Aligned drawings of planar graphs

Aligned Drawings of Planar Graphs*

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Abstract. Let G be a graph embedded in the plane and let $\mathcal{A}$ be an arrangement of pseudolines intersecting the drawing of G. An *aligned* drawing of G and $\mathcal{A}$ is a planar polyline drawing Γ of G with an arrangement A of lines so that Γ and A are homeomorphic to G and $\mathcal{A}$. We show that if $\mathcal{A}$ is stretchable and every edge e either entirely lies on a pseudoline or intersects at most one pseudoline, then G and $\mathcal{A}$ have a straight-line aligned drawing. In order to prove these results, we strengthen the result of Da Lozzo et al. [5], and prove that a planar graph G and a single pseudoline $\mathcal{L}$ have an aligned drawing with a prescribed convex drawing of the outer face. We also study the more general version of the problem where only a set of vertices is given and we need to determine whether they can be collinear. We show that the problem is $\mathcal{NP}$-hard but fixed-parameter tractable.

1 Introduction

Two fundamental primitives for highlighting structural properties of a graph in a drawing are alignment of vertices such that they are collinear and geometrically separating unrelated graph parts, e.g., separating them by a straight line. Not surprisingly, both these techniques have been previously considered from a theoretical point of view in the case of planar straight-line drawings.

Da Lozzo et al. [5] study the problem of producing a planar straight-line drawing of a given embedded graph $G = (V, E)$, i.e., G has a fixed combinatorial embedding and a fixed outer face, such that a given set $S \subseteq V$ of vertices is collinear. It is clear that if such a drawing exists, then the line containing the vertices in S is a simple curve starting and ending at infinity that for each edge e of G either fully contains e or intersects e in at most one point, which may be an endpoint. We call such a curve a *pseudoline* with respect to G. Da Lozzo et al. [5] show that this is a full characterization of the alignment problem, i.e., a straight-line drawing where the vertices in S are collinear exists if and only if there exists a pseudoline $\mathcal{L}$ with respect to G that contains the vertices in S. However testing whether such a pseudoline exists is an open problem, which we consider in this paper.

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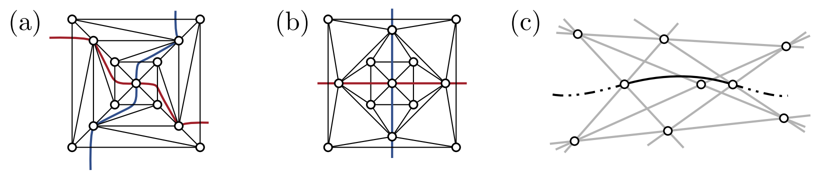

Fig. 1: Aligned Drawing (b) of a 2-aligned planar graph (a). The pseudolines $\mathcal{R}$ and $\mathcal{B}$ and the corresponding lines in the drawing are drawn red and blue, respectively. (c) A non-stretchable arrangement of 9 pseudolines, which can be seen as a stretchable arrangement of 8 pseudolines (grey) and an edge (black solid).

Likewise, for the problem of separation, Biedl et al. [1] considered so-called HH-drawings, where given an embedded graph $G = (V, E)$ and a partition $V = A \dot{\cup} B$, one seeks a planar drawing of G in which A and B can be separated by a line. Again, it turns out that such a drawing exists if and only if there exists a pseudoline $\mathcal{L}$ with respect to G such that the vertices in A and B are separated by $\mathcal{L}$ in the sense that they are in different *halfplanes*. Cano et al. [2] extend the result to planar straight-line drawings with a given star-shaped outer face.

In particular, the results of Da Lozzo et al. [5] show that given a pseudoline $\mathcal{L}$ with respect to G one can always find a planar straight-line drawing of G such that the vertices on $\mathcal{L}$ are collinear and the vertices contained in the halfplanes defined by $\mathcal{L}$ are separated by a line L. In other words, a topological configuration consisting of a planar graph G and a pseudoline with respect to G can always be stretched. In this paper we initiate the study of this stretchability problem with more than one given pseudoline.

More formally, a pair $(G, \mathcal{A})$ is a *k-aligned graph* if $G = (V, E)$ is a planar embedded graph and $\mathcal{A} = \{\mathcal{L}_1, \ldots, \mathcal{L}_k\}$ is an arrangement of (pairwise intersecting) pseudolines with respect to G. If the number k of curves is clear from the context, we drop it from the notation and simply speak of *aligned graphs*. For 1-aligned graphs we write $(G, \mathcal{L})$ instead of $(G, \{\mathcal{L}\})$. Let $A = \{L_1, \ldots, L_k\}$ be a line arrangement and Γ be planar drawing of G. A tuple (Γ, A) is an *aligned drawing of* $(G, \mathcal{A})$ if and only if the following properties hold; refer to Fig. 1(a-b). (i) The arrangement of A is homeomorphic to the arrangement of $\mathcal{A}$ (i.e., $\mathcal{A}$ is *stretchable*), (ii) Γ is homeomorphic to the planar embedding of G, (iii) each line L_i intersects in Γ the same vertices and edges as $\mathcal{L}_i$ in G, and it does so in the same order. We focus on straight-line aligned drawings. For brevity, unless stated otherwise, the term aligned drawing refers to a straight-line drawing throughout this paper.

Note that the stretchability of $\mathcal{A}$ is a necessary condition for the existence of an aligned drawing. Since testing stretchability is $\mathcal{NP}$-hard [14,15], we assume that a geometric realization A of $\mathcal{A}$ is provided. However, line arrangements of size up to 8 are always stretchable [11] and only starting from 9 lines non-stretchable arrangements exist; see the Pappus configuration [12] in Fig. 1(c). It is conceivable that in practical applications, e.g. stemming from user interactions, the number of lines to stretch is small. The same configuration illustrates an example of an 8-aligned graph with a single edge that does not have an aligned drawing.

The aligned drawing convention generalizes the problems studied by Da Lozzo et al. and Biedl et al. who concentrated on the case of a single line. We study a natural extension of their setting and ask for alignment on general line arrangements.

In addition to the strongly related work mentioned above, there are several other works that are related to the alignment of vertices in drawings. Dujmović [6] shows that every n-vertex planar graph $G = (V, E)$ has a planar straight-line drawing such that $\Omega(\sqrt{n})$ vertices are aligned, and Da Lozzo et al. [5] show that in planar treewidth-3 graphs, one can align $\Theta(n)$ vertices and that in treewidth-k graphs one can align $\Omega(k^2)$ vertices. Chaplik et al. [3] study the problem of drawing planar graphs such that all edges can be covered by k lines. They show that it is $\mathcal{NP}$-hard to decide whether such a drawing exists. Deciding whether there exists a drawing where all vertices lie on k lines is open [4]. Drawings of graphs on n lines where a mapping between the vertices and the lines is provided have been studied by Dujmović et al. [7,8].

Contribution & Outline. First we study the topological setting where we are given a planar graph G and set S of vertices to align in Section 3. We show that the problem is $\mathcal{NP}$-hard but fixed-parameter tractable (FPT) with respect to $|S|$. Afterwards in Section 4 we consider the geometric setting where we seek an aligned drawing of an aligned graph. In Section 4.2, we strengthen the result of Da Lozzo et al. and Biedl et al. and show that there exists a 1-aligned drawing of G with a given convex drawing of the outer face. In Section 4.3 we consider k-aligned graphs with a stretchable pseudoline arrangement, where every edge e either entirely lies on a pseudoline or intersects with at most one pseudoline, which can either be in the interior or an endpoint of e. We utilize the previous result to prove that every such k-aligned graph has an aligned drawing, for any value of k. The proofs of statements marked with ($\star$) can be found in the appendix.

2 Preliminaries

Let $\mathcal{A}$ be a pseudoline arrangement of a set of k pseudolines $\mathcal{L}_1, \ldots, \mathcal{L}_k$ and $(G, \mathcal{A})$ be an aligned graph. The set of cells in $\mathcal{A}$ is denoted by cells($\mathcal{A}$). A cell is *empty* if it does not contain a vertex of G. Removing from a pseudoline its intersections with other pseudolines gives a set of its *pseudosegments*.

Let $G = (V, E)$ be a planar embedded graph with a vertex set V and an edge set E. We call $v \in V$ *interior* if v does not lie on the boundary of the outer face of G. An edge $e \in E$ is *interior* if e does not lie entirely on the boundary of the outer face of G. An interior edge is a *chord* if it connects two vertices on the outer face. A point p of an edge e is an *interior* point of e if p is not an endpoint of e. A *triangulation* is a planar embedded graph whose inner faces are all triangles and whose outer face is bounded by a simple cycle. A *triangulation*

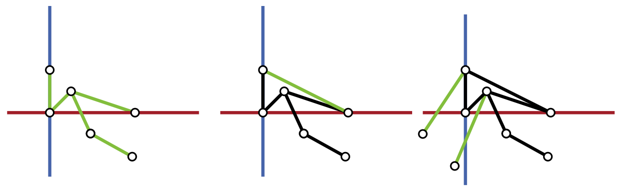

Fig. 2: Examples for the alignment complexity of an aligned graph.

of a graph G is a triangulation that contains G as a subgraph. For a graph G and an edge e of G, not being an edge of a separating triangle, the graph G/e is obtained from G by contracting e and merging the resulting multiple edges and removing self-loops. A *k-wheel* is a wheel graph W_k with k vertices on the outer face and one interior vertex. Let Γ be a drawing of G and let C be a cycle in G. We denote with $\Gamma[C]$ the drawing of C in Γ. A *k-aligned triangulation* of $(G, \mathcal{A})$ is a k-aligned graph $(G_T, \mathcal{A})$ with G_T being a triangulation of G.

