the expected profit in an interval is calculated using Eq. (15).

If an interval has not been eliminated, the first-order condition is applied in that interval (Line 7). Since the first-order condition is not in closed-form (see Eq. 10), a search by Newton’s method is performed. Since $\pi_0(Q_0, t, x)$ is strictly unimodal, if Newton’s method converges, then it converges to the local maximum. Note that if Newton’s method converges outside the interval $[t^B_{x+1}, t^B_x]$, then $t^s_{(x,Q_0)}$ is not the global maximum point. In particular, there exists an $x'$ that has a greater expected
profit for this AR timing. Newton’s method has the advantage of a quadratic convergence rate and little sensitivity to the selection of the initial point. Note that the condition that the function be differentiable, necessary for using Newton’s method, is satisfied (see Eq. 10). If Newton’s method fails to converge, one can use a golden section search instead. However, in a large experiment consisting of 2857 different problem instances, we observed that Newton’s method always converged.

5 Computational Experiments

Having specified our solution algorithm, we proceed by reporting numerical results that provide insights beyond what can be seen from the analytical results, thus completing the picture of our system with an AR. Moreover, we demonstrate the speed of our algorithm. Each problem instance examined is characterized by a combination of $\lambda$ and a critical ratio, $(p + V - c)/(p + V + h)$, since the optimal solution of our problem (and the NV problem as well) depends solely on these two values (Theorem 2).

Due to the fact that the expected profits of both the system with an AR and the NV system depend on $V$, $c$, $h$ and $p$ (and not only on the critical ratio), we decided to compare problem instances using a unitless measure that depends solely on $\lambda$ and the critical ratio. We termed this measure the Percent Reduction in Uncertainty Cost (PRUC). PRUC is defined as the portion of the costs due to uncertainty reduced by applying our AR model relative to a system with perfect information. It can thus serve as an indicator of the benefits achieved by an AR.

$$\text{PRUC} \equiv \frac{\pi_0(Q_0^*, t^*) - \pi_0(S^*(0), 0, 0)}{(V - c)\lambda - \pi_0(S^*(0), 0, 0)}.$$  (16)

Note that $\pi_0(S^*(0), 0, 0)$ is the expected profit of the standard NV problem, and $(V - c)\lambda$ is the profit if we knew the demand realization before ordering.

We conducted a full-factorial experiment involving ten different values of $\lambda$ and seven different values of the critical ratio, 70 instances in all. Specifically, we used $\lambda = 2^k$, $k \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $(p + V - c)/(p + V + h) \in \{0.05, 0.1, 0.3, 0.5, 0.7, 0.9, 0.95\}$.

In our implementation of the calculation of the objective function, we replaced the limits of the infinite sums (e.g., $\sum_{n=0}^{\infty}$) with two $\lambda$-dependent expressions. For a Poisson distribution with a rate of $\lambda$, the smallest number of events considered is $\max\{\lfloor \lambda - 5\sqrt{\lambda} \rfloor, 0\}$, and the largest is $\max\{\lceil \lambda + 5\sqrt{\lambda} \rceil, 20\}$. The largest total probability thus ignored for all our runs is $4.977 \cdot 10^{-6}$. The $\lambda$-dependent limits of the sum contribute to the fact that the run time increases in $\lambda$. Our results are reported in Table 1.
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The first two columns show the parameters, $\lambda$ and the critical ratio. The next three columns present the optimal solutions, i.e., providing the optimal initial order quantity, the optimal AR timing, and the optimal order-up-to level at the time of the AR. The sixth column contains the expected quantity ordered for both sub-periods (i.e., the optimal initial quantity plus the expected quantity replenished at the time of the AR). This quantity is given by $Q_0^* + S^*(t^*)P(N_{t^*} \geq Q_0^*) + \sum_{n=Q_0^*-S^*(t^*)}^{Q_0^*-1} P(N_{t^*} = n) [S^*(t^*) - (Q_0^* - n)]$. The seventh column provides the optimal solution of a standard NV problem with the same parameters. The right-most column presents the PRUC.

