MASTER

Derivation of a two-dimensional complete flux scheme in local flow adapted coordinates

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1 Introduction

Conservation laws occur frequently in science and engineering. They occur, among others, in disciplines like fluid mechanics, combustion theory, plasma physics and semiconductor physics. They often are of the advection-diffusion-reaction type, describing the interplay between different processes such as advection or drift, diffusion or conduction and (chemical) reaction or recombination/generation. Examples are laminar flames, gas discharges in plasma or equations for reacting flow.

To be able to calculate numerical solutions for these equations good space discretisation and time integration methods are needed, and in this paper we give a new approach for the spatial discretisation. There already are many methods available, such as finite element, finite difference, finite volume or spectral methods, but we restrict ourselves to finite volume methods (FVM). Finite volume methods are based on the integral formulation, so the conservation law is integrated over disjunct control volumes covering the domain. The resulting discrete conservation law contains fluxes at the interfaces of the control volumes and these fluxes need to be approximated.

The purpose of the paper is to derive a new way to approximate these fluxes. The method described in this paper is based on the complete flux scheme by Anthonissen and ten Thije Boonkkamp, as described in [4]. Just as in [4] we require the numerical flux to have the following properties. It should firstly be unconditionally second order accurate, which also has to hold when advection is highly dominant. Secondly the flux may only depend on its neighbouring values. This means that in 2 dimensions you get at most a 9-point scheme.

Our scheme is an extension of the complete flux scheme in [4]. We also derive an integral representation for the flux from the solution of a local boundary value problem (BVP) for the entire equation, including the source term, but we first rotate in the direction of the flow. This causes the grid to be locally aligned to the flow, which means that in one direction you have advection and diffusion, but perpendicular to this you only have diffusion. We will use this to provide a new scheme. We will also provide interpolation methods, needed to interpolate the solution since the required points for the BVP are no longer the same as the original grid points. The scheme is already described in [3], but we will give the complete derivation in this paper.

We have organised this paper as follows. In Chapter 2 we will discuss the original complete flux scheme, after which in Chapter 3 we will derive the complete flux scheme in flow adapted coordinates. In chapter 4 we will briefly discuss the extensions to three dimensions. In Chapter 5 we will give numerical results and perform convergence tests, after which in Chapter 6 we will give some concluding remarks and give recommendations for further research.
2 Derivation of the standard complete flux scheme

We first derive the original finite-volume complete flux scheme. The complete derivation of this method is given in [4], but here we will summarise this derivation. We will start by defining a finite volume method, then we will derive the one-dimensional complete flux scheme and then extend this to higher dimensions.

2.1 Finite volume method

We consider a stationary conservation law of advection-diffusion-reaction type, given by

\[ \nabla \cdot (u \phi - \varepsilon \nabla \phi) = s, \quad (2.1) \]

where \( u = u_x e_x + u_y e_y \) is a velocity, \( \varepsilon > 0 \) is a diffusion coefficient, and \( s \) a source term. The unknown \( \phi \) is, for example, the mass fraction or concentration of a species in a mixture. We now introduce the flux vector \( f \):

\[ f := u \phi - \varepsilon \nabla \phi, \quad (2.2) \]

so equation (2.1) can be written as \( \nabla \cdot f = s \). Integrating this equation over a fixed domain \( \Omega \) and applying Gauss’s theorem we obtain the integral form of the conservation law:

\[ \oint_{\Gamma} (f, \mathbf{n}) dS = \int_{\Omega} s dV, \quad (2.3) \]

where \( \mathbf{n} \) is the outward unit normal on the boundary \( \Gamma = \partial \Omega \). In a finite volume method we cover the domain with a finite number of disjunct control volumes and impose the integral form (2.3) for each of those volumes. For two-dimensional Cartesian coordinates we first choose the grid points \( x_{i,j} = (x_i, y_j) \) where the unknown \( \phi \) has to be approximated. Next we choose control volumes \( \Omega_{i,j} := (x_{i-1/2}, x_{i+1/2}) \times (y_{j-1/2}, y_{j+1/2}) \), where \( x_{i\pm 1/2} := \frac{1}{2}(x_i + x_{i\pm 1}) \) and \( y_{j\pm 1/2} := \frac{1}{2}(y_j + y_{j\pm 1}) \). The boundary of control volume \( \Omega_{i,j} \) then becomes the union of the four line segments \( \Gamma_{i\pm 1/2,j} := ((x_{i\pm 1/2}, y_{j-1/2}), (x_{i\pm 1/2}, y_{j+1/2})) \) and \( \Gamma_{i,j\pm 1/2} := ((x_{i+1/2}, y_{j\pm 1/2}), (x_{i-1/2}, y_{j\pm 1/2})) \). Taking \( \Omega = \Omega_{i,j} \) in the integral form of the conservation law (2.3) and approximating all integrals with the midpoint rule we find

\[ (F_{x,i+1/2,j} - F_{x,i-1/2,j}) \Delta y + (F_{y,i,j+1/2} - F_{y,i,j-1/2}) \Delta x \approx s_{i,j} \Delta x \Delta y. \quad (2.4) \]

Here \( F_x \) and \( F_y \) are the numerical fluxes which approximate the flux \( f = f_x e_x + f_y e_y \), and these approximations need to be determined to complete the FVM.
2.2 Integral representation of the flux

In this section the original complete flux scheme is presented, following the derivation of [4]. This is done by first deriving an expression for the numerical flux for a one-dimensional problem. In this case the flux is given by \( f = u\phi - \varepsilon \phi' \). This means that the derivation of the flux \( f_{j+1/2} \) at the cell edge \( x_{j+1/2} = \frac{1}{2}(x_j + x_{j+1}) \) is based on the following model BVP:

\[
\begin{align*}
  f' &= (u\phi - \varepsilon \phi')' = s, \quad x_j < x < x_{j+1}, \\
  \phi(x_j) &= \phi_j, \quad \phi(x_{j+1}) = \phi_{j+1}.
\end{align*}
\]  

(2.5a)

(2.5b)

In the following we need these variables:

\[
\begin{align*}
  \lambda &:= \frac{u}{\varepsilon}, \quad P := \lambda \Delta x, \\
  \Lambda(x) &:= \int_{x_j}^{x_{j+1/2}} \lambda(x')dx', \\
  S(x) &:= \int_{x_j}^{x_{j+1/2}} s(x')dx',
\end{align*}
\]

(2.6)

where \( \Delta x := x_{j+1} - x_j \) is constant. Here \( P \) is the Péclet function and \( \Lambda \) is the Péclet integral. If \( P \) is constant it is called the Péclet number. We now integrate (2.5a) from \( x_{j+1/2} \) to \( x \) to get the integral balance:

\[
  f(x) - f_{j+1/2} = S(x).
\]

(2.7)

Using the definition of \( \Lambda \) the flux can also be rewritten as:

\[
  f = -\varepsilon(\phi e^{-\Lambda})' e^{\Lambda}.
\]

(2.8)

Substituting this in (2.7) and integrating from \( x_j \) to \( x_{j+1} \) we obtain the following expression for the flux \( f_{j+1/2} \):

\[
\begin{align*}
  f_{j+1/2} &= f_{j+1/2}^h + f_{j+1/2}^i, \\
  f_{j+1/2}^h &= -\frac{\int_{x_j}^{x_{j+1/2}} e^{-\Lambda} \phi_{j+1} - e^{-\Lambda} \phi_j dx}{\int_{x_j}^{x_{j+1/2}} e^{-\Lambda} dx}, \\
  f_{j+1/2}^i &= -\frac{\int_{x_j}^{x_{j+1/2}} e^{-1} Sdx}{\int_{x_j}^{x_{j+1/2}} e^{-1} dx}.
\end{align*}
\]

(2.9a)

(2.9b)

(2.9c)

where \( f_{j+1/2}^h \) and \( f_{j+1/2}^i \) are the homogeneous and inhomogeneous part of the flux respectively.

In the special case where \( u, \varepsilon \) and \( s \) are constant on the interval \([x_j, x_{j+1}]\) we can determine all the integrals in (2.6). This gives for the flux:

\[
\begin{align*}
  f_{j+1/2}^h &= -\frac{\varepsilon}{\Delta x} (B(P)\phi_{j+1} - B(-P)\phi_j), \\
  f_{j+1/2}^i &= \left(1 - \frac{1}{2} - W(P)\right) s\Delta x,
\end{align*}
\]

(2.10a)

(2.10b)

where the functions \( B \) and \( W \) are defined by:

\[
\begin{align*}
  B(z) &:= \frac{z}{e^z - 1}, \quad W(z) := \frac{e^z - 1 - z}{z(e^z - 1)}.
\end{align*}
\]

(2.11)

Note that \( 0 \leq W(z) \leq 1, W(z) + W(-z) = 1 \), is a weighting function. In Figure 2.1 these
functions are plotted. We now introduce the standard inner product of two functions $a = a(x)$ and $b = b(x)$, on $(x_j, x_{j+1})$ by $\langle a, b \rangle$, which is given by:

$$\langle a, b \rangle := \int_{x_j}^{x_{j+1}} a(x)b(x)dx.$$  \hfill (2.12)