A vertex is *$\mathcal{L}_i$-aligned* (or simply *aligned* to $\mathcal{L}_i$) if it lies on the pseudoline $\mathcal{L}_i$. A vertex that is not aligned is *free*. An edge e is *$\mathcal{L}_i$-aligned* (or simply *aligned*) if it completely lies on $\mathcal{L}_i$. Let E_{aligned} be the set of all aligned edges. An *intersection vertex* lies on the intersection of two pseudolines $\mathcal{L}_i$ and $\mathcal{L}_j$. An edge is *i-anchored* $(i = 0, 1, 2)$ if i of its endpoints are aligned to distinct curves. Let E_i be the set of i-anchored edges; note that, the set of edges is the disjoint union $E_0 \uplus E_1 \uplus E_2$. A 0-anchored, 0-crossed, non-aligned edge is also called *free*. An edge e is (*at most*) *l-crossed* if (at most) l distinct pseudolines intersect e in its interior. A non-empty edge set $A \subset E$ is l-crossed if l is the smallest number such that every edge in A is at most l-crossed.

The *alignment complexity* of an aligned graph $\mathcal{G}$ in a way describes how "complex" the relationship between the graph G and the line arrangement $\mathcal{L}_1, \dots, \mathcal{L}_k$ is and formally is defined as a triple (l_0, l_1, l_2) , where l_i, $i = 0, 1, 2$, describes the "complexity of i-anchored edges", i.e. it indicates that E_i is at most l_i-crossed or has to be empty, if $l_i = \perp$. For example, an aligned graph where every vertex is aligned and every edge has at most l interior intersections has the alignment complexity $(\perp, \perp, l)$. For further examples, see Fig. 2.

3 Complexity and Fixed-Parameter Tractability

In this section we deal with the topological setting where we are given a planar embedded graph $G = (V, E)$ and a subset $S \subset V$ to be collinear. According to Da Lozzo et al. [5], this is equivalent to the existence of a pseudoline $\mathcal{L}(S)$ with respect to G passing exactly through the vertices in S. We refer to this problem as *pseudoline existence problem*. Using techniques similar to Fößmeier and Kaufmann [9], we can show that the pseudoline existence problem is $\mathcal{NP}$-hard; see Appendix A.1. In the following, we show that the pseudoline existence problem is FPT with respect to $|S|$. In the first step deal with biconnected planar embedded graphs only. Additionally to the set of vertices S we require the pseudoline to pass through a set of faces F. This trick allows us to combine two independent pseudolines of two biconnected components of a planar embedded graph. Thus, let additionally G be biconnected and F be a subset of faces of G. Notice that in case that the vertices of S are not independent, they can only form a *linear forest*, i.e. a set of paths, as otherwise there is no pseudoline through the vertices S with respect to G. Let $G^\star$ be the dual graph of G (see Fig. 3.a) and let $S' \subseteq S$ be the subset of vertices that form end-points of the paths induced by S. We denote by f a face of G and by $f^\star$ its dual vertex. An *extended* dual $G^\star_{\mathrm{e}}(S, F)$ is the graph obtained from $G^\star$ by the following steps. We omit the parameters (S, F) if they are clear from the context.

Step 1: For each path induced by S that contains exactly one edge, subdivide the edge by a vertex and add it to S,

Step 2: Place the vertices of S into the corresponding faces of $G^\star$ and the edges induced by them (red vertices and edges in Fig. 3(a)).

Step 3: connect the vertices of S' to all vertices of the dual face they lie in (red dashed edges in Fig. 3(a)).

Step 4: For each vertex $v \in S$ remove the edges dual to the primal edges incident to v.

Step 5: Remove all the dual vertices that have in degree zero or one; see Fig. 3(b).

Step 6: Replace each vertex $f^\star$ of $G^\star$ with a clique of the size equal the degree of $f^\star$, each vertex v of this clique corresponds to an edge e^* incident to a vertex $f^\star$, thus we call it a *clique vertex* corresponding to $e^\star$ and denote by $\mathrm{cl}(f, e^\star)$. For each edge of $G^\star_e$ that has survived, we connect $>$the two corresponding to it clique vertices.

Step 7: Recall that F is a set of faces of G. Assume we would like our pseudoline to pass through the faces of F. To check for existence of such a pseudoline we further augment the graph $G^\star_{\mathrm{e}}$ as follows. For each $f \in F$, additionally to the clique that is built on face vertices $\mathrm{cl}(f, e^\star_1) \dots \mathrm{cl}(f, e^\star_a)$, corresponding to the edges incident to the dual vertex f, we add a star with a new center vertex $\mathrm{cent}(f)$ that has $\mathrm{cl}(f, e^\star_1) \dots \mathrm{cl}(f, e^\star_a)$ as leaves. Finally, we set $\mathrm{cent}(F) = \{\mathrm{cent}(f) \mid f \in F\}$.

Lemma 1 ($\star$). *Let $G = (V, E)$ be a biconnected planar embedded graph, let $S \subset V$ be a set of vertices that induce a linear forest and let F be a set of faces of G. There exist an aligned graph $(G, \mathcal{L})$, where $\mathcal{L}$ passes through all vertices of S and faces F if and only if there exists a simple cycle C in the extended dual $G^\star_{\mathrm{e}}(S, F)$ through the vertices of $S \cup \mathrm{cent}(F)$.*

We utilize the following theorem.

Theorem 1 (Wahlström [16]). *Given a graph $G = (V, E)$ and a subset $S \subset V$, it can be tested in $O(2^{|S|}\mathrm{poly}(n))$ time whether a simple cycle through the vertices in S exists. In affirmative the cycle can be reported within the same asymptotic time.*

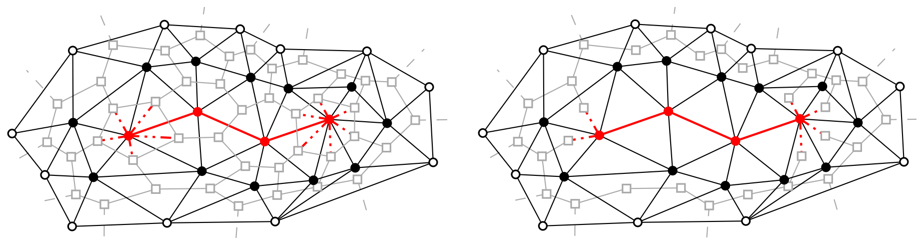

Fig. 3: A fragment of a graph G (black) and of $G^{\star}_{\mathrm{e}}(S, F)$ (gray and red) (a) after Step 3, (b) after Step 6.

Theorem 1 together with Lemma 1 gives a $O(2^{|S|}\mathrm{poly}(n))$ time algorithm to solve the pseudoline construction problem for the case of biconnected graphs. We next sketch how this can be extended to general planar graphs.

Lemma 2 ($\star$). *Let $G = (V, E)$ be a planar embedded graph, $S \subset V$ and c be a cut vertex of G separating G into subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Let $S_1 = S \cap V_1 \neq \emptyset$ and $S_2 = S \cap V_2 \setminus \{c\} \neq \emptyset$. Let f be the face of G_1 where G_2 lies. There exists an aligned graph $(G, \mathcal{L}(S))$ if and only if there exist aligned graphs $(G_1, \mathcal{L}(S_1))$ and $(G_2, \mathcal{L}(S_2))$, such that $\mathcal{L}(S_1)$ passes through face f.*

Utilizing Lemma 2, we can recursively decompose a graph into biconnected components, check for the pseudoline existence by applying Lemma 1 and Theorem 1 and glue the pseudolines if they exist. This implies the following:

Theorem 2 ($\star$). *Given a planar embedded graph $G = (V, E)$ and a subset $S \subset V$, it can be tested in $O(4^{|S|}\mathrm{poly}(n))$ time whether an aligned graph $(G, \mathcal{L}(S))$ exists.*

4 Drawing Aligned Graphs

We show that every aligned graph where each edge either entirely lies on a pseudoline or is intersected by at most one pseudoline, i.e., alignment complexity $(1, 0, \perp)$, has an aligned drawing. For 1-aligned graphs we show the stronger statement that every 1-aligned graph has an aligned drawing with a given aligned convex drawing of the outer face. We first present our proof strategy and then deal with 1- and k-aligned graphs.

4.1 Proof Strategy

Our general strategy for proving the existence of aligned drawings of an aligned graph $(G, \mathcal{A})$ is as follows. First, we show that we can triangulate $(G, \mathcal{A})$ by adding vertices and edges without invalidating its properties. We can thus assume that our aligned graph $(G, \mathcal{A})$ is an aligned triangulation. Second we show that, unless G is a specific graph (e.g., a k-wheel or a triangle), it contains a specific

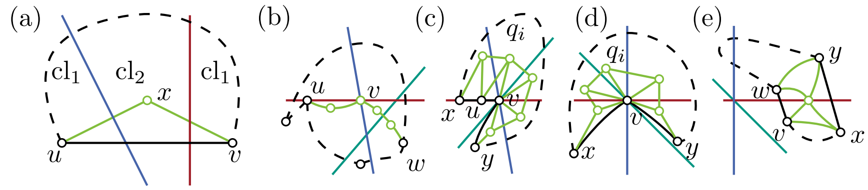

Fig. 4: Steps for triangulating aligned graphs (black) with 1-crossed edges (green).

type of edge, namely an edge that is contained in a pseudoline, or an edge that is not intersected by any of the pseudolines. Third, we exploit the existence of such an edge to inductively prove the existence of an aligned drawing of $(G, \mathcal{A})$. Depending on whether the edge is contained in a separating triangle or not, we either decompose along that triangle or contract the edge. In both cases the problem reduces to smaller instances that are almost independent. In order to combine solutions, it is, however, crucial to use the same arrangement of lines A for both of them.

