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Upscaling of Richards’ Equation in Fractured Porous Media

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Abstract

In this thesis, we consider fluid flow through a two-dimensional porous medium containing a fracture. In both the solid matrix blocks and the fracture, the flow is described by Richards’ equation, equipped with differing parametrisations of the hydraulic conductivity and the water content in the solid matrix and the fracture, respectively. For the coupling between the matrix blocks and the fracture, we impose the continuity of the pressure and of the flux at the interface. The geometry is chosen rectangular for the sake of simplicity.

At first, we fix the fracture width and apply Rothe’s method, that is an implicit Euler discretisation in time, in order to show the convergence of the time-discrete solutions towards the solution to the time-continuous problem as the time step vanishes. To do so, we make use of compactness arguments based on a priori estimates for the solution. It will prove crucial to overcome the difficulties posed by the nonlinearity of Richards’ equation. Rothe’s method yields the existence of a solution to the coupled problem provided that the time-discrete problem admits a solution. For the existence of solutions of the time-discrete problem, we refer the reader to previous works.

Then, we view the fracture width as a parameter and apply rigorous upscaling to prove the convergence towards solutions of different effective models as the fracture width goes to zero. The scaling of permeabilities and porosities with respect to the fracture width is decisive for determining the respective effective model. Finally, we present numerical simulations which confirm our theoretical upscaling results.

Zusammenfassung


Introduction

Water is a fundamental requirement of life on earth. Still, two-third of the world population face severe water scarcity at least one month of the year [38] and predictions foresee a rapid decline in the availability of drinking water in the coming decades caused by the increase of the world population, a rising water consumption per capita, and the climate change amongst other reasons. This scenario not only incites politics to act, but also encourages engineers and scientists to develop water-saving technologies and to gain a deeper understanding of flowing processes in order to optimise the positioning of wells, prevent groundwater from being polluted, and carry out water management on larger scales, to name a few challenges. During the recent decades, numerical simulations based on mathematical modelling have become a powerful tool to gain insight in numerous applications of fluid mechanics at high accuracy.

In this thesis, we consider a mathematical model for unsaturated subsurface flow in fractured porous media, in which the flow is governed by Richards’ equation. Fractures play a crucial role in subsurface flow since the permeability in fractures can significantly exceed the one in the surrounding soil and hence lead to massively altered flow patterns. Besides natural processes, fractured porous media occur in various technical applications such as carbon capture and storage (CCS), where CO₂ is permanently stored in geological formations to prevent it from being released into the atmosphere, or fracking, that is pumping a pressurised liquid into rock to create fractures that facilitate the flow of gas or petroleum.

However, since our mathematical model uses Richards’ equation for the description of the fluid flow, which is suitable for the modelling of liquid flow close to the earth’s surface, we focus on the application of water flow through fractured soil. We consider a single fracture embedded in two surrounding matrix blocks. The use of Richards’ equation in the fracture requires the fracture to be filled with a porous medium yielding sufficiently small flow velocities, which we assume to hold true herein.

When it comes to modelling of thin fractures in a d-dimensional domain, the fracture width is often neglected and the fracture is viewed as a (d−1)-dimensional quantity. It is interesting to justify these models mathematically as limit models of models containing a d-dimensional fracture in the limit when the fracture becomes infinitely thin. This is commonly done by so-called upscaling techniques, consisting in rescaling the fracture domain and the corresponding variables and then either employing a series expansion with respect to the fracture width as an ansatz for the solution (formal upscaling) or showing the convergence towards the limit model by strict means, harnessing results from functional analysis (rigorous upscaling). Here, we opt for rigorous upscaling. The central results of this work are the analysis of the model for fixed fracture width and the rigorous derivation of limit models for vanishing fracture width.

The structure will be as follows: Chapter 1 is dedicated to the mathematical modelling of unsaturated flow through porous media. We introduce the basic physical quantities related to porous media and we give the derivation of Richards’ equation. Then, we define our mathematical model for a two-dimensional porous medium crossed by a fracture.

In Chapter 2, we present the mathematical foundations, which will be required in the subsequent chapters. Since these results are well-known, we merely state the results and omit the proofs, which can be found in standard literature.

Chapter 3 is concerned with the existence of solutions. Rothe’s method is applied to the model equations leading to time-discrete equations. We proceed by proving essential bounds for the solution in $L^\infty$ and a priori estimates for the solution and its derivatives in $L^2$. Making use of compactness arguments, specifically the Aubin–Lions–Simon theorem, we obtain a strongly convergent subsequence as the time step vanishes and we show that the limit function is a solution to our model. The existence of solutions to the time-discrete model is discussed e.g. in [27] and will be assumed in this work.

In Chapter 4, we investigate the behaviour of solutions as the fracture width goes to zero. Similar estimates as in the previous chapter allow us to prove the convergence towards effective models in the limit by using rigorous upscaling techniques. Depending on the scaling of the permeability and the porosity with respect to
the fracture width, various effective models can arise, which strongly differ from each other: the fracture can become impermeable or remain permeable, and the subdomains on both sides of the fracture may completely decouple or stay connected. The remaining equation for the fracture in the effective model can be an interface condition or a differential equation.

In Chapter 5, we examine the upscaling results from the previous chapter by means of numerical simulations. We end this work with a discussion of the results and an outlook to further research and possible extensions.
Chapter 1

Modelling of subsurface flow

In this chapter, we introduce the fundamentals of mathematical modelling of subsurface flow through porous media. We define the basic physical quantities and derive Richards’ equation, which is widely employed in the modelling of unsaturated water flow near the surface, by the principle of mass conservation together with Darcy’s law for multiphase flow.

When it comes to subsurface flow, several entities play a role typically, such as a liquid phase, a gas phase and a rigid or deformable solid matrix interacting with the flowing fluids. The respective phases can be composed of several components, e.g. the gas phase could be made up of air and water steam. The interaction among the constituents on a microscopic level is highly complex and since the pore structure of the solid matrix is mostly unknown anyway, one commonly resorts to the description of the flow processes by averaged equations, not least because numerical simulation on the microscale is often not feasible due to computational limitations. We start with defining macroscopic quantities characterising subsurface flow.

1.1 Basics of subsurface flow

Subsurface flow in general describes fluid flow beneath the earth’s surface. Here, the considered fluid is water flowing in the vadose zone, that is the unsaturated zone between the earth’s surface and the groundwater level, or the groundwater filtering through water-conducting aquifers. The vadose zone and the aquifers consist of permeable rock or unconsolidated materials such as silt, loam, sand or gravel, which possess pores allowing the flow of water. Such media are called porous media.

In order to view a porous medium as a single entity without distinguishing the pore space and the solid matrix, a representative elementary volume (REV) is introduced. The REV should be chosen large enough to represent a typical sample of the heterogeneous medium and small enough to capture spatial inhomogeneities (see Figure 1.1). For comprehensive introductions to REVs, we refer the reader to [5, chap. 1] and [25, chap. 2].

![Figure 1.1: Definition of the REV](image)

On the REV scale, physical quantities that vary rapidly within the REV due to the heterogeneity of the medium can be meaningfully defined and the discontinuities at the transition between the pore space and the solid matrix are no longer observable. We specify the most important quantities in the context of subsurface flow in what follows. Note that we omit any temperature related quantities as we assume isothermal conditions.
Porosity

The ratio between the pore volume $V_{\text{pore}}$ [m$^3$], which can be occupied by a fluid, and the total volume $V_{\text{total}}$ [m$^3$] of a material sample is called the porosity $\phi$ [-] of a porous medium, in formulae

$$\phi := \frac{V_{\text{pore}}}{V_{\text{total}}}, \quad (1.1)$$

and thus lies between 0 and 1. Whereas the porosity oscillates rapidly if the sample under consideration is chosen too small, it varies slowly when a sample at the REV scale is translated in space (see Figure 1.2). As regards fluid flow, one usually considers the effective porosity, which disregards closed pores for the calculation of the pore volume.

Saturation and water content

When several immiscible phases are present in the pore space, the saturation $S$ [-] of a phase $\alpha$ measures the volume fraction of the pore space which is taken by the respective phase,

$$S_\alpha := \frac{V_\alpha}{V_{\text{pore}}}, \quad (1.2)$$

where $V_\alpha$ [m$^3$] denotes the volume occupied by phase $\alpha$. The system is then called a multiphase system. Recalling the definition of porosity the volume content $\theta_\alpha$ [-] of phase $\alpha$ can be defined as

$$\theta_\alpha := \frac{V_\alpha}{V_{\text{total}}} = \phi S_\alpha. \quad (1.3)$$

In the case of the water phase, the volume content is commonly referred to as the water content. Let the number of phases being present in the pore space be given as $n_{\text{phase}}$. Since the pore space is entirely filled, the following equivalent restrictions hold:

$$\sum_{\alpha=1}^{n_{\text{phase}}} V_\alpha = V_{\text{pore}},$$

$$\sum_{\alpha=1}^{n_{\text{phase}}} \theta_\alpha = \phi,$$

$$\sum_{\alpha=1}^{n_{\text{phase}}} S_\alpha = 1. \quad (1.4)$$

In general, the phases can be made up of several components. Moreover, in several applications phase transitions have to be taken into account, e.g. when a liquid evaporates.
Darcy velocity

Let the total volume discharge of phase \( \alpha \) be given by \( Q_\alpha \) [m\(^3\) s\(^{-1}\)], and let \( A \) [m\(^2\)] be the total flow cross-section (comprising solid matrix and pore space). The Darcy velocity (also discharge velocity, specific discharge) \( v_\alpha \) [m s\(^{-1}\)] of the phase is defined as

\[
v_\alpha := \frac{Q_\alpha}{A}.
\] (1.5)

In order to obtain the average flow velocity \( v_{av,\alpha} \) [m s\(^{-1}\)] of individual particles from this, one must relate the total discharge to the area accessible to fluid flow, that is

\[
v_{av,\alpha} := \frac{v_\alpha}{\Phi}.
\] (1.6)

All velocities and fluxes are vectorial if a multidimensional problem is considered.

Permeability and hydraulic conductivity

The intrinsic permeability \( k \) [m\(^2\)] is a macroscopic measure that characterises how easily fluids can pass through a porous medium. It depends only on the porous medium and is independent of the fluid. In the case of anisotropic media, the intrinsic permeability is a tensorial quantity. From the intrinsic permeability, the fluid density \( \rho_\alpha \) [kg m\(^{-3}\)] and the dynamic viscosity \( \mu_\alpha \) [kg m\(^{-1}\) s\(^{-1}\)] of a phase \( \alpha \), another property can be derived which is called the hydraulic conductivity \( K \) [m s\(^{-1}\)], defined as

\[
K_\alpha := k k_{r,\alpha}(\theta_\alpha) \frac{\rho_\alpha g}{\mu_\alpha}
\] (1.7)

Here, \( g \) [m s\(^{-2}\)] is the gravitational constant and \( k_{r,\alpha} = k_{r,\alpha}(\theta_\alpha) \) [-] is the relative permeability. The idea behind introducing the relative permeability is the following: when several phases occupy the pore space, the presence of one phase will impede the flow of the other phases and vice versa. This is due to a smaller cross-sectional area available for each phase and due to the increased tortuosity of the flow paths, since particles of a phase have to flow around accumulations of particles of the other phases. In order to express this microscopic observation on the macroscale, it is required to link the relative permeability to another macroscopic quantity. It turns out that the volume content (or the saturation) suits this purpose: the larger the volume content of a phase, the larger is the cross-sectional area through which this phase flows and the less must the particles of this phase give way to clusters of other phases. Thus, one can experimentally determine a parametrisation of the relative permeability as a function of the water content. Clearly, its values lay between 0 and 1, where 0 represents the case when the phase is not present in the pore space and 1 describes the case when the entire pore space is filled with the respective space, i.e. \( k_{r,\alpha}(\theta_\alpha = 0) = 0 \) and \( k_{r,\alpha}(\theta_\alpha = \phi) = 1 \). Typically, \( k_{r,\alpha}(\theta_\alpha) \) is a monotonically increasing, convex function. A common parametrisation due to Mualem and van Genuchten will be presented in Section 1.3.

Capillary pressure

We introduce the capillary pressure for the case of two fluids present in the system, the extension to three or more fluids is plain and can be found in [25, sec. 2.4]. When considering fluid flow through porous media on the microscale, the interfacial tensions between phases play a crucial role. They are caused by cohesive forces between molecules of the same phase and adhesive forces between molecules of different phases. The equilibrium of these forces results in a curved interface, which minimises the total potential energy of the system. Thus, the boundary angle in the equilibrium, which is referred to as the wetting angle \( \gamma \) [°], is in general not right-angled (see Figure 1.3). One calls the fluid with the acute wetting angle the wetting fluid (denoted with the subscript \( w \) here) and the fluid with the obtuse wetting angle the non-wetting fluid (denoted with the subscript \( n \) here). Notice that the notions of wetting and non-wetting depend on all the phases present in the system: oil is typically the wetting phase in gas-oil systems whereas it is typically the non-wetting phase in water-oil systems, to give an example. The jump in pressure \( p \) [kg m\(^{-1}\) s\(^{-2}\)] between the fluid phases which originates from the equilibrium of forces is named the capillary pressure \( p_c \) [kg m\(^{-1}\) s\(^{-2}\)]:

\[
p_c := p_n - p_w.
\] (1.8)
It can be computed from the capillary tube diameter $d$ [m], the wetting angle and the interfacial tension $\sigma$ [kg s$^{-2}$] by Laplace’s equation

$$p_c = \frac{4\sigma \cos \gamma}{d}. \quad (1.9)$$

A survey on the interfacial tensions for different fluid combinations can be found e.g. in [48]. Laplace’s equation (1.9) shows that the smaller the pores in the system are, the higher is the capillary pressure. This gives rise to the following observations: when the wetting phase is drained, it leaves the larger pores first and remains in the smaller pores where capillary action is the highest. On the other hand, when imbibition takes place and wetting phase fluid is injected, the wetting phase fills larger pores where the capillary pressure is lower.

This behaviour can be modelled on the macroscale by introducing a relation between the capillary pressure and the volume content (or the saturation respectively), which captures the findings on the microscale in the sense that a higher volume content of the wetting phase leads to a lower capillary pressure and vice versa:

$$p_c = p_c(\theta_w). \quad (1.10)$$

This relation cannot be obtained analytically due to the complexity of the pore space geometry, but several empirical relations have been proposed based on a multitude of experiments, the most famous among these being due to Leverett [34], Brooks and Corey [10], and Mualem and van Genuchten [54]. We will describe the van Genuchten–Mualem model in Section 1.3, which we will use in the numerical simulations at the end of this thesis.

The parametrisation of the capillary pressure only as a function of the water content as in equation (1.10) is limited to creeping flow, however, and deviations from experiments are observed when it comes to transient flow. To overcome this, the capillary pressure parametrisation may be supplemented with dynamic capillarity effects in order to obtain a model of the form

$$p_c = p_c(\theta_w, \partial_t \theta_w) = p_n - p_w + \tau(\theta_w) \partial_t \theta_w, \quad (1.11)$$

where $\tau = \tau(\theta_w)$ [kg m$^{-1}$ s$^{-1}$] is a non-equilibrium coefficient. Such models are used in e.g. [23, 28]. Besides, hysteresis effects expressing different soil behaviours for imbibition and drainage can be taken into account, see e.g. [13].

### 1.2 Richards’ equation

Richards’ equation was first presented in 1931 by Lorenzo A. Richards [45]. It is used for the macroscale modelling of unsaturated water flow close to the earth’s surface and is based upon the assumptions that the non-wetting gas phase in the porous medium, which is commonly air, is interconnected and connected to the atmosphere. Then, pressure fluctuations of the gas phase are small and can be neglected. In what follows, we will present the derivation of Richards’ equation from the balance of mass in conjunction with the extended Darcy law, which provides a suitable form of the momentum balance in porous media.