The impact of the critical ratio on $t^*$ for various values of $\lambda$ is illustrated in Figure 7. The impact of $\lambda$ on $t^*$ for various values of the critical ratio is illustrated in Figure 8 (note the logarithmic scale). Examining Table 1 and Figures 7 and 8, we observe the following:
1. The optimal AR timing can be either smaller or larger than 0.5, regardless of the critical ratio. We note that $t^* < 0.5$ only occurred for small demand rates (up to 16). We conjecture that this phenomenon was due to the discrete nature of demand, and that with continuous demand $t^*$ would always be greater than 0.5.

2. For a given $\lambda$, the optimal expected quantity ordered is increasing in the critical ratio. For a given critical ratio, the optimal expected quantity ordered is increasing in $\lambda$. Both trends are in line with intuition and are similar to what occurs in the NV problem.

3. The relation between the expected quantity ordered by our system and $S^*(0)$ is not clear-cut, especially for $\lambda \leq 8$. However, when the demand rate is large enough so that the normal approximation to the Poisson distribution holds, our system exhibits properties similar to inventory pooling: unless the critical ratio equals 0.5, our system, in expectation, orders a total quantity that is closer to the mean demand for the period, compared to the standard NV system. This is due to the reduction in uncertainty enabled by our AR model. For a critical ratio of 0.5, both systems will order practically the same quantity; the small difference is due to units being discrete.

4. In most cases, $t^*$ is weakly increasing in the critical ratio for a given $\lambda$. However, $t^*$ sometimes decreases with an increase in the critical ratio. A possible explanation for this phenomenon
is that in most cases, an increase in the critical ratio implies an increase in $Q^*_0$, which, in turn, implies an increase in $t^*$ due to a reduced risk of understocking in the first sub-period. In some cases (e.g., for $\lambda = 2$, when the critical ratio is increased from 0.5 to 0.7), however, increasing the critical ratio leaves $Q^*_0$ unchanged due to the units being discrete ($Q^*_0 = 2$ in this example). The now increased risk of understocking in the first sub-period is compensated for by reducing $t^*$ (from 0.4256 to 0.3197 in this example).

5. Similarly, in most cases $t^*$ is weakly increasing in $\lambda$ for a given critical ratio. Nevertheless, $t^*$ sometimes decreases with an increase in $\lambda$. It should be noted that the decrease in $t^*$ occurs only for critical ratios that are at least 0.5. A possible explanation for the non-monotonicity is that in most cases, an increase in $\lambda$ implies an increase in $Q^*_0$, which, in turn, implies an increase in $t^*$ due to a reduced risk of understocking in the first sub-period. In some cases (e.g., for a critical ratio of 0.9, when $\lambda$ is increased from 2 to 4), however, due to the units being discrete, the increase in $Q^*_0$ caused by increasing $\lambda$ is not sufficient to fully compensate for the increased risk of understocking in the first sub-period. This risk is then compensated for by reducing $t^*$ (from 0.5440 to 0.4888 in this example).

The run times depend mainly on $\lambda$. Table 2 shows the average (over the critical ratios), minimum and maximum run times for the different demand rates. Note that the range between the minimum
and the maximum times is small.

Table 2: Average, minimum and maximum run times for different values of $\lambda$

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<th>$\lambda$</th>
<th>1</th>
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<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
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<tr>
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<td>0.05[s]</td>
<td>0.14[s]</td>
<td>0.52[s]</td>
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<td>36.73[m]</td>
</tr>
<tr>
<td>Maximum</td>
<td>0.23[s]</td>
<td>0.44[s]</td>
<td>0.58[s]</td>
<td>1.13[s]</td>
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<td>8.31[m]</td>
<td>46.5[m]</td>
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</tbody>
</table>

6 Concluding Remarks

In this work we used analytical tools to optimize a two-phase NV problem with a possibility for an AR and replenishment. The novelty of our work is in simultaneously determining the optimal initial order quantity, the optimal timing for the AR and the optimal order-up-to level at the time of the AR.

We identify structural properties of the problem and propose an exact and tractable solution algorithm that scans $O(S^*(0))$ different intervals. For each interval, we scan $O(S^*(0))$ initial quantities. In this way, Algorithm 2 (and thus Newton’s search) is employed $O((S^*(0))^2)$ times. Although $S^*(0)$ can be large, for most practical problems, the proposed algorithm requires a reasonable computational effort. Moreover, this problem needs to be solved at the strategic level, only once for the entire planning horizon.

