Gap-planar graphs

arXiv:1708.07653v1 [cs.CG] 25 Aug 2017

Gap-planar Graphs*

Sang Won Bae[1], Jean-Francois Baffier[2], Jinhee Chun[3], Peter Eades[4], Kord Eickmeyer[5], Luca Grilli[6], Seok-Hee Hong[4], Matias Korman[3], Fabrizio Montecchiani[6], Ignaz Rutter[7], Csaba D. Tóth[8]

[1] Kyonggi University, Suwon, South Korea
swbae@kgu.ac.kr
[2] National Institute of Informatics, Tokyo, Japan
jf_baffier@nii.ac.jp
[3] Tohoku University, Sendai, Japan
{jinhee,mati}@dais.is.tohoku.ac.jp
[4] University of Sydney, Sydney, Australia
peter.eades@sydney.edu.au, shhong@it.usyd.edu.au
[5] TU Darmstadt, Darmstadt, Germany
eickmeyer@mathematik.tu-darmstadt.de
[6] University of Perugia, Perugia, Italy
{luca.grilli,fabrizio.montecchiani}@unipg.it
[7] TU Eindhoven, Eindhoven, The Netherlands
i.rutter@tue.nl
[8] California State University Northridge, Los Angeles, USA
csaba.toth@csun.edu

Abstract. We introduce the family of *k-gap-planar graphs* for $k \geq 0$, i.e., graphs that have a drawing in which each crossing is assigned to one of the two involved edges and each edge is assigned at most k of its crossings. This definition finds motivation in edge casing, as a k-gap-planar graph can be drawn crossing-free after introducing at most k local gaps per edge. We obtain results on the maximum density, drawability of complete graphs, complexity of the recognition problem, and relationships with other families of beyond-planar graphs.

1 Introduction

"Beyond-planar graphs" are informally defined as nonplanar graphs that can be represented with some forbidden edge crossing patterns (see, e.g., [29,30,36]). Research on this topic is attracting increasing attention within the communities of

* Research started at the NII Shonan Meeting "Algorithmics for Beyond Planar Graphs." The authors thank the organizers, and Yota Otachi for useful discussions. Bae was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2015R1D1A1A01057220). Baffier was supported by JST-ERATO Grant Number JPMJER1201, Japan. Eades and Hong were partially supported by ARC DP160104148. Korman was partially supported by MEXT KAKENHI No. 15H02665, 17K12635 and JST ERATO Grant Number JPMJER1305. Tóth was supported in part by the NSF awards CCF-1422311 and CCF-1423615.

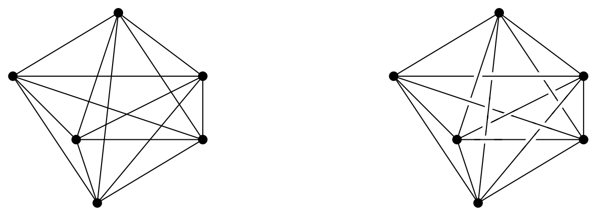


Fig. 1. (a) A drawing of a graph G and (b) its cased version where each edge is interrupted at most twice, i.e., a 2-gap-planar drawing of G.

graph theory, graph algorithms, graph drawing, and computational geometry, as these graphs represent a natural generalization of planar graphs, and their study can provide significant insights for the design of effective methods to visualize real-world networks. Indeed, the motivation for this line of research stems from both the interest raised by the combinatorial and geometric properties of these graphs, and experiments showing how the absence of particular edge crossing patterns has a positive impact on the readability of a graph drawing [31].

Among the investigated families of beyond-planar graphs are: *k-planar graphs* (see, e.g., [10,34,38]), which can be drawn with at most $k > 0$ crossings per edge; *k-quasiplanar graphs* (see, e.g., [2,3,21]), where there are no $k > 2$ pairwise crossing edges; *fan-planar graphs* (see, e.g., [8,11,32]), where no edge can be crossed by two indepedent edges; *fan-crossing-free graphs* [16], where crossings between an edge and two adjacent edges are forbidden; *planarly-connected graphs* [1], in which each pair of crossing edges is independent and there is a crossing-free edge that connects their endpoints; *RAC graphs* (refer, e.g., to [18]), which admit a straight-line (or polyline with few bends) drawing with right-angle crossings.

In this paper we introduce *k-gap-planar graphs*. Intuitively speaking, each crossing is assigned to one of the two involved edges and each edge is assigned at most k crossings (see Section 2). This definition generalizes that of k-planar graphs, and it is practically motivated by *edge casing*, a method commonly used to alleviate the visual clutter generated by crossing lines in a diagram [5,20]. In a *cased drawing* of a graph, each crossing is resolved by locally interrupting one of the two crossing edges. Clearly, minimizing the number of gaps per edge is one of the desirable goals in this situation, and a k-gap-planar graph can be equivalently defined as a graph that admits a cased drawing in which each edge has at most k gaps. Figure 1 shows a drawing of a graph and its version with edge casing. Eppstein et al. [20] studied many optimization problems related to edge casing, assuming the input to be a drawing (rather than a graph). In particular, the problem of minimizing the maximum number of gaps (called *tunnels*) for any edge of a drawing can be solved in polynomial time (see also Section 2). We also remark that a similar drawing paradigm is used by *partial edge drawings (PEDs)*, in which the central part of each edge is erased, while the two remaining stubs are required to be crossing-free (see, e.g., [14,15]).
Our results can be summarized as follows:

– Every k-gap-planar graph with n vertices has $O(\sqrt{k} \cdot n)$ edges (Section 3). If $k = 1$, a bound of $5n - 10$ edges is proved for 1-gap-planar multigraphs, which is tight as there exist k-gap-planar (simple) graphs with these many edges. Note that this density bound equals that of 2-planar graphs [38].
– The complete graph K_n is 1-gap-planar if and only if $n \leq 8$ (Section 4).
– Deciding whether a graph is 1-gap-planar is NP-complete, even when the input graph comes with a fixed rotation system that is part of the input (Section 5). We remark that analogous recognition problems for other families of beyond-planar graphs are also NP-hard (see, e.g., [7,8,11,12,25,35]), while polynomial algorithms are known only in restricted settings (see, e.g., [6,8,12,17,19,28,27]).
– We study relationships of the k-gap-planar family with other beyond-planar families. For all $k \geq 1$, the class of $2k$-planar graphs is properly included in the class of k-gap-planar graphs, which in turn is properly included in the $(2k + 2)$-quasiplanar graphs (Section 6). It is worth observing that recent papers proved that k-planar graphs are $(k + 1)$-quasiplanar [4,26].

For reasons of space some proofs and technicalities have been omitted and can be found in the appendix.

2 Preliminaries and basic results

A *drawing* Γ of a graph $G = (V, E)$ is a mapping of the vertices of V to distinct points of the plane, and of the edges of E to Jordan arcs connecting their corresponding endpoints but not passing through any other vertex. If two edges are incident to the same vertex, then they do not cross in Γ; else, they have at most one common interior point where they cross transversely. For a subset $E' \subseteq E$, $\Gamma[E']$ denotes the restriction of Γ to the curves representing the edges of E'. Drawing Γ is *planar* if no edge is crossed. The *crossing number* $\mathrm{cr}(G)$ of a graph G is the smallest number of edge crossings over all drawings of G. A graph is *planar* if it admits a planar drawing. A planar drawing subdivides the plane into topologically connected regions, called *faces*. The unbounded region is the *outer face*. A *planar embedding* of a planar graph G is an equivalence class of topologically equivalent drawings of G. A *plane graph* is a planar graph with a planar embedding. The *crossing graph* $C(\Gamma)$ of a drawing Γ is the graph having a vertex v_e for each edge e of G, and an edge (v_e, v_f) if and only if edges e and f cross in Γ. The *planarization* Γ^* of Γ is the plane graph formed from Γ by replacing each crossing by a *dummy vertex* . To avoid ambiguities, we call *real vertices* the vertices of Γ^* that are not dummy.