If we also introduce the average, $\bar{\Lambda}_{j+1/2} := \frac{1}{2}(\Lambda_j + \Lambda_{j+1})$ we can rewrite the homogeneous and inhomogeneous flux as

$$f_{j+1/2}^h = -e^{-\bar{\Lambda}_{j+1/2}} \frac{e^{-\langle \lambda, 1 \rangle/2} \phi_j - e^{\langle \lambda, 1 \rangle/2} \phi_{j+1}}{\langle \varepsilon^{-1}, e^{-\Lambda} \rangle},$$  \hfill (2.13a)

$$f_{j+1/2}^i = -\frac{\langle \varepsilon^{-1} S, e^{-\Lambda} \rangle}{\langle \varepsilon^{-1}, e^{-\Lambda} \rangle}.$$  \hfill (2.13b)

Note that (2.13a) can be rewritten as:

$$f_{j+1/2}^h = -\frac{\langle \lambda, e^{-\Lambda} \rangle}{\langle \varepsilon^{-1}, e^{-\Lambda} \rangle} \frac{\langle \lambda, 1 \rangle}{\langle \varepsilon^{-1}, e^{-\Lambda} \rangle} (B(\langle \lambda, 1 \rangle)\phi_{j+1} - B(-\langle \lambda, 1 \rangle)\phi_j),$$  \hfill (2.14)

and we will base our approximation on this. For the inhomogeneous flux we substitute the expression for $S$ in (2.9c) and change the order of integration. We assume $u$ and $\varepsilon$ are constant to obtain the following representation for the inhomogeneous flux:

$$f_{j+1/2}^i = \int_{0}^{1} G(\sigma; P)\tilde{s}(\sigma)d\sigma, \quad \sigma(x) := \frac{x-x_j}{\Delta x},$$  \hfill (2.15)

where $\sigma(x)$ is the normalised coordinate on $[x_j, x_{j+1}]$, and $G(\sigma; P)$ the Green’s function for the flux given by

$$G(\sigma; P) = \begin{cases} 
1 - e^{-P\sigma} & 0 \leq \sigma \leq \frac{1}{2}, \\
1 - e^{-P(1-\sigma)} & \frac{1}{2} < \sigma \leq 1,
\end{cases}$$  \hfill (2.16)

and we use this to determine the numerical approximation for the inhomogeneous flux.

### 2.3 Numerical flux

To get the final numerical approximation for the homogeneous flux we need to approximate the inner products from (2.13a). The detailed derivation can be found in [4]. Here the
2.4 Extension to higher dimensions

following averages are introduced:

\[ \bar{a}_{j+1/2} := \frac{1}{2} (a_j + a_{j+1}), \]  

\[ \tilde{a}_{j+1} := W(-\bar{P}) a_j + W(\bar{P}) a_{j+1}. \]  

Using these and the approximations of [4] we come to the following expression for the homogeneous flux:

\[ F_{j+1/2}^h = -\tilde{\varepsilon}_{j+1/2} \frac{\bar{\lambda}_{j+1/2}}{\lambda_{j+1/2}} (B(\bar{P}_{j+1/2}) \phi_{j+1} - B(-\bar{P}_{j+1/2}) \phi_j). \]  

(2.18)

For the inhomogeneous flux we assume \( \tilde{s} = s_j \) on \((x_j, x_{j+1/2})\) and \( \tilde{s} = s_{j+1} \) on \((x_{j+1/2}, x_{j+1})\). If we assume this we can integrate the Green’s function to find:

\[ F_{j+1/2}^i = C(-\bar{P}_{j+1/2}) s_j + C(\bar{P}_{j+1/2}) s_{j+1}, \]  

(2.19)

where \( C := C(z) \) is given by:

\[ C(z) := \frac{e^{\frac{1}{2} z} - 1 - \frac{1}{2} z}{ze^{z} - 1}. \]  

(2.20)

This together gives an approximation for the flux, which gives us our complete flux scheme.

2.4 Extension to higher dimensions

In this section we extend the derivation to two dimensions, and later to three dimensions. The key idea is to include the cross flux term \( \partial f_y / \partial y \) in the evaluation for the flux in the \( x \)-direction.

For two-dimensions the flux is given by:

\[ f = f_x e_x + f_y e_y = \left( u\phi - \varepsilon \frac{\partial \phi}{\partial x} \right) e_x + \left( v\phi - \varepsilon \frac{\partial \phi}{\partial y} \right) e_y. \]  

(2.21)

This leads to the following modified boundary value problem:

\[ \frac{\partial f_x}{\partial x} = \frac{\partial}{\partial x} \left( u\phi - \varepsilon \frac{\partial \phi}{\partial x} \right) = s_x, \quad x_i < x < x_{i+1}, y = y_j, \]  

(2.22a)

\[ \phi(x_i, j) = \phi_{i, j}, \quad \phi(x_{i+1}, j) = \phi_{i+1, j}. \]  

(2.22b)

Here the adjusted source term is defined by:

\[ s_x := s - \frac{\partial f_y}{\partial y}. \]  

(2.23)

The same is then done for the flux \( f_y \). The derivation of the expression for the numerical flux is essentially the same as for the flux (2.5a), with the main difference being the inclusion of the cross flux in the source term. To compute \( s_x \) we replace the cross flux \( \partial f_y / \partial y \) by its central difference approximation, and for \( f_y \) we then take the homogeneous numerical flux from before. For the flux in the \( y \)-direction the same approach is used.
The extension to three dimensions is described in [2], and it follows the same approach as the two-dimensional situation. We now have the flux:

\[ f = f_x e_x + f_y e_y + f_z e_z = \left( u\phi - \varepsilon \frac{\partial \phi}{\partial x} \right) e_x + \left( v\phi - \varepsilon \frac{\partial \phi}{\partial y} \right) e_y + \left( w\phi - \varepsilon \frac{\partial \phi}{\partial z} \right) e_z. \quad (2.24) \]

This leads to the following boundary value problem:

\[ \frac{\partial}{\partial x} \left( u\phi - \varepsilon \frac{\partial \phi}{\partial x} \right) = s_x, \quad x_i < x < x_{i+1}, \quad y = y_j, \quad z = z_k, \quad (2.25a) \]

\[ \phi(x_{i,j,k}) = \phi_{i,j,k}, \quad \phi(x_{i+1,j,k}) = \phi_{i+1,j,k}. \quad (2.25b) \]

Here the adjusted source term is defined by:

\[ s_x := \alpha s - \beta \left( \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} \right), \quad (2.26) \]

with \( \alpha \) and \( \beta \) coefficients we still have to determine. Taking \( \beta = 1 \) completely includes the cross fluxes, and taking \( \beta = 0 \) completely ignores them. Taking \( \beta > 0 \) decreases the diagonal elements. If \( \beta = 1 \) it is even possible that diagonal elements become zero for advection dominated problems. This means that usually a value between 0 and 1 is chosen.

To determine the value of \( \alpha \) we add the three one-dimensional problems:

\[ \frac{\partial f_x}{\partial x} = \alpha s - \beta \left( \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} \right), \quad \frac{\partial f_y}{\partial y} = \alpha s - \beta \left( \frac{\partial f_x}{\partial x} + \frac{\partial f_z}{\partial z} \right), \quad \frac{\partial f_z}{\partial z} = \alpha s - \left( \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} \right). \quad (2.27) \]

Adding these equations gives:

\[ \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} = 3\alpha s - 2\beta \left( \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} \right). \quad (2.28) \]

This means that we need to choose \( \alpha \) such that \( 1 + 2\beta = 3\alpha \), or \( \alpha = (1 + 2\beta)/3 \). The cross fluxes in the source term are then treated in the same way as in the two-dimensional case, using the homogeneous flux approximation. Since we want to avoid diagonal elements that are zero we will propose a new method to approximate the flux in the next chapter using locally rotated coordinates.
3 Flow adapted coordinates

In this chapter we will propose a new method to approximate the flux. This method is based on locally rotating the grid such that the grid is aligned with the flow. We will discuss the derivation of this scheme, by first giving the rotation, then we will derive the constant coefficient flux, after which we will discuss the interpolation needed to solve the problem. Finally we will derive the numerical flux for variable coefficients and for a non-uniform grid.

3.1 Flux representation in flow adapted coordinates

We now start the derivation by first locally rotating the grid such that it is aligned with the flow. This is a natural choice, since a lot of information is travelling in the direction of the flow. This also changes the original boundary value problem, providing a slightly altered numerical flux. To determine the numerical flux we first assume that $u$ and $\varepsilon$ are constant, and that $\Delta x = \Delta y := h$. We do this by first defining the angle between $u$ and $e_x$ by $\tan \alpha = \frac{v}{u}$.