We start by taking an arbitrary control volume $\Omega \subset \mathbb{R}^d$ for $d \in \{1, 2, 3\}$ with smooth boundary $\partial \Omega$. Note that in this derivation, we assume all properties to be sufficiently smooth for our purposes. The conservation of the water mass $m_w$ [kg] contained in $\Omega$ reads as

$$\partial_t m_w = \partial_t \int_\Omega \rho_w \phi S_w \, dV + \int_{\partial \Omega} (\rho_w \phi v_{av,w}) \cdot \vec{n} \, dA = \int_\Omega q_w \, dV, \quad (1.12)$$

where $\vec{n}$ is the outward pointing normal vector on $\partial \Omega$ and $q_w$ [kg m$^{-3}$ s$^{-1}$] is a mass source or sink term, for example due to wells or pumps in the domain. Interchanging the time derivative and the integral and applying
the divergence theorem gives
\[ \int_{\Omega} \partial_t (\rho_w \phi) \, dV + \int_{\Omega} \nabla \cdot (\rho_w \phi v_{\text{av},w}) \, dV = \int_{\Omega} q_w \, dV. \] (1.13)

As the control volume \( \Omega \) has been chosen arbitrarily, we demand the equality of the integrands to obtain
\[ \partial_t (\rho_w \phi) + \nabla \cdot (\rho_w \phi v_{\text{av},w}) = q_w. \] (1.14)

Making use of the relations \( \theta_w = \phi S_w \) and \( v_{\text{av},w} = \phi v_{\text{av},w} \), we can rewrite this as
\[ \partial_t (\rho_w \theta) + \nabla \cdot (\rho_w v_{\text{av},w}) = q_w. \] (1.15)

Now, we neglect the low compressibility of water and assume that the water density is temporally and spatially constant, namely \( \rho_w(t, \vec{x}) \equiv \rho_w > 0 \). Consequently, we get
\[ \partial_t \theta + \nabla \cdot v_{\text{av},w} = \frac{q_w}{\rho_w}. \] (1.16)

At this point, we are required to relate both the Darcy velocity \( v_{\text{av},w} \) and the water content \( \theta_w \) to the pressure. This is where Darcy’s law for multiphase flow comes into play. In 1856, Henry Darcy was concerned with the drafts for a water system in the French city of Dijon [14]. In this context, he discovered experimentally a proportionality between the Darcy velocity and the negative pressure gradient. Originally, Darcy stated the law only for saturated single-phase flow – Edgar Buckingham extended it to unsaturated flow, where the crucial supplement is the concept of relative permeability. By now, several mathematical derivations of Darcy’s law from the Navier–Stokes equation exist, based on volume averaging as in [56, chaps. 4–5], homogenisation as in [26, chap. 3], or thermodynamically constrained averaging theory as in [22, chap. 9].

Darcy’s law for multiphase flow through an isotropic medium writes as
\[ v_{\text{av},w} = -K_w(\theta_w) \nabla \psi_w, \] (1.17)
where \( \psi_w \) [m] is the pressure head, related to the pressure \( p_w \) [kg m\(^{-1}\) s\(^{-2}\)] by the equation
\[ \psi_w = \frac{p_w}{\rho_w g}. \] (1.18)

We neglect gravitational effects herein.

In order to obtain a relation between the water content and the pressure, recall the definition of the capillary pressure \( p_c(\theta_w) = p_n - p_w \). Taking now a constant air pressure \( p_n \equiv 0 \), justified by the assumption that the air is infinitely mobile and can equilibrate pressure fluctuations instantaneously because the flow domain is interconnected and connected to the surface [40, chap. 3], we get
\[ p_c(\theta_w) = -p_w. \] (1.19)

Since the macroscopic relation between the capillary pressure and the water content embodies the microscale behaviour, it is commonly strictly monotonically increasing, which allows us to invert equation (1.19). This provides us with an explicit parametrisation
\[ \theta_w = \theta_w(\psi_w). \] (1.20)

Equations (1.16), (1.17), and (1.20) leave us with Richards’ equation in the mixed pressure-saturation form
\[ \partial_t \theta_w(\psi_w) - \nabla \cdot [K_w(\theta_w(\psi_w)) \nabla \psi_w] = f_w, \] (1.21)
where we set \( f_w = \frac{q_w}{\rho_w} \).

### 1.3 Van Genuchten–Mualem parametrisation

The van Genuchten–Mualem model is one of the most widely employed empirical models for the relations between the capillary pressure and the volume content, and between the hydraulic conductivity and the water
content. The main difference in comparison with the Brooks–Corey model consists in the capillary pressure when the wetting phase saturation approaches 1: whereas the capillary pressure in the Brooks–Corey model takes a positive value (the so-called entry pressure) in the case of the porous medium being fully saturated with the wetting phase, the capillary pressure in the van Genuchten–Mualem model converges smoothly to 0 as the wetting phase saturation goes to 1.

In the setting of Richards’ equation where \( p_c(\theta_w) = -p_w \) and with \( p_c \) strictly monotonically increasing, we can express the water content as a function of the pressure head and hence the hydraulic conductivity as a function of the pressure head as well. The van Genuchten–Mualem parametrisation writes as

\[
\theta_w(\psi_w) = \begin{cases} 
\theta_R + (\theta_S - \theta_R) \left[ 1 + (\alpha \psi_w)^n \right]^{\frac{1}{n} - 1}, & \psi_w \leq 0, \\
\theta_S, & \psi_w > 0,
\end{cases}
\]

\[
K_w(\psi_w) = K_w(\theta_w(\psi_w)) = \begin{cases} 
K_S \theta_w(\psi_w)^{\frac{1}{2}} \left[ 1 - \left( 1 - \theta_w(\psi_w) \right)^{\frac{1}{n} - 1} \right], & \psi_w \leq 0, \\
K_S, & \psi_w > 0,
\end{cases}
\]

(1.22)

where \( \theta_S \) and \( K_S \) stand for the water content and the hydraulic conductivity of the fully saturated porous medium, respectively, \( \theta_R \) denotes the residual water content, and \( \alpha \) and \( n \) are curve fitting parameters expressing the soil properties. Figure 1.4 shows typical curves. When the pressure head approaches negative infinity, the water content converges to the residual water content \( \theta_R \) that remains in the pores and the hydraulic conductivity converges to \( K_w(\theta_R) \).

Note that in the fully saturated regime we retrieve the single-phase Darcy law from Richards’ equation, since the water content is then constant, and the hydraulic conductivity is independent of the pressure. This makes Richards’ equation a nonlinear degenerate elliptic-parabolic partial differential equation, which becomes elliptic wherever the flow is saturated.

![Figure 1.4: Typical profiles of \( \theta(\psi) \) and \( K(\theta(\psi)) \) as given by the van Genuchten–Mualem model](image)

### 1.4 Fractures in porous media

Fractures and fracture networks in porous media influence the flow properties significantly. They are formed when mechanical stresses in the porous medium become too high, causing the porous medium to crack or existing fractures to expand. Fractures divide the medium into several essentially disjoint matrix blocks. In the fracture, the permeability is typically higher than in the surrounding matrix blocks, for which reason the prerequisites for using Darcy’s law or Richards’ equation for the description of the flow in the fracture may not always be given: if the fracture is hollow and the water can flow freely through the fracture, Richards’ equation can certainly not be employed. In this case, the flow in the fracture is more accurately described by the (Navier–)Stokes equation or by the Hagen–Poiseuille equation (see e.g. [39] for the coupling of two-phase flow and free flow, and [47] for the coupling of Richards’ equation and free flow).
Herein, we assume the fracture to consist of a porous medium with possibly different properties than the porous medium in the matrix blocks, which still allows for the use of Richards’ equation for the fracture flow. Examples for this scenario are fractures in rock filled with silt or loam. Several approaches for the relation between relative permeability and volume content in fractures have been proposed (see e.g. [25, sec. 2.7],[46]), which have applications in simulations of fractured oil reservoirs [20]. In order to model the capillarity in fractures, a model for the relation between capillary pressure and volume content was presented by Pruess and Tsang [44], which is also introduced in [25, sec. 2.4]. For a single idealised fracture it is possible to explicitly calculate the equilibrium of forces [55] and to give a formula for the dependence of the volume content on the capillary pressure and the pore geometry. As a matter of fact, the interaction between the fracture and the matrix blocks is highly complex – for some aspects of it we refer the reader to [25, sec. 2.7]. For the sake of simplicity, we use the van Genuchten–Mualem model in the fracture for our simulations, equipped with van Genuchten parameters differing from the ones used for the matrix blocks.

1.5 Dimensional model

We consider a mathematical model of fluid flow through a fractured porous medium, in which the flow is governed by Richards’ equation in the fracture (denoted with the subscript $f$) and in the matrix blocks (denoted with the subscript $m$). For the ease of presentation, we resort to a simple geometry consisting of two square solid matrix blocks with edge length $L$ [m] separated by a fracture of width $l$ [m]. The geometry of the fracture is depicted in Figure 1.5. We use hats on top to indicate quantities associated with the dimensional model to distinguish them from the dimensionless quantities which will be introduced subsequently.

![Figure 1.5: Geometry of the fracture and the surrounding matrix blocks](image)

The model defined on this geometry is given by

$$
\begin{aligned}
\partial_t \hat{\theta}_m(\hat{\psi}_m) + \nabla \cdot \hat{\vartheta}_m &= \hat{f}_m, \\
\hat{\vartheta}_m &= -\hat{K}_m(\hat{\theta}_m(\hat{\psi}_m)) \nabla \hat{\psi}_m, \\
\hat{\vartheta}_f(\hat{\psi}_f) + \nabla \cdot \hat{\vartheta}_f &= \hat{f}_f, \\
\hat{\psi}_f &= 0, \\
\hat{\psi}_{m,j} \cdot \hat{n} &= \hat{\vartheta}_f \cdot \hat{n}, \\
\hat{\psi}_m(0) &= \hat{\psi}_{m,j}, \\
\hat{\psi}_f(0) &= \hat{\psi}_{f,j}
\end{aligned}
$$

for $i \in \{m_1, m_2, f\}$, $j \in \{1,2\}$, where $\hat{\Omega}_m := \hat{\Omega}_{m_1} \cup \hat{\Omega}_{m_2}$, $\hat{\Gamma} := \hat{\Gamma}_1 \cup \hat{\Gamma}_2$, $\hat{\Omega}_f := \hat{\Omega}_f \times (0, \hat{T})$ for a given time $\hat{T} > 0$, $\hat{n}$ is a normal vector (e.g. with unit length) pointing from $\hat{\Omega}_{m,j}$ into $\hat{\Omega}_f$, $\hat{\theta}$ is the water content, $\hat{\psi}$ is the flux as given by Darcy’s law for unsaturated flow, $\hat{\psi}_i$ is the pressure height, $\hat{K}$ is the hydraulic conductivity, $\hat{\psi}_{m,j}$ is a given initial condition, and $\hat{f}$ is a source/sink term.
In words, the model consists of Richards’ equation in the fracture and the matrix blocks, supplemented with the continuity of flux and pressure as interface conditions, and initial conditions. The continuity of the flux across the interfaces accounts for the conservation of mass, and the continuity of the pressures can be derived from the conservation of momentum (see e.g. [24]).

Note that we neglect gravitational forces in this thesis.

### 1.6 Dimensionless model

We carry out a non-dimensionalisation procedure in order to render problem $P_D$ dimensionless. We define the ratio of the fracture width to its length as

$$\varepsilon := \frac{l}{L}. \quad (1.23)$$

Moreover, we define reference length scales $\bar{x} = L$, $\bar{y} = L$, and introduce dimensionless spatial coordinates $x = \hat{x}/L, y = \hat{y}/L$. The partial derivatives transform according to $\partial_x = L \partial_{\hat{x}}, \partial_y = L \partial_{\hat{y}}$ by the chain rule.

Figure 1.6 shows the resulting geometry.

Since the pressure is continuous at the interfaces, we define a reference pressure head for the entire domain $\bar{\psi} = L$. For the matrix blocks and the fracture we use different reference hydraulic conductivities $\bar{K}_m$ and $\bar{K}_f$, respectively. As the reference time scale, we set

$$\bar{T} := \frac{L^2}{\bar{K}_m \bar{\psi}} = \frac{L}{\bar{K}_m}. \quad (1.24)$$

The dimensionless pressure heads are then given as $\bar{\psi}_{mj} = \hat{\psi}_{mj}/L$ and $\bar{\psi}_f = \hat{\psi}_f/L$, the dimensionless time as $t = \hat{t}/\bar{T}$, and the final time as $T = \hat{T}/\bar{T}$. The derivative with respect to the dimensionless time becomes $\partial_t = T \partial_{\hat{t}}$. As regards the source terms, let $f_{mj} = f_m T$ and $f_f = f_f T$. We define dimensionless hydraulic conductivities $\bar{K}_m = \bar{K}_m/\bar{K}_m$ and $\bar{K}_f = \bar{K}_f/\bar{K}_f$.

Substituting Darcy’s law for multiphase into the mass balance in Problem $P_D$, we obtain for the matrix blocks

$$\frac{1}{T} \partial_t (\phi_m S_m (\psi_{mj})) - \frac{1}{L} \nabla \cdot \left( \bar{K}_m \bar{K}_m (\theta_m (\psi_{mj})) \frac{1}{L} \nabla (L \psi_{mj}) \right) = \frac{1}{T} f_{mj}. \quad (1.25)$$

Making use of the relation in equation (1.24), this yields

$$\partial_t (\phi_m S_m (\psi_{mj})) - \nabla \cdot \left( \bar{K}_m (\theta_m (\psi_{mj})) \nabla \psi_{mj} \right) = f_{mj}. \quad (1.26)$$
We relate the reference values for the porosity and the hydraulic conductivity in the fracture and the matrix blocks to each other by setting

\[
\frac{\phi_f}{\phi_m} = \epsilon^\kappa, \quad \frac{K_f}{K_m} = \epsilon^\lambda.
\] (1.27)

This is where the scaling parameters \(\kappa, \lambda \in \mathbb{R}\) come into play, which will prove crucial in determining effective models in the limit \(\epsilon \to 0\), as we will see later. The value of \(\kappa\) determines the storage capacity of the fracture: for \(\kappa < 0\), the reference porosity of the fracture increases for decreasing \(\epsilon\) as compared to the reference porosity of the matrix blocks, and for \(\kappa\) being sufficiently small, the fracture can maintain its ability to store water as \(\epsilon\) goes to zero. For \(\kappa = 0\), no scaling occurs, and for \(\kappa > 0\), the fracture can become so impervious that a discontinuity in pressure arises at the fracture in the effective models. The choice \(\lambda = 0\) implies no scaling of the flux term in the fracture, and for \(\lambda > 0\), the fracture can become so impervious that a discontinuity in pressure arises at the fracture in the effective models, or the matrix blocks can fully decouple from each other.

From the fracture equations, we get

\[
\frac{1}{L} \partial_t (\phi_f S_f(\psi_f)) - \frac{1}{L} \nabla \cdot \left( K_f (\theta_f(\psi_f)) \frac{1}{L} \nabla (L \psi_f) \right) = \frac{1}{T} f_f,
\] (1.28)

and inserting equations (1.24) and (1.27) yields

\[
\partial_t (\phi_m \epsilon^\kappa S_f(\psi_f)) - \nabla \cdot \left( \epsilon^\lambda K_f (\theta_f(\psi_f)) \nabla \psi_f \right) = f_f.
\] (1.29)

The flux interface condition transforms into

\[
K_m (\theta_m(\psi_m)) \nabla \psi_m \cdot \bar{n} = \epsilon^\lambda K_f (\theta_f(\psi_f)) \nabla \psi_f \cdot \bar{n} \quad \text{on } \Gamma_j.
\] (1.30)

For the sake of simplicity, we will slightly abuse notation and write \(\theta_m = \phi_m S_m\) and \(\theta_f = \phi_f S_f\) in the following.