Let Γ be a drawing of a graph G. We shall assume that exactly two edges of G cross in one point p of Γ, and we say that these two edges are *responsible* for p. A *k-gap assignment* of Γ maps each crossing point of Γ to one of its two responsible edges so that each edge is assigned with at most k of its crossings; see, e.g., Fig. 1(b). The *gap* of an edge is the number of crossings assigned to it. An edge with at least one gap is *gapped*, or a *gap edge*, else it is *gap free*. A drawing is *k-gap-planar* if it admits a k-gap assignment. A graph is *k-gap-planar* if it has a k-gap-planar drawing. Note that the 0-gap-planar graphs coincide with the planar graphs, and that k-gap-planarity is a monotone property: every subgraph of a k-gap-planar graph is k-gap-planar. From the pigeonhole principle we have:

Property 1. Let Γ be a k-gap-planar drawing of a graph $G = (V, E)$. For any $E' \subseteq E$, $\Gamma[E']$ contains at most $k \cdot |E'|$ crossings.

A k-gap assignment of a drawing Γ corresponds to orienting the edges of the crossing graph $C(\Gamma)$ such that each vertex has indegree at most k (intuitively, orienting a crossing towards an edge means we assign the crossing to that edge). Since finding a lowest indegree orientation of a graph corresponds to finding its pseudoarboricity [22,39], Property 2 follows. A *pseudoforest* is a graph in which every connected component has at most one cycle, and the *pseudoarboricity* of a graph is the smallest number of pseudoforests needed to cover all its edges.

Property 2. A graph is k-gap-planar if and only if it admits a drawing whose crossing graph has pseudoarboricity at most k.

Given a drawing Γ of a graph $G = (V, E)$, finding the minimum k such that Γ is k-gap-planar can be solved in $O(|E|^4)$ time, due to the fact that finding a lowest indegree orientation of $C(\Gamma)$ can be solved in time quadratic in the number of edges of $C(\Gamma)$ [41].

3 Density of k-gap-planar graphs

We begin with an upper bound on the number of edges of k-gap-planar graphs.

Theorem 1. *A k-gap-planar graph on $n \geq 3$ vertices has $O(\sqrt{k} \cdot n)$ edges.*

Proof. The crossing number of a graph G with n vertices and m edges is bounded by $\mathrm{cr}(G) \geq \frac{1024}{31827} \cdot m^3/n^2$ when $m \geq \frac{103}{6}n$ [37]. Combined with the bound $\mathrm{cr}(G) \leq k \cdot m$ (Property 1), we obtain

$$\frac{1024}{31827} \cdot \frac{m^3}{n^2} \leq \mathrm{cr}(G) \leq km,$$

which implies $m \leq \max(5.58\sqrt{k}, 17.17) \cdot n$, as required. □

Better upper bounds are possible for small values of k, in particular for $k = 1$. Pach et al. [37] proved that a graph G with $n \geq 3$ vertices satisfies $\mathrm{cr}(G) \geq \frac{7}{3}m - \frac{25}{3}(n-2)$. Combined with the bound $\mathrm{cr}(G) \leq k \cdot m$, we have

$$m \leq \frac{25(n-2)}{7-3k}.$$

For $k = 1$ (i.e., for 1-gap-planar graphs), this gives $m \leq 6.25n - 12.5$. We now show how to improve this bound to $m \leq 5n - 10$. The idea is to follow a strategy developed by Pach and Tóth [38] and Bekos et al. [10] on the density of 2- and 3-planar graphs, with several important differences.

We start by stating the assumptions and notations for the proof of Theorem 2. In order to accommodate the elementary operations in the proof, we work on a broader class of graphs, namely multigraphs admitting a drawing without homotopic[9] parallel edges.

i) For any $n \in \mathbb{N}$, $n \geq 3$, let $G = (V, E)$ be a 1-gap-planar multigraph with n vertices that has the maximum number of edges possible over all n-vertex 1-gap-planar multigraphs without homotopic parallel edges; ii) let Γ be a 1-gap-planar drawing of G with the minimum number of edge crossings over all possible 1-gap-planar drawings of G with non-homotopic parallel edges; and iii) let $H = (V, E')$ be a sub-multigraph of G, where $E' \subseteq E$ is a multiset of edges that are pairwise noncrossing in $\Gamma[E']$. iv) We assume that over all choices of G, Γ, and H described above, the multigraph H is maximum and, in case of ties, has the fewest connected components.

Our proof is based on the next technical lemma.

Lemma 1. *The multigraph H is a triangulation, that is, a plane multi-graph in which every face is bounded by a walk with three vertices and three edges.*

We can now show that $|E| \leq 5n - 10$.

Theorem 2. *A 1-gap-planar graph on $n \geq 3$ vertices has at most $5n-10$ edges.*

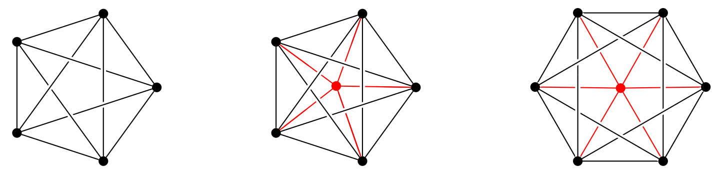
Proof. By Lemma 1, we know that $H = (V, E')$ is a triangulation. By Euler's polyhedron theorem, it has $3n - 6$ edges and $2n - 4$ triangular faces. Consider the edges in $E'' = E \setminus E'$. It remains to show that $|E''| \leq 2n - 4$.

The embedding of edge $e \in E''$ is a Jordan arc that visits two or more triangle faces of H. We call the first and last triangles along e the *end triangles* of e. For an end triangle Δ, the connected component of $e \cap \Delta$ incident to a vertex of Δ is called an *end portion*. We use the following charging scheme.

Each edge $e \in E''$ charges one unit to a triangle face of H. If e has an end portion that has a gap neither in the interior nor on the boundary of the corresponding end triangle Δ, then e charges one unit to Δ. (If neither end portions of e has a gap in the interior or on the boundary of its end triangle, then e charges one arbitrary end triangle.) Otherwise the two end portions of e lie in two adjacent triangles, say, Δ_1 and Δ_2, and e uses its gap to cross the common edge on the boundary between them; in this case e charges one unit to Δ_1 or Δ_2 as follows: If the gap of the common edge between Δ_1 and Δ_2 is used for an end portion of $e' \cap \Delta_1$ for another edge $e' \in E''$ and e' charges Δ_1, then e charges Δ_2, otherwise it charges Δ_1.

We claim that each face of H receives at most one unit of charge. Let $\Delta = \Delta abc$ be a face in H. Note that if Δ receives positive charge from an edge $e \in E''$, then an end portion of e lies in Δ, and does not use any gap in the interior of Δ. Consequently if Δ received positive charge from edges $e_1, e_2 \in E''$, then the end portions of e_1 and e_2 in Δ cannot cross, and they are incident to the same

[9] Two parallel edges are *homotopic* if at least one of the two regions defined by these two edges contains no vertex in its interior.



(a) (b) (c)

Fig. 2. Patterns that produce 1-gap-planar graphs with n vertices and $5n-\Theta(1)$ edges.

vertex of Δ. Therefore, all edges in E'' that charge Δ are incident to the same vertex of Δ, say a, and cross the edge of Δ opposite to a, namely (b, c). Let $\Delta' = \Delta' bcd$ be the face of the plane graph H on the opposite side of (b, c).

The gap of edge (b, c) can be used for at most one crossing along (b, c). If the gap of (b, c) is used for a crossing with one of the end portions in Δ, then e sends 1 unit charge to Δ. The only other edge that could possibly send a charge to Δ is the edge $(a, d) \in E''$ that uses its own gap to cross (b, c). However, in this case, (a, d) charges one unit to Δ' in our charging scheme. If the gap of (b, c) is not used for any of these end portions in Δ, then the edge (a, d) may send 1 unit charge to Δ. Overall, Δ receives at most 1 unit of charge. Consequently, $|E''|$ is bounded above by the number of faces of H, which is $2n - 4$, as required. □

We now show that the bound of Theorem 2 is worst-case optimal. A 2-planar graph with n vertices and $5n - 10$ edges is also 1-gap-planar (see Lemma 6). Pach and Tóth [38] construct such a graph by starting with a plane graph with pentagonal faces (e.g., using nested copies of an icosahedron), and then add all five diagonals in each pentagonal face; see Fig. 2(a). This construction yields a 1-gap-planar graph with n vertices and $m = 5n - 10$ edges for all $n \geq 20$, $n \equiv 5 \pmod{15}$.