The fixed cost for the initial ordering is assumed to be zero in this work, but can easily be incorporated by finding the optimal policy and comparing its objective to not ordering. Although in many settings the fixed cost associated with the additional replenishment is negligible, it would be useful to extend the model proposed in this study to account for this fixed cost. Clearly, this would make the model applicable to a wider range of businesses.

Another interesting direction for future research is to extend the setting to multiple centrally-managed locations, for which, due to fixed cost considerations, the AR is performed at the same time. This is economical if, for example, the locations share the same production facility or are supplied by the same supplier, as is often the case.
References


**Appendix**

**Proof of Theorem 3:** By definition, as $t \to t_x^B^-$, the optimal order-up-to level is $x$ and as $t \to t_x^{B^+}$, the optimal order-up-to level is $x - 1$ (see Eq. 1). The result follows from the fact that the objective function (Eq. 2) is clearly continuous. □

**Proof of Lemma 1:** Conditioning on the number of arrivals in the first sub-period, there are three possible outcomes at time $t$:

1. Shortage at time $t$. Replenishing one unit more at time 0 would result in increasing the profit by $p + V - c$. The probability of this outcome is $P(N_t > Q_0)$.

2. Zero inventory or surplus below the optimal order-up-to level at time $t$. Replenishing one unit more at time 0 would result in replenishing one unit less at time $t$. In this case, the change in the profit is zero.

3. Surplus above or exactly equal to the optimal order-up-to level at time $t$. Replenishing one unit more at time 0 would result in increasing $A^*(t)$ by one unit. Given $n$ arrivals in the first sub-period, one more unit of surplus at time 1 (with probability $P(N_{1-t} \leq Q_0 - n)$)
would reduce the profit by \(c + h\), while one less unit of shortage at time 1 (with probability \(P(N_{1-t} > Q_0 - n)\)) would increase the profit by \(p + V - c\).

\[ \Box \]

**Proof of Lemma 2:** We prove Lemma 2 by showing that the finite difference, Eq. (7), is decreasing in \(Q_0\). Since \(P(N_t > Q_0)\) is decreasing in \(Q_0\) and \(p + V - c > 0\), the first term in Eq. (7) is decreasing in \(Q_0\). For all \(n = 0,...,Q_0\), clearly \(P(N_{1-t} > Q_0 - n)\) and \(-P(N_{1-t} \leq Q_0 - n)\) are decreasing in \(Q_0\), and together with \(c + h > 0\), we find that the entire expression in square brackets in Eq. (7) is decreasing in \(Q_0\). By increasing \(Q_0\), one adds a non-positive term to the sum, and thus the sum decreases. This term is non-positive since it represents a situation in which \((Q_0 - n)^+ \geq S^*(t)\), and by Theorem 1, it is non-positive. Therefore, the second term in Eq. (7) is decreasing in \(Q_0\).

\[ \Box \]

**Proof of Theorem 4:** Define \(\tilde{\pi}_0(Q,t,n)\) as the expected profit for the period when ordering \(Q\) items at time 0, performing the AR at time \(t\) and replenishing optimally, conditioned on \(n\) arrivals in the first sub-period. Define \(\pi_{0}^NV(Q,t,n)\) as the expected profit for the period without an AR, when \(Q\) units are ordered at time 0, conditioned on \(n\) arrivals in the time interval corresponding to the first sub-period \([0,t]\) in the system with an AR. The proof is based on showing that regardless of the number of arrivals in the first sub-period, we have \(\Delta \tilde{\pi}_0(Q,t,n) \leq \Delta \pi_{0}^NV(Q,t,n)\). There are three different cases, depending on \(n\).

1. **Shortage at time \(t\), namely \(n > Q_0\).** The finite difference in the proposed system is \((p + V - c)\), which is also the finite difference in the NV system.

2. **Surplus below the order-up-to level at time \(t\).** The finite difference in the proposed system is zero. However, in the NV system there is no AR, and therefore, replenishing an additional unit at time 0 would result in a positive finite difference of \((p + V - c)P(N_{1-t} > Q - n) - (c + h)P(N_{1-t} \leq Q - n)\).

3. **Surplus above or exactly equal to the order-up-to level at time \(t\).** The finite difference in the proposed system is \((p + V - c)P(N_{1-t} > Q - n) - (c + h)P(N_{1-t} \leq Q - n)\), which is also the finite difference in the NV system.