In the following we introduce the necessary tools used for all three steps on k-aligned graphs of alignment complexity $(1, 0, \perp)$. Recall, that for this class (i) every free edge is at most 1-crossed, (ii) every 1-anchored edge has no intersections, and (iii) there is no edge with its endpoints on two pseudolines. Lemma 3 shows that every aligned graph has an aligned triangulation with almost the same alignment complexity. If G contains a separating triangle, Lemma 4 shows that $(G, \mathcal{A})$ admits an aligned drawing if both split components have an aligned drawing. Finally, with Lemma 5 we obtain a drawing of $(G, \mathcal{A})$ from a drawing of the aligned graph $(G/e, \mathcal{A})$ where one special edge e is contracted. For simplicity we assume the input graph to be 2-connected, general graphs allow similar techniques; see Lemma B.2 in the Appendix.

Lemma 3 ($\star$). *Let $(G, \mathcal{A})$ be a biconnected k-aligned graph of alignment complexity $(2, 0, 0)$. There exists a k-aligned triangulation $(G_T = (V_T, E_T), \mathcal{A})$ of $(G, \mathcal{A})$ whose size is $O(k^4 n)$. Each edge in $E_T \setminus E(G)$ is at most 1-crossed and 0-anchored, or 0-crossed and 1-anchored.*

Proof sketch. To triangulate $(G, \mathcal{A})$, we exhaustively apply each of the following steps.

1. If f is a non-triangular face whose boundary contains a 2-crossed edge uv, we build a triangle by inserting a vertex x in the intermediate cell, as shown in Fig 4(a). This step ensures that every edge of a non-triangular face is at most 1-crossed.
2. If f is a non-triangular face whose interior contains the intersection I of a set of pseudolines, we place a vertex v on I and connect v to two disjoint vertices on f with two simple paths, where every vertex of the path, which is not an endpoint, is free; compare Fig. 4(b).

3. If f is a non-triangular face with an aligned edge $e = uv$ we can split f into two faces f' and f'' (as shown in Fig. 4(c)) such that f' contains e on its boundary. Then we can triangulate f' with 1-crossed edges. A similar approach works for aligned vertices; see Fig. 4(d).
4. If f is a non-triangular face whose interior contains a pseudosegment $\mathcal{S}$, then we find two edges vw, xy as shown in Fig. 4(e) and we can triangulate by inserting a vertex on $\mathcal{S}$ and 1-crossed edges.
5. If none of the cases above applies, then no non-triangular face contains a pseudosegment. Thus all remaining non-triangular faces can be triangulated with free edges.

□

In order to simplify the constructions we augment the input graph with an additional cycle in the outer face, so that no two pseudolines intersect in the outer face. More formally, let $\mathcal{A}$ be an arrangement of pseudolines $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_k$. Let $U_1, U_2, \dots, U_t \in \text{cells}(\mathcal{A})$ be the set of unbounded cells in the arrangement of $\mathcal{A}$ such that U_i, U_{i+1} are adjacent cells with $U_{t+1} = U_1$. For $k > 1$, a k-aligned graph is *proper* (i) if the boundary of the outer face is a 0-anchored 1-crossed cycle of length t such that every unbounded region U_i contains exactly one vertex of the cycle, and (ii) every aligned edge in $(G, \mathcal{A})$ is 0-crossed. Observe that for every k-aligned graph $(G, \mathcal{A})$ there is proper k-aligned triangulation $(G', \mathcal{A})$ containing G as a minor; see Lemma B.3 in the Appendix.

The following two lemmas show that we can reduce the size of the aligned graph and obtain a drawing by merging two drawings or by unpacking an edge.

Lemma 4 (⋆). *Let $(G, \mathcal{A})$ be a k-aligned triangulation. Let T be a separating triangle splitting G into subgraphs $G_{\text{in}}, G_{\text{out}}$ so that $G_{\text{in}} \cap G_{\text{out}} = T$ and G_{out} contains the outer face of G. Then, (i) $(G_{\text{out}}, \mathcal{A})$ and $(G_{\text{in}}, \mathcal{A})$ are k-aligned triangulations, and (ii) $(G, \mathcal{A})$ has an aligned drawing if and only if there exists a common line arrangement A such that $(G_{\text{out}}, \mathcal{A})$ has an aligned drawing (Γ_{out}, A) and $(G_{\text{in}}, \mathcal{A})$ has an aligned drawing (Γ_{in}, A) with the outer face drawn as $\Gamma_{\text{out}}[T]$.*

Lemma 5 (⋆). *Let $(G, \mathcal{A})$ be a k-aligned triangulation of alignment complexity $(1, 0, \perp)$ and let e be a 0-anchored aligned edge or a free edge of G that is not an edge of a separating triangle. Then $(G/e, \mathcal{A})$ is a k-aligned triangulation of alignment complexity $(1, 0, \perp)$. Further, $(G, \mathcal{A})$ has an aligned drawing if $(G/e, \mathcal{A})$ has an aligned drawing.*

Proof sketch. Since u and v either both lie in the same cell or both in the interior of a pseudosegment, $(G/e, \mathcal{A})$ is an aligned triangulation.

Let c be the vertex in G/e obtained by contracting $e = uv$ and let f be the face obtained by removing the vertex c from the aligned drawing (Γ', A) of $(G/e, \mathcal{A})$. We place u at the position of c. This leaves a unique face f' to place v in. Since G/e is a triangulation, f' is star-shaped. Thus we can either place v close to u within its cell or on its line. □

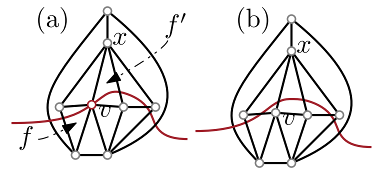

Fig. 5: Consistent transformation from a red vertex (a) to a gray vertex (b).

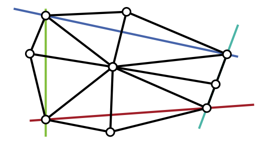

Fig. 6: All possible variations of vertices and edges in Lemma 7.

4.2 One Pseudoline

We show that every 1-aligned graph $(G, \mathcal{R})$ has an aligned drawing (Γ, R), where $\mathcal{R}$ is a single pseudoline and R the corresponding straigh-line.

Lemma 6. *Let $(G, \mathcal{R})$ be a 1-aligned triangulation with k vertices on the outer face without a chord. If G is neither a triangle nor a k-wheel, then $(G, \mathcal{R})$ contains an interior aligned or an interior free edge.*

Proof. We first prove two useful claims.

Claim 1. Consider the order in which $\mathcal{R}$ intersects the vertices and edges of G. If vertices u and v are consecutive on $\mathcal{R}$, then the edge uv is aligned.

Observe that the edge uv can be inserted into G without creating crossings. Since G is a triangulation, it therefore contains uv, and, further, since no 1-crossed edge can have both its endpoints on $\mathcal{R}$, it follows that indeed uv is aligned. This proves the claim.

Claim 2. Let $(G, \mathcal{R})$ be an aligned triangulation without aligned edges and let x be an interior free vertex of G, then x is incident to a free edge.

Assume for a contradiction that all neighbors of x lie either on $\mathcal{R}$ or on the other side of $\mathcal{R}$. First, we slightly modify $\mathcal{R}$ to a curve $\mathcal{R}'$ that does not contain any vertices. Assume v is an aligned vertex; see Fig.5. Since there are no aligned edges, $\mathcal{R}$ enters v from a face f incident to v and leaves it to a different face f' incident to v. We then reroute $\mathcal{R}$ from f to f' locally around v. If v is incident to x, we choose the rerouting such that it crosses the edge vx.

Notice that if e is intersected by $\mathcal{R}$ in its endpoints, then $\mathcal{R}'$ either does not intersect it, or intersects it in an interior point. Moreover, e cannot be intersected by $\mathcal{R}'$ twice as in such case $\mathcal{R}$ would pass through both its endpoints. Therefore $(G, \mathcal{R}')$ is an aligned graph without any aligned vertices. Now, since G is a triangulation, $\mathcal{R}'$ is a simple cycle in the dual $G^\star$ of G, and hence corresponds to a cut C of G. Let H denote the connected component of $G - C$ that contains x and note that all edges of H are free. By the assumption and the construction of $\mathcal{R}'$, x is the only vertex in H. Thus, $\mathcal{R}'$ intersects only the faces incident to x which are interior. This contradicts the assumption that $\mathcal{R}'$ passes through the outer face of G. This finishes the proof of the claim.