Figure 3.1: Representation of the rotation
Flow adapted coordinates

and then rotating in that direction to obtain the orthonormal coordinates $\xi$ and $\eta$, where $\xi$ is aligned with the flow and $\eta$ is perpendicular to it, see Figure 3.1. We define the rotation matrix by:

$$
R(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix},
$$

and then define $\xi = \xi e_\xi + \eta e_\eta$ by:

$$
x - x_e = R(\alpha)\xi.
$$

This causes the local origin to be at $x_e$. Here $x_e = (x_e, y_e)$ is the location of the middle of the eastern boundary, as can be seen in Figure 3.1. We can invert equation (3.2), giving:

$$
\begin{align*}
\xi &= R(-\alpha)(x - x_e) \\
\eta &= -(x - x_e)\sin \alpha + (y - y_e)\cos \alpha.
\end{align*}
$$

We also have:

$$
f = f_x e_x + f_y e_y = f_\xi e_\xi + f_\eta e_\eta,
$$

implying that:

$$
\begin{pmatrix} f_x \\ f_y \end{pmatrix} = R(\alpha) \begin{pmatrix} f_\xi \\ f_\eta \end{pmatrix}.
$$

Moreover we have:

$$
\nabla \phi = \frac{\partial \phi}{\partial \xi} e_\xi + \frac{\partial \phi}{\partial \eta} e_\eta, \quad \nabla^2 \phi = \frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2}.
$$

We then obtain the following flux in rotated coordinates:

$$
f(x_e) = u\phi - \varepsilon \nabla \phi = U \phi e_\xi - \varepsilon \left( \frac{\partial \phi}{\partial \xi} e_\xi + \frac{\partial \phi}{\partial \eta} e_\eta \right) = \left( U \phi - \varepsilon \frac{\partial \phi}{\partial \xi} \right) e_\xi - \varepsilon \frac{\partial \phi}{\partial \eta} e_\eta = f_\xi e_\xi + f_\eta e_\eta,
$$

where $U = |u|$. Note that $U \geq 0$ always, because it is the velocity in the $\xi$–direction. From the conservation law we then get:

$$
\nabla \cdot f = s \iff \frac{\partial f_\xi}{\partial \xi} + \frac{\partial f_\eta}{\partial \eta} = s \iff \frac{\partial f_\xi}{\partial \xi} = s + \varepsilon \frac{\partial^2 \phi}{\partial \eta^2} =: s_\xi.
$$

Now, to calculate $f_\xi$ we define a local boundary value problem in the rotated coordinates:

$$
\frac{\partial f_\xi}{\partial \xi} = \frac{\partial}{\partial \xi} \left( U \phi - \varepsilon \frac{\partial \phi}{\partial \xi} \right) = s_\xi, \quad -h/2 < \xi < h/2, \quad \phi(-h/2) = \phi_L, \quad \phi(h/2) = \phi_R.
$$

Here $\phi_L$ and $\phi_R$ are the values of $\phi$ at the points $L$ and $R$ respectively. Now the derivation for the numerical flux is essentially the same as before. We integrate (3.9) from 0 to $\xi$ to obtain the following expression:

$$
f(\xi) - f_{\xi,e} = S(\xi) := \int_0^\xi s_\xi(\xi')d\xi'.
$$

Define $\lambda = \frac{U}{\varepsilon}$, then note that $f_\xi$ can be written as:

$$
f_\xi = U \phi - \varepsilon \frac{\partial \phi}{\partial \xi} = -\varepsilon \frac{\partial}{\partial \xi} \left( e^{-\lambda \xi} \phi \right) e^{\lambda \xi}.
$$
3.1. Flux representation in flow adapted coordinates

We now substitute (3.11) in (3.10) and integrate this from \( \xi_L = -h/2 \) to \( \xi_R = h/2 \) to obtain:

\[
-\frac{1}{\lambda} f_{\xi,e} \left( e^{-\lambda \xi_R} - e^{-\lambda \xi_L} \right) = -\varepsilon \left( e^{-\lambda \xi_R \phi_R} - e^{-\lambda \xi_L \phi_L} \right) - \int_{\xi_L}^{\xi_R} e^{-\lambda \xi} S(\xi) \, d\xi. \tag{3.12}
\]

This can then be rewritten as:

\[
f_{\xi,e} = f^{\text{h}}_{\xi,e} + f^{\text{i}}_{\xi,e}, \tag{3.13a}
\]

\[
f^{\text{h}}_{\xi,e} = U e^{-\lambda \xi_R \phi_R} - e^{-\lambda \xi_L \phi_L}, \tag{3.13b}
\]

\[
f^{\text{i}}_{\xi,e} = \lambda \int_{\xi_L}^{\xi_R} e^{-\lambda \xi} S(\xi) \, d\xi \tag{3.13c}
\]

Note the similarities to the original representations for the homogeneous and inhomogeneous flux in (2.9). The main difference is the change from the grid coordinates to the rotated coordinates. This also means that we obtain the same expression for the numerical flux, but once again on the rotated coordinates.

We can now write the homogeneous flux as follows:

\[
f^{\text{h}}_{\xi,e} = \frac{\varepsilon}{h} \left( B(P) \phi_R - B(-P) \phi_L \right), \tag{3.14}
\]

where \( P = \frac{U_h}{\tau} \). This is the expression we will use for the homogeneous flux.

To determine the inhomogeneous flux we first define:

\[
I = \int_{\xi_L}^{\xi_R} e^{-\lambda \xi} S(\xi) \, d\xi. \tag{3.15}
\]

Note that now:

\[
f^{\text{i}}_{\xi,e} = a \frac{I}{e^{-\lambda \xi_R} - e^{-\lambda \xi_L}}. \tag{3.16}
\]

We now split the integral in two parts:

\[
I = \int_{\xi_L}^{\xi_R} e^{-\lambda \xi} \int_{0}^{\xi} s_\xi(\xi') \, d\xi' \, d\xi + \int_{0}^{\xi_R} e^{-\lambda \xi} \int_{0}^{\xi} s_\xi(\xi') \, d\xi' \, d\xi. \tag{3.17}
\]

If we now change the order of integration and evaluate the inner integrals we get:

\[
I = -\frac{1}{\lambda} \left( e^{-\lambda \xi'} - e^{-\lambda \xi_L} \right) \int_{0}^{\xi_R} s_\xi(\xi') \, d\xi' + \frac{1}{\lambda} \left( e^{-\lambda \xi'} - e^{-\lambda \xi_R} \right) \int_{0}^{\xi_L} s_\xi(\xi') \, d\xi'. \tag{3.18}
\]

This implies that the inhomogeneous flux is equal to:

\[
f^{\text{i}}_{\xi,e} = \int_{\xi_L}^{0} e^{-\lambda \xi'} - e^{-\lambda \xi_L} s_\xi(\xi') \, d\xi' + \int_{\xi_R}^{0} e^{-\lambda \xi'} - e^{-\lambda \xi_R} s_\xi(\xi') \, d\xi'. \tag{3.19}
\]

We now define \( \sigma = \frac{\varepsilon - \xi_L}{\xi_R - \xi_L} = \frac{\xi - \xi_L}{h} \) as a scaled coordinate and define the Green’s function:

\[
G(\sigma; P) = \begin{cases} 
1 - \frac{1}{\tau} P(1 - \sigma) & 0 \leq \sigma \leq \frac{1}{2}, \\
1 - \frac{1}{\tau} P(1 - \sigma) & \frac{1}{2} < \sigma \leq 1.
\end{cases} \tag{3.20}
\]
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If we substitute this into the expression for the inhomogeneous flux we get:

\[ f_{\xi,e}^{i} = \int_{0}^{1} G(\sigma; P) s_{\xi}(\sigma) d\sigma. \]

If we now integrate this and assume \( \tilde{s} \) is equal to \( s_{L} \) on the interval \((0, 1/2)\) and equal to \( s_{R} \) on \((1/2, 1)\) we get the following expression for the inhomogeneous flux:

\[ f_{\xi,e}^{i} = h \left( C(-P) s_{L} - C(P) s_{R} \right), \]

We now need to approximate the \( \eta \)-flux. This is done in a similar way. We first put the \( \xi \)-flux together with the source term to obtain an adjusted source term:

\[ s_{\eta} := s - \frac{\partial f_{\xi}}{\partial \xi}. \]

We obtain a similar boundary value problem:

\[ \frac{\partial f_{\eta}}{\partial \eta} = -\varepsilon \frac{\partial^{2} \phi}{\partial \eta^{2}} = s_{\eta}, \quad -h/2 < \eta < h/2, \]

\[ \phi(-h/2) = \phi_{B}, \quad \phi(h/2) = \phi_{T}. \]

This can be directly integrated twice to obtain the following expression:

\[ h f_{\eta,e} = -\varepsilon (\phi_{T} - \phi_{B}) - \int_{x_B}^{x_T} \int_{0}^{\tilde{\eta}} s_{\eta} \partial \eta \partial \tilde{\eta}. \]

Now define, by using \( x_B = -\frac{1}{2} h \) and \( x_T = \frac{1}{2} h \)

\[ I := \int_{-h/2}^{h/2} \int_{0}^{\tilde{\eta}} s_{\eta} \partial \eta \partial \tilde{\eta}. \]

We split the integral and change the order of integration to obtain:

\[ I = -\int_{h/2}^{0} \int_{-h/2}^{\eta} s_{\eta}(\eta) d\eta d\tilde{\eta} + \int_{0}^{h/2} \int_{\eta}^{h/2} s_{\eta}(\eta) d\eta d\eta. \]

We simplify this equation to obtain:

\[ I = -\int_{-h/2}^{0} (\eta + \frac{h}{2}) s_{\eta}(\eta) d\eta + \int_{0}^{h/2} (\frac{h}{2} - \eta) s_{\eta}(\eta) d\eta. \]

If we define \( \sigma = \frac{\eta + h}{h} \) we can simplify this to:

\[ I = -h^{2} \int_{0}^{1/2} \sigma s_{\eta}(\sigma) d\sigma + h^{2} \int_{1/2}^{1} (1 - \sigma) s_{\eta}(\sigma) d\sigma. \]

Now define the new Green’s function:

\[ \tilde{G}(\sigma) = \begin{cases} \sigma & 0 \leq \sigma \leq \frac{1}{2}, \\ \frac{1}{2} - \sigma & \frac{1}{2} < \sigma \leq 1, \end{cases} \]

and note that this is the limit for \( P \to 0 \) from (3.20), and we can write the inhomogeneous \( \eta \) flux as follows:

\[ f_{\eta,e}^{i} = h \int_{0}^{1} \tilde{G}(\sigma) s_{\eta}(\sigma) d\sigma. \]
3.2. Interpolation methods