We can modify this construction by inserting a new vertex in one or more pentagons, and connecting it to the 5 vertices of the pentagon; see Fig. 2(b). Every new edge crosses exactly one diagonal of the pentagon, so the new crossings can be charged to the new edges. Since every new vertex has degree 5, the equation $m = 5n - 10$ prevails. By inserting a suitable number of vertices into pentagons, we obtain constructions for $n \in \mathbb{N}$ such that $20 \leq n \leq 32$ or $n \geq 38$. A similar construction is based on hexagonal faces; see Fig. 2(c). Start with a *fullerene*, that is, a 3-regular, plane graph G_0 with n_0 vertices, 12 pentagon faces, and $n_0/2 - 10$ hexagon faces (including the external face). Add diagonals in each face to connect a vertex to their second neighbors (the graph is 2-planar so far); finally insert a new vertex in each face of G_0, and connect them to all vertices of that face. We obtain a 1-gap-planar graph G. The number of vertices is $n = n_0 + 12 + (n_0/2 - 10) = \frac{3}{2}n_0 + 2$, and the number of edges is $m = \frac{3}{2}n_0 + 10 \cdot 12 + 12 \cdot (n_0/2 - 10) = \frac{15}{2}n_0 = 5n - 10$. Fullerenes exist for $n_0 = 20$ and for all even integers $n_0 \geq 24$ [13]. This yields a lower bound of $5n - 10$ for $n = 32$ and for all $n \geq 35$ where $n \equiv 2 \mod 3$. However, similarly to the previous construction, the equation $m = 5n - 10$ prevails if we *delete* up to 12 vertices inserted into pentagons. Consequently, the upper bound $5n - 10$ in Theorem 2 is tight for all $n \geq 20$.

Theorem 3. *For every $n \geq 20$ there exists a 1-gap-planar graph G with n vertices and $5n - 10$ edges.*

4 1-gap-planar drawings of complete graphs

Theorem 4. *The complete graph K_n is 1-gap-planar if and only if $n \leq 8$.*

Proof. Figure 3(a) shows a 1-gap-planar drawing of K_8, and by monotonicity the graphs $K_1, \dots, K_7$ are 1-gap-planar as well. We now prove that K_9 is not 1-gap-planar, which again by monotonicity settles all cases K_n for $n \geq 9$.

Since K_9 has 36 edges and $\mathrm{cr}(K_9) = 36$, a 1-gap-planar drawing of K_9 can only arise from assigning exactly one gap to each edge in a crossing-minimal drawing of K_9 (cf. Property 1). We obtain a contradiction by showing that in every crossing-minimal drawing of K_9 some edge has no crossing at all.

Let Γ^* be the planarization of such a crossing-minimal drawing Γ. Note that Γ^* has $n^* = 45$ vertices and $m^* = 108$ edges (since it has 9 real vertices of degree 8 and 36 dummy vertices of degree 4), so by Euler's formula, the number of faces of Γ^* is $f^* = m^* - n^* + 2 = 108 - 45 + 2 = 65$. For a real vertex u of Γ^*, we denote by $F(u)$ the set of faces of Γ^* that are incident to u. We claim that Γ^* is biconnected and $|F(u)| = 8$ for every real vertex u of Γ^*.

Suppose, for a contradiction, that Γ^* is not biconnected. Then it contains a cut-vertex c, which is either a dummy or a real vertex. If c is a dummy vertex, note that it is adjacent to exactly two connected components of $\Gamma^* \setminus \{c\}$. Then we can reflect the drawing of one of the two components, thereby eliminating the crossing at c, which contradicts the crossing-minimality of Γ. We now show that no real vertex is a cut-vertex in Γ^*. Every 3-cycle in K_9 forms a simple cycle in Γ^* (since Γ is a simple drawing and thus adjacent edges do not cross). On the other hand, any three real vertices in Γ^* are part of a 3-cycle in K_9, and thus part of a simple cycle in Γ^*. Hence, no real vertex is a cut-vertex in Γ^*. Finally, $|F(u)| = 8$ because every real vertex u has degree 8 and Γ^* is biconnected.

It follows that there are real vertices u, v which share a face (i.e. $F(u) \cap F(v) \neq \emptyset$), as otherwise there would have to be $\sum_u |F(u)| = 9 \cdot 8 = 72 > 65 = f^*$ faces. But now the edge (u, v) can be redrawn inside this face, and since Γ was assumed to be crossing-minimal this edge can not have had any crossing to begin with. $\square$

5 Recognizing 1-gap-planar graphs

We use 1GapPlanarity to denote the problem of deciding whether a given graph G is 1-gap-planar. We show that 1GapPlanarity is NP-complete, and we use a reduction from 3Partition. Recall that an instance of 3Partition

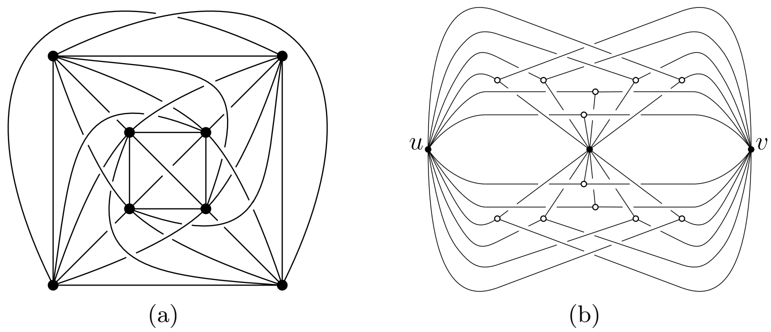


Fig. 3. A 1-gap-planar drawing of (a) K_8 and (b) $K_{3,12}$.

consists of a multiset $A = \{a_1, a_2, \ldots, a_{3m}\}$ of $3m$ positive integers in the range $(B/4, B/2)$, where B is an integer such that $B = 1/m \cdot \sum_{i=1}^{3m} a_i$, and asks whether A can be partitioned into m subsets $A_1, A_2, \ldots, A_m$, each of cardinality 3, such that the sum of integers in each subset is B. This problem is strongly NP-hard [24], and thus we may assume that B is bounded by a polynomial in m.
The fact that 1GapPlanarity is in NP can easily be shown by exploiting Property 2.
Our reduction is reminiscent to the reduction used in [8]. However, the proof in [8] holds only for the case in which a clockwise order of the edges around each vertex is part of the input, i.e., only if the *rotation system* of the input graph is fixed. A similar reduction is also used in [9], in which the rotation system assumption is not used. However, the gadgets used in [9] have a unique embedding. We do not use the fixed rotation system assumption, nor we can easily derive a unique embedding for our gadgets, and thus have to deal with additional challenges in our proof. In what follows we define a "blob" graph that will be used to enforce an ordering among the edges adjacent to certain vertices. Consider the complete bipartite graph $K_{3,12}$, whose crossing number is 30 [33,42]. We exhibit a 1-gap-planar drawing of $K_{3,12}$ with exactly 30 gaps in Fig. 3(b). Note that two degree-12 vertices, u and v, are drawn on the outer face. Since $K_{3,12}$ has 36 edges, the next lemma easily follows.

Lemma 2. *Every* 1*-gap-planar drawing of* $K_{3,12}$ *has at most* 6 *gap-free edges.*

A *blob* B is a copy of $K_{3,12}$. A *gapped chain* $\mathcal{C}$ of a 1-gap-planar drawing is a closed, possibly nonsimple, curve such that any point of $\mathcal{C}$ either belongs to a gap edge or it corresponds to a vertex.