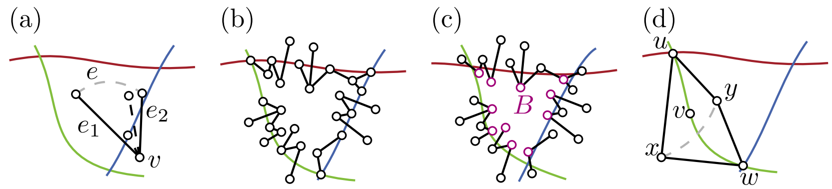

Fig. 7: Illustrations for the proof Lemma 7

We now prove the lemma. Assume that G is neither a triangle nor a k-wheel. Thus, G contains at least two interior vertices. If one of both vertices is free, we find a free edge by Claim 2. Otherwise, there is no free internal vertex, therefore the only edge which can intersect $\mathcal{R}$ is a chord of G. Since G does not contain any chord, there is a pair of aligned vertices consecutive along $\mathcal{R}$. Thus by Claim 1 the instance $(G, \mathcal{R})$ has an aligned edge.

□

Theorem 3 (⋆). *Let $(G, \mathcal{R})$ be an aligned graph and let (Γ_O, R) be a convex aligned drawing of the aligned outer face $(O, \mathcal{R})$ of G. There exists an aligned drawing (Γ, R) of $(G, \mathcal{R})$ with the same line R and the outer face drawn as Γ_O.*

Proof sketch. Given an arbitrary aligned graph $(G, \mathcal{R})$, we first triangulate it using Lemma 3. As long as it has a free or an aligned edge e we do the following. If e is contained in a separating triangle, we decompose the graph using Lemma 4. Otherwise we simply contract e (Lemma 5). If no such edge exists, $(G, \mathcal{R})$ is either a triangle or a k-wheel (Lemma 6) and has an obvious straight-line aligned drawing. We obtain an aligned drawing of (G, R) by reversing the sequence contraction (Lemma 5) and decompositions along the separating triangles (Lemma 4). □

4.3 Alignment Complexity $(1, 0, \perp)$

Let $(G, \mathcal{A})$ be a k-aligned graph of alignment complexity $(1, 0, \perp)$, i.e., every edge has at most one interior intersection and 2-anchored edges are forbidden. In this section, we prove that every such k-aligned graph has an aligned drawing. Fig. 6 illustrates the statement of the following lemma.

Lemma 7. *Let $(G, \mathcal{A})$ be a proper k-aligned triangulation of alignment complexity $(1, 0, \perp)$ that does neither contain an interior free edge, nor a 0-anchored aligned edge, nor a separating triangle. Then (i) every intersection contains a vertex, (ii) every cell of the pseudoline arrangement contains exactly one free vertex, (iii) every pseudosegment is either covered by two aligned edges or intersected by an edge.*

Proof. The statement follows trivially from the following sequence of claims. We refer to an aligned vertex that is not an intersection vertex as a *flexible aligned* vertex.

Claim 1. Every pseudosegment alternately intersects flexible aligned vertices and edges.

Let $\mathcal{S}$ be a pseudosegment in the pseudoline arrangement A. As in the proof of Lemma 6 one can argue that if two vertices occur consecutively along $\mathcal{S}$, then we find an aligned edge. Assume that $\mathcal{S}$ intersects two edges e_1, e_2 consecutively as depicted in Fig 7(a). Since G is a triangulation, it follows that e_1 and e_2 share an endpoint v. Every 1-crossed edge in G is 0-anchored, thus all endpoints of e_1 and e_2 must be free. Further e_1 and e_2 are consecutive in the circular order of edges around v as otherwise we would either find an intersection with $\mathcal{S}$ between e_1 and e_2 or a free edge. Thus, e_1 and e_2 bound a face and are 1-crossed, hence their endpoints distinct from v are in the same cell and connected by an edge e, which is thus free. In a proper graph, the edges on the outer face are 1-crossed, thus, e is an interior edge, contradicting our assumptions.

Claim 2. Every cell contains at least one free vertex.

Observe that every triangle T containing the intersection of two pseudolines has at least one l-crossed edge, with $l \geq 2$. Since by definition $(G, \mathcal{A})$ does not contain 2-anchored aligned edges, T cannot contain an empty cell in its interior. Further, since G is proper, the outer face of G contains the intersection of every pair of pseudolines in its interior. Thus, since G is triangulated every cell contains at least one vertex.

Claim 3. Every cell contains at most one free vertex.

The following proof is similar to Claim 2 in the proof of Lemma 6. Let $\mathcal{C}$ be a cell and assume for the sake of a contradiction that $\mathcal{C}$ contains more than one vertex in the interior; see Fig. 7(b). These vertices are connected by edges to adjacent cells. If $\mathcal{C}$ contains a vertex v on its boundary, we reroute the corresponding pseudolines close to v such that v is now outside of $\mathcal{C}$; refer to Fig. 7(c). Let $\mathcal{C}'$ be the resulting cell, it represents a cut in the graph with two components A and B, where $\mathcal{C}'$ contains B in its interior. It is not difficult to see that the modified pseudolines are still pseudolines with respect to G. Since $(G, \mathcal{A})$ neither contains l-anchored edges nor l-crossed edges, $l \geq 2$, every edge of $(G, \mathcal{A}')$ intersects the boundary of $\mathcal{C}'$ at most once. Hence, B is connected and since it contains at least two vertices it also contains at least one free edge, contradicting our initial assumption.

Claim 4. Every flexible aligned vertex is incident to two 1-anchored aligned edges.

Let v be a vertex lying in the interior of a pseudosegment $\mathcal{S}$. Let u and w be the anchored vertices incident to $\mathcal{S}$. Further, let x and y be the vertices in the interior of the two cells incident to $\mathcal{S}$. Our instance $(G, \mathcal{A})$ is triangulated and every edge is at most 1-crossed. Thus, the vertices u, x, w, y build a quadrangle containing v in its interior. Since G does not contain a separating triangles, it cannot contain the edge xy. Moreover, $\mathcal{S}$ contains exactly v in its interior, otherwise we would find a free aligned edge. Finally, since $(G, \mathcal{A})$ is an aligned triangulation, the vertex v is connected to all four vertices and, thus incident to two 1-anchored aligned edges.

Since $(G, \mathcal{A})$ is an aligned triangulation, Property (iii) immediately follows from Claims 3 and 4. □

Lemma 8. *Let $(G, \mathcal{A})$ be a proper k-aligned triangulation of alignment complexity $(1, 0, \perp)$ that does neither contain an interior free edge, nor a 0-anchored aligned edge, nor a separating triangle. Let A be a line arrangement homeomorphic to the pseudoline arrangement $\mathcal{A}$. Then $(G, \mathcal{A})$ has an aligned drawing (Γ, A).*

Proof. We obtain a drawing (Γ, A) by placing a free vertex in its cell, an aligned vertex on its pseudosegment and an intersection vertex on its intersection. According to Lemma 7 every cell and every intersection contains exactly one vertex and each pseudosegment is either crossed by an edge or it is covered by two aligned edges. Observe that the union of two adjacent cells of the arrangement A is convex. Thus, this drawing of G has an homeomorphic embedding to $(G, \mathcal{A})$ and every edge intersects in (Γ, A) the line $L \in A$ corresponding to the pseudoline $\mathcal{L} \in \mathcal{A}$ in $(G, \mathcal{A})$ □

The following theorem can be proven along the same lines as Theorem 3.

Theorem 4 ($\star$). *Every k-aligned graph $(G, \mathcal{A})$ of alignment complexity $(1, 0, \perp)$ with a stretchable pseudoline arrangement $\mathcal{A}$ has an aligned drawing.*

5 Conclusion

In this paper we showed that if $\mathcal{A}$ is stretchable, then every k-aligned graph $(G, \mathcal{A})$ of alignment complexity $(1, 0, \perp)$ has a straight-line aligned drawing. As an intermediate result we showed that a 1-aligned graph $(G, \mathcal{R})$ has an aligned drawing with a fixed convex drawing of the outer face. We showed that the less restricted version of this problem, where we are only given a set of vertices to be aligned, is $\mathcal{NP}$-hard but fixed-parameter tractable.

The case of more general alignment complexities is widely open. Our techniques imply the existence of one-bend aligned drawings of general 2-aligned graphs [13]. However, the existence of straight-line aligned drawings is unknown even if in addition to 1-crossed edges, we only allow 2-anchored edges, i.e., in the case of alignment complexity $(1, 0, 0)$. In particular, there exist 2-aligned graphs that neither contain a free edge nor an aligned edge but their size is unbounded in the size of the arrangement; see Appendix C. It seems that further reductions are necessary to arrive at a base case that can easily be drawn. This motivates the following questions.

1) What are all the combinations of line numbers k and alignment complexities C such that for every k-aligned graph $(G, \mathcal{A})$ of alignment complexity C there exists a straight-line aligned drawing provided $\mathcal{A}$ is stretchable.
2) Given a k-aligned graph $(G, \mathcal{A})$ and a line arrangement A homeomorphic to $\mathcal{A}$, what is the complexity of deciding whether $(G, \mathcal{A})$ admits a straight-line aligned drawing (Γ, A)?