If we assume $s_\eta$ to be constant and equal to $s_{\eta,B}$ and $s_{\eta,T}$ we can evaluate the integral, and obtain:

$$f^i_{\eta,e} = -\frac{h^2}{8}s_{\eta,B} + \frac{h}{8}s_{\eta,T} = \frac{h^2}{8}(s_B - s_T) + \frac{h^2}{8}\left(\frac{\partial f_{\xi,T}}{\partial \xi} - \frac{\partial f_{\xi,B}}{\partial \xi}\right). \tag{3.31}$$

For the flux derivatives we take a central difference scheme and approximate them by the homogeneous flux. Putting this all together the following scheme is obtained:

$$f_{\xi,e} = -\frac{\varepsilon}{h} (B(P)\phi_R - B(-P)\phi_L) + C(-P) (s_L + \varepsilon\delta_{\eta\eta}\phi_L) h - C(P) (s_R + \varepsilon\delta_{\eta\eta}\phi_R) h$$

$$f_{\eta,e} = -\frac{\varepsilon}{h} (\phi_T - \phi_B) + \frac{h}{8} (s_B - s_T) + \frac{h}{8} (\delta_{\xi f_{\xi,T}} - \delta_{\xi f_{\xi,B}})$$

$$f_{x,e} = f_{\xi,e} \cos \alpha - f_{\eta,e} \sin \alpha.$$

We now need to approximate the second derivative and the flux derivatives. For the second derivatives we use finite differences on a circle with radius $h$ around point $R$ and $L$, as can be seen in Figure 3.2. For these finite differences we get for example:

$$\partial_{\eta\eta}\phi_R = \left(\frac{\partial^2 \phi}{\partial \eta^2}\right)_R \approx \frac{\phi_{RN} - 2\phi_R + \phi_{RS}}{h^2}.$$ \tag{3.32}

The same approximation is then applied to $\delta_{\eta\eta}\phi_L$. We approximate the flux derivative by using a central difference method, and then replace the flux with the homogeneous flux $f^i_{\xi,e}$ as in (3.14).

If we apply the same procedure for the north flux we get the following scheme. Note that here $R, L, T$ and $B$ are the locations relative to the north wall of the control volume, and are different from the locations before.

$$f_{\xi,n} = -\frac{\varepsilon}{h} (B(P)\phi_R - B(-P)\phi_L) + h (C(-P) (s_L + \varepsilon\delta_{\eta\eta}\phi_L) - C(P) (s_R + \varepsilon\delta_{\eta\eta}\phi_R))$$

$$f_{\eta,n} = -\frac{\varepsilon}{h} (\phi_T - \phi_B) + \frac{h}{8} (s_B - s_T) + \frac{h}{8} (\delta_{\xi f_{\xi,T}} - \delta_{\xi f_{\xi,B}})$$

$$f_{y,n} = f_{\xi,n} \sin \alpha + f_{\eta,n} \cos \alpha.$$

### 3.2 Interpolation methods

To be able to calculate the solution we need to interpolate $\phi$, since we need the value of $\phi$ at the rotated points. We introduce two ways of interpolating $\phi$, both ensure that the final scheme has at most a 9-point coupling.

We first interpolate over the 6 points depicted in Figure 3.3. This is done by applying linear interpolation in the $x$-direction and quadratic interpolation in the $y$-direction, when computing $F_{x,e}$ and $F_{x,w}$, so we will call this quadratic-linear interpolation. For $F_{y,n}$ and $F_{y,s}$ this is reversed, and we will interpolate linearly in the $y$-direction and quadratically in the $x$-direction. In order to obtain the interpolation over the 6 points we first introduce the scaled coordinates: $\sigma_x := \frac{x-x_1}{x_2-x_1} = \frac{x-x_1}{\Delta x}$ and $\sigma_y := \frac{y-y_1}{y_2-y_1} = \frac{y-y_1}{\Delta y}$, so $0 \leq \sigma_x \leq 1$ and
Flow adapted coordinates

Figure 3.2: Schematic representation of how the second derivative is calculated

$-1 \leq \sigma_y \leq 1$. We need to interpolate over the function values $f_{1,1}, f_{1,2}, \text{etc}$. We now first interpolate the $y$-direction quadratically. This means that the $y$-interpolation becomes:

$$
p(x_1, y) = -\frac{\sigma_y(1 - \sigma_y)}{2} f_{1,3} + (1 - \sigma_y)(1 + \sigma_y)f_{1,2} + \frac{\sigma_y(1 + \sigma_y)}{2} f_{1,1},
$$

$$
p(x_2, y) = -\frac{\sigma_y(1 - \sigma_y)}{2} f_{2,3} + (1 - \sigma_y)(1 + \sigma_y)f_{2,2} + \frac{\sigma_y(1 + \sigma_y)}{2} f_{2,1}.
$$

(3.33)

We now get the total interpolation by interpolating the $x$-direction linearly using these:

$$
p(x, y) = (1 - \sigma_x)p(x_1, y) + \sigma_x p(x_2, y)
$$

$$
= (1 - \sigma_x) \left[ -\frac{\sigma_y(1 - \sigma_y)}{2} f_{1,3} + (1 - \sigma_y)(1 + \sigma_y)f_{1,2} + \frac{\sigma_y(1 + \sigma_y)}{2} f_{1,1} \right] +
$$

$$
\sigma_x \left[ -\frac{\sigma_y(1 - \sigma_y)}{2} f_{2,3} + (1 - \sigma_y)(1 + \sigma_y)f_{2,2} + \frac{\sigma_y(1 + \sigma_y)}{2} f_{2,1} \right]
$$

$$
= [(1 - \sigma_x) \sigma_x] \left[ \begin{array}{ccc}
-\frac{\sigma_y(1 - \sigma_y)}{2} f_{1,3} + (1 - \sigma_y)(1 + \sigma_y)f_{1,2} + \frac{\sigma_y(1 + \sigma_y)}{2} f_{1,1} \\
-\frac{\sigma_y(1 - \sigma_y)}{2} f_{2,3} + (1 - \sigma_y)(1 + \sigma_y)f_{2,2} + \frac{\sigma_y(1 + \sigma_y)}{2} f_{2,1}
\end{array} \right]
$$

(3.34)

For the second approach bilinear interpolation is used. We will later test the effectiveness of both interpolation methods. We now need to interpolate over the area shown in Figure 3.4. We once again introduce the scaled coordinates $\sigma_x := \frac{x - x_1}{x_2 - x_1}$ and $\sigma_y := \frac{y - y_1}{y_2 - y_1}$, so now so
3.3 Final method

We now have all the necessary components to define the complete numerical method. We first define the homogeneous and inhomogeneous flux for the $x$-direction, then we define the homogeneous and inhomogeneous flux for the $y$-direction. Note that the $R, L, T, B,$ and other indices used here are local for each flux, and change for each boundary of the control volume. For the east flux we get the following approximations:

\[ f_{x,e}^h = -\frac{\varepsilon \cos \alpha}{h} (B(P)\phi_R - B(-P)\phi_L) + \frac{\varepsilon \sin \alpha}{h} (\phi_T - \phi_B), \quad (3.36a) \]

\[ p(x, y) = [ (1 - \sigma_x) \quad \sigma_x ] \begin{pmatrix} f_{2,1}^h & f_{2,2}^h \\ f_{1,1}^h & f_{1,2}^h \end{pmatrix} \begin{pmatrix} (1 - \sigma_y) \\ \sigma_y \end{pmatrix}. \quad (3.35) \]
Flow adapted coordinates

\[ f_{x,e}^i = \frac{\varepsilon C(-) \cos \alpha}{h^2} (\phi_{RN} - 2\phi_R + \phi_{RS}) - \frac{\varepsilon C(-) \cos \alpha}{h^2} (\phi_{LN} - 2\phi_L + \phi_{LS}) + \frac{h \sin \alpha}{8} (\delta \xi f_{\xi,T} - \delta \xi f_{\xi,B}). \]  

(3.36b)

For the north flux we now get:

\[ f_{y,n}^h = -\frac{\varepsilon \sin \alpha}{h} (B(P) \phi_R - B(-P) \phi_L) - \frac{\varepsilon \cos \alpha}{h} (\phi_T - \phi_B), \]  

(3.37a)

\[ f_{y,n}^i = \frac{\varepsilon C(P) \sin \alpha}{h^2} (\phi_{RN} - 2\phi_R + \phi_{RS}) - \frac{\varepsilon C(P) \sin \alpha}{h^2} (\phi_{LN} - 2\phi_L + \phi_{LS}) - \frac{h \cos \alpha}{8} (\delta \xi f_{\xi,T} - \delta \xi f_{\xi,B}). \]  

(3.37b)

If we define flux differences like:

\[ \delta_x F_{x,C} := \frac{1}{h} (F_{x,e} - F_{x,w}), \quad \delta_y F_{y,C} := \frac{1}{h} (F_{y,n} - F_{y,x}), \]  

(3.38)

it is clear that we can write the discrete conservation law as:

\[ \delta_x F_{x,C} + \delta_y F_{y,C} = s_C. \]  

(3.39)

These fluxes can now be split in homogeneous and inhomogeneous flux differences, which gives us:

\[ \delta_x F_{x,C}^h + \delta_x F_{x,C}^i + \delta_y F_{y,C}^h + \delta_y F_{y,C}^i = s_C \]

\[ -C(-P) \cos \alpha s_{L(e)} + C(P) \cos \alpha s_{R(e)} + C(-P) \cos \alpha s_{L(w)} - C(P) \cos \alpha s_{R(w)} \]

\[ -C(-P) \sin \alpha s_{L(n)} + C(P) \sin \alpha s_{R(n)} + C(-P) \sin \alpha s_{L(s)} - C(P) \sin \alpha s_{R(s)} \]

\[ -\frac{\sin(\alpha)}{8} (s_{B(e)} - s_{T(e)} - s_{B(w)} + s_{T(w)}) + \frac{\cos(\alpha)}{8} (s_{B(n)} - s_{T(n)} - s_{B(s)} + s_{T(s)}). \]  

(3.40)

Here \( s_{R(e)} \) means the value of the source term \( s \) at location \( R(e) \), which is the location of \( R \) for the eastern flux, \( s_{R(w)} \) the value at location \( R \) for the western flux, etc.