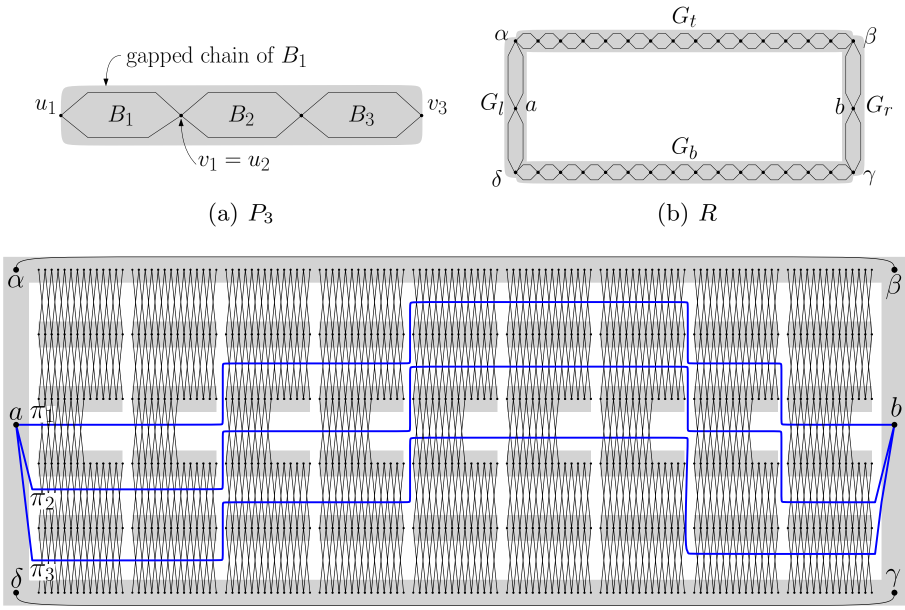
Lemma 3. *Let u and v be any two degree-*12 *vertices of B. Every* 1*-gap-planar drawing Γ of B contains a gapped chain $\mathcal{C}$ containing u and v.*

Sketch of proof. Let Γ^* be the planarization of Γ. Let Γ' be the subgraph of Γ^* consisting only of those edges that correspond to or belong to gap edges of Γ.

We prove that Γ' contains two edge-disjoint paths from u to v. Note that these two edge-disjoint paths may meet at real vertices and at dummy vertices (i.e., a crossing between two gap edges). A curve that goes through these two paths is the desired gapped chain. According to Menger's theorem, two such paths exist if and only if every (u, v)-cut of Γ' has size at least 2, where a (u, v)-cut of Γ' is a set of edges of Γ' whose removal disconnects u and v. Such edge cuts correspond to cycles in the dual, which in turn correspond to curves that separate u and v by crossing a set of edges. By Lemma 2, one can show that any such curve crosses at least two gap edges in the original drawing Γ. $\square$

We are now ready to show how to transform an instance A of 3Partition into an instance G_A of 1GapPlanarity. We start by defining some gadgets for our construction. A *path gadget* P_k is a graph obtained by merging a sequence of $k > 0$ blobs as follows. Denote by $B_1, B_2, \ldots, B_k$, k blobs such that u_i and v_i are two vertices of degree 12 in B_i. We let $v_i = u_{i+1}$ for $i = 1, \ldots, k-1$, each of these vertices is an *attaching vertex*. Thus, P_k has $k + 1$ attaching vertices. A 1-gap-planar drawing of P_k is such that any two gapped chains of any two blobs B_i and B_j $(i < j)$ do not share points, except at a possible common attaching vertex. A schematization of P_k (for $k = 3$) is shown in Fig. 4(a). A *top beam* G_t is a path gadget P_k with $k = 3m(\lceil B/2 \rceil + 2) + 1$. Recall that G_t has $3m(\lceil B/2 \rceil + 2) + 2$ attaching vertices. A *right wall* G_r is a path gadget P_k with $k = 2$. Symmetrically, a *bottom beam* G_r is a path gadget with $k = 3m(\lceil B/2 \rceil + 2) + 1$, and a *left wall* G_l is a path gadget with $k = 2$. A *global ring* R is obtained by merging G_t, G_r, G_b, and G_l in a cycle as in Fig. 4(b). Again, in any 1-gap-planar drawing Γ_R of R, the gapped chains of any two blobs B_i and B_j do not share points, except at a possible common attaching vertex. Thus, Γ_R contains a gapped chain C_R that is the union of all the gapped chains of the blobs of R.

We start the construction of G_A with a global ring R. Let α, β, γ, δ be the attaching vertices shared by G_l and G_t, G_t and G_r, G_r and G_b, G_b and G_l, respectively (see also Fig. 4(b)). First we add the edges (α, β) and (γ, δ). Denote as R^+ the resulting graph, and consider a 1-gap-planar drawing of this graph. The gapped chain of R subdivides the plane into a set of connected regions, such that only two of them contain all of α, β, γ, and δ on their boundaries. We denote these two regions as r_1 and r_2. For ease of illustration, we assume that one of them is infinite (as in Fig. 4(b)), say r_2. Since the drawing is 1-gap-planar, each of (α, β) and (γ, δ) is drawn entirely in one of these two regions. We assume that both these two edges are drawn in the same region, say r_2, and we will later show that this is the only possibility in any 1-gap-planar drawing of the final graph G_A. We continue by connecting the top and bottom beams by a set of $3m$ *columns*; refer to Fig. 4(c). Each column consists of $2m - 1$ *cells*; a cell consists of a set of pairs of crossing edges, called its *vertical pairs*. In particular, there are $m - 1$ *bottom cells*, one *central cell* and $m - 1$ *top cells*. Cells of the same column are separated by $2m - 2$ path gadgets, called *floors*. Note that we are assuming a particular left-to-right order for the attaching vertex of a floor, we will see that this is the only possible order in a 1-gap-planar drawing. The central cells (we have $3m$ of them in total) have a number of vertical pairs depending on



(a) P_3 (b) R

(c) G_A

Fig. 4. (a) Schematization of a path gadget P_3. (b) A global ring R. (c) Schematization of the instance G_A with $m = 3$, $A = \{7, 7, 7, 8, 8, 8, 8, 9, 10\}$ and $B = 24$. Transversal paths are routed according to the following solution of 3Partition $A_1 = \{7, 7, 10\}$, $A_2 = \{7, 8, 9\}$ and $A_3 = \{8, 8, 8\}$. For simplicity, the gapped chains of the various blobs are not shown, as well as vertex w and all the degree-2 vertices of the transversal paths.

the elements of A. Specifically, the central cell C_i of the i-th column contains a_i vertical pairs connecting its delimiting floors ($i \in \{1, 2, \ldots, 3m\}$). Each of the remaining cells each has $\lceil B/2 \rceil + 1$ vertical pairs. Hence, a noncentral cell contains more edges than any central cell. Further, the number of attaching vertices of a floor can be computed based on how many vertical pairs must be connected to the gadget. It is now straightforward to see that it is not possible to draw both a column and one of (α, β) and (γ, δ) in r_1 or r_2 without violating 1-gap-planarity. Hence, we shall assume that both (α, β) and (γ, δ) are in r_2 and that all the columns are in r_1. Consider now a 1-gap-planar drawing of a column. If we invert the left-to-right order of the attaching vertices of a floor (i.e., we mirror its drawing), then the resulting drawing is not 1-gap-planar, since each floor delimits at least one noncentral cell with $\lceil B/2 \rceil + 1$ vertical pairs. Moreover, since each vertical pair has a gap edge, two vertical pairs cannot cross each other in a 1-gap-planar drawing, and thus the drawings of the columns are disjoint one another. Finally, let a and b be the attaching vertices of the left and right walls distinct from α, β, γ, and δ. We connect a and b with m

pairwise internally disjoint paths, called *transversal paths*; each transversal path has exactly $(3m-3)(\lceil B/2 \rceil + 1) + B$ edges. The routing of these paths will be used to determine a solution of A, if it exists. Thus, we aim at forcing the transversal paths to be inside r_1 in a 1-gap-planar drawing of the graph. For this purpose, adding a vertex w connected to all the attaching vertices of G_t and G_b will suffice. Due to the presence of the columns in r_1, vertex w must be in r_2 and, due to the edges (α, β) and (δ, γ) in r_2, all its incident edges (except at most two) are gapped. Thus, the transversal paths must be drawn in r_1. This concludes the construction of G_A.