References

1. Biedl, T.C., Kaufmann, M., Mutzel, P.: Drawing planar partitions II: HH-drawings. In: Hromkovič, J., Sýkora, O. (eds.) WG'98. pp. 124–136. Springer (1998)
2. Cano, J., Tóth, C.D., Urrutia, J.: Upper bound constructions for untangling planar geometric graphs. SIAM Journal on Discrete Mathematics 28(4), 1935–1943 (2014)
3. Chaplick, S., Fleszar, K., Lipp, F., Ravsky, A., Verbitsky, O., Wolff, A.: Drawing graphs on few lines and few planes. In: Hu, Y., Nöllenburg, M. (eds.) GD'16. pp. 166–180. Springer (2016)
4. Chaplick, S., Fleszar, K., Lipp, F., Ravsky, A., Verbitsky, O., Wolff, A.: The complexity of drawing graphs on few lines and few planes. In: Ellen, F., Kolokolova, A., Sack, J.R. (eds.) Algorithms and Data Structures: 15th International Symposium, WADS 2017, St. John's, NL, Canada, July 31 – August 2, 2017, Proceedings. pp. 265–276. Springer, Cham (2017)
5. Da Lozzo, G., Dujmović, V., Frati, F., Mchedlidze, T., Roselli, V.: Drawing planar graphs with many collinear vertices. In: Hu, Y., Nöllenburg, M. (eds.) GD'16. pp. 152–165. Springer (2016)
6. Dujmović, V.: The utility of untangling. In: Di Giacomo, E., Lubiw, A. (eds.) GD'15. pp. 321–332. Springer (2015)
7. Dujmović, V., Evans, W., Kobourov, S., Liotta, G., Weibel, C., Wismath, S.: On graphs supported by line sets. In: Brandes, U., Cornelsen, S. (eds.) GD'10. pp. 177–182. Springer (2011)
8. Dujmović, V., Langerman, S.: A center transversal theorem for hyperplanes and applications to graph drawing. Discrete & Computational Geometry 49(1) (2013)
9. Fößmeier, U., Kaufmann, M.: Nice drawings for planar bipartite graphs. In: Algorithms and Complexity, Third Italian Conference, CIAC '97, Rome, Italy, March 12-14, 1997, Proceedings. pp. 122–134 (1997)
10. Garey, M.R., Johnson, D.S., Tarjan, R.: The planar Hamiltonian circuit problem is NP-complete. SIAM Journal on Computing 5(4), 704–714 (1976)
11. Goodman, J.E., Pollack, R.: Proof of Grünbaum's conjecture on the stretchability of certain arrangements of pseudolines. Journal of Combinatorial Theory, Series A 29(3), 385 – 390 (1980)
12. Levi, F.: Die Teilung der projektiven Ebene durch Geraden oder Pseudogerade. Ber. Math.-Phys. Kl. Sächs. Akad. Wiss 78, 256–267 (1926)
13. Mchedlidze, T., Radermacher, M., Rutter, I.: Aligned drawings of planar graphs. In: Proceedings of the 33rd European Workshop on Computational Geometry (EuroCG 2017) (2017)
14. Mnëv, N.E.: The universality theorems on the classification problem of configuration varieties and convex polytopes varieties. In: Viro, O.Y., Vershik, A.M. (eds.) Topology and Geometry — Rohlin Seminar. LNCS, vol. 1346, pp. 527–543. Springer (1988)
15. Shor, P.: Stretchability of pseudolines is NP-hard. In: Applied Geometry and Discrete Mathematics–The Victor Klee Festschrift, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, vol. 4, pp. 531–554. Amer. Math. Soc. (1991)
16. Wahlström, M.: Abusing the Tutte matrix: An algebraic instance compression for the k-set-cycle problem. In: 30th International Symposium on Theoretical Aspects of Computer Science, STACS 2013, February 27 - March 2, 2013, Kiel, Germany. pp. 341–352 (2013)

A Omitted Proofs of Section 3

A.1 Complexity

Let $G^\star + V$ be the graph obtained from the dual graph $G^\star = (V^\star, E^\star)$ of $G = (V, E)$ by placing every vertex $v \in V$ in its dual face $v^\star$ and connect it with a set of edges to every vertex on the boundary of the face $v^\star$.

Lemma A.1. *Let $G = (V, E)$ be a 3-connected 3-regular planar graph. There exists a pseudoline through V with respect to the graph $G^\star + V$ if and only if G is Hamiltonian.*

Proof. Recall that the dual of a 3-connected 3-regular graph is a triangulation.

"$\Rightarrow$": Assume that there exists a pseudoline $\mathcal{L}$ through V with respect to $G^\star + V$, then the order of appearance of the vertices of $G^\star + V$ on $\mathcal{L}$ defines a sequence of faces in $G^\star$, i.e. vertices of the primal G, connected via primal edges. This yields a Hamiltonian cycle in G.

"$\Leftarrow$": Let C be a Hamiltonian cycle of G and consider a simultaneous embedding of G and $G^\star + V$ on the plane, where each pair of primal and its dual edge intersect exactly once. Thus, the cycle C crosses each dual edge e at most once and passes through exactly the vertices V. There is a vertex v on the cycle C such that v lies in unbounded face of $G^\star + V$. Thus, the cycle C can be interpreted as a pseudoline $\mathcal{L}(V)$ in $G^\star + V$ through all vertices in V by splitting it in the unbounded face of $G^\star + V$.

□

Since computing a Hamiltonian cycle in 3-connected 3-regular planar graphs is $\mathcal{NP}$-complete [10], we get the following

Theorem A.1. *The pseudoline existence problem is $\mathcal{NP}$-hard.*

A.2 FPT

Lemma 1. *Let $G = (V, E)$ be a biconnected planar embedded graph, let $S \subset V$ be a set of vertices that induce a linear forest and let F be a set of faces of G. There exist an aligned graph $(G, \mathcal{L})$, where $\mathcal{L}$ passes through all vertices of S and faces F if and only if there exists a simple cycle C in the extended dual $G^\star_{\mathrm{e}}(S, F)$ through the vertices of $S \cup \mathrm{cent}(F)$.*

Proof. Assume that there exists an aligned graph $(G, \mathcal{L})$, where $\mathcal{L}$ passes through all vertices of S and F. This curve can be cut into pieces such that each piece falls into one of three categories. For each of them we explain how to substitute it by an edge of $G^\star_{\mathrm{e}}(S, F)$ that all together form the cycle C. To identify the used edges we refer to the steps of construction of $G^\star_{\mathrm{e}}(S, F)$.

- A piece that enters a face f of G through an edge e and leaves it through edge e_1. Since the edge e (reps. e_1) is neither incident to a vertex in S, nor it is induced by vertices of S, after Step 4, graph $G^\star_{\mathrm{e}}$ still contains an edge

$e^\star$ (resp. $e_1^\star$) dual to e (resp. e_1). The dual vertex $f^\star$ was substituted by a clique and the clique vertices $\mathrm{cl}(f, e^\star)$ and $\mathrm{cl}(f, e_1^\star)$, representing $e^\star$ and $e_1^\star$, are connected by an edge e' of this clique. In case $f \in F$, vertices $\mathrm{cl}(f, e^\star)$ and $\mathrm{cl}(f, e_1^\star)$ also belong to a star with the center $\mathrm{cent}(f)$. Observe that $\mathcal{L}$ may visit f several times, if $f \in F$ then one of this visits is substituted by the edges $e^\star$, $(\mathrm{cl}(f, e^\star), \mathrm{cent}(f))$, $(\mathrm{cent}(f), \mathrm{cl}(f, e_1^\star))$ and $e_1^\star$ of $G_\mathrm{e}^\star$ all the other visits are substituted by the edges $e^\star$, e' and $e_1^\star$ of $G_\mathrm{e}^\star$.
- A piece connecting a face f of G and an incident vertex v of S. Since $\mathcal{L}$ is a pseudoline through G, v is an end-vertex of a path in S. It can be substituted by an edge of $G_\mathrm{e}^\star$ connecting v and the vertex representing f (refer to Step 3).
- A piece that visits edges induced by S. It is substituted by the edges themselves (Step 2).

Since the curve $\mathcal{L}$ intersects every edge e at most once, and since each face f of G has a unique clique vertex representing e, each clique vertex of $G_\mathrm{e}^\star$ appears at most once in C. Each vertex of S also appears once on $\mathcal{L}$, thus C is simple and passes through all vertices of S. Finally $\mathcal{L}$ visits each face of F and therefore, by the previous substitution, it also visits the vertices $\mathrm{cent}(F)$.

Conversely, assume that there exists a simple cycle C in the extended dual $G_\mathrm{e}^\star$ that visits all vertices of $S \cup \mathrm{cent}(F)$. The curve $\mathcal{L}$ is just routed along this cycle. Notice that, if C visits a vertex v of G, it cannot cross an edge e incident to it, as the edge dual to e is not part of $G_\mathrm{e}^\star$, see Step 4. Cycle C is simple and therefore no dual edge is used twice, thus $\mathcal{L}$ cannot cross an edge of G twice. If $\mathcal{L}$ visits an end-point of a non-trivial path induced by S, then it traverses the path until the other end-point, since each path induced by S has an internal vertex that can only be reached from the end-vertices (see Step 1). Thus $\mathcal{L}$ visits all vertices of S in the order they appear along the paths induced by S. Since C contains vertices $\mathrm{cent}(F)$, curve $\mathcal{L}$ visits each of the faces F. Note that the curve is not yet a pseudoline, since it may have self intersections. We iteratively remove a self-intersection I as follows; compare Fig. 8. Every intersection is due to the replacement of $f^\star$ with a clique within a face f. Hence, there exists a circle D around I neither containing a second self-intersection nor a vertex nor an edge of G. Further, let $\alpha, \beta, \gamma, \delta$ the intersection of D with $\mathcal{L}$. We replace the piece $\alpha\gamma$ with a piece $\alpha\beta$ and $\beta\delta$ with a piece $\gamma\delta$ and route them through the interior of D such that both segments do not intersect. Thus, $\mathcal{L}$ is a pseudoline with respect to G that passes through vertices S and faces F.