We now have the complete numerical scheme, but we still need to calculate the interpolated values. We can calculate these if we now the location of the desired points. These locations are known when \( \alpha \) is known. The locations are given in Table 3.1 and 3.2. Note that for RN,RS,LN and LS the points can lie outside of the interpolation area. If this happens we use the same interpolation formula, but extrapolate the solution.

### 3.4 Variable coefficients

When \( u \) is not constant the discretization scheme can still be used. We still keep \( \varepsilon \) constant in the rest of this derivation. First the rotation is applied, but now locally at each boundary. This means that the local boundary value problem is given as in (3.9), but now with \( U = |u(x_e)| \) changing for every interface. We now define, once again:

\[ \lambda := \frac{U}{\varepsilon}, \quad P = P(\lambda) := \lambda h, \quad \Lambda(\xi) := \int_0^\xi \lambda(\xi') d\xi', \quad S(\xi) := \int_0^\xi s(\xi') d\xi', \]  

(3.41)

so now \( P \) is a function and no longer constant. This means that \( f_{\xi,e} \) can now be rewritten to:

\[ f_{\xi,e} = -\varepsilon (\phi e^{-\lambda})' e^\lambda. \]  

(3.42)
### 3.4. Variable coefficients

<table>
<thead>
<tr>
<th>Location</th>
<th>x - xe</th>
<th>y - ye</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>( \frac{h(1 + \cos(\alpha))}{2} )</td>
<td>( \frac{h (1 + \sin(\alpha))}{2} )</td>
</tr>
<tr>
<td>L</td>
<td>( \frac{h(1 - \cos(\alpha))}{2} )</td>
<td>( -\frac{h \sin(\alpha)}{2} )</td>
</tr>
<tr>
<td>T</td>
<td>( \frac{h(1 - \sin(\alpha))}{2} )</td>
<td>( \frac{h \cos(\alpha)}{2} )</td>
</tr>
<tr>
<td>B</td>
<td>( \frac{h(1 + \sin(\alpha))}{2} )</td>
<td>( -\frac{h \cos(\alpha)}{2} )</td>
</tr>
<tr>
<td>RN</td>
<td>( \frac{h(1 + \cos(\alpha))}{2} ) - \sin(\alpha)h</td>
<td>( \frac{h \sin(\alpha)}{2} ) + \cos(\alpha)h</td>
</tr>
<tr>
<td>RS</td>
<td>( \frac{h(1 + \cos(\alpha))}{2} ) + \sin(\alpha)h</td>
<td>( \frac{h \sin(\alpha)}{2} ) - \cos(\alpha)h</td>
</tr>
<tr>
<td>LN</td>
<td>( \frac{h(1 - \cos(\alpha))}{2} ) - \sin(\alpha)h</td>
<td>( -\frac{h \sin(\alpha)}{2} ) - \cos(\alpha)h</td>
</tr>
<tr>
<td>LS</td>
<td>( \frac{h(1 - \cos(\alpha))}{2} ) + \sin(\alpha)h</td>
<td>( -\frac{h \sin(\alpha)}{2} ) - \cos(\alpha)h</td>
</tr>
<tr>
<td>TR</td>
<td>( \frac{h(1 - \sin(\alpha))}{2} ) + \cos(\alpha)h</td>
<td>( \frac{h \cos(\alpha)}{2} ) + \sin(\alpha)h</td>
</tr>
<tr>
<td>TL</td>
<td>( \frac{h(1 + \sin(\alpha))}{2} ) - \cos(\alpha)h</td>
<td>( -\frac{h \cos(\alpha)}{2} ) - \sin(\alpha)h</td>
</tr>
<tr>
<td>BR</td>
<td>( \frac{h(1 + \sin(\alpha))}{2} ) + \cos(\alpha)h</td>
<td>( -\frac{h \cos(\alpha)}{2} ) - \sin(\alpha)h</td>
</tr>
<tr>
<td>BL</td>
<td>( \frac{h(1 + \sin(\alpha))}{2} ) - \cos(\alpha)h</td>
<td>( -\frac{h \cos(\alpha)}{2} ) - \sin(\alpha)h</td>
</tr>
</tbody>
</table>

**Table 3.1:** Location of points for x-flux

<table>
<thead>
<tr>
<th>Location</th>
<th>x - xn</th>
<th>y - yn</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>( \frac{h \cos(\alpha)}{2} ) - \sin(\alpha)h</td>
<td>( \frac{h (1 + \sin(\alpha))}{2} )</td>
</tr>
<tr>
<td>L</td>
<td>( -\frac{h \cos(\alpha)}{2} ) + \sin(\alpha)h</td>
<td>( \frac{h (1 - \sin(\alpha))}{2} )</td>
</tr>
<tr>
<td>T</td>
<td>( -\frac{h \sin(\alpha)}{2} ) - \cos(\alpha)h</td>
<td>( \frac{h (1 + \sin(\alpha))}{2} )</td>
</tr>
<tr>
<td>B</td>
<td>( \frac{h \sin(\alpha)}{2} ) + \sin(\alpha)h</td>
<td>( \frac{h (1 - \sin(\alpha))}{2} )</td>
</tr>
<tr>
<td>RN</td>
<td>( \frac{h \cos(\alpha)}{2} ) - \sin(\alpha)h</td>
<td>( \frac{h (1 + \sin(\alpha))}{2} ) + \cos(\alpha)h</td>
</tr>
<tr>
<td>RS</td>
<td>( \frac{h \cos(\alpha)}{2} ) + \sin(\alpha)h</td>
<td>( \frac{h (1 - \sin(\alpha))}{2} ) - \cos(\alpha)h</td>
</tr>
<tr>
<td>LN</td>
<td>( -\frac{h \cos(\alpha)}{2} ) - \cos(\alpha)h</td>
<td>( \frac{h (1 + \sin(\alpha))}{2} ) + \cos(\alpha)h</td>
</tr>
<tr>
<td>LS</td>
<td>( -\frac{h \sin(\alpha)}{2} ) + \cos(\alpha)h</td>
<td>( \frac{h (1 - \sin(\alpha))}{2} ) - \cos(\alpha)h</td>
</tr>
<tr>
<td>TR</td>
<td>( -\frac{h \cos(\alpha)}{2} ) + \cos(\alpha)h</td>
<td>( \frac{h (1 + \sin(\alpha))}{2} ) + \sin(\alpha)h</td>
</tr>
<tr>
<td>TL</td>
<td>( -\frac{h \sin(\alpha)}{2} ) - \cos(\alpha)h</td>
<td>( \frac{h (1 - \sin(\alpha))}{2} ) - \sin(\alpha)h</td>
</tr>
<tr>
<td>BR</td>
<td>( \frac{h \sin(\alpha)}{2} ) + \cos(\alpha)h</td>
<td>( \frac{h (1 + \sin(\alpha))}{2} ) + \sin(\alpha)h</td>
</tr>
<tr>
<td>BL</td>
<td>( \frac{h \sin(\alpha)}{2} ) - \cos(\alpha)h</td>
<td>( \frac{h (1 - \sin(\alpha))}{2} ) - \sin(\alpha)h</td>
</tr>
</tbody>
</table>

**Table 3.2:** Location of points for y-flux
Flow adapted coordinates

Substituting this and integrating from $\xi_L := -\frac{1}{2}h$ to $\xi_R := \frac{1}{2}h$ we now get the following expressions for the flux:

$$f_{\xi,e}^h = f_{\xi,e}^h + f_{\xi,e}^i,$$

$$f_{\xi,e}^h = -e^{-\Lambda_R \phi_R} - e^{-\Lambda_L \phi_L},$$

$$f_{\xi,e}^i = \frac{\int_{\xi_L}^{\xi_R} e^{-\Lambda} d\xi}{\langle e^{-\lambda}, e^{-\lambda} \rangle},$$

$$f_{\xi,e}^i = \frac{\int_{\xi_L}^{\xi_R} e^{-\Lambda} d\xi}{\langle e^{-\lambda}, e^{-\lambda} \rangle}.$$ (3.43a, 3.43b, 3.43c)

We now introduce the usual inner product of two functions $a = a(\xi)$ and $b = b(\xi)$ on $(\xi_L, \xi_R)$:

$$\langle a, b \rangle = \int_{\xi_L}^{\xi_R} a(\xi)b(\xi)d\xi.$$ (3.44)

If we now define $\bar{\lambda}_{j+1/2} := \frac{1}{2}(\lambda_L + \lambda_R)$ and use $\lambda_R - \lambda_L = (\lambda, 1)$, we can rewrite the homogeneous and inhomogeneous flux using inner products:

$$f_{\xi,e}^h = -e^{-\bar{\lambda}_{j+1/2} e^{\langle (\lambda, 1)/2 \phi_R - e^{\langle (\lambda, 1)/2 \phi_L}} \langle e^{-1}, e^{-\lambda} \rangle,}$$

$$f_{\xi,e}^i = \frac{\langle e^{-1}, e^{-\lambda} \rangle}{\langle e^{-1}, e^{-\lambda} \rangle}.$$ (3.45a, 3.45b)

We can now see that the homogeneous flux can be written as:

$$f_{j+1/2}^h = -\frac{\langle \lambda, e^{-\lambda} \rangle}{\langle 1, e^{-\lambda} \rangle} (B(\langle \lambda, 1 \rangle) \phi_R - B(\langle \lambda, 1 \rangle) \phi_L).$$ (3.46)

For the inhomogeneous flux we once again assume $s_\xi = s_L$ on $(\xi_L, 0)$ and $s_\xi = s_R$ on $(0, \xi_R)$, so we get the same expression for the integral. For the homogeneous flux we use the same approximation as in [4], and we use the same average $\bar{a}_{j+1/2}$ and weighted average $\bar{a}_{j+1/2}$ as before.