We treat the dimensionless fracture width \(\epsilon > 0\) as a model parameter. For each \(\epsilon\), the model reads

\[
\begin{align*}
\text{Problem } P : & \quad \partial_t \theta_m(\psi_m^\epsilon) + \nabla \cdot v_m^\epsilon = f_m & \text{in } \Gamma_m^\epsilon, \\
& \quad v_m^\epsilon = -K_m(\theta_m(\psi_m^\epsilon)) \nabla \psi_m^\epsilon & \text{in } \Gamma_m^\epsilon, \\
& \quad \partial_t (\epsilon^\kappa \theta_f(\psi_f^\epsilon)) + \nabla \cdot v_f^\epsilon = f_f^\epsilon & \text{in } \Gamma_f^\epsilon, \\
& \quad v_f^\epsilon = -\epsilon^\lambda K_f(\theta_f(\psi_f^\epsilon)) \nabla \psi_f^\epsilon & \text{in } \Gamma_f^\epsilon, \\
& \quad \psi_f^\epsilon = 0 & \text{on } \partial \Omega \setminus \Gamma \times (0, T], \\
& \quad v_m^\epsilon \cdot \bar{n} = v_f^\epsilon \cdot \bar{n} & \text{on } \Gamma_j \times (0, T], \\
& \quad \psi_m^\epsilon = \psi_f^\epsilon & \text{on } \Gamma_j \times (0, T], \\
& \quad \psi_f^\epsilon(0) = \psi_f^{\epsilon, I} & \text{in } \Omega_f.
\end{align*}
\]
Chapter 2

Mathematical foundations

In this chapter, we gather mathematical results, in particular from the field of functional analysis, which we will employ in the subsequent chapters when it comes to the existence of a solution for fixed fracture width and to the upscaling results. Since these results are standard and can be found in a multitude of introductory books, we omit the proofs here and refer the interested reader to [9, 16, 17].

2.1 Function spaces

First of all, we introduce Lebesgue and Sobolev spaces, which are fundamental in the theory of partial differential equation. For the sake of simplicity, we consider real vector spaces here, the extension to complex vector spaces is plain.

2.1.1 Lebesgue spaces

Let \( 1 \leq p \leq \infty \) and let \((\Omega, \mathcal{A}, \mu)\) be a measure space. Then, the set

\[
L^p(\Omega, \mathcal{A}, \mu) := \{ f : \Omega \to \mathbb{R} : f \text{ is measurable, } \| f \|_{L^p(\Omega)} < \infty \}
\]  

is a vector space with seminorm \( \| \cdot \|_{L^p(\Omega)} \), defined as

\[
\| f \|_{L^p(\Omega)} := \left\{ \begin{array}{ll}
\left( \int_{\Omega} |f|^p \, d\mu \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\
\text{ess sup}_{x \in \Omega} |f(x)|, & p = \infty.
\end{array} \right.
\]  

(2.2)

The map \( \| \cdot \|_{L^p(\Omega)} \) is only a seminorm because every function which is zero everywhere except on a set of measure zero has norm zero. In order to obtain a normed space, one identifies functions that are equal almost everywhere with each other. To be more specific, we define the subspace

\[
\mathcal{N} := \{ f \in L^p(\Omega, \mathcal{A}, \mu) : f = 0 \mu\text{-a.e. (almost everywhere)} \} = \{ f \in L^p(\Omega, \mathcal{A}, \mu) : \| f \|_{L^p(\Omega)} = 0 \},
\]  

(2.3)

and introduce the quotient space \( L^p(\Omega, \mathcal{A}, \mu) = L^p(\Omega, \mathcal{A}, \mu) / \mathcal{N} \), called \( L^p \) space or Lebesgue space. All functions that differ only on a set of measure zero from each other are represented by one element in \( L^p(\Omega, \mathcal{A}, \mu) \). One can show that the space \( L^p(\Omega, \mathcal{A}, \mu) \) with the norm \( \| \cdot \|_{L^p(\Omega)} \) is complete and therefore a Banach space.

The space \( L^2(\Omega, \mathcal{A}, \mu) \) plays a special role since it is the only Hilbert space among the \( L^p \) spaces, equipped with the inner product

\[
(f, g)_{L^2(\Omega)} := \int_{\Omega} fg \, d\mu.
\]  

(2.4)

If \((\Omega, \mathcal{A})\) is a separable measurable space, the space \( L^p(\Omega, \mathcal{A}, \mu) \) is separable as well for \( 1 \leq p < \infty \), whereas \( L^\infty(\Omega, \mathcal{A}, \mu) \) is not separable in general.

For \( 1 < p < \infty \) the dual space of an \( L^p \) space is again an \( L^p \) space, namely

\[
L^p(\Omega, \mathcal{A}, \mu)^* = L^q(\Omega, \mathcal{A}, \mu),
\]  

(2.5)
write with Lipschitz boundary \( \partial \Omega \). In this thesis, the measure space \((\Omega, \mathcal{A}, \mu)\) is composed of a bounded open domain \(\Omega \subset \mathbb{R}^d\) for \(d \in \{1, 2, 3\}\) with Lipschitz boundary \(\partial \Omega\), the Lebesgue \(\sigma\)-algebra \(\Sigma\) on \(\Omega\) and the Lebesgue measure \(\mu = \lambda\). Then, we just write \(L^p(\Omega) = L^p(\Omega, \Sigma, \lambda)\).

### 2.1.2 Sobolev spaces

The \(L^p\) spaces can be viewed as a special case of a larger class of spaces, the so-called Sobolev spaces. Sobolev spaces are the natural spaces for the solutions of partial differential equations. In order to introduce them, we first need to define the concept of weak derivatives. Let \(\Omega \subset \mathbb{R}^d\) be an open set, let \(L^1_{\text{loc}}(\Omega)\) be the set of locally integrable functions, that is

\[
L^1_{\text{loc}}(\Omega) := \{ f : \Omega \to \mathbb{R} : f \text{ is measurable, } f|_K \in L^1(\Omega) \text{ for all compact } K \subset \Omega \},
\]

(2.6)

and let \(\alpha \in \mathbb{N}_0^d\) be a multi-index. Then we call \(g \in L^1_{\text{loc}}(\Omega)\) the weak \(\alpha\)-derivative of \(f \in L^1_{\text{loc}}(\Omega)\) if

\[
\int_{\Omega} f D^\alpha \phi \, d\lambda = (-1)^{|\alpha|} \int_{\Omega} g \phi \, d\lambda,
\]

(2.7)

for all \(\phi \in C^\infty_c(\Omega)\), which is the space of all infinitely differentiable functions with compact support in \(\Omega\). Here, we set

\[
D^\alpha = \frac{\partial^{\lvert \alpha \rvert}}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_d} x_d} \quad \text{and} \quad |\alpha| = \sum_{i=1}^{d} a_i.
\]

(2.8)

Then, we write \(g = D^\alpha f\), which is justified as it is easy to show that the weak derivative is unique in the sense of an \(L^p\) function. The weak derivative is an extension of the classical derivative in view of the fact that for every classically differentiable function the weak derivative exists and equals the classical derivative almost everywhere. Exploiting the concept of weak derivatives, we define for \(1 \leq p \leq \infty\) the Sobolev spaces

\[
W^{k,p}(\Omega) := \{ f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega) \text{ for all } |\alpha| \leq k \},
\]

(2.9)

and we endow these spaces with the norm

\[
\| \cdot \|_{W^{k,p}(\Omega)} : f \mapsto \left\{ \begin{array}{ll}
\left( \sum_{|\alpha| \leq k} \| \partial^\alpha f \|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\
\max_{|\alpha| \leq k} \| \partial^\alpha f \|_{L^\infty(\Omega)}, & p = \infty.
\end{array} \right.
\]

(2.10)

The Sobolev spaces are Banach spaces, and moreover separable for \(1 \leq p < \infty\) and reflexive for \(1 < p < \infty\). The space \(W^{1,2}(\Omega)\) is a Hilbert space, where the inner product is given by

\[
(f,g)_{W^{1,2}(\Omega)} := \sum_{|\alpha| \leq 1} \left( \partial^\alpha f, \partial^\alpha g \right)_{L^2(\Omega)}.
\]

(2.11)

When it comes to boundary value problems, we are required to evaluate functions on the boundary. Since the boundary \(\partial \Omega\) is of lower dimension than the set \(\Omega\) itself, \(\partial \Omega\) is a null set with respect to the Lebesgue measure \(\lambda\), and evaluating \(L^p\) functions on \(\partial \Omega\) makes no sense initially. On Sobolev spaces, the trace operator remedies this issue. We present the trace theorem in a form that we will use later for the case of the Hilbert space \(W^{1,2}(\Omega)\), but generalisations to other Sobolev spaces are straightforward and can be found in literature:

**Theorem 2.1.1 (Trace theorem)** Let \(\Omega \subset \mathbb{R}^d\) be a bounded domain with Lipschitz boundary \(\partial \Omega\). Then, there exists a bounded linear operator

\[
T : W^{1,2}(\Omega) \to L^2(\partial \Omega),
\]

(2.12)

such that

\[
Tf = f|_{\partial \Omega}, \quad \text{for all } f \in C^\infty(\overline{\Omega}).
\]

(2.13)

Furthermore, there exists a \(C = C(\Omega)\) such that for all \(f \in W^{1,2}(\Omega)\)

\[
\|Tf\|^2_{L^2(\partial \Omega)} \leq C \|f\|^2_{L^2(\Omega)} \|f\|_{W^{1,2}(\Omega)},
\]

(2.14)
The above inequality can be proven using the interpolation inequality (see e.g. [8, 21]). We will henceforth omit the trace operator \( T \) in the notation when it is clear that a Sobolev function is evaluated in the sense of traces.

Moreover, let \( W^{k,p}(\Omega) \) be the closure of \( C^\infty(\Omega) \) in \( W^{k,p}(\Omega) \). One can show that if \( \Omega \) has a Lipschitz boundary \( \partial \Omega \) the statement \( f|_{\partial \Omega} = 0 \) is equivalent to \( f \in W^{1,p}_0(\Omega) \), that is, the functions in \( W^{1,p}_0(\Omega) \) are exactly the functions in \( W^{1,p}(\Omega) \) with zero boundary conditions in the sense of traces.

The dual space of \( W^{1,p}_0(\Omega) \) is known as the space \( W^{-1,2}(\Omega) \). It is equipped with the usual dual space norm

\[
\|f\|_{W^{-1,2}(\Omega)} := \sup_{g \in W^{1,2}_0(\Omega)} \left| \langle f, g \rangle_{W^{-1,2}(\Omega), W^{1,2}_0(\Omega)} \right|
\]

(2.15)

where \( \langle \cdot, \cdot \rangle_{W^{-1,2}(\Omega), W^{1,2}_0(\Omega)} \) denotes the dual pairing.

For functions that vanish on parts of the boundary in the sense of traces, the following useful variant of Poincaré’s inequality holds:

**Lemma 2.1.1 (Poincaré’s inequality)** Let \( \Omega \subset \mathbb{R}^d \) be a bounded and connected set with Lipschitz boundary \( \partial \Omega \) such that \( \partial \Omega \) can be written as the disjoint union of two sets \( \partial \Omega = \Gamma_1 \cup \Gamma_2 \), where \( \Gamma_1 \) has positive Lebesgue measure. Then, there exists a constant \( C_p = C_p(\Omega, \Gamma_1) \) such that

\[
\|f\|_{L^2(\Omega)} \leq C_p \|\nabla f\|_{L^2(\Omega)},
\]

\[
\left\| g - \frac{1}{\lambda(\Omega)} \int_{\Omega} g \, d\Omega \right\|_{L^2(\Omega)} \leq C_p \|\nabla g\|_{L^2(\Omega)},
\]

(2.16)

for all \( f \in W^{1,2}(\Omega) \) such that \( f = 0 \) on \( \Gamma_1 \) in the sense of traces, and for all \( g \in W^{1,2}(\Omega) \).

There exists a variety of results concerning the continuous or compact embedding of Sobolev spaces into other Sobolev spaces or into Hölder spaces, some of which can be found e.g. in [33, chap. 11].

### 2.1.3 Bochner spaces

In the context of partial differential equations, we can view a function \( f(t,x) \) as a map from the time \( t \) to a function \( (f(t))(x) \) in an appropriate Sobolev space. In order to make this idea rigorous, the definition of Bochner spaces turns out beneficial. Let \( (X, \| \cdot \|_X) \) be a Banach space, let \((I, B, \nu)\) be a measure space, and \( 1 \leq p \leq \infty \). Then, we can define the vector space

\[
L^p(I, B, \nu; X) := \left\{ f : I \to X : f \text{ is measurable}, \|f\|_{L^p(I; X)} < \infty \right\},
\]

(2.17)

where we defined

\[
\| \cdot \|_{L^p(I; X)} : f \mapsto \left( \int_I \|f\|_X^p \, d\nu \right)^{\frac{1}{p}}.
\]

(2.18)

Note that the measurability in equation (2.17) is with respect to the Borel \( \sigma \)-algebra of the norm topology generated by \( \| \cdot \|_X \) on \( X \). Again, the map \( \| \cdot \|_{L^p(I; X)} \) defines a only a seminorm on \( L^p(I, B, \nu; X) \), for which reason one introduces the Bochner space \( L^p(I, B, \nu; X) \) as the quotient space which results when functions that are equal on null sets are identified with each other, just as for the Lebesgue spaces.

The dual space of \( L^p(I, B, \nu; X) \) is given by

\[
L^p(I, B, \nu; X)^* = L^q(I, B, \nu; X^*),
\]

(2.19)

where \( \frac{1}{p} + \frac{1}{q} = 1 \). Since \( X \) is a Banach space it follows that \( L^p(I, B, \nu; X) \) is a Banach space, too. Properties such as separability and reflexivity hold under the same conditions as for Lebesgue spaces, provided that the space \( X \) has the respective property as well.

Let \( \nu \) be the Lebesgue measure \( \lambda \). Then, one defines weak derivatives of Bochner functions similarly as for Lebesgue functions by replacing \( \Omega \) by \( I \), and the space \( L^1_{loc}(\Omega) \) by \( L^1_{loc}(I; X) \) for the functions in equation (2.7). We will use the Bochner spaces for the following setting: the domain \( I \) is a time interval \([0, T]\) and the Banach space \( X \) is a Lebesgue or Sobolev space. An important space that we will use in particular is the space
\[ L^2(0, T; X), \text{ where } X = L^2(\Omega) \text{ or } X = W^{1,2}(\Omega). \] This space is a Hilbert space with the inner product

\[ (f, g)_{L^2(0, T; X)} := \int_0^T (f(t), g(t))_X \, dt, \] (2.20)

since \( X \) is a Hilbert space.

In the course of this work, we will sometimes consider closed subspaces of Hilbert spaces in order to account for the boundary conditions – these subspaces are then Hilbert spaces, too, with the same inner product restricted to the subspace. Moreover, let us mention that since in our case the domain \( I = [0, T] \) is one-dimensional the space \( W^{1, p}(0, T; X) \) consists of time-continuous functions. To make this more precise, we define the space \( C(0, T; X) \) as the set of all measurable functions \( f : [0, T] \to X \) such that for any \( t \in [0, T] \) and any sequence \( \{ t_n \}_{n \in \mathbb{N}} \subset [0, T] \) with \( t_n \to t \) it holds

\[ \lim_{n \to \infty} \| f(t) - f(t_n) \|_X = 0. \] (2.21)

This space becomes a normed space with the norm

\[ \| x \|_{C(0, T; X)} : f \mapsto \max_{t \in [0, T]} \| f(t) \|_X. \] (2.22)

We state the following lemma:

**Lemma 2.1.2** Let \( 1 \leq p \leq \infty \) and \( f \in W^{1, p}(0, T; X) \). Then, there exists an \( \tilde{f} \in C(0, T; X) \) such that

\[ f = \tilde{f} \quad \text{a.e.} \] (2.23)

In this case, we will tacitly redefine the function \( f \) on a null set and write \( f \in C(0, T; X) \).