□

Lemma 2. *Let $G = (V, E)$ be a planar embedded graph, $S \subset V$ and c be a cut vertex of G separating G into subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Let $S_1 = S \cap V_1 \neq \emptyset$ and $S_2 = S \cap V_2 \setminus \{c\} \neq \emptyset$. Let f be the face of G_1 where G_2 lies. There exists an aligned graph $(G, \mathcal{L}(S))$ if and only if there exist aligned graphs $(G_1, \mathcal{L}(S_1))$ and $(G_2, \mathcal{L}(S_2))$, such that $\mathcal{L}(S_1)$ passes through face f.*

Proof. For the necessity, assume that $\mathcal{L}(S_1)$ and $\mathcal{L}(S_2)$ exist, such that $\mathcal{L}(S_1)$ passes through face f, Figure 9.a. Let $\gamma - \gamma'$ be a piece of $\mathcal{L}(S_1)$ that lies inside f of G_1. Let $\delta - \delta'$ be a piece of $\mathcal{L}(S_2)$ that lies on the outer face of G_2. By

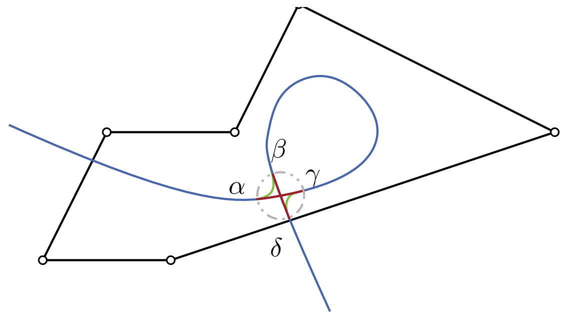

Fig. 8: Resolving a crossing

cutting the curve $\mathcal{L}(S_1)$ at γ, γ' and $\mathcal{L}(S_2)$ at δ, δ' and joining these points so that no crossing is introduced, we construct a curve $\mathcal{L}(S)$. In case $c \in S$, since $c \notin S_2$ the overall curve $\mathcal{L}(S)$ does not pass through c twice.

For the sufficiency, assume that $\mathcal{L}(S)$ exists, Figure 9.b. Let $\gamma_i - \gamma_i'$, $i = 1, \ldots, a$ be a maximal piece of $\mathcal{L}(S)$ that lies inside f without crossing elements of G_1 or G_2. We cut $\mathcal{L}(S)$ at γ_i, γ_i', $i = 1, \ldots, a$ and join the consecutive peaces that pass through G_1 (reps. G_2) into $\mathcal{L}(S_1)$ (resp. $\mathcal{L}(S_2)$). Notice that if $c \in S$, then the curve $\mathcal{L}(S_2)$ can be rerouted through an edge incident to c. $\square$

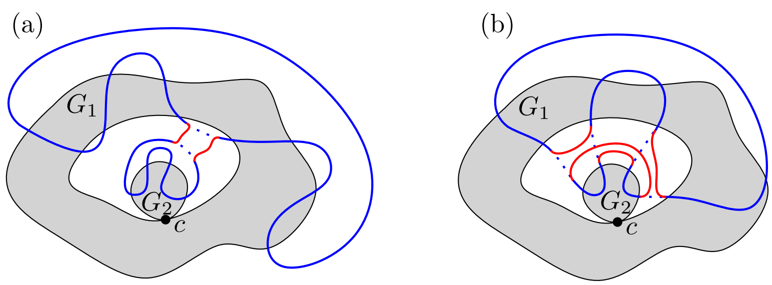

Fig. 9: (a) Curves $\mathcal{L}(S_1)$ and $\mathcal{L}(S_2)$ (blue) are joined into $\mathcal{L}(S)$ (by red segments). (b) Curve $\mathcal{L}(S)$ (blue) is separated into $\mathcal{L}(S_1)$ and $\mathcal{L}(S_2)$ (using the reg segments).

Theorem 2. *Given a planar embedded graph $G = (V, E)$ and a subset $S \subset V$, it can be tested in $O(4^{|S|}\text{poly}(n))$ time whether an aligned graph $(G, \mathcal{L}(S))$ exists.*

Proof. Decompose G into biconnected components $G_1, \ldots, G_l$, set $S_i = V(G_i) \cap S$. Let B the block-cut tree representing this decomposition. A single vertex v_i

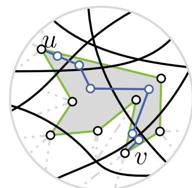

Fig. 10: Connecting two vertices in black pseudoline arrangement $\mathcal{A}$ and the interior of a green face f.

in B corresponds to a biconnected component G_i, an edge between v_i and v_j means that G_i and G_j have a common cut vertex.

A vertex v_i is *marked* if a child of v_i is marked or G_i contains vertices in S. If v_i is marked, G_i is marked as well. For each G_i let F_i be the set of faces that contain the marked child components. By Lemma 1 and Theorem 1 we can test in $O(2^{|S_i|+|F_i|}\text{poly}(n_i))$ time whether G_i has a curve $\mathcal{L}_i$ passing through vertices S_i and faces F_i. With a post order traversal of B we recursively combine curves $\mathcal{L}_i$ via Lemma 2.

Since the number of marked faces $|F_i| \leq |S|, l \leq n$ and

$$\sum_{i=1}^{l} 2^{|S_i|+|F_i|} \leq \sum_{i=1}^{l} 2^{|S_i|+|S|} \leq \sum_{i=1}^{l} 2^{2|S|} \leq l4^{|S|}$$

the overall time complexity is $O(4^{|S|}\text{poly}(n))$.

□

B Omitted Proofs of Section 4.1

Lemma B.1. *Let $(G, \mathcal{A})$ be a k-aligned graph with two distinct vertices u, v incident to a common face f. Then there is a k-aligned graph $(G', \mathcal{A})$ containing G as a subgraph with a path P from u to v in the interior of f. The edges in $E(P)$ are at most 1-crossed and 0-anchored, or 0-crossed and 1-anchored. The size of P is in $O(k^2 n)$.*

Proof. We obtain a k-aligned graph $(G', \mathcal{A})$ by routing path P from u to V through the interior of f; compare Fig. 10. Let P be a path from u to v in the interior of f that has a vertex in the interior of a cell. As each of the $O(k^2)$ cells in $\mathcal{A}$ may be subdivided into $O(n)$ smaller cells by f, a shortest path from u to v contains $O(k^2 n)$ vertices. □

Lemma B.2. *Let $(G, \mathcal{A})$ be a k-aligned graph. Then there exists a 2-connected k-aligned graph $(G', \mathcal{A})$ that contains G as a subgraph. The set $E(G') \setminus E(G)$ is at most 1-crossed and 1-anchored, or 0-crossed and 1-anchored. The size of $E(G') \setminus E(G)$ is in $O(k^2 n^2)$.*

Proof. While G is not connected, we use Lemma B.1 to connect two vertices from distinct connected components. Afterwards, we use Lemma B.1 to iteratively connect vertices from different connected components. This results in the claimed biconnected k-aligned graph $(G', \mathcal{A})$. Since there are $O(n)$ connected components and $O(n)$ cut vertices, it follows that Lemma B.1 is applied $O(n)$ times, and hence G' has size $O(k^2n^2)$. □

Lemma 3. *Let $(G, \mathcal{A})$ be a biconnected k-aligned graph of alignment complexity $(2,0,0)$. There exists a k-aligned triangulation $(G_T = (V_T, E_T), \mathcal{A})$ of $(G, \mathcal{A})$ whose size is $O(k^4n)$. Each edge in $E_T \setminus E(G)$ is at most 1-crossed and 0-anchored, or 0-crossed and 1-anchored.*

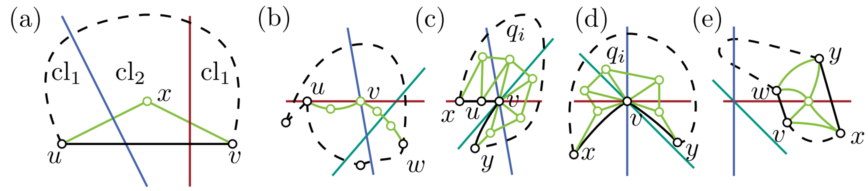

Fig. 11: (a) Isolate a 2-crossed edge from the face. (b) Isolate an intersection. (c-d) Isolate an aligned edge or vertex. (e) Isolate a pseudosegment. Green edges and vertices are inserted to the (black) face.

Proof. We call a face non-triangular, if its boundary contains more than three vertices. An aligned vertex v or an aligned edge e is *isolated* if all faces with v or e on its boundary are triangles. A pseudosegment s is *isolated* if s does not intersect the interior of a non-triangular face.

Our proof is structured as follows. First, if a non-triangular face contains a 2-crossed edge, we insert two 1-crossed edge in order to separate the edge from this face. In the actual triangulation step, this guarantees that every edge of a non-triangular face is at most 1-crossed. The second step, the actual triangulation, distinguishes between four cases. Each case is applied exhaustively in this order. Let f be a non-triangular face.

1. If the interior of f contains the intersection of two or more pseudolines, we split the face, so that there is a vertex that lies on the intersection.
2. If the boundary of a face has an aligned vertex or an aligned edge, we isolate the vertex or the edge from f.
3. If the interior of a face f is intersected by a pseudoline $\mathcal{L}$ we isolate the respective parts of the pseudolines.
4. Finally, if none of the previous cases applies, i.e., neither the boundary nor the interior of f contain parts of a pseudoline, the face f can be triangulated with a set of additional free edges.