We can prove the following.

Theorem 5. *The* 1GapPlanarity *problem is* NP*-complete.*

We conclude by observing that our proof can be easily adjusted for the setting in which the rotation system of the input graph is fixed. We call this problem 1GapPlanarityWithRotSys. It suffices to choose a rotation system for G_A that guarantees the existence of a 1-gap-planar drawing ignoring the transversal paths (we already discussed the details of this drawing), and such that the transversal paths are attached to a and b with the ordering of their edges around a reversed with respect to the ordering around b. The membership of the problem to NP can be easily verified. Thus, the next theorem follows.

Theorem 6. *The* 1GapPlanarityWithRotSys *problem is* NP*-complete.*

6 Relationship between k-gap-planar graphs and other families of beyond planar graphs

In this section we prove the following theorem.

Theorem 7. *For every integer $k \geq 1$, the following relationships hold.*

$$(2k)\text{-PLANAR} \subsetneq k\text{-GAP-PLANAR} \subsetneq (2k+2)\text{-QUASIPLANAR}$$

We begin by showing the following.

Lemma 4. *For all $k \geq 1$, every k-gap-planar drawing is $(2k+2)$-quasiplanar.*

Proof. Recall that a graph G is q-quasiplanar, for $q \in \mathbb{N}$, if it admits a drawing in which there is no subset of q pairwise crossing edges, or equivalently if every subset of q edges has less than $\binom{q}{2} = q(q-1)/2$ crossings. On the other hand, in a k-gap-planar drawing there are at most kq crossings among any q edges (Property 1). Consequently, a k-gap-planar graph is $(2k+2)$-quasiplanar. □

We also need to show that for every $k \in \mathbb{N}$ there is a $(2k+2)$-quasiplanar graph that is not k-gap-planar. We prove a stronger statement:

Lemma 5. *For all $k \geq 1$, there is a 3-quasiplanar graph G_k that is not k-gap-planar.*

Proof. Let $k \in \mathbb{N}$. We construct a graph $G_k = (V, E)$ as follows. Start with $K_{3,3}$ and replace each edge by $t = 19k$ edge-disjoint paths of length 2. Note that the total number of edges is $|E| = 9 \cdot 2t = 18t$. Graph G_k is 3-quasiplanar. Since $\mathrm{cr}(K_{3,3}) = 1$, it admits a drawing with precisely one crossing. The paths of length 2 can be drawn close to the edges of $K_{3,3}$ such that two paths cross if and only if the two corresponding edges of $K_{3,3}$ cross. Consequently any two crossing edges in this drawing are part of two paths that correspond to two crossing edges of $K_{3,3}$, which in turn implies that no three edges of G_k pairwise cross.

Suppose that G_k admits a k-gap-planar drawing Γ. Then the total number of crossings is at most $k|E| = 18kt$. We derive a contradiction by showing that $\mathrm{cr}(G_k) \geq 19kt$. If we choose one of the t paths for each of the 9 edges of $K_{3,3}$ independently, then we obtain a subdivision of $K_{3,3}$, therefore there is a crossing between at least one pair of paths. There are t^9 ways to choose a path for each of the 9 edges of $K_{3,3}$. Each crossing between two paths in Γ is counted $t^{9-2} = t^7$ times. Consequently, the total number of crossings in Γ is at least $t^2 = 19kt$. $\square$

We now show that every $2k$-planar drawing is k-gap-planar. We note that a similar result can be also derived from [15] (Lemma 10) for the case $k = 1$. A bipartite graph with vertex sets A and B is denoted as $H = (A, B, E)$. A *matching* from A into B is a set $M \subseteq E$ such that every vertex in A is incident to exactly one edge in M and every vertex in B is incident to at most one edge in M. The *neighborhood* of a subset $A' \subseteq A$ is the set of all vertices in B that are adjacent to a vertex in A', and is denoted as $N(A')$. We recall that, by Hall's theorem, the graph H has a matching from A into B if and only if for each set $A' \subseteq A$ it is $|N(A')| \geq |A'|$.

Lemma 6. *For all $k \geq 1$, every $(2k)$-planar drawing is k-gap-planar.*

Proof. Let G be a $(2k)$-planar graph, for any $k \geq 1$, and let Γ be a $2k$-planar drawing of G. Let $H = (A \cup B, E_H)$ be a bipartite graph obtained as follows. The set A has a vertex $a_{e,f}$ for each crossing in Γ between two edges e and f of G. For each edge e of G there are k vertices $b_e^1, \ldots, b_e^k$ in B. For every pair of edges e, f of G that cross in Γ, graph H contains edges $(a_{e,f}, b_e^1), \ldots, (a_{e,f}, b_e^k)$ and $(a_{e,f}, b_f^1), \ldots, (a_{e,f}, b_f^k)$ in H. Notice that if H admits a matching of A in B, then each crossing of Γ between an edge e and an edge f of G can be assigned to either e or f, and no edge of G is assigned with more than k crossings.

Consider any subset A' of A, and let $B' = N(A')$ be the neighborhood of A' in B. We claim that $|A'| \leq |B'|$. Let $E' \subseteq E_H$ denote the edges between A' and B'. By construction every vertex in A has degree $2k$, and hence $|E'| \geq 2k|A'|$. On the other hand, every vertex in B has degree at most $2k$ as every edge of G has at most $2k$ crossings, and hence $|E'| \leq 2k|B'|$. Hence $|A'| \leq |B'|$ as claimed.

By Hall's theorem, it now follows that H admits a matching from A into B, which corresponds to an assignment of gaps in Γ such that no edge has more than k gaps, i.e., Γ is a k-gap-planar drawing. $\square$

To conclude the proof of Theorem 7, we should prove that for every $k \geq 1$, there is a k-gap-planar graph that is not $2k$-planar. A stronger result holds:

Lemma 7. *For every $k \geq 1$, there exists a* 1*-gap-planar graph G_k that is not k-planar.*

7 Conclusions and open problems

We introduced k-gap-planar graphs, our results give rise to several questions for future research. Among them are: i) We proved that k-gap-planar graphs have $O(\sqrt{k} \cdot n)$ edges, and that 1-gap-planar graphs have at most $5n - 10$ edges, which is a tight bound. Can we establish a tight bound also for 2-gap-planar graphs? ii) We proved that K_n is 1-gap-planar if and only if $n \leq 8$. A similar characterization could be studied also for complete bipartite graphs. Note that $K_{5,7}$ is not 1-gap-planar since it has crossing number greater than its number of edges, while we can exhibit a 1-gap-planar drawing of $K_{5,6}$.It is open whether $K_{6,6}$ is 1-gap-planar. Similarly, $K_{3,12}$ (Fig. 3(b)) and $K_{4,8}$ are 1-gap-planar, while we ask if this is true also for $K_{3,13}$ and $K_{4,9}$. iii) We proved that deciding whether a graph is 1-gap-planar is NP-complete, even if the rotation system is fixed. Does the problem become polynomial for drawings in which all vertices are on the outer boundary? iv) We proved that a drawing with at most $2k$ crossings per edge is k-gap-planar, and that a k-gap-planar drawing does not contain $2k+2$ pairwise crossing edges. Do 1-gap-planar graphs have RAC drawings with at most 1 or 2 bends per edge?

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Appendix

A Additional Material for Section 3

A.1 Proof of Lemma 1

We start with a few basic observations.

Lemma 8. *Graph $G = (V, E)$ is connected.*

Proof. Suppose, to the contrary, that G is disconnected. Let $G_1 = (V_1, E_1)$ be one component, and let $G_2 = (V_2, E_2)$, where $V_2 = V \setminus V_1$ and $E_2 = E \setminus E_1$. For $i = 1, 2$, let Γ_i be the drawing of G_i inherited from G, and let Γ_i^* be its planarization.