### 2.2 Theorems from functional analysis

We will state some useful theorems from functional analysis in what follows. For a finite-dimensional Euclidean space, the Bolzano–Weierstrass theorem ensures that every bounded sequence possesses a convergent subsequence. This theorem does not generalise to infinite-dimensional spaces, however, in reflexive Banach spaces, one obtains at least the existence of a weakly convergent subsequence by the following theorem:

**Theorem 2.2.1 (Eberlein–Smulian)** Let \( X \) be a reflexive Banach space and let \( \{ x_n \} \) be a bounded sequence in \( X \). Then, the following statements hold:

1. There exists a subsequence \( \{ x_{n_k} \} \) of \( \{ x_n \} \) and an \( x \in X \) such that

\[ x_{n_k} \rightharpoonup x \quad \text{weakly in } X \text{ as } k \to \infty. \] (2.24)

2. If every weakly convergent subsequence of \( \{ x_n \} \) has the same limit \( x \), then the entire sequence \( \{ x_n \} \) converges weakly to \( x \).

In some cases, it is possible to extract a strongly convergent subsequence in an \( L^p \) space by establishing uniform bounds for the sequence in a Sobolev norm. A classical compact embedding theorem providing this is:

**Theorem 2.2.2 (Rellich–Kondrachov)** Let \( \Omega \subset \mathbb{R}^d \) be an open, bounded Lipschitz domain. Then, the embedding

\[ W^{1, p}(\Omega) \hookrightarrow L^q(\Omega), \quad 1 \leq q < p^*, \] (2.25)

is compact for

\[ p^* = \begin{cases} \frac{dp}{d-p}, & \text{for } p < d, \\ \infty, & \text{for } p \geq d. \end{cases} \] (2.26)

The following compactness result for Bochner spaces from [4, 49] will prove useful:

**Theorem 2.2.3 (Aubin–Lions–Simon)** Let \( X, B, Y \) be Banach spaces such that \( X \) is compactly embedded in \( B \) and \( B \) is continuously embedded in \( Y \). Suppose \( \{ f_n \} \) is a sequence of functions in \( L^2(0, T; B) \), and assume that \( \{ f_n \} \) is bounded
in $L^2(0, T; X)$ and that the (weak) derivatives $\partial_t f_n$ are uniformly bounded in $L^2(0, T; Y)$. Then, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$f_{n_k} \rightarrow f, \quad \text{strongly in } L^2(0, T; B).$$

(2.27)

In order to handle products of (weakly) convergent sequences, we state the following result:

**Theorem 2.2.4 (Strong-weak convergence)** Let $1 < p < \infty$, let $\{f_n\}$ be a sequence in $L^p(\Omega)$ and $\{g_n\}$ a sequence in $L^{p'}(\Omega)$ for $\frac{1}{p} + \frac{1}{p'} = 1$. Suppose that

$$f_n \rightharpoonup f, \quad \text{weakly in } L^p(\Omega),$$

$$g_n \rightarrow g, \quad \text{strongly in } L^{p'}(\Omega).$$

Then, we have

$$f_n g_n \rightharpoonup fg, \quad \text{weakly in } L^1(\Omega).$$

(2.29)

2.3 Inequalities

We continue with some fundamental inequalities which will be harnessed to obtain estimates in the subsequent chapters:

**Lemma 2.3.1 (Triangle inequality)** Let $(X, \|\cdot\|)$ be a normed vector space. Then, it holds for all $x, y \in X$

$$\|x + y\| \leq \|x\| + \|y\|.$$

(2.30)

**Lemma 2.3.2 (Cauchy–Schwarz inequality)** Let $(X, (\cdot, \cdot))$ be an inner product space. Then, it holds for all $x, y \in X$

$$(x, y) \leq \|x\| \|y\|,$$

where $\|x\| := \sqrt{(x, x)}$ is the norm induced by the inner product.

**Lemma 2.3.3 (Young’s inequality)** Let $a$ and $b$ be non-negative real numbers. Then, we have for all $\varepsilon > 0$

$$ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}.$$

(2.32)

**Lemma 2.3.4 (Jensen’s inequality)** Let $f$ be an integrable function on a probability space $(X, A, \mu)$, and let $g$ be a real-valued function which is convex in the range of $f$. Then, it holds

$$g \left( \int_X f \, d\mu \right) \leq \int_X g \circ f \, d\mu.$$

(2.33)

2.4 Notations

We will briefly introduce notation which will be used in the subsequent chapters. Since the space $L^2$ is of outstanding importance, we set for a domain $\Omega \subset \mathbb{R}^d$ and a time interval $[0, T]$

$$\|\cdot\|_{\Omega} := \|\cdot\|_{L^2(\Omega)}, \quad \text{and} \quad \|\cdot\|_{\Omega^T} := \|\cdot\|_{L^2(0, T; L^2(\Omega))},$$

and

$$(\cdot, \cdot)_{\Omega} := (\cdot, \cdot)_{L^2(\Omega)}, \quad \text{and} \quad (\cdot, \cdot)_{\Omega^T} := (\cdot, \cdot)_{L^2(0, T; L^2(\Omega))}.$$

(2.34)

In view of the particular Problem $P$, we use the following conventions:

- $i$ is the index for the subdomain and takes values in $\{m_1, m_2, f\}$,
- $j$ is the index to specify the matrix block subdomain and takes values in $\{1, 2\}$,
- for functions $f$ which are the same in both matrix block subdomains (such as $\theta, K, \ldots$), we use the notation $f_{m_1} = f_{m_2} =: f_m$, thus allowing to write e.g. $\theta_i(\psi_i)$.

Moreover, $C \geq 0$ is a generic constant.
Chapter 3

Existence

This chapter is concerned with the existence of a (weak) solution to Problem $P$ for a fixed fracture width $\varepsilon > 0$. For the Richards equation considered on a single domain, several results concerning the existence and uniqueness of solutions exist, amongst others in the much-quoted article by Alt and Luckhaus [2]. For works discussing the existence of solutions, we mention [3, 52], and uniqueness results can be found in e.g. [41, 42]. Analysis for the full two-phase framework is carried out in e.g. [11, 12, 18, 30].

In view of the analysis, difficulties arise from the non-linearity and from the degeneracy of Richards’ equation. Degeneracy of Richards’ equation occurs in the slow diffusion case, that is when $K(\theta(\psi)) \to 0$, and in the fast diffusion case, when $\theta'(\psi) = 0$. Since we focus on the upscaling, we exclude degeneracy in this work, which will be expressed by the assumptions below.

In order to prove the existence of solutions of Problem $P$, we apply the Rothe method (for an introduction on the application of the Rothe method to time-dependent differential equations see e.g. [29]), and discretise the time derivative using an implicit Euler scheme. Under appropriate smoothness assumptions, the existence and uniqueness of a solution to the discretised problem follow from general works on coupled elliptic differential equations, see e.g. [27]. With the discrete solution at hand, time-interpolated functions can be defined, which are shown to converge towards a solution to Problem $P$ as the time step $\Delta t$ approaches zero. To do so, we make use of compactness arguments. Moreover, an $L^\infty$ estimate is proven for the time-discrete solution, which then translates to an $L^\infty$ estimate for the time-continuous solution to Problem $P$. We proceed in the spirit of [43], but observe that a linear partial differential equation is considered in the latter work.

For the sake of readability, we drop the superscript $\varepsilon$ since it is fixed throughout this chapter.

3.1 Assumptions

For the analysis, we assume in this work that the following conditions are satisfied:

$(A_{D_f})$ $f_i \in C(0, T; L^2(\Omega_i))$.

$(A_f)$ There exists $M_f > 0$ such that $|f_i| \leq M_f$ a.e. in $\Omega_i^T$.

$(A_{D_{\theta}})$ $\theta_i \in C^1(\mathbb{R})$.

$(\theta)$ There exist $m_{\theta}, M_{\theta} > 0$ such that $0 < m_{\theta} \leq \theta'_i(\psi_i) \leq M_{\theta}$ for all $\psi_i \in \mathbb{R}$.

$(A_{D_K})$ $K_i \in C^2(\mathbb{R})$ and $K_i'(\theta_i) > 0$ for all $\theta_i \in \mathbb{R}$.

$(A_{K})$ There exist $m_K, M_K > 0$ such that $0 < m_K \leq K_i(\theta_i) < M_K$ for all $\theta_i \in \mathbb{R}$.

$(A_I)$ There exists $M_I > 0$ such that $|\psi_{i,I}| \leq M_I$ a.e. in $\Omega_i$.

Remark 3.1.1 (Assumptions)

- The time continuity of the source term in Assumption $(A_{D_f})$ allows us to evaluate it at all times $t \in (0, T]$.

- Note that due to Assumption $(\theta)$, we consider the strict parabolic case only here as $\theta'_i(\psi_i)$ is bounded away from zero, which corresponds to a situation where the porous medium is unsaturated everywhere at all times because $\theta'_i(\psi_i) = 0$ in the saturated regime.
When applying the Kirchhoff transformation to Problem $P$, the flux term becomes linear at the cost of a non-linear Dirichlet condition at the interface. The latter non-linearity is required to be a $C^3$-function for the existence result in [27]. This regularity is provided by Assumption ($A_{D_j}$). However, the smoothness assumptions on the boundary of the domain in [27] are not satisfied in our setting, for which reason we can only show the existence of a weak solution to Problem $P$ if the existence of a weak solution to the time-discrete problem is assumed.

Assumption ($A_{K}$) excludes the slow diffusion case and guarantees the existence of a minimum positive permeability everywhere.

Moreover, for the sake of presentation, we make the assumption $\theta_i(0) = 0$, which can easily be achieved by redefining $\tilde{\theta}_i(u) = \theta_i(u) - \theta_i(0)$.

### 3.2 Weak solution

First, we establish a suitable notion of a solution to Problem $P$. For this purpose we define the function spaces

$$V_{m_j} := \{ u \in W^{1,2}(\Omega_{m_j}) : u = 0 \text{ on } \partial \Omega_{m_j} \setminus \Gamma_j \},$$

$$V_f := \{ u \in W^{1,2}(\Omega_f) : u = 0 \text{ on } \partial \Omega_f \setminus \Gamma \},$$

for $j \in \{1, 2\}$. Note that the function spaces depend on the fixed fracture width $\varepsilon$.

We obtain the definition of a weak solution by substituting Darcy’s law into the mass balance equation in Problem $P$ for every subdomain, multiplying with appropriate test functions and applying integration by parts. It is straightforward to show that every strong solution to Problem $P$ is also a weak solution, and conversely every weak solution which is sufficiently regular is a strong solution.

**Definition 3.2.1 (Weak solution)** A triple $(\psi_{m_1}, \psi_{m_2}, \psi_f) \in L^2(0, T; V_{m_1}) \times L^2(0, T; V_{m_2}) \times L^2(0, T; V_f)$ is called a weak solution to Problem $P$ if

$$\psi_{m_1} = \psi_f \text{ on } \Gamma_1 \quad \text{and} \quad \psi_{m_2} = \psi_f \text{ on } \Gamma_2 \quad \text{for a.e. } t \in [0, T],$$

in the sense of traces, and

$$-(\theta_m(\psi_{m_1}), \partial_t \phi_{m_1})_{\Omega_{m_1}} - (\theta_m(\psi_{m_2}), \partial_t \phi_{m_2})_{\Omega_{m_2}} - \varepsilon \left( \theta_f(\psi_f), \partial_t \phi_f \right)_{\Omega_f}$$

$$+ (K_m(\theta_m(\psi_{m_1})) \nabla \psi_{m_1}, \nabla \phi_{m_1})_{\Omega_{m_1}} + (K_m(\theta_m(\psi_{m_2})) \nabla \psi_{m_2}, \nabla \phi_{m_2})_{\Omega_{m_2}} + \varepsilon \left( K_f(\theta_f(\psi_f)) \nabla \psi_f, \nabla \phi_f \right)_{\Omega_f}$$

$$= (f_{m_1}, \phi_{m_1})_{\Omega_{m_1}} + (f_{m_2}, \phi_{m_2})_{\Omega_{m_2}} + \left( f_f, \phi_f \right)_{\Omega_f}$$

$$+ (\theta_m(\psi_{m_1}), \phi_{m_1}(0))_{\Omega_{m_1}} + (\theta_m(\psi_{m_2}), \phi_{m_2}(0))_{\Omega_{m_2}} + \varepsilon \left( \theta_f(\psi_f), \phi_f(0) \right)_{\Omega_f},$$

for all $(\phi_{m_1}, \phi_{m_2}, \phi_f) \in W^{1,2}(0, T; V_{m_1}) \times W^{1,2}(0, T; V_{m_2}) \times W^{1,2}(0, T; V_f)$ satisfying

$$\phi_{m_1} = \phi_f \text{ on } \Gamma_1 \quad \text{and} \quad \phi_{m_2} = \phi_f \text{ on } \Gamma_2 \quad \text{for a.e. } t \in [0, T],$$

and

$$\phi_i(T) = 0, \quad \text{for } i \in \{m_1, m_2, f\}.$$  

Note that in the above definition, it makes sense to evaluate the test functions $\phi_i$ at the time $t = T$ since the space $W^{1,p}(0, T; V_f)$ is embedded in $C([0, T]; V_f)$ due to Lemma 2.1.2.

Define in each subdomain the Kirchhoff transform by

$$K_i : \mathbb{R} \rightarrow \mathbb{R}, \quad u_i := K_i(\psi_i) = \int_0^{\psi_i} K_i(\theta_i(\varphi)) \, d\varphi.$$  

Due to the assumptions on $K_i$ and $\theta_i$, the Kirchhoff transformation is invertible and we can define the property

$$b_i(u_i) := \theta_i(K_i^{-1}(u_i)).$$
By the chain rule we have
\[ \nabla u_i = K_i(\partial_i(\psi_i)) \nabla \psi_i, \] (3.8)
which transforms Problem \( \mathcal{P} \) into a semi-linear problem. For an introduction on the Kirchhoff transform for Richards’ equation see e.g. [7]. Since \( K_i \) is Lipschitz continuous, the transformed problem is equivalent to the original problem [36]. Note in particular that \( \psi_i \equiv 0 \) if and only if \( u_i \equiv 0 \).

By definition of the Kirchhoff transform and Assumption (A\( K \)), we obtain bounds from both sides for the Kirchhoff transformed function and its derivative, provided that the function is sufficiently regular. The necessary regularity assumptions will be proven later.