Let f be a non-triangular face with a 2-crossed edge uv. Due to the alignment complexity of the graph, this edge is 0-anchored. Thus, let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \in \text{cells}(\mathcal{A})$

be the three cells in f that occur in this order along uv. Then place a vertex x in the interior of $\mathcal{C}_2$ and insert the two edges ux, xv and route both edges close to uv; see Fig. 11(a).

Thus, in the following we assume that only triangular face contain may have 2-crossed edges. Let $\mathcal{A}_f$ be the arrangement of $\mathcal{A}$ and f restricted to the interior of f.

1. Let f be a non-triangular face whose interior contains and intersection of a set of pseudolines; see Fig. 11(b). We place a vertex x on the intersection. We obtain a 2-connected aligned graph with the application of Corollary B.2. There are at most $O(k^2)$ intersections. Isolating each uses at most $O(k^2 n)$ additional vertices and edges. Thus, in total the isolation of intersections increases the graph size by $O(k^4 n)$.
2. Let f be a non-triangular face with an aligned vertex or an aligned edge uv on its boundary. Further, the interior of f does not contain the intersection of some pseudolines; see Fig. 11(c-d). In case of an aligned vertex we simply assume $u = v$. Since G is biconnected, there exist two edges xu, vy on the boundary of f.
   There are at most $2k$ cells $\mathcal{C}_1, \ldots, \mathcal{C}_l \in \text{cells}(\mathcal{A}_f), l \leq 2k$ with u or v on their boundary, such that $\mathcal{C}_i$ is adjacent to $\mathcal{C}_{i+1}, i < l$. We construct a graph $(G', \mathcal{A})$ from $(G, \mathcal{A})$ as follows. We place a vertex q_i in the interior of each cell $\mathcal{C}_i, i \leq l$. Let $q_0 = x$ and $q_{k+1} = y$. We insert edges $e_i = q_i q_{i+1}, i = 0, \ldots, k$ in the interior of f and so that interior of e_i crosses the boundary of $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ exactly once and no other boundary. Thus, each edge e_i is an 1-crossed edge. This splits f into two face f', f'' with a unique faces f' containing u and v on its boundary. If $w \in \{u, v\}$ is on the boundary of cell $\mathcal{C}_i$ we insert an edge wq_i. Let $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ be two cells incident to w Then, the vertices w, q_i, q_{i+1} form a triangle. If $u \neq v$, there is a unique cell $\mathcal{C}_i$ incident to u and v. Hence, the vertices u, v, q_i form a triangle. Moreover, for $1 \leq i \leq k$, every edge uq_i and vq_i is incident to two triangles. Therefore, f' is triangulated. The edges e_i are either 1-crossed and 0-anchored, or 0-crossed and 1-anchored, and thus, the number of aligned edges and aligned vertices of f'' is one less compared to f. Hence, we can inductively proceed on f''.
   Since a vertex is incident to $O(k)$ cells and lies in on the boundary of at most $\deg(v)$ faces, we can isolate a vertex or edge with $O(k + \deg(v))$ additional vertices and edges. Since there are $O(k^2)$ intersections in the pseudoline arrangement. We get in total $O(k^2(k + n))$ additional edges and vertices. The same bounds hold for an aligned edge.
3. Let f be a non-triangular face whose interior intersects a pseudoline $\mathcal{L}$ and has no aligned edge and no aligned vertex. Further, the interior of f does not contain the intersection of two or more pseudolines. Then the face f subdivides $\mathcal{L}$ into a set of pseudosegments $\mathcal{S}$; see Fig. 11(e). We iteratively isolate such a pseudosegment $\mathcal{S}$. Let E' be the set of edges intersecting a pseudosegment $\mathcal{S}$ (at the endpoints of $\mathcal{S}$). Let $\mathcal{C}_1 \in \text{cells}(\mathcal{A}_f)$ and $\mathcal{C}_2 \in \text{cells}(\mathcal{A}_f)$ be the cells with $\mathcal{S}$ on their boundary. Since f does neither contain an aligned vertex nor an aligned edge and G is 2-connected, there are exactly

two edges $e_1 = vw$ and $e_2 = xy$ with the endpoints of s on the interior of these edges. Since f does not contain an intersection of two pseudolines, the vertices v (w) and x (y) lie on the boundary of the same cell, and v and w on the boundary of different cells. Since f does not have a 2-crossed edge, the vertices v, w, x, y are free vertices. We construct a graph G' by placing a vertex u on s and insert edges uv, uw, ux uy, vx and wy. We route each edge so that it the interior of an edge does not intersect the boundary of a region Q_i. Thus, the edges vx and wy are free and the others are 1-anchored and 0-crossed.

Every edge in f is at most 1-crossed, thus the number of pseudosegments is linear in the size of G. Thus, we add only a number of vertices and edges linear in the size of G in order to isolate all pseudosegments.

4. If non of the cases above apply to a non-triangular face f, then neither the interior nor the boundary of the face intersects a pseudoline $\mathcal{L}_i$. Thus, we can triangulate f with a number of free edges linear in the size of f.

Thus, in total there is an aligned triangulation $(G_t, \mathcal{A})$ of G whose size is in $O(k^4 n + k^2(k+n) + n + n) = O(k^4 n)$. The additional set of edges is at most 1-anchored and at most 1-crossed. □

Lemma B.3. *For $k \geq 2$ and every k-aligned graph $(G, \mathcal{A})$ of alignment complexity $(2, 0, \perp)$ there exists a proper k-aligned triangulation $(G', \mathcal{A})$ of alignment complexity $(2, 0, \perp)$ containing G as a minor whose size is in $O(k^4 n^2)$. The edges $E(G') \setminus E(G)$ are at most 1-crossed 0-anchored, or 0-crossed and 1-anchored.*

Proof. We first construct an aligned graph $(G^1, \mathcal{A})$ whose outer face is a simple cycle C so that every unbounded cell $\mathcal{A}$ contains exactly one vertex of C and every edge of C is 1-crossed. By Lemma B.2 and Lemma 3 we obtain a k-aligned triangulation $(G^2, \mathcal{A})$. We construct a proper k-aligned triangulation from $(G^2, \mathcal{A})$ by placing subdivision vertices on the intersection of an aligned edge with a pseudoline in $\mathcal{A}$.

We briefly describe how to construct $(G^1, \mathcal{A})$. Let $U_1, U_2, \ldots, U_t \in \text{cells}(\mathcal{A})$ be the set of unbounded cells of $\mathcal{A}$ such that U_i and U_{i+1} are adjacent cells, with $U_{t+1} = U_1$. Let $(G, \mathcal{A})$ be a k-aligned graph, for $k > 1$. In this case, place a vertex v_i in each unbounded cell U_i and connect two distinct vertices v_i and v_j if the respective cells are adjacent. □

Lemma 4. *Let $(G, \mathcal{A})$ be a k-aligned triangulation. Let T be a separating triangle splitting G into subgraphs $G_{\text{in}}, G_{\text{out}}$ so that $G_{\text{in}} \cap G_{\text{out}} = T$ and G_{out} contains the outer face of G. Then, (i) $(G_{\text{out}}, \mathcal{A})$ and $(G_{\text{in}}, \mathcal{A})$ are k-aligned triangulations, and (ii) $(G, \mathcal{A})$ has an aligned drawing if and only if there exists a common line arrangement A such that $(G_{\text{out}}, \mathcal{A})$ has an aligned drawing (Γ_{out}, A) and $(G_{\text{in}}, \mathcal{A})$ has an aligned drawing (Γ_{in}, A) with the outer face drawn as $\Gamma_{\text{out}}[T]$.*

Proof. Its easy to verify that $(G_{\text{out}}, \mathcal{A})$ and $(G_{\text{in}}, \mathcal{A})$ are aligned triangulations. An aligned drawing (Γ, A) of $(G, \mathcal{A})$ immediately implies the existence of an aligned drawing (Γ_{out}, A) of $(G_{\text{out}}, \mathcal{A})$ and (Γ_{in}, A) of $(G_{\text{in}}, \mathcal{A})$.

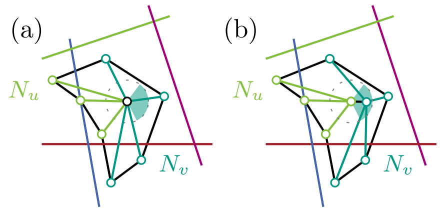

Fig. 12: Unpacking an edge in a drawing Γ' of G/e (a) to obtain a drawing Γ of G (b).