We now use these to approximate the inner products as in [4]. This gives, when omitting second order terms, the following approximations:

$$\langle \lambda, 1 \rangle \approx \bar{P}_{1/2},$$

$$\langle \lambda, e^{-\lambda} \rangle \approx \bar{\lambda}_{1/2}.$$ (3.47a, 3.47b)

Combining this gives the following approximation:

$$\langle \lambda, e^{-\lambda} \rangle \approx \bar{\lambda}_{1/2}.$$ (3.48)

which gives the following for the homogeneous flux:

$$F_{\xi}^h = -\frac{\varepsilon}{h} \bar{\lambda}_{j+1/2} (B(\bar{P}_{j+1/2}) \phi_R - B(\bar{P}_{j+1/2}) \phi_L).$$ (3.49)

For the inhomogeneous flux we use the same procedure as before, but now with $\bar{P}_{j+1/2}$, which gives us:

$$F_{\xi,e}^i = h (C(-\bar{P}_{j+1/2}) \bar{s}_L - C(\bar{P}_{j+1/2}) \bar{s}_R).$$ (3.50)
3.5 Non-uniform grid

So far we assumed that $\Delta x = \Delta y$, but this is not necessary. Here we assume $\Delta x \neq \Delta y$. This means that, instead of having a circle around the locations of the flux we now have an ellipse. This is shown in Figure 3.5. The rotation can still be calculated in the same manner, but we now have to determine the locations of $R, L, T, B$ and the distance to these points in a different way. The way this is done here is by intersecting the formula for the ellipse with the formula for the lines that represent $\xi$ and $\eta$ in the not rotated coordinates, which are:

\[
\left( \frac{x-x_e}{\Delta x} \right)^2 + \left( \frac{y-y_e}{\Delta y} \right)^2 = 1, \quad y - y_e = \tan(\alpha)(x - x_e).
\] (3.51)

Substituting this gives the points shown in Table 3.3 and Table 3.4. We now need to define $\Delta \xi$, the distance from L to R, and $\Delta \eta$, the distance from B to T. To determine these we take twice the 2-norm from the center of the ellipse to the edge of the ellipse. This gives, if we define $q := \frac{\Delta y}{\Delta x}$:

\[
\Delta \xi := 2\|x_R - x_e\| := 2\sqrt{x_R^2 + y_R^2} = \frac{g\sqrt{1+\tan^2 \alpha}}{\sqrt{\tan^2 \alpha + q^2}},
\]
\[
\Delta \eta := 2\|x_T - x_e\| := 2\sqrt{x_T^2 + y_T^2} = \frac{g\sqrt{1+\tan^2 \alpha}}{\sqrt{1+q^2\tan^2 \alpha}}.
\] (3.52)

This, however, means that the BVP for the $\xi$- and $\eta$-fluxes also change. These now
Flow adapted coordinates

\[
\begin{array}{|c|c|c|}
\hline
R & x - x_\epsilon = \frac{\Delta x}{2} + \frac{q}{2\sqrt{\tan^2\alpha + q^2}} & y - y_\epsilon = (x - x_\epsilon) \tan \alpha \\
L & x - x_\epsilon = \frac{\Delta x}{2} + \frac{1}{2\sqrt{\tan^2\alpha + q^2}} & y - y_\epsilon = (x - x_\epsilon) \tan \alpha \\
T & x - x_\epsilon = \frac{\Delta x}{2} + \frac{q\tan \alpha}{2\sqrt{1 + q^2 \tan^2\alpha}} & y - y_\epsilon = (x - x_\epsilon) \tan \alpha \\
B & x - x_\epsilon = \frac{\Delta x}{2} + \frac{q\tan \alpha}{2\sqrt{1 + q^2 \tan^2\alpha}} & y - y_\epsilon = (x - x_\epsilon) \tan \alpha \\
\hline
\end{array}
\]

Table 3.3: Location of the eastern coordinates with irregular grid

\[
\begin{array}{|c|c|c|}
\hline
R & x - x_n = \frac{q}{2\sqrt{\tan^2\alpha + q^2}} & y - y_n = \frac{\Delta y}{2} + (x - x_n) \tan \alpha \\
L & x - x_n = \frac{-q}{2\sqrt{\tan^2\alpha + q^2}} & y - y_n = \frac{\Delta y}{2} + (x - x_n) \tan \alpha \\
T & x - x_n = \frac{-q\tan \alpha}{2\sqrt{1 + q^2 \tan^2\alpha}} & y - y_n = \frac{\Delta y}{2} + (x - x_n) \tan \alpha \\
B & x - x_n = \frac{-q\tan \alpha}{2\sqrt{1 + q^2 \tan^2\alpha}} & y - y_n = \frac{\Delta y}{2} + (x - x_n) \tan \alpha \\
\hline
\end{array}
\]

Table 3.4: Location of the northern coordinates with irregular grid

become:

\[
\frac{\partial f_{\xi}}{\partial \xi} = \frac{\partial}{\partial \xi} \left( U \phi - \varepsilon \frac{\partial \phi}{\partial \xi} \right) = s_{\xi}, \quad -\Delta \xi/2 < \xi < \Delta \xi/2, \tag{3.53a}
\]

\[
\phi(-\Delta \xi/2) = \phi_L, \quad \phi(\Delta \xi/2) = \phi_R. \tag{3.53b}
\]

\[
\frac{\partial f_{\eta}}{\partial \eta} = -\varepsilon \frac{\partial^2 \phi}{\partial \eta^2} = s_{\eta}, \quad -\Delta \eta/2 < \eta < \Delta \eta/2, \tag{3.54a}
\]

\[
\phi(-\Delta \eta/2) = \phi_B, \quad \phi(\Delta \eta/2) = \phi_T. \tag{3.54b}
\]

These are essentially the same BVPs as before, so we can use the same approximation of the numerical flux. An alternative is to choose \( h := \min(\Delta x, \Delta y) \). This causes the method to remain the same, thus making it easier to compute. It however loses the property that, if \( \alpha \) is a multiple of \( \frac{1}{2}\pi \), the points of the interpolated grid align completely with the non-rotated grid.

### 3.6 Limit cases

When either \( U \to 0 \) or \( \varepsilon \downarrow 0 \) it is possible to calculate the exact form of the numerical method. We first look at the case of \( \varepsilon \downarrow 0 \). Then \( P = \frac{Uh}{2} \to \infty \), so \( B(P) \to 0 \) and \( \frac{B(-P)}{P} \to P \). We also have \( C(P) = 0 \) and \( C(-P) = \frac{1}{2} \). This means that the flux becomes:

\[
f_{\xi,e} = \frac{\xi}{h} P \phi_L + \frac{\varepsilon h}{2} \delta_{\eta} \phi_L = \frac{\varepsilon U}{h} \xi \phi_L = U \phi_L. \tag{3.55}
\]

This means that the FVM gives:

\[
U \cos \alpha (\phi_{L,e} - \phi_{L,w}) + U \sin \alpha (\phi_{L,n} - \phi_{L,s}) = s, \tag{3.56}
\]

which, since \( \cos \alpha = \frac{u}{\sqrt{u^2 + v^2}} \), \( \sin \alpha = \frac{v}{\sqrt{u^2 + v^2}} \) and \( U = \sqrt{u^2 + v^2} \), is also equal to:

\[
u (\phi_{L,e} - \phi_{L,w}) + v (\phi_{L,n} - \phi_{L,s}) = s, \tag{3.57}
\]
3.6. Limit cases

which is the upwind method on the rotated coordinates. Secondly we consider the other limit case \( U = 0 \) and \( P = 0 \). If \( u \) and \( v \) are approaching 0 you need to choose \( \alpha \), since it is not defined. Here \( \alpha \) is chosen to be equal to 0. since \( P = 0 \), we have: \( B(\pm P) = 1 \) and \( C(\pm P) = 1/8 \).