**Lemma 3.2.1** For almost all \( t \in (0, T] \), the following estimates hold in each subdomain \( \Omega_i \):

\[
\begin{align*}
\| \psi_i(t) \|_{\Omega_i} &\leq \| u_i(t) \|_{\Omega_i} \leq M \| \psi_i(t) \|_{\Omega_i}, \\
\| \nabla \psi_i(t) \|_{\Omega_i} &\leq \| \nabla u_i(t) \|_{\Omega_i} \leq M \| \nabla \psi_i(t) \|_{\Omega_i}, \\
\| \partial_i \psi_i(t) \|_{\Omega_i} &\leq \| \partial_i u_i(t) \|_{\Omega_i} \leq M \| \partial_i \psi_i(t) \|_{\Omega_i}.
\end{align*}
\] (3.9)

Let \( u_{i,t} := \mathcal{K}(\psi_{i,t}) \). The weak formulation of the problem in terms of the Kirchhoff transformed variables writes as

**Definition 3.2.2** (Weak solution Kirchhoff transformed) A triple \((u_{m_1}, u_{m_2}, f) \in L^2(0, T; \mathcal{V}_{m_1}) \times L^2(0, T; \mathcal{V}_{m_2}) \times L^2(0, T; \mathcal{V}_f)\) is called a weak solution to the Kirchhoff transformed problem if

\[
\mathcal{K}_{m_1}^{-1}(u_{m_1}) = \mathcal{K}_{m_2}^{-1}(u_{m_2}) \text{ on } \Gamma_1 \quad \text{and} \quad \mathcal{K}^{-1}(f) \text{ on } \Gamma_2 \text{ for a.e. } t \in [0, T], \quad (3.10)
\]

in the sense of traces, and

\[
\begin{align*}
- (b_m(u_{m_1}), \partial_i \phi_{m_1})_{\Omega_{m_1}} &+ (b_m(u_{m_2}), \partial_i \phi_{m_2})_{\Omega_{m_2}} - \varepsilon^k \left( b_f(u_f), \partial_i \phi_f \right)_{\Omega_f} \\
+ \langle \nabla u_{m_1}, \nabla \phi_{m_1} \rangle_{\Omega_{m_1}} &+ \langle \nabla u_{m_2}, \nabla \phi_{m_2} \rangle_{\Omega_{m_2}} + \varepsilon^k \left( \nabla u_f, \nabla \phi_f \right)_{\Omega_f} \\
= (f_{m_1}, \phi_{m_1})_{\Omega_{m_1}} &+ (f_{m_2}, \phi_{m_2})_{\Omega_{m_2}} + \left( f_f, \phi_f \right)_{\Omega_f} \\
+ (b_m(u_{m_1, f}), \phi(0))_{\Omega_{m_1}} &+ (b_m(u_{m_2, f}), \phi(0))_{\Omega_{m_2}} + \varepsilon^k \left( b_f(u_f, \phi), \phi_f \right)_{\Omega_f}
\end{align*}
\] (3.11)

for all \((\phi_{m_1}, \phi_{m_2}, \phi_f) \in W^{1,2}(0, T; \mathcal{V}_{m_1}) \times W^{1,2}(0, T; \mathcal{V}_{m_2}) \times W^{1,2}(0, T; \mathcal{V}_f)\) satisfying

\[
\phi_{m_1} = \phi_f \text{ on } \Gamma_1 \quad \text{and} \quad \phi_{m_2} = \phi_f \text{ on } \Gamma_2 \text{ for a.e. } t \in [0, T], \quad (3.12)
\]

and

\[
\phi_i(T) = 0 \quad \text{for } i \in \{m_1, m_2, f\}. \quad (3.13)
\]

The advantage of the Kirchhoff transformed formulation is the linear flux term, which comes at the cost of a non-linear interface condition. We will use either formulation depending on what seems more suitable for a specific purpose.

### 3.3 Time discretisation

In what follows, we discretise the problem in time using an implicit Euler approach, which gives elliptic equations at every discrete time \( t_k = k \Delta t \), for \( k \in \{0, \ldots, N\} \), where \( N \in \mathbb{N} \). We assume without loss of generality that \( N \Delta t = T \). Here, \( \Delta t > 0 \) denotes the fixed time step. Choose \( \psi^0_i = \psi_{i,1} \) and let the sequence of solutions to the time-discrete problem be given as \( \{\psi^k_i\} \). Let moreover \( f^k_i := f_i(t_k) \). We define the notion of a weak solution to the time-discrete problem in the original formulation — the definition of a weak solution to the Kirchhoff transformed problem is analogous and one takes \( u^k_i = \mathcal{K}(\psi^k_{i,1}) \) initially.

**Definition 3.3.1** Let \( k > 0 \) and let \((\psi^k_{m_1}, \psi^k_{m_2}, \psi^k_f) \in \mathcal{V}_{m_1} \times \mathcal{V}_{m_2} \times \mathcal{V}_f \) be given. We call \((\psi^k_{m_1}, \psi^k_{m_2}, \psi^k_f) \in \mathcal{V}_{m_1} \times \mathcal{V}_{m_2} \times \mathcal{V}_f \) a weak solution to the time-discrete problem at time \( t_k \) if it satisfies

\[
\psi^k_{m_1} = \psi^k_f \text{ on } \Gamma_1 \quad \text{and} \quad \psi^k_{m_2} = \psi^k_f \text{ on } \Gamma_2 \quad (3.14)
\]
in the sense of traces, and
\[
\left( \theta_m(\psi_{m1}^k), \phi_{m1} \right)_{\Omega_{m1}} + \left( \theta_m(\psi_{m2}^k), \phi_{m2} \right)_{\Omega_{m2}} + \varepsilon^k \left( \theta_f(\psi_f^k), \phi_{m1} \right)_{\Omega_f} \\
+ \Delta t \left( K_m(\theta_m(\psi_{m1}^k)) \nabla \psi_{m1}^k, \nabla \phi_{m1} \right)_{\Omega_{m1}} + \Delta t \left( K_m(\theta_m(\psi_{m2}^k)) \nabla \psi_{m2}^k, \nabla \phi_{m2} \right)_{\Omega_{m2}} + \varepsilon^k \Delta t \left( K_f(\theta_f(\psi_f^k)) \nabla \psi_f^k, \nabla \phi_{f1} \right)_{\Omega_f} \\
= \Delta t \left( f_{m1}^k, \phi_{m1} \right)_{\Omega_{m1}} + \Delta t \left( f_{m2}^k, \phi_{m2} \right)_{\Omega_{m2}} + \Delta t \left( f_{f}^k, \phi_{f1} \right)_{\Omega_f} \\
+ \left( \theta_m(\psi_{m1}^{k-1}), \phi_{m1} \right)_{\Omega_{m1}} + \left( \theta_m(\psi_{m2}^{k-1}), \phi_{m2} \right)_{\Omega_{m2}} + \varepsilon^k \left( \theta_f(\psi_f^{k-1}), \phi_{m1} \right)_{\Omega_f},
\]
for all \((\phi_{m1}, \phi_{m2}, \phi_f) \in \mathcal{V}_{m1} \times \mathcal{V}_{m2} \times \mathcal{V}_f\) satisfying \(\phi_{m1} = \phi_f\) on \(\Gamma_f\) for \(j \in \{1, 2\} \).

### 3.4 A priori estimates

Define in each domain the energy functional
\[
\mathcal{W}_i(\psi_i) = \int_0^{\psi_i} \theta'_i(\phi) \phi \ d\phi,
\]
which we will require in the proof of the following a priori estimate. First, we gather some properties of \(\mathcal{W}_i\) in a simple lemma, which is based on Assumption \((A_0)\):

**Lemma 3.4.1** The function \(\mathcal{W}_i\) satisfies the following inequalities:
\[
\mathcal{W}_i(\psi_i) \geq 0, \\
\mathcal{W}_i(\psi_i) - \mathcal{W}_i(\zeta_i) \leq \psi_i(\theta_i(\psi_i) - \theta_i(\zeta_i)), \\
m_\theta \frac{\psi_i^2}{2} \leq \mathcal{W}_i(\psi_i) \leq M_\theta \frac{\psi_i^2}{2},
\]
for all \(\psi_i, \zeta_i \in \mathbb{R}\).

**Proof.** The non-negativity follows immediately from \(\theta'_i(\psi_i) > 0\) due to Assumption \((A_0)\): for \(\psi_i \geq 0\), we have
\[
\mathcal{W}_i(\psi_i) = \int_0^{\psi_i} \theta'_i(\phi) \phi \ d\phi \geq 0,
\]
and in case \(\psi_i < 0\)
\[
\mathcal{W}_i(\psi_i) = -\int_{\psi_i}^{0} \theta'_i(\phi) \phi \ d\phi \geq 0.
\]
The second inequality holds because of
\[
\mathcal{W}_i(\psi_i) - \mathcal{W}_i(\zeta_i) = \int_{\psi_i}^{\zeta_i} \theta'_i(\phi) \phi \ d\phi \leq \psi_i \int_{\psi_i}^{\zeta_i} \theta'_i(\phi) \ d\phi = \psi_i(\theta_i(\psi_i) - \theta_i(\zeta_i)),
\]
for \(\psi_i \geq \zeta_i\) and
\[
\mathcal{W}_i(\psi_i) - \mathcal{W}_i(\zeta_i) = -\int_{\zeta_i}^{\psi_i} \theta'_i(\phi) \phi \ d\phi \leq -\psi_i \int_{\zeta_i}^{\psi_i} \theta'_i(\phi) \ d\phi = \psi_i(\theta_i(\psi_i) - \theta_i(\zeta_i)),
\]
for \(\psi_i < \zeta_i\). Finally, we have
\[
m_\theta \frac{\psi_i^2}{2} = m_\theta \int_0^{\psi_i} \phi \ d\phi \leq \int_0^{\psi_i} \theta'_i(\phi) \phi \ d\phi = \mathcal{W}_i(\psi_i) \leq M_\theta \int_0^{\psi_i} \phi \ d\phi = M_\theta \frac{\psi_i^2}{2}.
\]

We obtain the following estimate for the time-discrete solution:
Lemma 3.4.2 (A priori estimate) The solution \((\psi_{m_1}^k, \psi_{m_2}^k, \psi_f^k)\) to the time-discrete problem in Definition 3.3.1 satisfies

\[
\begin{align*}
\max_{l \in \{0, \ldots, N\}} \int_{\Omega_{m_1}} \mathcal{W}_m(\psi_{m_1}^l) \, dx + \max_{l \in \{0, \ldots, N\}} \int_{\Omega_{m_2}} \mathcal{W}_m(\psi_{m_2}^l) \, dx + \varepsilon^k \max_{l \in \{0, \ldots, N\}} \int_{\Omega_f} \mathcal{W}_f(\psi_f^l) \, dx \\
+ \frac{\Delta t m_K}{2} \sum_{k=1}^{N} ||\nabla \psi_{m_1}^k||_{\Omega_{m_1}}^2 + \frac{\Delta t m_K}{2} \sum_{k=1}^{N} ||\nabla \psi_{m_2}^k||_{\Omega_{m_2}}^2 + \varepsilon \frac{\Delta t m_K}{2} \sum_{k=1}^{N} ||\nabla \psi_f^k||_{\Omega_f}^2 \\
\leq \int_{\Omega_{m_1}} \mathcal{W}_m(\psi_{m_1,l}) \, dx + \int_{\Omega_{m_2}} \mathcal{W}_m(\psi_{m_2,l}) \, dx + \varepsilon \int_{\Omega_f} \mathcal{W}_f(\psi_{f,l}) \, dx \\
+ \frac{\Delta t C_{p_{m_1}}}{2m_K} \sum_{k=1}^{N} ||f_{m_1}^k||_{\Omega_{m_1}}^2 + \frac{\Delta t C_{p_{m_2}}}{2m_K} \sum_{k=1}^{N} ||f_{m_2}^k||_{\Omega_{m_2}}^2 + \varepsilon \frac{\Delta t C_{p_f}}{2m_K} \sum_{k=1}^{N} ||f_f^k||_{\Omega_f}^2.
\end{align*}
\tag{3.18}
\]

(3.18)

\[\text{Proof.}\] We test equation (3.15) with the triple \((\phi_{m_1}, \phi_{m_2}, \phi_f) = (\psi_{m_1}^k, \psi_{m_2}^k, \psi_f^k)\), which yields

\[\left(\theta_m(\phi_{m_1}^k) - \theta_m(\phi_{m_1}^{k-1}), \theta_m(\phi_{m_1}^k), \phi_{m_1}^k\right)_{\Omega_{m_1}} + \left(\theta_m(\phi_{m_2}^k), \theta_m(\phi_{m_2}^{k-1}), \phi_{m_2}^k\right)_{\Omega_{m_2}} + \varepsilon^k \left(\theta_f(\phi_f^k) - \theta_f(\phi_f^{k-1}), \phi_f^k\right)_{\Omega_f}
\]

\[+ \Delta t (K_m(\theta_m(\phi_{m_1}^k)) \nabla \phi_{m_1}^k, \nabla \phi_{m_1}^k)_{\Omega_{m_1}} + \Delta t (K_m(\theta_m(\phi_{m_2}^k)) \nabla \phi_{m_2}^k, \nabla \phi_{m_2}^k)_{\Omega_{m_2}} + \varepsilon \Delta t (K_f(\theta_f(\phi_f^k)) \nabla \phi_f^k, \nabla \phi_f^k)_{\Omega_f}
\]

\[= \Delta t (f_{m_1}^k, \phi_{m_1}^k)_{\Omega_{m_1}} + \Delta t (f_{m_2}^k, \phi_{m_2}^k)_{\Omega_{m_2}} + \Delta t (f_f^k, \phi_f^k)_{\Omega_f}.
\]

Poincaré’s inequality gives

\[||\nabla \phi_f^k||_{\Omega_f}^2 \geq \frac{||\nabla \phi_f^k||_{\Omega_f}^2}{2} + \frac{||\phi_f^k||_{\Omega_f}^2}{2C_{p_f}},\]

for \(i \in \{m_1, m_2, f\}\), where \(C_{p_i} > 0\) denotes the Poincaré constant of the respective subdomains. Note that the geometries of \(\Omega_{m_1}\) and \(\Omega_{m_2}\) are the same, and so are the Poincaré constants hence, for which reason we set \(C_{p_{m_1}} := C_{p_{m_2}} = C_{p_f}\).

Making use of this together with Assumption \((A_K)\) and equation (3.17)2 in Lemma 3.4.1, we estimate

\[\sum_{j=1}^{2} \int_{\Omega_{m_1}} \mathcal{W}_m(\phi_{m_1}^k) \, dx + \varepsilon \int_{\Omega_f} \mathcal{W}_f(\phi_f^k) \, dx + \frac{\Delta t m_K}{2} \sum_{j=1}^{2} ||\nabla \phi_{m_1}^k||_{\Omega_{m_1}}^2 + \varepsilon \frac{\Delta t m_K}{2} ||\nabla \phi_f^k||_{\Omega_f}^2
\]

\[+ \frac{\Delta t m_K}{2} \sum_{j=1}^{2} ||\phi_{m_1}^k||_{\Omega_{m_1}}^2 + \varepsilon \frac{\Delta t m_K}{2} ||\phi_f^k||_{\Omega_f}^2
\]

\[\leq \sum_{j=1}^{2} \int_{\Omega_{m_1}} \mathcal{W}_m(\phi_{m_1}^{k-1}) \, dx + \varepsilon \int_{\Omega_f} \mathcal{W}_f(\phi_f^{k-1}) \, dx + \Delta t \sum_{j=1}^{2} (f_{m_1}^k, \phi_{m_1}^k)_{\Omega_{m_1}} + \Delta t (f_f^k, \phi_f^k)_{\Omega_f}
\]

\[\leq \sum_{j=1}^{2} \int_{\Omega_{m_1}} \mathcal{W}_m(\phi_{m_1}^{k-1}) \, dx + \varepsilon \int_{\Omega_f} \mathcal{W}_f(\phi_f^{k-1}) \, dx + \Delta t \sum_{j=1}^{2} (f_{m_1}^k, \phi_{m_1}^k)_{\Omega_{m_1}} + \Delta t \sum_{j=1}^{2} ||f_{m_1}^k||_{\Omega_{m_1}}^2 + \varepsilon \frac{\Delta t C_{p_{m_1}}}{2m_K} ||f_{m_1}^k||_{\Omega_{m_1}}^2
\]

\[+ \frac{\Delta t m_K}{2} \sum_{j=1}^{2} ||\phi_{m_1}^{k-1}||_{\Omega_{m_1}}^2 + \varepsilon \frac{\Delta t m_K}{2} ||\phi_f^{k-1}||_{\Omega_f}^2
\]

\[+ \frac{\Delta t m_K}{2} \sum_{j=1}^{2} ||\phi_{m_1}^{k-1}||_{\Omega_{m_1}}^2 + \varepsilon \frac{\Delta t m_K}{2} ||\phi_f^{k-1}||_{\Omega_f}^2,
\]

where we applied the Cauchy–Schwarz inequality and Young’s inequality. Summing over \(k\) from 1 to \(l\) for \(1 \leq l \leq N\) leaves us with

\[\sum_{j=1}^{2} \int_{\Omega_{m_1}} \mathcal{W}_m(\phi_{m_1}^j) \, dx + \varepsilon \int_{\Omega_f} \mathcal{W}_f(\phi_f^j) \, dx + \frac{\Delta t m_K}{2} \sum_{j=1}^{l} ||\nabla \phi_{m_1}^j||_{\Omega_{m_1}}^2 + \varepsilon \frac{\Delta t m_K}{2} \sum_{j=1}^{l} ||\nabla \phi_f^j||_{\Omega_f}^2
\]

\[\leq \sum_{j=1}^{2} \int_{\Omega_{m_1}} \mathcal{W}_m(\phi_{m_1,l}) \, dx + \varepsilon \int_{\Omega_f} \mathcal{W}_f(\phi_{f,l}) \, dx + \Delta t \sum_{j=1}^{l} (f_{m_1}^j, \phi_{m_1}^j)_{\Omega_{m_1}} + \Delta t \sum_{j=1}^{l} ||f_{m_1}^j||_{\Omega_{m_1}}^2 + \varepsilon \frac{\Delta t C_{p_{m_1}}}{2m_K} ||f_{m_1}^j||_{\Omega_{m_1}}^2
\]

which finishes the proof.