Let (Γ_{out}, A) be an aligned drawing of $(G_{\text{out}}, \mathcal{A})$. Since (Γ_{out}, A) is an aligned drawing, $(\Gamma_{\text{out}}[T], A)$ is an aligned drawing of $(T, \mathcal{A})$. Let Γ be the drawing obtained by merging the drawing Γ_{out} and Γ_{in}. Since, (Γ_{out}, A) and (Γ_{in}, A) are aligned drawings on the same line arrangement A, (Γ, A) is an aligned drawing of $(G, \mathcal{A})$. □

Lemma 5. *Let $(G, \mathcal{A})$ be a k-aligned triangulation of alignment complexity $(1, 0, \perp)$ and let e be a 0-anchored aligned edge or a free edge of G that is not an edge of a separating triangle. Then $(G/e, \mathcal{A})$ is a k-aligned triangulation of alignment complexity $(1, 0, \perp)$. Further, $(G, \mathcal{A})$ has an aligned drawing if $(G/e, \mathcal{A})$ has an aligned drawing.*

Proof. Let c be the vertex in G/e obtained from contracting an edge $e = uv$. We place c at the position of u. Thus, all edges incident to u keep their topological properties. Thus, we argue that the edges vw have the same intersections as the corresponding edges cw in $(G/e, \mathcal{A})$. Recall that G/e is simple and triangulated since e is not an edge of a separating triangle. If e is free, u is in the same cell as v and thus every edge vw has the same intersections in $(G, \mathcal{A})$ as cw in $(G/e, \mathcal{A})$. Thus, let e be a 0-anchored $\mathcal{L}$-aligned edge, for a pseudoline $\mathcal{L} \in \mathcal{A}$. Since G is proper and e is 0-anchored, $e = uv$ does not contain the intersection of two pseudolines in its interior. Thus, c is $\mathcal{L}$-aligned and lies on the same pseudosegment as e. Therefore, every edge vw in $(G, \mathcal{A})$ has the same intersections with a pseudoline as the corresponding edge uw in $(G/e, \mathcal{A})$. Hence, $(G/e, \mathcal{A})$ is k-aligned graph with the alignment complexity $(1, 0, \perp)$.

Let (Γ', A) be an aligned drawing of $(G/e, \mathcal{A})$. Let Γ'' denote the drawing obtained from Γ' by removing c together with its incident edges and let f denote the face of Γ'' where c used to lie. Since, G/e is triangulated, f is star-shaped and c lies inside the kernel of f; see Fig. 12. We construct a drawing Γ of G as follows. First, we place u at the position of c and insert all edges incident to u; if one of both vertices lie on the outer face or on the intersection of two pseudolines, we assume, without loss of generality, that vertex to be u. Note that since G is a proper triangulation and e is 0-anchored, there is no aligned edge incident to the intersection of two pseudolines or to the outer face. This leaves a uniquely defined face f' in which we have to place v. Since, we only removed a fan of triangles from f to obtain f', the face f' is star-shaped and furthermore, all points inside f' sufficiently close to c lie in the kernel of f'. If e is a free edge, the vertex v has to be placed in the same cell as u. We then place v inside f' sufficiently close to c so that it lies inside the kernel of f' and in the same cell as u. Every edge in $(G/e, \mathcal{A})$ is at most 1-crossed, thus no triangle contains the intersection of two pseudoline. Hence, f' does not contain the intersection of two pseudolines. Thus, (Γ, A) is an aligned drawing of $(G, \mathcal{A})$. Likewise, if e is an $\mathcal{L}$-aligned edge for a pseudoline $\mathcal{L} \in \mathcal{A}$, then v has to be placed on the corresponding line $L \in A$. But in this case also c lies on L, and hence we can place f on L sufficiently close to c.

Since G is triangulated and $e = uv$ is $\mathcal{L}$-aligned, it is an interior edge of G. Hence, there exist two triangles uv, vx, xu and uv, vy, yu sharing the edge uv. Since $(G, \mathcal{A})$ is a k-aligned triangulation, $\mathcal{L}$ intersects exactly two times with the boundary u, x, v, y of the triangles. Thus, x and y lie in different cells, therefore the face f' contains a segment of the line L of positive length. As u lies on L there is space to place v in f' on L. □

B.1 Omitted Proofs from Section 4.2

Theorem 3. *Let $(G, \mathcal{R})$ be an aligned graph and let (Γ_O, R) be a convex aligned drawing of the aligned outer face $(O, \mathcal{R})$ of G. There exists an aligned drawing (Γ, R) of $(G, \mathcal{R})$ with the same line R and the outer face drawn as Γ_O.*

Proof. Given an arbitrary aligned graph $(G, \mathcal{R})$, we first triangulate it using Lemma 3.

We prove this by induction on the size of G. If G is just a triangle, then clearly (Γ_O, R) is the desired drawing. If G is the k-wheel, placing the vertex on the line in the interior of Γ_O yields an aligned drawing of G. This finishes the base case.

If G contains a chord e, then e splits $(G, \mathcal{R})$ into two graphs G_1, G_2 with $G_1 \cap G_2 = e$. It is easy to verify that $(G_i, \mathcal{R})$ is an aligned graph. Let $(\Gamma_{O,i}, R)$ be a drawing of the face of $\Gamma_O \cup e$ whose interior contains G_i. By the inductive hypothesis, there exists an aligned drawing of (Γ_i, R) with the outer face drawn as (Γ_O, R). We obtain a drawing Γ by merging the drawings Γ_1 and Γ_2. The fact that both (Γ_1, R) and (Γ_2, R) are aligned drawings imply that (Γ, R) is an aligned drawing of $(G, \mathcal{R})$.

If G contains a separating triangle T, let G_{in} and G_{out} be the respective split components. By Lemma 4 the graphs $(G_{\text{in}}, \mathcal{R})$ and $(G_{\text{out}}, \mathcal{R})$ are aligned graphs. By the induction hypothesis there exists an aligned drawing (Γ_{out}, R) of the aligned graphs $(G_{\text{out}}, \mathcal{R})$ with the outer face drawn as (Γ_O, R). Let $\Gamma[T]$ be the drawing of T in Γ_{out}. Further, $(G_{\text{in}}, \mathcal{R})$ and $(G_{\text{in}}, \mathcal{R})$ has per induction hypothesis an aligned drawing with the outer face drawn as $\Gamma[T]$. Thus, by Lemma 4 we obtain an aligned drawing of $(G, \mathcal{R})$.

If G does not contain a separating triangle but a free or an aligned edge $e = uv$ that does not lie on the boundary of the outer face O and is not a chord of O, then consider $(G/e, \mathcal{R})$. Then, by Lemma 4 we have that $(G/e, \mathcal{R})$ is an aligned graph and by induction hypothesis there is an aligned drawing (Γ', R)

with the outer face drawing as Γ_O. Then, again by Lemma 4 there is the is an aligned drawing of G/e. □

B.2 Omitted Proofs from Section 4.3

Theorem 4. *Every k-aligned graph $(G, \mathcal{A})$ of alignment complexity $(1, 0, \perp)$ with a stretchable pseudoline arrangement $\mathcal{A}$ has an aligned drawing.*

Proof. Given an arbitrary aligned graph $(G, \mathcal{A})$, we first triangulate it using Lemma 3.

We prove this by induction on the size of instance $(G, \mathcal{A})$. If $(G, \mathcal{A})$ does neither contain an interior free edge, a 0-anchored aligned edge nor a separating triangle, then, by Lemma 8 there is an aligned drawing (Γ, A).

If G contains a separating triangle T, let G_{in} and G_{out} be the respective split components. Every triangle T intersects at most one pseudoline $\mathcal{L}$ in its interior. It follows that $(G_{\text{out}}, \mathcal{A})$ is a k-aligned triangulation and that $(G_{\text{in}}, \mathcal{L})$ is a 1-aligned triangulation.

By the inductive hypothesis there exists an aligned-drawing (Γ_{out}, A) of the k-aligned graph $(G_{\text{out}}, \mathcal{A})$. Let $\Gamma_{\text{out}}[T]$ be the drawing of T in Γ_{out}. By Theorem 3, we obtain an aligned drawing (Γ_{in}, L) with T drawn as $\Gamma_{\text{out}}[T]$. Moreover, since the drawing of T is fixed and does not contain the intersection of two (or more) pseudolines in its interior (Γ_{in}, A) is an aligned drawing. Then according to Lemma 4 there exists an aligned drawing of $(G, \mathcal{A})$.

If G contains no separating triangles but contains either an interior free edge or a 0-anchored aligned edge e, let G/e be the graph after the contraction of e. By the inductive hypothesis there exists an aligned drawing of the k-aligned graph $(G/e, \mathcal{A})$. By Lemma 5 there exists an aligned drawing of G. □

C   *k*-aligned Graphs with 2-anchored edges

Fig. 13 sketches a 2-aligned triangulation without aligned or free edges, the triangulation can be completed as indicated by the black edges The size of the graph can be arbitrarily increased and certainly be generalized to k-aligned graphs.

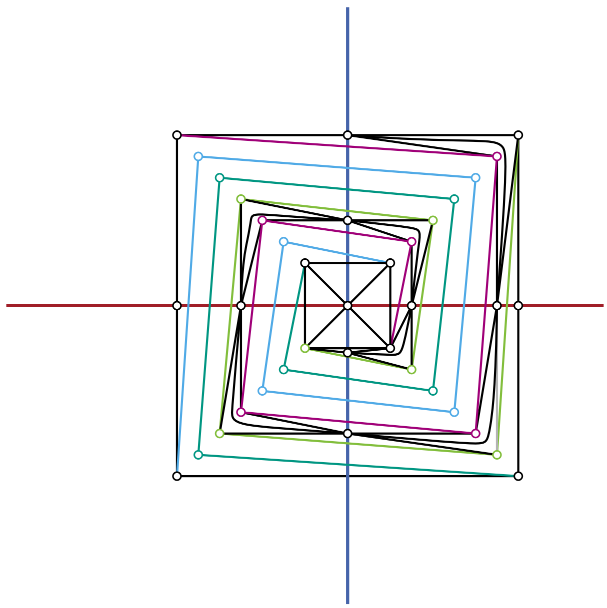

Fig. 13: 2-aligned graph with 2-anchored edges.