Since \(\Delta \tilde{\pi}_0(Q,t,n) \leq \Delta \pi_{0}^NV(Q,t,n)\) for every realization of demand in the first sub-period, it holds that \(\Delta \pi_0(Q,t) \leq \Delta \pi_0(Q,0)\). The proof is completed by noting that in both systems, the expected profits are discrete concave in the initial replenishment decision (Theorem 1 for \(t = 0\) and Lemma 2). Since \(\Delta \pi_0(Q,t) \leq \Delta \pi_0(Q,0)\), the objective function in our problem attains its maximum no later than the objective function in the NV system does, so \(Q_0^* \leq S^*(0)\).

\[ \Box \]
Proof of Lemma 3: Differentiating \( P(N_t = n) \) with respect to \( t \), we obtain:

\[
\frac{dP(N_t = n)}{dt} = \frac{d}{dt} e^{-\lambda t}(\lambda t)^n = \frac{n \cdot e^{-\lambda t}(\lambda t)^{n-1}}{n!} + e^{-\lambda t}(\lambda t)^n (-\lambda) = n \cdot e^{-\lambda t}(\lambda t)^{n-1} - \lambda e^{-\lambda t}(\lambda t)^n.
\]

Equating to zero and solving for \( t \), we obtain \( t = n/\lambda \). Another stationary point is \( t = 0 \), but we note that \( P(N_t = n) = 0 \) for all \( n > 0 \) and the two points are the same when \( n = 0 \). If \( n/\lambda \leq 1 \), then \( t = n/\lambda \) is a local maximum since \( \lim_{t \to n/\lambda} (dP(N_t = n)/dt) > 0 \) and \( \lim_{t \to n/\lambda}^+ (dP(N_t = n)/dt) < 0 \). If \( n/\lambda > 1 \), the maximum is attained at the boundary, \( t = 1 \). Since \( P(N_t = n) \) is continuously differentiable with respect to \( t \), it is unimodal and the maximum is attained either at \( t = n/\lambda \) or \( t = 1 \), whichever is smaller.

Proof of Theorem 6: By Theorem 4, we have \( Q_0^* \leq S^*(0) \). The upper bound on the finite difference, \( \Delta \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B]) \), is decreasing in \( Q_0 \) by arguments similar to those found in the proof of Lemma 2 for Eq. (7), since Eq. (8) has the same structure with respect to \( Q_0 \) as Eq. (7). Thus, the first value for which \( \Delta \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B]) \leq 0 \) is an upper bound on \( Q_0^* \) when \( t \in [t_{x+1}^B, t_x^B] \). By Theorem 5, we have \( Q_0^* \geq x \). The lower bound on the finite difference, \( \Delta \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B]) \), is decreasing in \( Q_0 \) by arguments similar to the upper bound. The first value for which \( \Delta \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B]) \leq 0 \) is a lower bound on \( Q_0^* \) when \( t \in [t_{x+1}^B, t_x^B] \).

Proof of Theorem 7: It suffices to show that for all \( Q_0 \), it holds that \( \lim_{t \to t_x^B^-} \partial \pi_0(Q_0, t)/\partial t < \lim_{t \to t_x^B^+} \partial \pi_0(Q_0, t)/\partial t \). Using Eq. (10), due to continuity we have:

\[
\lim_{t \to t_x^B^-} \frac{\partial \pi_0(Q_0, t)}{\partial t} = \frac{\partial \pi_0(Q_0, t, x)}{\partial t} \bigg|_{t = t_x^B} = -\lambda (p + V - c) P(N_{t_x^B} \geq Q_0) - \lambda P(N_{t_x^B} \geq Q_0 - x)
\]

\[
= -\lambda (p + V - c) P(N_{t_x^B} \geq Q_0) - \lambda P(N_{t_x^B} \geq Q_0 - x) (c + h) P(N_{1-t_x^B} < x) - (p + V - c) P(N_{1-t_x^B} \geq x)
\]

\[
< -\lambda (p + V - c) P(N_{t_x^B} \geq Q_0) - \lambda P(N_{t_x^B} \geq Q_0 + 1 - x) (c + h) P(N_{1-t_x^B} < x - 1) - (p + V - c) P(N_{1-t_x^B} \geq x)
\]

\[
= -(p + V - c) P(N_{1-t_x^B} \geq x - 1) = \frac{\partial \pi_0(Q_0, t, x - 1)}{\partial t} \bigg|_{t = t_x^B} = \lim_{t \to t_x^B^+} \frac{\partial \pi_0(Q_0, t)}{\partial t}.
\]