\[\square\]

Remark 3.4.1 (Non-degenerate case) Note that we did not use Assumption \((A_\theta)\) in the proof above. Here in the strictly parabolic case, where we have an \(m_\theta > 0\) such that \(0 < m_\theta \leq \theta_f(\psi)\) and all \(\psi \in \mathbb{R}\), we immediately
obtain an $L^2$ bound for $\phi_i^k$ from the first two terms in Lemma 3.4.2 by Lemma 3.4.1:

$$\int_{\Omega_{mk}} \mathcal{W}_m(\phi_{mk}^i) \, dx \geq m_0 \frac{\|\phi_{mk}^i\|_{L^2}^2}{2}, \quad \text{and} \quad \int_{\Omega_f} \mathcal{W}_f(\phi_{mk}^i) \, dx \geq m_0 \frac{\|\phi_{mk}^i\|_{L^2}^2}{2}.$$ 

\section{3.5 Interpolation in time}

Now, we define functions on a continuous time domain by interpolating the solutions of the time-discrete problem in time. Since Richards’ equation contains the time derivative of the water content, we bound the time derivative of the water content and transfer the results later to the pressure head. In order to obtain time-continuous functions, we introduce piecewise linearly interpolated functions in addition to the piecewise constant functions: for almost every $t \in (t_{k-1}, t_k)$ set

$$\bar{\Psi}_{\Delta t}^i(t) := \phi_i^k,$$

$$\bar{\Theta}_{\Delta t}^i(t) := \theta(\phi_i^k),$$

$$\bar{\Theta}_{\Delta t}^i(t) := \theta(\phi_i^{k-1}) + \frac{t - t_{k-1}}{\Delta t} (\theta(\phi_i^k) - \theta(\phi_i^{k-1})). \quad (3.19)$$

Moreover, we need the piecewise constant interpolation of the source term

$$f_{\Delta t}^i(t) = f_i^k. \quad (3.20)$$

In view of the a priori estimate in Lemma 3.4.2, we obtain the following result for the interpolated functions:

**Lemma 3.5.1** For each $\varepsilon > 0$ fixed, the functions $\Psi_{\Delta t}^i$, $\Theta_{\Delta t}^i$, and $\bar{\Theta}_{\Delta t}^i$ are bounded uniformly with respect to $\Delta t$ in $L^\infty(0, T; L^2(\Omega_t)) \cap L^2(0, T; V_t)$.

**Proof.** We present the proof for the piecewise linear functions $\bar{\Theta}_{\Delta t}^i$, the proof for $\bar{\Theta}_{\Delta t}^i$ and $\Psi_{\Delta t}^i$ is similar and easier. The first statement follows from the a priori estimate in Lemma 3.4.2 from

$$\text{ess sup}_{t \in (0, T)} \|\bar{\Theta}_{\Delta t}^i(t)\|_{\Omega_t} = \max_{k \in \{0, \ldots, N\}} \|\theta(\phi_i^k)\|_{\Omega_t} \leq M_\theta \max_{k \in \{0, \ldots, N\}} \|\phi_i^k\|_{\Omega_t} \leq C(\varepsilon, T),$$

where we used the Lipschitz continuity of $\theta$, and the technical assumption $\theta(0) = 0$. Notice that $C(\varepsilon, T)$ does in particular not depend on $\Delta t$ due to the time-continuity of the source term $f_i$.

Moreover, we estimate

$$\|\bar{\Theta}_{\Delta t}^i\|_{L^2(0, T; \Omega_t)}^2 = \int_0^T \|\bar{\Theta}_{\Delta t}^i(t)\|_{\Omega_t}^2 \, dt \leq \max_{k \in \{0, \ldots, N\}} \|\theta(\phi_i^k)\|_{\Omega_t}^2 T \leq C(\varepsilon, T),$$

and

$$\|\nabla \bar{\Theta}_{\Delta t}^i\|_{L^2(0, T; \Omega_t)}^2 \leq M_\theta^2 \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \left( \|\nabla \phi_i^{k-1}\|_{\Omega_t} + \|\nabla \phi_i^k\|_{\Omega_t} \right)^2 \, dt \leq 2M_\theta^2 \Delta t \sum_{k=1}^N \left( \|\nabla \phi_i^{k-1}\|_{\Omega_t}^2 + \|\nabla \phi_i^k\|_{\Omega_t}^2 \right) \leq C(\varepsilon, T),$$

where we used Assumption $(A_\theta)$, the triangle inequality, $(a + b)^2 \leq 2(a^2 + b^2)$, and Lemma 3.4.2. \hfill $\Box$

By the Eberlein–Šmulian theorem 2.2.1, Lemma 3.5.1 gives rise to the existence of a $\Psi_i \in L^2(0, T; V_t)$ and a subsequence of $\Delta t \to 0$ along which we obtain the weak convergence

$$\{\Psi_{\Delta t}^i\}_{\Delta t} \to \Psi_i \quad \text{in} \ L^2(0, T; V_t). \quad (3.21)$$

In order to get strong convergence in $L^2(0, T; L^2(\Omega_t))$, which we require to show the convergence of the nonlinear function $\theta(\Psi_{\Delta t}^i)$ to the desired limit function as $\Delta t \to 0$, we need the following estimate for the time derivative of the water content:
Lemma 3.5.2 For each $\varepsilon > 0$, there exists a constant $C = C(\varepsilon, T)$ such that
\[ \|\partial_t \hat{\Theta}_i^{\Delta t}\|_{L^2(0,T;W^{-1,2}(\Omega_i))} \leq C. \] (3.22)

Proof. Since the function $\hat{\Theta}_i^{\Delta t}(t)$ is piecewise linear, its weak time derivative exists, is piecewise constant, and for almost every $t \in (t_{k-1}, t_k]$ given by
\[ \partial_t \hat{\Theta}_i^{\Delta t}(t) = \frac{\theta_i(\psi_i^k) - \theta_i(\psi_i^{k-1})}{\Delta t}. \]

We view $\partial_t \hat{\Theta}_i^{\Delta t}$ as an element of $L^2(0,T;W^{-1,2}(\Omega_i))$, where $W^{-1,2}(\Omega_i)$ is the dual of $W_0^{1,2}(\Omega_i)$ (the latter space containing the $W^{1,2}$-functions on $\Omega_i$ with vanishing trace on the entire boundary $\partial \Omega_i$). This is clearly possible due to the natural inclusion $L^2(\Omega_i) \hookrightarrow W^{-1,2}(\Omega_i)$ given by $u \mapsto \langle u, \cdot \rangle_{\Omega_i}$.

Testing equation (3.15) with arbitrary $\phi_i \in W_0^{1,2}(\Omega_i)$ and $\phi_s \equiv 0$ for $s \neq i$ gives the estimate
\[ \left\| \left( \partial_t \hat{\Theta}_i^{\Delta t}(t), \phi_i \right)_{W^{-1,2}(\Omega_i),W_0^{1,2}(\Omega_i)} \right\| \leq \left\| K_i(\theta_i(\psi_i^k)) \nabla \psi_i^k, \nabla \phi_i \right\|_{\Omega_i} + \left\| f_i^k, \phi_i \right\|_{\Omega_i} \leq \|\phi_i\|_{W^{1,2}(\Omega_i)} \left( M_i \|\nabla \psi_i^k\|_{L^2} + \|f_i^k\|_{L^2} \right). \]

Using the a priori estimate in Lemma 3.4.2, we obtain
\[ \|\partial_t \hat{\Theta}_i^{\Delta t}\|_{L^2(0,T;W^{-1,2}(\Omega_i))} \leq C(\varepsilon, T), \]
which finishes the proof. \( \square \)

The strong convergence of the piecewise linear functions $\hat{\Theta}_i^{\Delta t}$ in $L^2$ will be provided by the Aubin–Lions-Simon theorem 2.2.3:

Lemma 3.5.3 There exists a $\Theta_i \in L^2(0,T;L^2(\Omega_i))$ and a subsequence $\Delta t \to 0$ along which we get
\[ \{\hat{\Theta}_i^{\Delta t}\}_{\Delta t} \to \Theta_i \quad \text{in} \quad L^2(0,T;L^2(\Omega_i)). \] (3.23)

Proof. In view of the estimate in Lemma 3.5.2 for the time derivative, the boundedness of $\{\hat{\Theta}_i^{\Delta t}\}_{\Delta t}$ in $L^2(0,T;\mathcal{V}_i)$ and Lemma 3.5.1, the Aubin–Lions–Simon theorem 2.2.3 provides the existence of a subsequence $\Delta t \to 0$ along which $\hat{\Theta}_i^{\Delta t} \to \Theta_i$ in $L^2(0,T;L^2(\Omega_i))$ by choosing $X = \mathcal{V}_i, B = L^2(\Omega_i), Y = W^{-1,2}(\Omega_i)$. Note that $L^2(\Omega_i)$ is continuously embedded in $W^{-1,2}(\Omega_i)$, and $\mathcal{V}_i$ is compactly embedded in $L^2(\Omega_i)$ by Theorem 2.2.2. \( \square \)

The following lemma is taken from [32] and transfers the strong convergence result to the piecewise constantly interpolated function:

Lemma 3.5.4 Let $X$ be a Hilbert space. Then, the strong convergence
\[ \{\hat{\Theta}_i^{\Delta t}\}_{\Delta t} \to \Theta_i \quad \text{in} \quad L^2(0,T;X) \]
implies the strong convergence
\[ \{\hat{\Theta}_i^{\Delta t}\}_{\Delta t} \to \Theta_i \quad \text{in} \quad L^2(0,T;X). \]

From this, we obtain immediately

Lemma 3.5.5 Along a subsequence $\Delta t \to 0$, we get
\[ \{\hat{\Theta}_i^{\Delta t}\}_{\Delta t} \to \Theta_i \quad \text{in} \quad L^2(0,T;L^2(\Omega_i)). \] (3.24)

It remains to carry over this strong convergence result for the water content to the pressure head. This is done in the following way: due to Assumption $(A_d)$, the inverse function $\theta_i^{-1}$ exists and is continuous. Note that by definition of the piecewise constantly interpolated functions, we have
\[ \Psi_i^{\Delta t} = \theta_i^{-1}(\hat{\Theta}_i^{\Delta t}). \] (3.25)
From the strong convergence in equation (3.24) and the Lipschitz continuity of $\theta^{-1}_i$, we deduce:

**Lemma 3.5.6** Along a subsequence $\Delta t \to 0$, we get

$$\{\Psi_{i,\Delta t}\} \to \Psi_i \quad \text{in } L^2(0, T; L^2(\Omega)).$$

(3.26)

**Remark 3.5.1** Applying the previous considerations to the weakly convergent subsequence in equation (3.21), one obtains actually a subsubsequence (still denoted as $\{\Psi_{i,\Delta t}\}$ with a slight abuse of notation), which converges strongly in $L^2(0, T; L^2(\Omega))$ and weakly in $L^2(0, T; V_i)$. The limit function $\Psi_i$ in equation (3.26) is indeed the same as the limit function in equation (3.21) for which the same notation has been used: since $\{\Psi_{i,\Delta t}\} \to \Psi_i$ in $L^2(0, T; V_i)$, this weak convergence holds true in particular in $L^2(0, T; L^2(\Omega))$, and as strong convergence implies weak convergence to the same limit function, the two limits are identical by the uniqueness of the weak limit. From now on, we omit these standard arguments.

It remains to show that the limit functions are a weak solution:

**Theorem 3.5.1** The limit $(\Psi_{m_1}, \Psi_{m_2}, \Psi_f)$ is a weak solution to Problem $\mathcal{P}$ in the sense of Definition 3.2.1.

*Proof.* Let $t \in (t_{k-1}, t_k)$ and $(\phi_{m_1}, \phi_{m_2}, \phi_f) \in V_{m_1} \times V_{m_2} \times V_f$. Summing equation (3.15) from 1 to $k$ yields

$$\sum_{j=1}^{k} \left( \theta_m(\Psi_{m_j}(t), \phi_{m_j}(t)) \right)_{\Omega_{m_j}} + \epsilon \int_{0}^{t} \left( \theta_f(\Psi_{f,j}(t), \phi_f(t)) \right)_{\Omega_f} - \sum_{j=1}^{k} \left( \theta_m(\Psi_{m,\Delta t}(t), \phi_{m_j}(t)) \right)_{\Omega_{m_j}} - \epsilon \int_{0}^{t} \left( \theta_f(\phi_{f,j}(t), \phi_f(t)) \right)_{\Omega_f}$$

$$+ \sum_{j=1}^{k} \int_{0}^{t} \left( \theta_m(\Psi_{m,\Delta t}(t), \phi_{m_j}(t)) \right)_{\Omega_{m_j}} \frac{d\tau}{d\tau} + \epsilon \int_{0}^{t} \left( \theta_f(\Psi_{f,\Delta t}(t), \phi_f(t)) \right)_{\Omega_f} \frac{d\tau}{d\tau}$$

$$= \sum_{j=1}^{k} \int_{0}^{t} \left( \theta_m(\Psi_{m,\Delta t}(t), \phi_{m_j}(t)) \right)_{\Omega_{m_j}} \frac{d\tau}{d\tau} - \epsilon \int_{0}^{t} \left( \theta_f(\phi_{f,\Delta t}(t), \phi_f(t)) \right)_{\Omega_f} \frac{d\tau}{d\tau}$$