Also note that \( \cos \alpha = 1 \) and \( \sin \alpha = 0 \). If we now substitute this in the flux we get:

\[
\begin{align*}
    f_{x,e} &= \frac{\varepsilon}{h} (\phi_L - \phi_R) + \frac{\varepsilon h}{8} (\partial_{\eta\eta} \phi_L - \partial_{\eta\eta} \phi_R), \\
    f_{y,n} &= \frac{\varepsilon}{h} (\phi_B - \phi_T) - \frac{h}{8} \left( \frac{\partial f_{\xi,T}}{\partial \xi} - \frac{\partial f_{\xi,B}}{\partial \xi} \right).
\end{align*}
\]

(3.58)

This means that, if we only take the homogeneous part of the flux the discrete conservation law becomes:

\[
\frac{\varepsilon}{h} (-\phi_N - \phi_L + 4\phi_C - \phi_R - \phi_S) = s,
\]

(3.59)

which is exactly the 2-dimensional central difference scheme for the equations \(-\varepsilon \nabla^2 \phi = s\). If the inhomogeneous flux is taken into account this formula becomes a bit more complicated, but it can be given in matrix form by:

\[
\frac{\varepsilon}{h} \begin{pmatrix}
    -1 & -\frac{1}{2} & -\frac{1}{4} \\
    -\frac{1}{2} & 3 & -\frac{3}{4} \\
    -\frac{1}{4} & -\frac{1}{2} & -\frac{1}{4}
\end{pmatrix} \phi = s,
\]

(3.60)

which is a variant on the central difference method.
4 Extension to three dimensions

In this chapter we will outline the extension to 3 dimensions. The method as described here is not yet implemented, but should perform approximately the same. In 3 dimensions it is a bit harder to describe rotations, since you first have to determine a rotation axis. In order to rotate $e_x = (1, 0, 0)$ to $u = (u, v, w)$ we use Rodrigues rotation formula, as described in [1].

First define $k$ as an axis of rotation and let $l$ be a vector which we want to rotate around $k$ by an angle $\theta$. We first split $l$ in its normal and perpendicular component:

$$l = l_{\parallel} + l_{\perp}, \quad (4.1)$$

where the component parallel to $k$ is equal to:

$$l_{\parallel} = (l \cdot k)k. \quad (4.2)$$

This is called the vector projection of $l$ on $k$. The component perpendicular to $k$ is equal to:

$$l_{\perp} = l - l_{\parallel} = l - (k \cdot l)k = -k \times (k \times l), \quad (4.3)$$

and this is called the vector rejection of $l$ from $k$. We will now rotate these two components around the axis. The component parallel to the axis will not change magnitude or directions due to the rotation, so:

$$l_{\parallel}^{\text{rot}} = l_{\parallel}. \quad (4.4)$$

The component perpendicular will change its direction but retain its magnitude. This gives the rotation:

$$l_{\perp}^{\text{rot}} = \cos \theta l_{\perp} + \sin \theta (k \times l_{\perp}). \quad (4.5)$$

This rotation is correct, since $k \times l_{\perp}$ is $l_{\perp}$ rotated anticlockwise through $90^\circ$, after which a normal rotation is applied which preserves length. Now note that:

$$k \times l_{\perp} = k \times (l - l_{\parallel}) = k \times l - k \times l_{\parallel} = k \times l, \quad (4.6)$$

since $k$ and $l_{\parallel}$ are parallel, thus $k \times l_{\parallel} = 0$. If we now sum the two rotated components we get the full rotated vector:

$$l^{\text{rot}} = (l - l_{\parallel}) + \cos \theta l_{\perp} + \sin \theta (k \times l), \quad (4.7a)$$

$$= l + (\cos \theta - 1)l_{\perp} + \sin \theta (k \times l), \quad (4.7b)$$

$$= l + (\cos \theta - 1)(-k \times (k \times l)) + \sin \theta (k \times l), \quad (4.7c)$$

$$= l + (1 - \cos \theta)k \times (k \times l) + \sin \theta (k \times l). \quad (4.7d)$$
We now get the matrix notation by using the fact the cross product can be expressed as a matrix product:

\[
\begin{bmatrix}
(k \times l)_x \\
(k \times l)_y \\
(k \times l)_z
\end{bmatrix} =
\begin{bmatrix}
k_y l_z - k_z l_y \\
k_z l_x - k_x l_z \\
k_x l_y - k_y l_x
\end{bmatrix} =
\begin{bmatrix}
0 & -k_z & k_y \\
k_z & 0 & -k_x \\
-k_y & k_x & 0
\end{bmatrix}
\begin{bmatrix}
l_x \\
l_y \\
l_z
\end{bmatrix} := Kl. \tag{4.8}
\]

Using this definition of \( K \) we can write the rotation formula as:

\[
l_{\text{rot}} = l + (\sin \theta)Kl + (1 - \cos \theta)K^2l, \tag{4.9}
\]

so we can write this using a rotation matrix, which means:

\[
l_{\text{rot}} = Rl, \tag{4.10}
\]

where

\[
R = I + (\sin \theta)K + (1 - \cos \theta)K^2. \tag{4.11}
\]

To now apply the Rodrigues rotation we start by normalizing \( u \), so we define:

\[
\hat{u} := (\hat{u}, \hat{v}, \hat{w}) := \frac{u}{|u|}. \tag{4.12}
\]

Now we determine the axis of rotation by taking the cross product between \( e_x \) and \( \hat{u} \):

\[
\hat{u} \times e_x = \begin{pmatrix}
0 & \hat{w} \\
\hat{v} & 0 \\
\end{pmatrix}. \tag{4.13}
\]

We also need to determine the angle of rotation, and this is done by taking the inner product between \( e_x \) and \( \hat{u} \). The angle is given by:

\[
\cos(\alpha) = (e_x, \hat{u}) = \hat{u}. \tag{4.14}
\]

The Rodrigues rotation matrix is now defined by:

\[
R(\alpha) = I + \sin \alpha n_x + (1 - \cos \alpha)n_x^2, \tag{4.15}
\]

where \( n_x \) is given by:

\[
\begin{bmatrix}
0 & \hat{v} & \hat{w} \\
-\hat{v} & 0 & 0 \\
-\hat{w} & 0 & 0
\end{bmatrix}. \tag{4.16}
\]

We now define \( \xi = \xi e_\xi + \eta e_\eta + \zeta e_\zeta \) by:

\[
x - x_e = R(\alpha)\xi. \tag{4.17}
\]

This causes \( e_\xi \) to be aligned with the flow and \( \eta \) and \( \zeta \) perpendicular to it. From now on the derivation is almost the same as before, since we now have:

\[
f = f_x e_x + f_y e_y + f_z e_z = f_\xi e_\xi + f_\eta e_\eta + f_\zeta e_\zeta, \tag{4.18}
\]

and we have:

\[
\nabla \phi = \frac{\partial \phi}{\partial \xi} e_\xi + \frac{\partial \phi}{\partial \eta} e_\eta + \frac{\partial \phi}{\partial \zeta} e_\zeta, \quad \nabla^2 \phi = \frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} + \frac{\partial^2 \phi}{\partial \zeta^2}. \tag{4.19}
\]
We now obtain the following flux in rotated coordinates:

\[
x_e = u\phi - \varepsilon \nabla \phi = U\phi e_\xi - \varepsilon \left( \frac{\partial \phi}{\partial \xi} e_\xi + \frac{\partial \phi}{\partial \eta} e_\eta + \frac{\partial \phi}{\partial \zeta} e_\zeta \right) = \left( U\phi - \varepsilon \frac{\partial \phi}{\partial \xi} \right) e_\xi - \varepsilon \frac{\partial \phi}{\partial \eta} e_\eta - \varepsilon \frac{\partial \phi}{\partial \zeta} e_\zeta.
\]

From the conservation law we now get:

\[
\nabla \cdot f = s \iff \frac{\partial f_\xi}{\partial \xi} + \frac{\partial f_\eta}{\partial \eta} + \frac{\partial f_\zeta}{\partial \zeta} = s \iff \frac{\partial f_\xi}{\partial \xi} = s + \varepsilon \frac{\partial^2 \phi}{\partial \eta^2} + \varepsilon \frac{\partial^2 \phi}{\partial \zeta^2} = : s_\xi.
\]

To determine the numerical \(\xi\) flux we now get:

\[
\frac{\partial f_\xi}{\partial \xi} = \frac{\partial}{\partial \xi} \left( U\phi - \varepsilon \frac{\partial \phi}{\partial \xi} \right) = s_\xi, \quad -h/2 < \xi < h/2,
\]

\[
\phi(-h/2) = \phi_L, \quad \phi(h/2) = \phi_R,
\]

and now the same procedure can be used as before.

We also need to determine the cross flux coefficients. Just as before we can write the adjusted source terms as:

\[
\frac{\partial f_\xi}{\partial \xi} = \alpha s - \beta \left( \frac{\partial f_\eta}{\partial \eta} + \frac{\partial f_\zeta}{\partial \zeta} \right), \quad \frac{\partial f_\eta}{\partial \eta} = \alpha s - \beta \left( \frac{\partial f_\xi}{\partial \xi} + \frac{\partial f_\zeta}{\partial \zeta} \right), \quad \frac{\partial f_\zeta}{\partial \zeta} = \alpha s - \beta \left( \frac{\partial f_\xi}{\partial \xi} + \frac{\partial f_\eta}{\partial \eta} \right)
\]

Adding these will give the same relation for \(\alpha\) as before, namely \(\alpha = (1 + 2\beta)/3\). We will however show in the next chapter using the numerical results that, in the two dimensional case, it is not needed to include the cross fluxes. This is equivalent to saying \(\beta\) equals 0. This would mean that there is no choice needed for \(\beta\), and that \(\alpha = 1/3\).

Interpolation can also be done in a similar manner. Where first linear-quadratic interpolation was used now bilinear-interpolation has to be used, where you have to interpolate linearly in the extra third dimension.
5 Numerical results

In this chapter we discuss the numerical results obtained from this method. We first discuss the results for a constant flow problem, and perform convergence tests for several different possibilities, where we either include or exclude parts of the inhomogeneous flux. After this we discuss a problem which includes a rotating flow, with and without a source term.