Both inequalities follow from the fact that \( P(Z \geq z) > P(Z \geq z + 1) \) for the Poisson distribution.
Proof of Lemma 6: Differentiating Eq. (10) with respect to $t$, we obtain

\[
\frac{\partial^2 \pi_0(Q_0,t,x)}{\partial t^2} = -\lambda(p + V - c) \frac{d}{dt} P(N_t \geq Q_0) - \lambda(c + h) \frac{d}{dt} [P(N_t \geq Q_0 - x)P(N_{1-t} < x)] \\
+ \lambda(p + V - c) \frac{d}{dt} [P(N_t \geq Q_0 - x)P(N_{1-t} \geq x)]
\]

\[
= -\lambda^2(p + V - c)P(N_t = Q_0 - 1) - \lambda^2(c + h)P(N_{1-t} < x)P(N_t = Q_0 - x - 1) \\
- \lambda^2(c + h)P(N_t \geq Q_0 - x)P(N_{1-t} = x - 1) \\
+ \lambda^2(p + V - c)P(N_t \geq Q_0 - x)P(N_{1-t} = x - 1) \\
- \lambda^2(p + V - c)P(N_t = Q_0 - 1) \\
- \lambda^2(p + V + h)P(N_t \geq Q_0 - x)P(N_{1-t} = x - 1) \\
- \lambda^2P(N_t = Q_0 - x - 1)(c + h)P(N_{1-t} < x) - (p + V - c)P(N_{1-t} \geq x)].
\]

We used a result from Gallager (2013). □

Proof of Theorem 8: Note that the first-order condition (equating Eq. 10 to zero) can be written as $P(N_{1-t} < x) = \frac{pV - c}{pV + c}P(N_t < Q_0 \mid N_t \geq Q_0 - x)$. If $x = 0$, the first-order condition holds true for all $t > 0$. Thus, for $x = 0$ the expected profit is constant with respect to $t$. Otherwise, since $P(N_{1-t} < x)$ is strictly increasing in $t$, it suffices to show that $P(N_t < Q_0 \mid N_t \geq Q_0 - x)$ is strictly decreasing in $t$. Alternatively, we will show that $P(N_t \geq Q_0 \mid N_t \geq Q_0 - x)$ is strictly increasing in $t$ for all $t > 0$ for $Q_0 \geq 0$, and this will complete the proof.

If $x > 0$, first, assume $Q_0 = x > 0$. Then $P(N_t \geq x \mid N_t \geq 0) = P(N_t \geq x)$. We know (e.g., from Gallager 2013) that $dP(N_t \geq x)/dt = \lambda P(N_t = x - 1)$, which is positive for all $t > 0$, implying that $P(N_t \geq x \mid N_t \geq 0)$ is strictly increasing in $t$ for all $t > 0$. Next, assume $Q_0 > x > 0$. Then we have $P(N_t \geq Q_0 \mid N_t \geq Q_0 - x) = P(N_t \geq Q_0)/P(N_t \geq Q_0 - x)$. Differentiating the latter expression with respect to $t$ (see, e.g., Gallager 2013), we obtain:

\[
\frac{d}{dt} P(N_t \geq Q_0) = \frac{P(N_t \geq Q_0 - x) \frac{d}{dt} P(N_t \geq Q_0) - P(N_t \geq Q_0) \frac{d}{dt} P(N_t \geq Q_0 - x)}{P(N_t \geq Q_0 - x)[P(N_t \geq Q_0 - x)]^2} \\
= \frac{P(N_t \geq Q_0 - x)\epsilon P(N_t = Q_0 - 1) - P(N_t \geq Q_0)\epsilon P(N_t = Q_0 - x - 1)}{[P(N_t \geq Q_0 - x)]^2} \\
= \frac{\epsilon P(N_t = Q_0 - x - 1)}{[P(N_t \geq Q_0 - x)]^2} \left[ \sum_{k=Q_0-x}^{\infty} e^{-\lambda t} \frac{\lambda^k}{k!} \frac{(Q_0 - x - 1)!}{(Q_0 - 1)!} - \sum_{k=Q_0}^{\infty} e^{-\lambda t} \frac{\lambda^k}{k!} \right]
\]