The terms on the right hand side correct the error made on the left hand side by integrating to $t$ instead of $t_k$. Now, we choose test functions $(\phi_{m_1}, \phi_{m_2}, \phi_f) \in V_{m_1} \times V_{m_2} \times V_f$ fulfilling $\phi_{m_j} = \phi_f$ on $\Gamma_f$ and integrate in time from 0 to $T$ to get

$$\sum_{j=1}^{k} \int_{0}^{T} \left( \theta_m(\Psi_{m_j}(t), \phi_{m_j}(t)) \right)_{\Omega_{m_j}} dt + \epsilon \int_{0}^{T} \left( \theta_f(\Psi_{f,j}(t), \phi_f(t)) \right)_{\Omega_f} dt$$

$$- \sum_{j=1}^{k} \int_{0}^{T} \left( \theta_m(\phi_{m_j}, \phi_{m_j}(t)) \right)_{\Omega_{m_j}} dt - \epsilon \int_{0}^{T} \left( \theta_f(\phi_{f,j}, \phi_f(t)) \right)_{\Omega_f} dt$$

$$+ \epsilon \int_{0}^{T} \int_{0}^{t} \left( \theta_m(\Psi_{m,\Delta t}(t), \phi_{m_j}(t)) \right)_{\Omega_{m_j}} d\tau dt$$

$$- \sum_{j=1}^{k} \int_{0}^{T} \int_{0}^{t} \left( \theta_m(\phi_{m_j}, \phi_{m_j}(t)) \right)_{\Omega_{m_j}} d\tau dt - \int_{0}^{T} \int_{0}^{t} \left( \phi_f(t), \phi_f(t) \right)_{\Omega_f} d\tau dt$$

$$= \sum_{j=1}^{k} \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \left( \theta_m(\Psi_{m_j}(t), \phi_{m_j}(t)) \right)_{\Omega_{m_j}} dt + \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \int_{0}^{t_l} \left( \phi_f(t), \phi_f(t) \right)_{\Omega_f} d\tau dt$$

(3.3)}
From the strong $L^2$ convergence from equation (3.24) and the Lipschitz continuity of $\theta_m$, we infer that

$$
\int_0^T \left( \theta_m(\Psi_{m,t}(t)), \phi_{m,t}(t) \right)_{\Omega_{m,t}} dt \to \int_0^T \left( \bar{\theta}_m(\Psi_{m,t}(t)), \phi_{m,t}(t) \right)_{\Omega_{m,t}} dt.
$$

Furthermore, we have to show that

$$
\int_0^T \int_0^1 \left( K_m(\theta_m(\Psi_{m,t}(t))) \nabla \Psi_{m,t}(\tau), \nabla \phi_{m,t}(t) \right)_{\Omega_{m,t}} d\tau dt \to \int_0^T \int_0^1 \left( K_m(\bar{\theta}_m(\Psi_{m,t}(t))) \nabla \Psi_{m,t}(\tau), \nabla \phi_{m,t}(t) \right)_{\Omega_{m,t}} d\tau dt.
$$

This follows from the following considerations: due to the boundedness of $K_m$, we know that there exists a $\bar{\xi}_{m,t} \in \left( L^2(\Omega_{m,t}) \right)^d$ such that

$$
K_m(\theta_m(\Psi_{m,t}(\tau))) \nabla \Psi_{m,t}(\tau) \to \bar{\xi}_{m,t}, \quad \text{weakly in} \left( L^2(\Omega_{m,t}) \right)^d.
$$

Due to the finite measure of $\Omega_{m,t}$, this also implies

$$
K_m(\theta_m(\Psi_{m,t}(\tau))) \nabla \Psi_{m,t}(\tau) \to \bar{\xi}_{m,t}, \quad \text{weakly in} \left( L^1(\Omega_{m,t}) \right)^d.
$$

From the Strong-weak convergence theorem 2.2.4, we infer that

$$
K_m(\theta_m(\Psi_{m,t}(\tau))) \nabla \Psi_{m,t}(\tau) \to K_m(\theta_m(\Psi_{m,t}(\tau))) \nabla \Psi_{m,t}(\tau), \quad \text{weakly in} \left( L^1(\Omega_{m,t}) \right)^d,
$$

in view of the strong convergence from equation (3.24), the Lipschitz continuity of $\theta_m$ and $K_m$, and the weak convergence from equation (3.21). From the uniqueness of the weak limit in $\left( L^1(\Omega_{m,t}) \right)^d$, we deduce that

$$
\bar{\xi}_{m,t} = K_m(\theta_m(\Psi_{m,t}(\tau))) \nabla \Psi_{m,t}(\tau),
$$

which proves equation $(\ast)$. Moreover, the time-continuity of $f$ gives

$$
\int_0^T \int_0^1 \left( f_{m,t}(\tau), \phi_{m,t}(t) \right)_{\Omega_{m,t}} d\tau dt \to \int_0^T \int_0^1 \left( f_{m,t}(\tau), \phi_{m,t}(t) \right)_{\Omega_{m,t}} d\tau dt.
$$

For the fracture solution, we obtain analogous results with the same arguments.

In what follows, we show that the terms on the right hand side in equation $(\Delta)$ vanish as $\Delta t$ approaches zero. For the terms involving the source term $f_{m,t}$, we obtain

$$
\left| \sum_{k=1}^N \int_{l_k-1}^{l_k} \int_t^T \left( f_{m,t}(\tau), \phi_{m,t}(t) \right)_{\Omega_{m,t}} d\tau dt \right| \leq \frac{(\Delta t)^2}{2} \sum_{k=1}^N \| f_{m^k} \|_{\Omega_{m,t}}^2 + \frac{\Delta t}{2} \int_0^T \| \phi_{m,t} \|_{\Omega_{m,t}}^2 \leq C\Delta t,
$$

where we used the Cauchy–Schwarz inequality and Young’s inequality.

Furthermore, we get

$$
\left| \sum_{k=1}^N \int_{l_k-1}^{l_k} \int_t^T \left( K_m(\theta_m(\Psi_{m,t}(\tau))) \nabla \Psi_{m,t}(\tau), \nabla \phi_{m,t}(t) \right)_{\Omega_{m,t}} d\tau dt \right| \leq \frac{(\Delta t)^2 M_k}{2} \sum_{k=1}^N \| \nabla \psi_{m^k} \|_{\Omega_{m,t}}^2 + \frac{\Delta t}{2} \int_0^T \| \nabla \phi_{m,t} \|_{\Omega_{m,t}}^2 \leq C\Delta t,
$$

using the a priori estimate in Lemma 3.4.2. Estimates for the correspondent terms for the fracture solution
follow similarly. Therefore, in the limit $\Delta t \to 0$, we are left with

$$\frac{2}{M} \int_0^T \left( \theta_m(\Psi_m(t)), \phi_n(t) \right)_{\Omega_m} dt + \epsilon^* \int_0^T \left( \theta_f(\Psi_f(t)), \phi_f(t) \right)_{\Omega_f} dt$$

$$+ \sum_{j=1}^2 \int_0^T \int_0^t \left( K_m(\theta_m(\Psi_m(\tau))), \nabla \Psi_m(\tau), \nabla \phi_m(t) \right)_{\Omega_m} \tau d\tau dt$$

$$+ \epsilon^* \int_0^T \int_0^t \left( K_f(\theta_f(\Psi_f(\tau))), \nabla \Psi_f(t), \nabla \phi_f(t) \right)_{\Omega_f} \tau d\tau dt$$

$$= \sum_{j=1}^2 \int_0^T \int_0^t \left( f_m(\tau), \phi_m(t) \right)_{\Omega_m} \tau d\tau dt + \int_0^T \int_0^t \left( f_f(\tau), \phi_f(t) \right)_{\Omega_f} \tau d\tau dt$$

$$+ \sum_{j=1}^2 \int_0^T \left( \theta_m(\Psi_{m,j}), \phi_m(t) \right)_{\Omega_m} dt + \epsilon^* \int_0^T \left( \theta_f(\Psi_{f,j}), \phi_f(t) \right)_{\Omega_f} dt,$$

for all $(\phi_{m,1}, \phi_{m,2}, \phi_f) \in L^2(0, T; \mathcal{V}_m(1)) \times L^2(0, T; \mathcal{V}_m(2)) \times L^2(0, T; \mathcal{V}_f)$ such that $\phi_{m,j} = \phi_f$ on $\Gamma_j$ for $j \in \{1, 2\}$. Note that when choosing test functions $(\phi_{m,1}, \phi_{m,2}, \phi_f) \in W^{1,2}(0, T; \mathcal{V}_m(1)) \times W^{1,2}(0, T; \mathcal{V}_m(2)) \times W^{1,2}(0, T; \mathcal{V}_f)$ satisfying $\phi_{m,1}(T) = \phi_{m,2}(T) = \phi_f(T) = 0$, integration by parts yields

$$\int_0^T \int_0^t \left( f_m(\tau), \partial_t \phi_m(t) \right)_{\Omega_m} \tau d\tau dt = -\int_0^T \left( f_m(t), \phi_m(t) \right)_{\Omega_m} dt,$$

$$\int_0^T \int_0^t \left( K_m(\theta_m(\Psi_m(\tau))), \nabla \Psi_m(\tau), \nabla \phi_m(t) \right)_{\Omega_m} \tau d\tau dt = -\int_0^T \left( K_m(\theta_m(\Psi_m(\tau))), \nabla \Psi_m(t), \nabla \phi_m(t) \right)_{\Omega_m} dt,$$

and similarly for the terms associated with the fracture.

Thus, selecting $\phi_j = \partial_t \phi_j$ in equation (5) gives

$$\frac{2}{M} \int_0^T \left( \theta_m(\Psi_m(t)), \partial_t \phi_m(t) \right)_{\Omega_m} dt + \epsilon^* \int_0^T \left( \theta_f(\Psi_f(t)), \partial_t \phi_f(t) \right)_{\Omega_f} dt$$

$$- \sum_{j=1}^2 \int_0^T \left( K_m(\theta_m(\Psi_m(\tau))), \nabla \Psi_m(\tau), \nabla \phi_m(t) \right)_{\Omega_m} dt$$

$$- \epsilon^* \int_0^T \left( K_f(\theta_f(\Psi_f(\tau))), \nabla \Psi_f(t), \nabla \phi_f(t) \right)_{\Omega_f} dt$$

$$= -\sum_{j=1}^2 \int_0^T \left( f_m(\tau), \phi_m(t) \right)_{\Omega_m} dt - \int_0^T \left( f_f(\tau), \phi_f(t) \right)_{\Omega_f} dt$$

$$- \sum_{j=1}^2 \left( \theta_m(\Psi_{m,j}), \phi_m(0) \right)_{\Omega_m} - \epsilon^* \left( \theta_f(\Psi_{f,j}, \phi_f(0) \right)_{\Omega_f}.$$

Therefore, equation (3.3) holds true for all appropriate test functions. In order to show that the interface conditions are satisfied, we estimate

$$\|\Psi_m - \Psi_f\|_{L^2} \leq \|\Psi_m - \Psi_m\|_{L^2} + \|\Psi_m - \Psi_f\|_{L^2} + \|\Psi_f - \Psi_f\|_{L^2} \quad (\ast)$$

and consider the terms on the right hand side individually. The second term is zero by definition of the discrete weak solution. For the first term, we obtain by the trace inequality

$$\|\Psi_m - \Psi_m\|_{L^2} \leq C(\Omega_m) \|\Psi_m - \Psi_m\|_{L^2} \left( \|\nabla \Psi_m - \nabla \Psi_m\|_{L^2} + \|\Psi_m - \Psi_m\|_{L^2} \right).$$

By the weak convergence in equation (3.21), the term in brackets is bounded, and from the strong convergence in equation (3.24), we get that $\|\Psi_m - \Psi_m\|_{L^2} \to 0$ for $\Delta t \to 0$. The third term on the right hand side of equation (5) vanishes with an analogous argument, which finishes the proof.
3.6 Essential bounds

In what follows, we prove the $L^\infty$-stability of the solution. For this, we employ techniques in the spirit of [15, 43]. We define the non-negative and non-positive cuts of a function $u \in W^{1,2}(\Omega)$ by

\[
[u]_+ := \max \{u, 0\}, \quad [u]_- := \min \{u, 0\}.
\]

One can show that $[u]_+, [u]_- \in W^{1,2}(\Omega)$, and

\[
\nabla [u]_+ = \begin{cases} \nabla u, & u > 0, \\ \{0, & u \leq 0, \end{cases}
\]

\[
\nabla [u]_- = \begin{cases} \{0, & u \geq 0, \\ \nabla u, & u < 0, \end{cases}
\]

for a proof see e.g. [19, Lemma 7.6]. First, we consider the time-discrete solution:

**Lemma 3.6.1** For each $\Delta t > 0$ and $k \in \{1, \ldots, N\}$, it holds

\[
\|\psi^k_{m_1}\|_{L^\infty(\Omega_{m_1})} + \|\psi^k_{m_2}\|_{L^\infty(\Omega_{m_2})} + \|\psi^k_f\|_{L^\infty(\Omega_f)} \leq 3M_\psi (k\Delta t + 1),
\]

where

\[
M_\psi := \max \left\{ M_{I_f}, \frac{M_f}{m_0} \right\}.
\]

**Proof.** The proof is done by induction. For $k = 0$, the statement holds due to Assumption $(A_I)$. Assume now that

\[
\|\psi^{k-1}\|_{L^\infty(\Omega_i)} < M_\psi ((k-1)\Delta t + 1).
\]

First, we show that $\psi^k_i \leq M_\psi (k\Delta t + 1)$ almost everywhere in $\Omega_i$.

We test equation (3.15) with $\phi_i = [\psi^k_i - M_\psi (k\Delta t + 1)]_+$. These test functions satisfy the required interface conditions because $\psi^k_{m_j} = \psi^k_f$ on $\Gamma_f$. Adding some terms on both sides of the equation, we obtain

\[
\begin{align*}
\sum_{j=1}^2 \left( \theta_m (\psi^k_{m_j}) - \theta_m (M_\psi (k\Delta t + 1)) \right) \left( \psi^k_{m_j} - M_\psi (k\Delta t + 1) \right)_+ & \Omega_{m_j} \\
+ \varepsilon^k \left( \theta_f (\psi^k_f) - \theta_f (M_\psi (k\Delta t + 1)) \right) \left( \psi^k_f - M_\psi (k\Delta t + 1) \right)_+ & \Omega_f \\
+ \Delta t \sum_{j=1}^2 \left( K_m (\theta_m (\psi^k_{m_j})) \nabla \left( \psi^k_{m_j} - M_\psi (k\Delta t + 1) \right) \nabla \left[ \psi^k_{m_j} - M_\psi (k\Delta t + 1) \right]_+ \right) & \Omega_{m_j} \\
+ \varepsilon^k \Delta t \left( K_f (\theta_f (\psi^k_f)) \nabla \left( \psi^k_f - M_\psi (k\Delta t + 1) \right) \nabla \left[ \psi^k_f - M_\psi (k\Delta t + 1) \right]_+ \right) & \Omega_f \\
= \sum_{j=1}^2 \left( \theta_m (\psi^{k-1}_{m_j}) - \theta_m (M_\psi (k\Delta t + 1)) \right) \left( \psi^{k-1}_{m_j} - M_\psi (k\Delta t + 1) \right)_+ & \Omega_{m_j} \\
+ \varepsilon^k \left( \theta_f (\psi^{k-1}_f) - \theta_f (M_\psi (k\Delta t + 1)) \right) \left( \psi^{k-1}_f - M_\psi (k\Delta t + 1) \right)_+ & \Omega_f \\
+ \Delta t \sum_{j=1}^2 \left( f^k_{m_j} \left[ \psi^{k-1}_{m_j} - M_\psi (k\Delta t + 1) \right]_+ \right) & \Omega_{m_j} + \Delta t \left( f^k_f \left[ \psi^{k-1}_f - M_\psi (k\Delta t + 1) \right]_+ \right) & \Omega_f.
\end{align*}
\]
From Assumptions \((A_\theta)\) and \((A_K)\), and in particular the monotonicity of \(\theta_i\), we deduce

\[
m_\theta \sum_{j=1}^2 \left\| \frac{1}{\Omega_{mj}} \left( \left[ \psi_{mj}^k - M_\psi(k\Delta t + 1) \right] + \left[ \psi_{mj}^k - M_\psi(k\Delta t + 1) \right] - \left[ \psi_{mj}^k - M_\psi(k\Delta t + 1) \right] \right) \right\|^2_{\Omega_{mj}} + \epsilon^\theta m_\theta \sum_{j=1}^2 \left\| \nabla \left[ \psi_{mj}^k - M_\psi(k\Delta t + 1) \right] + \left[ \psi_{mj}^k - M_\psi(k\Delta t + 1) \right] - \left[ \psi_{mj}^k - M_\psi(k\Delta t + 1) \right] \right\|^2_{\Omega_{mj}} + \epsilon^\psi m_\theta \sum_{j=1}^2 \left\| \nabla \left[ \psi_{mj}^k - M_\psi(k\Delta t + 1) \right] + \left[ \psi_{mj}^k - M_\psi(k\Delta t + 1) \right] - \left[ \psi_{mj}^k - M_\psi(k\Delta t + 1) \right] \right\|^2_{\Omega_{mj}}
\]

\[
+ \Delta t m_K \sum_{j=1}^2 \left\| \psi_{mj}^k - M_\psi(k\Delta t + 1) \right\|^2_{\Omega_{mj}} + \epsilon^\psi \Delta t m_k \left\| \psi_{mj}^k - M_\psi(k\Delta t + 1) \right\|^2_{\Omega_{mj}}
\]

\[
\leq \sum_{j=1}^2 \left( \theta_m (\psi_{mj}^{k-1}) - \theta_m (M_\psi((k-1)\Delta t + 1)), \left[ \psi_{mj}^k - M_\psi(k\Delta t + 1) \right] \right)_{\Omega_{mj}} + \epsilon^\psi \left( \theta_f (\psi_f^{k-1}) - \theta_f (M_\psi((k-1)\Delta t + 1)), \left[ \psi_f^k - M_\psi(k\Delta t + 1) \right] \right)_{\Omega_f} + \Delta t \sum_{j=1}^2 \left( f_{mj}^k - m_\theta M_\psi, \left[ \psi_{mj}^k - M_\psi(k\Delta t + 1) \right] \right)_{\Omega_{mj}} + \Delta t \left( f_f^k - m_\theta M_\psi, \left[ \psi_f^k - M_\psi(k\Delta t + 1) \right] \right)_{\Omega_f},
\]

where we used \(\theta_i(u) \geq \theta_i(v) + m_\theta(u - v)\) on the right hand side in order to get

\[
\theta_i (M_\psi((k\Delta t + 1)) \geq \theta_i (M_\psi((k-1)\Delta t + 1)) + m_\theta M_\psi \Delta t.
\]