Let f_2 be a face in Γ_2^* incident to some vertex $v_2 \in V_2$. Apply a projective transformation to Γ_1 so that the outer face is incident to some vertex $v_1 \in V_1$; followed by an affine transformation that maps Γ_1 into the interior of face f_2. Now we can add a new edge (v_1, v_2), contradicting the maximality of G. $\square$

Since G is connected, every face in the planarization Γ^* of Γ has a connected boundary. The *boundary walk* of a face f is a closed walk $(a_1, a_2, \ldots, a_m)$ in Γ^* such that f lies on the left hand side of each edge (a_i, a_{i+1}) along the walk; and every two consecutive edges of the walk, (a_{i-1}, a_i) and (a_i, a_{i+1}), are also consecutive in the counterclockwise rotation of all edges incident to a_i. Let F_0 denote the set of faces in the planarization Γ^* that are not incident to any vertex in V.

Lemma 9. *If $f \in F_0$, then the boundary walk of f is*

1. *a simple cycle (i.e., has no repeated vertices) with at least 3 vertices;*
2. *disjoint from the boundary walk of any other face in F_0.*

Proof. 1. Let $f \in F_0$, and let $w = (a_1, a_2, \ldots, a_\ell)$ be its boundary walk for some $\ell \geq 3$. Let $C_f = \{a_1, \ldots, a_\ell\}$ be the set of vertices in w; and let $E_f \subset E$ be the set of edges in G that contain some edge of w. It suffices to show that $|C_f| = \ell$, and then w has no repeated vertices, hence it is a simple cycle.

Suppose, to the contrary, that the vertices in w are not distinct. Since $f \in F_0$, all vertices in w are crossings in the drawing Γ, consequently they all have degree 4 in the planarization Γ^*. If $a_i = a_j$, $i \neq j$, then a_i and a_j cannot be consecutive vertices in w, and two pairs of edges from (a_{i-1}, a_i), (a_i, a_{i+1}), (a_{j-1}, a_j), (a_j, a_{j+1}) are part of the same edge in E. If $|C_f| = \ell - k$, for some $k \in \mathbb{N}$, then $|E_f| \leq \ell - 2k$. This implies $|E_f| < |C_f|$. That is, the edges in E_f are involved in more than $|E_f|$ crossings, contradicting the assumption that Γ is a 1-gap-planar drawing.

2. Let $f_1, f_2 \in F_0$ be two faces, with boundary walks $w_1 = (a_1, \ldots, a_\ell)$ and $w_2 = (b_1, \ldots, b_{\ell'})$. Both w_1 and w_2 are simple cycles by part 1. For $i = 1, 2$, let C_i be the set of vertices in w_i, and $E_i \subseteq E$ the set of edges of G that contain the edges of the walk w_i.

Note that w_1 and w_2 cannot share two consecutive edges, say (a_{i-1}, a_i) and (a_i, a_{i+1}), since the middle vertex a_i has degree 4 in Γ^*. When w_1 and w_2 have a common edge, say $(a_i, a_{i+1}) = (b_{j+1}, b_j)$, then three pairs of edges from (a_{i-1}, a_i), (a_i, a_{i+1}), (a_{i+1}, a_{i+2}), (b_{j-1}, b_j), (b_j, b_{j+1}) (b_{j+1}, b_{j+2}) are part of the same edge in E. When w_1 and w_2 have a common vertex $a_i = b_j$ but no common edge incident to $a_i = b_j$, then two pairs of edges from (a_{i-1}, a_i), (a_i, a_{i+1}), (b_{j-1}, b_j), (b_j, b_{j+1}) are part of the same edge in E. This implies $|E_1 \cup E_2| < |C_1 \cup C_2|$. That is, the edges in $E_1 \cup E_2$ are involved in more than $|E_1 \cup E_2|$ crossings, contradicting the assumption that Γ is 1-gap-planar. $\square$

Lemma 10. *Graph $H = (V, E')$ is connected.*

Proof. Suppose, to the contrary, that H is disconnected. Let $H_1 = (V_1, E'_1)$ be one component, and let $H_2 = (V_2, E'_2)$, where $V_2 = V \setminus V_1$ and $E'_2 = E' \setminus E'_1$.

Consider the faces in the planarization Γ^* of Γ. Notice that there is no face in Γ^* incident to a vertex $v_1 \in V_1$ and a vertex $v_2 \in V_2$, otherwise we could either add a new edge (v_1, v_2) (contradicting the maximality of G), or redraw an existing edge (v_1, v_2) to pass through the interior of this face, contradicting the maximality of E'.

Consequently, we can partition the faces in Γ^* into three categories: For $i = 1, 2$, let F_i be the set of faces incident to a vertex in V_i; and let F_0 be the set of faces incident to neither V_1 nor V_2. By Lemma 9, the region obtained by removing all faces in F_0 (i.e., $\mathbb{R}^2 \setminus \bigcup_{f \in F_0}$) is connected. Consequently, there exist some faces $f_1 \in F_1$ and $f_2 \in F_2$ that have a common edge in Γ^*. Let $v_1 \in V_1$ and $v_2 \in V_2$ be incident to $f_1 \in F_1$ and $f_2 \in F_2$. Let $e \in E$ be the edge on the common boundary of f_1 and f_2, and denote its endpoints by $a, b \in V$.

We consider three possible edges (some of which may be homotopic to an existing edge in G): let $e_0 = (v_1, v_2)$ such that it lies in $f_1 \cup f_2$; let $e_1 = (v_1, a)$ (resp., $e_2 = (v_1, b)$) such that it starts in f_1 and follow edge e from f_1 to its endpoint a (resp., b).

- If $e \notin E'$, then replace edge $e = (a, b)$ by a new edge $e_0 = (v_1, v_2)$ in G, and add this new edge to H. This modification contradicts the assumption that H has the minimum number of components.
- Assume $e \in E'$. Note that e_1 and e_2 form a path between a and b, consequently at most one of these edges may be present in G (as a homotopic copy), otherwise we could modify E' by replacing e with these edges, contradicting the maximality of E'. Now we can increase E by replacing e with e_1 or e_2 (whichever is not already present), contradicting the choice of H.

Both cases lead to a contradiction. $\square$

In the proof of Lemma 1, we shall use Sperner's Lemma [40], a well-known discrete analogue of Brouwer's fixed point theorem.

Lemma 11. (Sperner [40]) *Let K be a geometric simplicial complex in the plane, where the union of faces is homeomorphic to a disk. Assume that each vertex is assigned a color from the set $\{1, 2, 3\}$ such that three vertices $v_1, v_2, v_3 \in$*

∂K are colored 1, 2, and 3, respectively, and for any pair $i, j \in \{1, 2, 3\}$, the vertices on the path between v_i and v_j along ∂K that does not contain the 3rd vertex are colored with $\{i, j\}$. Then K contains a triangle whose vertices have all three different colors.

We are now ready to prove Lemma 1.

Lemma 1. *The multigraph H is a triangulation, that is, a plane multi-graph in which every face is bounded by a walk with three vertices and three edges.*

Proof. We need to show that the multigraph H is a triangulation. Suppose, to the contrary, that H is not a triangulation. Then H has a face f whose boundary walk $w = (v_1, v_2, \ldots, v_m)$ has more than three vertices ($m \geq 4$). To simplify notation, we assume that w is a simple cycle; this assumption is not essential for the proof.

Let P_f be the subgraph of Γ^* formed by all edges and vertices lying in the interior or on the boundary of f; let V_f denote the set of vertices of P_f (it consists of $v_1, \ldots, v_m$ and all crossings in the interior or on the boundary of f); and let F denote the set of faces of Γ^* that lie in f. Let $F_0 \subseteq F$ be the set of faces that are not incident to any vertex; and for $i = 1, \ldots, m$, let $F_i \subseteq F$ be the set of faces incident to v_i.

Note that a face in F cannot be incident to two nonconsecutive vertices v_i and v_j, $j \notin \{i-1, i, i+1\}$, otherwise we could add a new edge v_iv_j, contradicting the maximality of G. A vertex $c \in V_f$ cannot be incident to two faces $f_1 \in F_i$ and $f_2 \in F_j$ such that $j \notin \{i-1, i, i+1\}$, otherwise two edges $e_1, e_2 \in E \setminus E'$ cross at c, and we can replace edge e_1 with a new edge v_iv_j that lies in $f_1 \cup f_2$ that uses one gap to cross edge e_2—the new edge can be inserted into E', contradicting the maximality of H.