31
Hence, the upper bound on the expected profit in the interval in question is

\[\lambda P(N_t = Q_0 - x - 1) = \frac{\lambda P(N_t = Q_0 - x - 1)}{[P(N_t \geq Q_0 - x)]^2} \left[ \sum_{k=Q_0}^{\infty} e^{-\lambda t} (\lambda t)^k \frac{(Q_0 - x - 1)!}{(k-x)! (Q_0-1)!} - \sum_{k=Q_0}^{\infty} e^{-\lambda t} (\lambda t)^k \frac{1}{k!} \right].\]

The denominator is clearly positive. Since we are under the case \(Q_0 > x\), \(\lambda P(N_t = Q_0 - x - 1)\) is also positive. It thus suffices to show that \((Q_0 - x - 1)!/(k-x)! (Q_0-1)! > 1/k!\) for all \(k \geq Q_0\).

Note that

\[\frac{(Q_0 - x - 1)!}{(k-x)! (Q_0-1)!} = \frac{1}{(k-x)! \prod_{i=Q_0-x}^{Q_0-1} i} > \frac{1}{(k-x)! \prod_{i=k-x+1}^{\infty} i} = \frac{1}{k!}.\]

Note that the products have the same number of terms and since \(k \geq Q_0\), we have that \(k-x+1 \geq Q_0-x+1\), i.e., \(k-x+1 > Q_0-x\). Hence, the derivative of \(P(N_t \geq Q_0 \mid N_t \geq Q_0 - x)\) under \(Q_0 > x\) is positive for all \(t > 0\), implying that \(P(N_t \geq Q_0 \mid N_t \geq Q_0 - x)\) is strictly increasing in \(t\) for all \(t > 0\), thus completing the proof.

**Proof of Theorem 9:** Suppose that over the interval \([t_{x+1}^B, t_x^B]\), the objective function increases at a rate equal to \(\pi_0(Q_0, t \in [t_{x+1}^B, t_x^B])\). The corresponding linear function passes through \(t = t_{x+1}^B\) and can be expressed as

\[\pi_0^U(Q_0, t \in [t_{x+1}^B, t_x^B]) = \pi_0(Q_0, t_{x+1}^B, x) + \frac{\partial \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B])}{\partial t} (t - t_{x+1}^B).\]  

(17)

Suppose that over the interval \([t_{x+1}^B, t_x^B]\), the objective function decreases at a rate equal to \(\pi_0(Q_0, t \in [t_{x+1}^B, t_x^B])\). The corresponding linear function passes through \(t = t_x^B\) and can be expressed as

\[\pi_0^L(Q_0, t \in [t_{x+1}^B, t_x^B]) = \pi_0(Q_0, t_x^B, x) + \frac{\partial \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B])}{\partial t} (t - t_x^B).\]  

(18)

The expected profit in \([t_{x+1}^B, t_x^B]\) cannot exceed \(\pi_0^U(Q_0, t_\cap)\), where \(t_\cap\) is the intersection point between the two linear functions in Eqs. (17) and (18). Solving \(\pi_0^U(Q_0, t_\cap) = \pi_0^L(Q_0, t_\cap)\) for \(t_\cap\), we obtain

\[t_\cap = \frac{\partial \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B])}{\partial t} t_{x+1}^B - \partial \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B]) t_x^B + \pi_0(Q_0, t_x^B, x) - \pi_0(Q_0, t_{x+1}^B, x) \quad \frac{\partial \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B])}{\partial t} - \partial \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B])}{\partial t} (t_\cap - t_x^B).\]

Hence, the upper bound on the expected profit in the interval in question is

\[\pi_0(Q_0, t \in [t_{x+1}^B, t_x^B]) \equiv \pi_0^U(Q_0, t_\cap) = \pi_0(Q_0, t_{x+1}^B, x) + \frac{\partial \pi_0(Q_0, t \in [t_{x+1}^B, t_x^B])}{\partial t} (t_\cap - t_x^B),\]  

(19)

which completes the proof.