5.1 Constant velocity field

The first problem implemented is a problem with Dirichlet boundary conditions, where we set the solution to be equal to 0 at the boundaries, and a source term, with a constant flow velocity and diffusion. The source term used here is a local quadratic profile, given by:

\[ s = (x - 1)(x + 1)y(y - 1), \]  

(5.1)
on \(-1 \leq x \leq 1\) and \(0 \leq y \leq 1\). In Figure 5.1 the solution computed by the homogeneous flux scheme is shown with \(\varepsilon = 10^8\) and \(u = v = 1\). Since no exact solution is available for this boundary value problem we use Richardson extrapolation to compute the order of convergence. Let

\[ \phi \left( \frac{1}{2}, \frac{1}{2} \right) = \phi_h + e_h = \phi_{h/2} + e_{h/2} = \phi_{h/4} + e_{h/4}, \quad h = \Delta x, \]

(5.2)
where \(\phi_h\) is the numerical approximation of \(\phi \left( \frac{1}{2}, \frac{1}{2} \right)\) with grid size \(h\) and \(e_h\) the discretisation error. Assuming the following error expansion

\[ e_h = Ch^p + O(h^q), \quad q > p, \]

(5.3)
we can derive the following relation for the order of accuracy \(p\):

\[ 2^p \approx \frac{\phi_{h/2} - \phi_h}{\phi_{h/4} - \phi_{h/2}} =: r_h. \]

(5.4)
In the tables below the value for \(r_h\) is given for different methods, which gives us the order of convergence for these methods.

We tested 4 different possibilities for the flux. Method one included the complete method, with all the terms given in (3.36) and (3.37), while also including all the terms from (3.40). Method two included the complete flux in \(\xi\)-direction, but only the homogeneous flux in the \(\eta\)-direction, meaning that the flux derivatives were omitted. Method three once again included the complete flux in the \(\xi\)-direction, and the homogeneous flux for the \(\eta\)-direction but now the cross flux is omitted for the \(\xi\)-flux. This means that the second derivative in \(\eta\)-direction in the \(\xi\)-flux is omitted, but the terms still appear in the source term. For the fourth method only the homogeneous fluxes were included.
In Table 5.1 the results are shown for method one. Since $r_h$ converges to 4 it is apparent that the full method is second order convergent. The results of method two are shown in Table 5.2. This shows that the method still remains second order convergent even when the inhomogeneous $\eta$-flux is omitted. Table 5.3 shows the results for method 3, and it shows that method 3 also is second order convergent. In Table 5.4 the results are shown when only the homogeneous flux is included. This method is first order convergent for dominant advection. This means that method three is the best method, since this is the simplest method which remains second order convergent always. In further experiments this method will thus also be used.

This method is also second order convergent for all directions of flow, so for all possible angles $\alpha$. This is shown in tables 5.5 and 5.6, where two different angles are tested both with the flow in a different direction. There also is no difference in bilinear interpolation or quadratic-linear interpolation, as can be seen in Table 5.7, where the interpolation is done using bilinear interpolation instead of the quadratic-linear interpolation used so far. This means that in the rest of this paper quadratic-linear interpolation will be used.
5.2 Rotating flow

Here we test the method on two different problems with a rotating flow, both also treated in [4]. The first problem has an inlet flow, and the boundary value problem is given by:

\[
\nabla \cdot (u\phi - \varepsilon \nabla \phi) = 0, \quad -1 < x < 1, 0 < y < 1, \quad (5.5a)
\]

\[
\phi(x, 0) = 1 + \tanh(\alpha(2x + 1)), \quad -1 \leq x \leq 0 \quad \text{(inlet)}, \quad (5.5b)
\]

\[
\frac{\partial \phi}{\partial y}(x, 0) = 0, \quad 0 < x \leq 1 \quad \text{(outlet)}, \quad (5.5c)
\]

\[
\phi(x, y) = 1 - \tanh(\alpha), \quad (x = \pm 1, 0 \leq y \leq 1) \quad \text{and} \quad (-1 \leq x \leq 1, y = 1), \quad (5.5d)
\]

where the velocity field is given by:

\[
u := (u, v) = (2y(1 - x^2), -2x(1 - y^2)). \quad (5.6)
\]

For \(\alpha = 10\) this problem has a steep layer at the inlet, and this steep profile should be advected with the rotating flow if \(\varepsilon\) is small enough. The results of this are shown in Figure 5.2. On the left the solution is shown, while on the right the solution is shown at \(x = 0\). Here you can see the inflow on the left, and the outflow on the right. Since the problem is advection dominant since we take \(\varepsilon = 10^{-8}\), so the outflow profile should be the mirror image of the inflow profile, which it is as can be seen in the figure, which means steep profiles are well preserved.

The second problem has no steep inlet, but instead uses a source term. This boundary value problem is given by:

\[
\nabla \cdot (u\phi - \varepsilon \nabla \phi) = s, \quad -1 < x < 1, 0 < y < 1, \quad (5.7a)
\]

\[
\phi(x, 0) = 0, \quad -1 \leq x \leq 0, \quad (5.7b)
\]

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<tr>
<th>(h^{-1})</th>
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<th>(\varepsilon = 10^{-2})</th>
<th>(\varepsilon = 10^{-8})</th>
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Table 5.2: Values for \(r_h\) with method two for \(u = \sqrt{3}, v = \sqrt{2}\)
Numerical results

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Table 5.3: Values for $r_h$ with method three for $u = \sqrt{3}, v = \sqrt{2}$

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Table 5.4: Values for $r_h$ with method four for $u = \sqrt{3}, v = \sqrt{2}$

\[
\frac{\partial \phi}{\partial y}(x,0) = 0, \quad 0 < x \leq 1 \quad \text{(outlet)}, \quad (5.7c)
\]

\[
\phi(x,y) = 0, \quad (x = \pm 1, 0 \leq y \leq 1) \text{ and } (-1 \leq x \leq 1, y = 1), \quad (5.7d)
\]

with the source term defined by:

\[
s(x,y) = \frac{1}{2} \frac{s_{\text{max}}}{1 + s_{\text{max}}(x')} \left( 1 - \text{tanh}^2 \left( \frac{\beta}{2} \sqrt{2} - y' \right) \right),
\]

\[
(x',y') := \frac{1}{2} \sqrt{2}(x + y, -x + y).
\]

(5.8)

Note that $s$ has a maximum $s_{\text{max}}$ in $(-\frac{1}{2}, \frac{1}{2})$, and that the source term has a sharp peak here when $\beta = 10$. We also choose $s_{\text{max}} = 10^2$. $s$ is virtually equal to 0 in $[0, 1] \times [0, 1]$, meaning that only the advection or diffusion causes $\phi$ to change there. For $u$ we once again choose the expression in (5.6). We should see steep profiles when $\varepsilon$ is small enough, but here the profiles are caused by the source term instead of the inlet. The results are shown in Figure 5.3. It is once again apparent that the solution still has a sharp profile. In the figure on the right the outflow is shown together with the profile at $y = 0$. Because the problem is advection dominated, since $\varepsilon = 10^{-8}$ again, these two profiles should be the same, which they are as can be seen in this figure.

The solutions shown in Figure 5.2 and 5.3 are both computed using method three as described before, which included the complete flux only in the $\xi$-direction without including the cross fluxes. The solutions retained the steep profiles, which means that this method performs well to keep steep profiles in advection dominated problems.
5.2. Rotating flow

Figure 5.2: Solution of BVP (5.5) for $\alpha = 10$ and $\varepsilon = 10^{-8}$ computed with $\Delta x = \Delta y = 2.5 \times 10^{-2}$.

Figure 5.3: Solution of BVP (5.7) for $\beta = 10$, $s_{\text{max}} = 10^2$ and $\varepsilon = 10^{-8}$ computed with $\Delta x = \Delta y = 2.5 \times 10^{-2}$.
### Table 5.5: Values for $r_h$ using method 3 for $u = -\sqrt{3}, v = -\sqrt{5}$

<table>
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### Table 5.6: Values for $r_h$ using method 3 for $u = -\sqrt{2}, v = \sqrt{5}$

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### Table 5.7: Values for $r_h$ with bilinear interpolation instead of quadratic-linear interpolation using method 3 for $u = \sqrt{3}, v = \sqrt{2}$

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</tr>
</tbody>
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6 Conclusions and recommendations

We have derived the finite volume complete flux method for local flow adapted coordinates, and implemented and tested it for two dimensional problems. For constant flow the method proved to be second order convergent and worked well. It even showed that it is not necessary to include the cross fluxes. The method then remained second order convergent, and steep profiles were preserved.

When this method was applied to boundary value problems with a rotating flow the solution still preserved the steep profiles when only the homogeneous flux was included. This means that, while excluding the cross flux in $\xi$- and $\eta$-direction, the scheme will still perform as wanted.

Several extension have already been outlined, but have not been tested in detail. The first is the extension to three dimensions. In Chapter 4 we outlined the idea, and it most likely is possible to implement the same method in three dimensions without a lot of trouble. This has not been done in this paper, but could be of interest in future research.

A second extension is time dependent problems. In this paper we only considered stationary advection-diffusion equations, but often these equations are time dependent. There are many different time integration methods, and different methods may perform better for the complete flux scheme.

We furthermore also only looked at constant diffusion. In Chapter 2 we already derived the complete flux scheme for non-constant diffusion, but this is not yet extended to the complete flux in flow adapted coordinates. It is very likely that the same approach can be used as with the standard complete flux scheme, but this has not yet been tested.
Bibliography