We distinguish two cases.

Case 1. For every $i \in \{1, \ldots, m\}$, the edge (v_i, v_{i+1}) is incident to faces in $F_0 \cup F_i \cup F_{i+1}$ only. We use Sperner's Lemma [40] for a triangulation K of the *dual graph* on the faces $F_1 \cup \ldots \cup F_m$, that we define here. We first create the standard dual graph of all faces in F: The nodes correspond to the faces in F; and two nodes are adjacent if and only if the corresponding faces are adjacent in Γ^*. We then triangulate the standard dual graph as follows. If a crossing $c \in V_f$ is incident to four faces in F, then the adjacency graph forms a 4-cycle in the standard dual. By Lemma 9(2), at least three of those faces are in $F \setminus F_0$, and we triangulate the 4-cycle by an arbitrary diagonal between two faces in $F \setminus F_0$. Note that the faces in F_0 still form an independent set by Lemma 9(2)). Finally, remove all nodes corresponding to F_0, and triangulate the chain of adjacent nodes arbitrarily to obtain a triangulation K. The condition in Case 1 implies that K is a geometric simplicial complex, where the union of faces is homeomorphic to a disk.

We now define a 3-coloring of K (the coloring need not be proper). Assign color 1 to all faces in F_1. For $i = 2, \ldots, m$, assign color 2 to all faces in $F_i \setminus \bigcup_{j<i} F_j$ if i is even, and color 3 if i is odd.

By Sperner's Lemma, K has a triangle whose nodes have all three different colors, say $f_1 \in F_i$, $f_2 \in F_j$, and $f_3 \in F_k$. Without loss of generality, assume that $j \notin \{i-1, i+1\}$. We add a new edge (v_i, v_j), as follows. There are three cases depending on how the edge $f_i f_j$ in K was created:

- Faces f_1 and f_2 are adjacent in Γ^*. Then we can add a new edge (v_i, v_j) to G such that (v_i, v_j) lies in $f_i \cup f_j$ and uses a gap to cross the boundary between these faces. This contradicts the maximality of G.
- A vertex $c \in V_f$ is incident to both f_1 and f_2. Then two edges $e_1, e_2 \in E \setminus E'$ cross at c. We can replace edge e_1 with a new edge (v_i, v_j) that lies in $f_1 \cup f_2$ and that crosses edge e_2 at c. The new edge can be inserted into both G and H, contradicting the maximality of H.
- A face $f_0 \in F_0$ is adjacent to both f_1 and f_2. Then two edges $e_1, e_2 \in E \setminus E'$ are on the common boundary of the adjacent pairs f_1, f_0 and f_0, f_2. We can replace edge e_1 with a new edge (v_i, v_j) that lies in $f_1 \cup f_0 \cup f_2$ that crosses edge e_2. The new edge can be inserted into both G and H, contradicting the maximality of H.

Case 2. There is an index $i \in \{1, \ldots, m\}$, such that (v_i, v_{i+1}) is incident to a face in F_j for some $j \neq 0, i, i+1$. Without loss of generality, we may assume that edge (v_1, v_m) is incident to a face in F_j for some $1 < j < m$. Note that edge (v_1, v_m) must be incident to some face in F_j for *all* $1 \leq j \leq m$; otherwise $v_1 v_m$ would be incident to two faces, $f_i \in F_i$ and $f_j \in F_j$, $j \notin \{i-1, i, i+1\}$, that are either adjacent to each other or both adjacent to some face $f_0 \in F_0$; and then we could add a new edge (v_i, v_j) lying in $f_i \cup f_j$ or $f_i \cup f_0 \cup f_j$.

It follows that there are faces $f_2 \in F_2$ and $f_3 \in F_3$ that are incident to some point $c \in (v_1, v_2)$; or both are adjacent to some common face $f_0 \in F_0$ that is incident to $v_1 v_2$.

Consider the face f' of H on the opposite side of (v_1, v_m), and let F' be the set of faces in the planarization Γ^* contained in f'. Let $f'' \in F'$ be a face incident to $c \in (v_1, v_m)$ or adjacent to face f_0. By Lemma 9(2), we may assume that f'' is incident to a vertex v_k on the boundary of the face f'. It is possible that $v_k = v_1$ or $v_k = v_m$.

- If $v_k = v_1$, then we modify G, Γ, and H as follows: remove the edge that crosses (v_1, v_m) at c, and add a new edge (v_3, v_1) that lies in $f_3 \cup f''$ or $f_3 \cup f_0 \cup f''$, and crosses (v_1, v_m) at a point c. Then redraw the edges (v_1, v_m) and (v_1, v_3) by exchanging their initial arcs between v_1 and c, and eliminating the crossing at c. Both (v_1, v_m) and (v_1, v_3) can be added to E', contradicting the maximality of E'.
- If $v_k = v_m$ and $v_{m-1} = v_3$, we make similar changes replacing edge e with (v_2, v_m).
- Otherwise we similarly modify G, Γ, and H as follows: first replace the edge (v_1, v_m) with two new edges (v_2, v_k) and (v_3, v_k), that lie in $f_2 \cup f''$ and $f_3 \cup f''$, respectively, and one of them may cross some edge at c. Both (v_2, v_k) and (v_3, v_k) can be added to E', contradicting the maximality of E'.

All cases lead to a contradiction. Therefore, our initial assumption must be dropped, consequently the multigraph H is a triangulation, as claimed. $\square$

A.2 Lower bound constructions

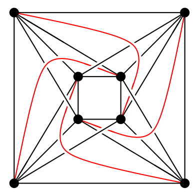

Fig. 5. One more pattern that produces 1-gap-planar graphs with n vertices and $5n - \Theta(1)$ edges.

We mention a third, slightly weaker construction, which is based on a sequence of nested squares. Fig. 5 shows how to add 16 edges between two consecutive squares such that the 16 crossings are assigned to distinct edges. We can add two diagonals in the external face and the innermost square. Using s squares, we have $n = 4s$, and $m = 4s + 16(s-1) + 2 \cdot 2 = 20s - 12 = 5n - 12$. In particular, for $s = 2$ this yields a drawing of K_8; see Fig. 3(a).

B Additional Material for Section 5

Lemma 3. *Let u and v be any two degree-12 vertices of B. Every 1-gap-planar drawing Γ of B contains a gapped chain $\mathcal{C}$ containing u and v.*

Proof. Let Γ^* be the planarization of Γ. Let Γ' be the subgraph of Γ^* consisting only of those edges that correspond to or belong to gap edges of Γ. We prove that Γ' contains two edge-disjoint paths from u to v. Note that these two edge-disjoint paths may meet at real vertices and at dummy vertices (i.e., a crossing between two gap edges). A curve that goes through these two paths is the desired gapped chain. According to Menger's theorem, two such paths exist if and only if every (u,v)-cut of Γ' has size at least 2, where a (u,v)-cut of Γ' is a set of edges of Γ' whose removal disconnects u and v. It is well-known that such edge cuts correspond to cycles in the dual, which in turn correspond to curves that separate u and v by crossing a set of edges. We now consider one such curve, and claim this curve crosses at least two gap edges in the original drawing Γ (after a slight perturbation we can assume that it does not pass through a vertex). More precisely, let ℓ be a simple closed curve such that: it does not pass through any vertex of Γ; it crosses each edge of Γ^* at most once; it divides the plane into two nonempty topologically connected regions, one containing u and the other containing v. Let L denote the set of edges of G that are crossed by ℓ. Note that

G contains 12 edge-disjoint paths from u to v, which induce 12 edge-disjoint paths from u to v in Γ^*. It follows that $|L| \geq 12$. Also, Γ has at most 6 gap free edges by Lemma 2. Hence, L contains at least $12 - 6 > 2$ gap edges. □

Lemma 4. *The problem* 1GapPlanarity *is in* NP.

Proof. Given a planarization Γ^* of a drawing Γ, we can check whether it is 1-gap-planar in polynomial time by using Property 2. A nondeterministic algorithm to generate all planarizations of a graph with k crossings, where $0 \leq k < \binom{m}{2}$, evaluates all possible k pairs of edges that cross (and the order of the crossings along the edges) with a technique similar to the one in [23]. Then it replaces crossings with dummy vertices and tests whether the resulting graph is planar, i.e., whether it is a planarization of a drawing of G, and whether it is 1-gap-planar. Hence, the problem belongs to NP. □

Theorem 5. *The* 1GapPlanarity *problem is* NP*-complete.*

Proof. The 1GapPlanarity problem is in NP by Lemma 4.

We now prove that an instance A of 3Partition is a positive instance if and only if the graph G_A is a positive instance of 1GapPlanarity.

Suppose first that G_A is a positive instance of 1GapPlanarity. From what we said it is clear that each traversing path must be routed through exactly three central cells and $3m-3$ noncentral cells. In particular, each path has $(3m-3)(\lceil B/2 \rceil + 1) + B$ edges, and hence can traverse at most these many vertical pairs. Since each noncentral cell consists of $\lceil B/2 \rceil + 1$ vertical pairs, it must be that the 3 central cells contain B vertical pairs in total. Thus, we can construct a solution for A by looking at the central cells traversed by the m paths.

Suppose now that A is a positive instance of 3Partition. Note that a 1-gap-planar drawing of G_A can be always computed if one omits all the transversal paths (see also Fig. 4(c)). To draw the paths, let $\{A_1, A_2, ..., A_m\}$ be a solution for A. Then we route the paths similarly as in [8], that is, in such a way that: (1) they do not cross each other; (2) they do not cross any blob; (3) each path passes through exactly 3 central cells with B vertical pairs in total, and $3m-3$ noncentral cells; and (4) each cell is traversed by at most one path. Consider a subset A_j of the solution of instance A of 3Partition and assume without loss of generality that $A_j = \{a_\kappa, a_\lambda, a_\mu\}$, where $1 \leq \kappa, \lambda, \mu \leq 3m$. Then, in the computed drawing path π_j will cross the κ-th, λ-th and μ-th columns of G_A through central cells. Path π_j will cross the remaining columns of G through noncentral cells. Hence, requirement (3) is satisfied. Consider now the routing of the remaining transversal paths through the κ-th column; the corresponding routings though the λ-th and μ-th columns of G_A are symmetric. By construction, there must exist exactly $m-1$ available cells above and exactly $m-1$ available cells below the central cell of the κ-th column. This implies that there exist at least as many available noncentral cells as transversal paths to route at each side of the central cell of the κ-th column. Hence, we can route the remaining transversal paths through the κ-th column so that all other requirements are satisfied. □

C Additional Material For Section 6

Lemma 7. *For every $k \geq 1$, there exists a 1-gap-planar graph G_k that is not k-planar.*

Proof. Let $k \in \mathbb{N}$. We construct a graph $G_k = (V, E)$ together with its 1-gap-planar drawing as follows. Start with an edge (a, b) crossed by $k+1$ disjoint edges (c_i, d_i), for $i = 1, \dots, k+1$. The $2k+2$ vertices lie in a common face, and we can connect them by a Jordan curve, which forms a cycle $C = (a, c_1, \dots, c_{k+1}, b, d_{k+1}, \dots, d_1)$. Add a new vertex v_0 in the exterior of the cycle, and connect it to all vertices of C. The cycle C and v_0 is a wheel W, which has $m = 4k+4$ edges. Finally, replace each edge of the wheel by t edge-disjoint paths of length 2, where $t \geq k$ is a suitable parameter that we shall specify shortly. This completes the construction of $G_k = (V, E)$. Note that the total number of edges is bounded above by

$$|E| = 1 + (k+1) + (4k+4)2t = 1 + (k+1)(8t+1) < 10(k+1)t.$$

It is clear that G_k is 1-gap-planar, since the crossing between (a, b) and (c_i, d_i) can be charged to (c_i, d_i) for all $i = 1, \dots, k+1$.

Suppose that G_k admits a k-planar drawing Γ. Since each edge crosses at most k other edges, the total number of crossings is at most $k|E|/2 < 5k(k+1)t$.

We claim that for each edge of the wheel W, we can choose $k+1$ of the t paths such that no two chosen paths that correspond to different edges of the wheel cross in the drawing Γ. We prove the claim by contradiction. Since we choose $k+1$ out of t paths for each of the m edges of the wheel independently, there are $\binom{t}{k+1}^m$ possible choices. Suppose, for the sake of contradiction, that every choice produces a graph that has at least one crossing in Γ between paths corresponding to different edges of W. Each crossing between two such paths is counted $\binom{t-1}{k}^2 \binom{t}{k+1}^{m-2}$ times. Consequently, the total number of crossings in Γ is at least $t^2/(k+1)^2$. If we put $t = 5(k+1)^4$, then we would have at least $t^2/(k+1)^2 = t \cdot 5(k+1)^4/(k+1)2 = 5(k+1)^2 t > 5k(k+1)t$ crossings, a contradiction. This completes the proof of the claim.

Let G'_k be a subgraph of G_k that consists of $k+1$ paths corresponding to each edge of W such that the paths corresponding to different edges of W do not cross in Γ; and let Γ' be the restriction of Γ to G'_k. Note that any $k+1$ paths that correspond to the same edge of W are homotopic to each other in Γ'. If we pick one of the $k+1$ paths, for each edge of W, the Jordan arc along these paths provide a planar drawing of W. Since W is 3-connected, it has a combinatorially unique embedding, which we denote by $\Gamma(W)$. As noted above, every edge of W in the drawing $\Gamma(W)$ is homotopic to $k+1$ paths in the drawing Γ'.

As the combinatorial embedding of W is unique, the vertices a, b, and c_i, d_i, for $i = 1, \dots, k+1$, lie on the boundary of a single face, which we denote by F. If edges (a, b) and (c_i, d_i), for $i = 1, \dots, k+1$, are homotopic to Jordan arcs that lie in F, then (a, b) crosses (c_i, d_i), for all $i = 1, \dots, k+1$. If any of these edges is not homotopic to a Jordan arc in F, then it crosses a bundle of $k+1$ paths corresponding to some edge of W. In both cases, one of the edges crosses $k+1$ other edges in Γ, contradicting our assumption that Γ is a k-planar drawing. $\square$

D 1-Gap-Planar Drawings of Complete Bipartite Graphs

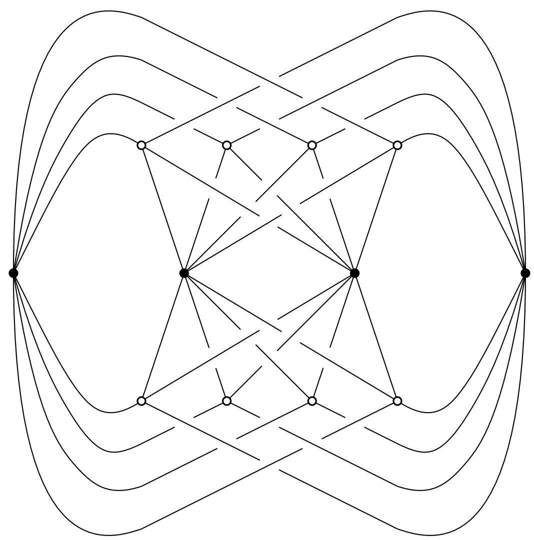

Fig. 6. A 1-gap-planar drawing of $K_{4,8}$.

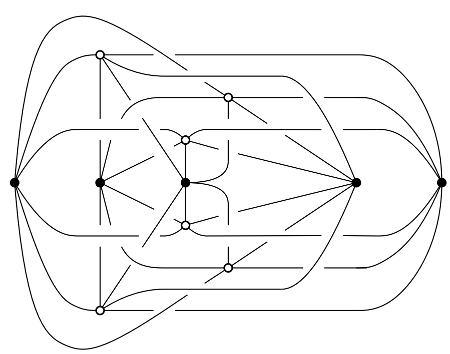

Fig. 7. A 1-gap-planar drawing of $K_{5,6}$.