Note that the first two terms on the right hand side in equation \((*)\) are non-positive due to the induction assumption and the monotonicity of \(\theta_i\). Moreover, since \(M_\psi \geq \frac{M_f}{M_\psi}\), the last two terms are non-positive as well, from which we infer that \(\psi^k \leq M_\psi(k\Delta t + 1)\) almost everywhere in \(\Omega_i\).

Similarly, we test equation (3.15) with \(\phi_i = \left[ \psi_i^k + M_\psi(k\Delta t + 1) \right]_\Omega\) in order to show that \(\psi^k \geq -M_\psi(k\Delta t + 1)\) almost everywhere in \(\Omega_i\).

This gives

\[
\sum_{j=1}^2 \left( \theta_m (\psi_{mj}^{k-1}) - \theta_m (M_\psi((k-1)\Delta t + 1)), \left[ \psi_{mj}^k - M_\psi(k\Delta t + 1) \right] \right)_{\Omega_{mj}} + \epsilon^\psi \left( \theta_f (\psi_f^{k-1}) - \theta_f (M_\psi((k-1)\Delta t + 1)), \left[ \psi_f^k - M_\psi(k\Delta t + 1) \right] \right)_{\Omega_f} + \Delta t \sum_{j=1}^2 \left( K_m(\theta_m(\psi_{mj}^k)) \nabla \left[ \psi_{mj}^k + M_\psi(k\Delta t + 1) \right], \nabla \left[ \psi_{mj}^k + M_\psi(k\Delta t + 1) \right] \right)_{\Omega_{mj}} + \epsilon^\psi \Delta t \left( K_f(\theta_f(\psi_f^k)) \nabla \left[ \psi_f^k + M_\psi(k\Delta t + 1) \right], \nabla \left[ \psi_f^k + M_\psi(k\Delta t + 1) \right] \right)_{\Omega_f}
\]

\[
= \sum_{j=1}^2 \left( \theta_m (\psi_{mj}^{k-1}) - \theta_m (M_\psi((k-1)\Delta t + 1)), \left[ \psi_{mj}^k + M_\psi(k\Delta t + 1) \right] \right)_{\Omega_{mj}} + \epsilon^\psi \left( \theta_f (\psi_f^{k-1}) - \theta_f (M_\psi((k-1)\Delta t + 1)), \left[ \psi_f^k + M_\psi(k\Delta t + 1) \right] \right)_{\Omega_f} + \Delta t \sum_{j=1}^2 \left( f_{mj}^k, \left[ \psi_{mj}^k + M_\psi(k\Delta t + 1) \right] \right)_{\Omega_{mj}} + \Delta t \left( f_f^k, \left[ \psi_f^k + M_\psi(k\Delta t + 1) \right] \right)_{\Omega_f}.
\]
With the same arguments as above, we obtain
\[
m \theta \sum_{j=1}^2 \left\| \nabla \left[ \psi_k^m + M \phi(k \Delta t + 1) \right] - \nabla \psi_f^m \right\|_{\Omega_j}^2
\]
\[
+ \Delta t mK \sum_{j=1}^2 \left\| \nabla \left[ \psi_k^m + M \phi(k \Delta t + 1) \right] - \nabla \psi_f^m \right\|_{\Omega_j}^2
\]
\[
\leq 2 \sum_{j=1}^2 \left( \theta_m(\psi_k^{m-1}) - \theta_m(-M \phi((k-1) \Delta t + 1)) \right) \left[ \psi_k^m + M \phi(k \Delta t + 1) \right]_{\Omega_j}
\]
\[
+ \epsilon_k \left( \theta_f(\psi_f^{k-1}) - \theta_f(-M \phi((k-1) \Delta t + 1)) \right) \left[ \psi_f^k + M \phi(k \Delta t + 1) \right]_{\Omega_f}
\]
\[
+ \Delta t \sum_{j=1}^2 \left( f_j^{k-1} + mK \phi \right) \left[ \psi_k^m + M \phi(k \Delta t + 1) \right]_{\Omega_j} + \Delta t \left( f_j^k + mK \phi \right) \left[ \psi_f^k + M \phi(k \Delta t + 1) \right]_{\Omega_f}.
\]

Again, the first two terms on the right hand side are non-positive in view of the induction assumption and the last two terms are non-positive due to the choice of $M \phi$. This shows that $\psi_f^k \geq -M \phi(k \Delta t + 1)$ almost everywhere in $\Omega$. Altogether, we conclude that
\[
\| \psi_f^k \|_{L^\infty(\Omega)} \leq M \phi(k \Delta t + 1),
\]
which finishes the proof. 

These estimates can be translated to the time-continuous solution, as the following lemma shows:

**Lemma 3.6.2** For almost all $t \in (0, T]$, it holds that
\[
\| \Psi_{m_1}(t) \|_{L^\infty(\Omega_{m_1})} + \| \Psi_{m_2}(t) \|_{L^\infty(\Omega_{m_2})} + \| \Psi_f(t) \|_{L^\infty(\Omega_f)} \leq 3M \phi(t + 1). \tag{3.31}
\]

**Proof.** Let $\Delta t > 0$ and $t \in (t_{k-1}, t_k]$. Due to Lemma 3.6.1, we have for the piecewise constantly interpolated function
\[
\| \Psi^M_{\Delta t}(t) \|_{L^\infty(\Omega)} = \| \psi_f^k \|_{L^\infty(\Omega)} \leq M \phi(k \Delta t + 1) \leq M \phi(t + 1) + M \phi \Delta t.
\]
In the limit $\Delta t \to 0$, the strong convergence in Lemma 3.5.5 gives the desired estimate. 

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33
Chapter 4

Upscaling

An appealing approach for the modelling of thin fractures is to embed the fractures as a \((d-1)\)-dimensional manifold into a \(d\)-dimensional porous medium domain. In the fracture domain, a differential equation is solved or physical restrictions are imposed, such as no flow across the fracture (see e.g. [1, 37]). In this chapter, we show that such limit models can be rigorously derived from the original model with positive fracture width \(\varepsilon > 0\) in the limit of vanishing \(\varepsilon\), provided that the ratios of porosity and permeability between the fracture and the matrix blocks scale appropriately with \(\varepsilon\). We follow ideas from [53], where the upscaling was considered in the context of crystal dissolution and precipitation, and [43], which is concerned with a reactive transport model. In order to avoid averaging the non-linearity in the flux term, we consider the Kirchhoff transformed version in Definition 4.1.1.

4.1 Upscaling formulation

We rescale the fracture in horizontal direction by defining \(z = x/\varepsilon\), and introduce the following notation for this chapter:

\[
\begin{align*}
\bar{u}_f(t,z,y) &= u_f(t, z\varepsilon, y), \\
\bar{u}_{f,I}(z,y) &= u_{f,I}(z\varepsilon,y) = K_f(\psi_{f,I})(z\varepsilon,y), \\
\bar{f}_f(t,z,y) &= f_f(t, z\varepsilon, y).
\end{align*}
\]  

(4.1)

To unify the notation, we set \(z = x\) in the matrix blocks. For the notation of the domains, we use the following conventions: since the solid matrix subdomains are merely translated when \(\varepsilon\) varies, we omit the \(\varepsilon\) in the notation and write \(\Omega_{mj}\) without a superscript. The fracture domain will be denoted as \(\Omega_f := (-\frac{1}{2}, \frac{1}{2}) \times (0,1)\), and the shorthand notation \(\Omega_f := \Omega_f^1\) will be used. Moreover, we introduce the one-dimensional fracture domain for the effective models \(\Gamma_f = (0,1)\).

Figure 4.1 illustrates the geometry of the problem in rescaled variables and the geometry of the effective models, in which the fracture has become one-dimensional.

Figure 4.1: Geometry with two-dimensional fracture in rescaled variables (left) and upscaled geometry with one-dimensional fracture (right)

35
The solution in each domain chapter: term for certain choices of the parameter \( \lambda \), be seen later when it comes to the a priori estimates, we require an assumption on the decline of the source across the fracture in these models.

For the ease of presentation, we restrict ourselves to the case of \( \varepsilon \)-independent initial conditions \( K_\varepsilon(\psi_\varepsilon) \). As will be seen later when it comes to the a priori estimates, we require an assumption on the decline of the source term for certain choices of the parameter \( \lambda \), in addition to the assumptions at the beginning of the previous chapter:

\[
(A_\lambda) \quad \begin{cases} 
\| f_j \|^2_{\Omega_j^T} \leq C, & \text{if } \lambda \leq 1, \\
\| f_j \|^2_{\Omega_j^T} \leq C\varepsilon\lambda, & \text{if } \lambda > 1.
\end{cases}
\]

The solution in each domain \( u_j^\varepsilon \) will be endowed with a superscript \( \varepsilon \) to emphasise the \( \varepsilon \)-dependence.

We directly state the weak formulation in terms of the rescaled variables, which allow to eliminate the \( \varepsilon \)-dependence of the geometry. Instead, the \( x \)-argument of the functions associated with the fracture becomes \( \varepsilon \)-dependent:

**Definition 4.1.1 (Weak solution for upscaling)** A triple \((u_{m_1}^\varepsilon, u_{m_2}^\varepsilon, \tilde{u}_f^\varepsilon) \in L^2(0,T;\mathcal{V}_{m_1}) \times L^2(0,T;\mathcal{V}_{m_2}) \times L^2(0,T;\mathcal{V}_f)\) is called a weak solution to the Kirchhoff transformed upscaling problem if

\[
\mathcal{K}_m^{-1}(u_{m_1}^\varepsilon) = \mathcal{K}_f^{-1}(\tilde{u}_f^\varepsilon) \quad \text{on } \Gamma_1 \quad \text{and} \quad \mathcal{K}_m^{-1}(u_{m_2}^\varepsilon) = \mathcal{K}_f^{-1}(\tilde{u}_f^\varepsilon) \quad \text{on } \Gamma_2 \quad \text{for a.e. } t \in [0,T],
\]

in the sense of traces, and

\[
\begin{align*}
- (b_m(u_{m_1}^\varepsilon), \partial_t \phi_{m_1}\big|_{\Omega_{m_1}^T}) + (b_m(u_{m_2}^\varepsilon), \partial_t \phi_{m_2}\big|_{\Omega_{m_2}^T}) & \leq \varepsilon^{e+1} \left( \beta_f(\tilde{u}_f^\varepsilon) \right)_{\Omega_f^T} \\
+ (\nabla u_{m_1}^\varepsilon, \nabla \phi_{m_1}\big|_{\Omega_{m_1}^T}) + (\nabla u_{m_2}^\varepsilon, \nabla \phi_{m_2}\big|_{\Omega_{m_2}^T}) & \leq \varepsilon^{1-e} \left( \partial_y \tilde{u}_f^\varepsilon, \partial_y \phi_f\big|_{\Omega_f^T} \right) + \varepsilon^{e+1} \left( \partial_y \tilde{u}_f^\varepsilon, \partial_y \phi_f\big|_{\Omega_f^T} \right) \\
= (f_{m_1}, \phi_{m_1}\big|_{\Omega_{m_1}^T}) + (f_{m_2}, \phi_{m_2}\big|_{\Omega_{m_2}^T}) & + \varepsilon \left( \tilde{f}_f, \phi_f\big|_{\Omega_f^T} \right) \\
+ (b_m(u_{m_1}, t), \phi_{m_1}(0)\big|_{\Omega_{m_1}}) + (b_m(u_{m_2}, t), \phi_{m_2}(0)\big|_{\Omega_{m_2}}) & \leq \varepsilon^{e+1} \left( \beta_f(\tilde{u}_f, t), \phi_f(0)\big|_{\Omega_f} \right)\end{align*}
\]

for all \((\phi_{m_1}, \phi_{m_2}, \phi_f) \in W^{1,2}(0,T;\mathcal{V}_{m_1}) \times W^{1,2}(0,T;\mathcal{V}_{m_2}) \times W^{1,2}(0,T;\mathcal{V}_f)\) satisfying

\[
\phi_{m_1} = \phi_f \quad \text{on } \Gamma_1 \quad \text{and} \quad \phi_{m_2} = \phi_f \quad \text{on } \Gamma_2 \quad \text{for a.e. } t \in [0,T],
\]

and

\[
\phi_i(T) = 0 \quad \text{for } i \in \{m_1, m_2, f\}.
\]

**Remark 4.1.1 (Transformation of the terms)** When rescaling the fracture domain to \( \Omega_f = (0,1) \times (0,1) \), the terms in equation (3.2.2) transform due to \( z = x/\varepsilon \) and the chain rule (yielding \( \partial_z = \varepsilon \partial_x \)) into

\[
\begin{align*}
(b_f(u_f^\varepsilon), \partial_z \phi_f)_{\Omega_f^T} & = \varepsilon (b_f(\tilde{u}_f^\varepsilon), \partial_z \phi_f)_{\Omega_f^T}, \\
(\partial_y u_f^\varepsilon, \partial_y \phi_f)_{\Omega_f^T} & = \varepsilon (\partial_y \tilde{u}_f^\varepsilon, \partial_y \phi_f)_{\Omega_f^T}, \\
(\partial_z u_f^\varepsilon, \partial_z \phi_f)_{\Omega_f^T} & = \varepsilon^{1-e} (\partial_z \tilde{u}_f^\varepsilon, \partial_z \phi_f)_{\Omega_f^T}, \\
(f_f, \phi_f)_{\Omega_f^T} & = \varepsilon (\tilde{f}_f, \phi_f)_{\Omega_f^T}, \\
(b_f(u_{f, t}), \phi_f(0)\big|_{\Omega_f}) & = \varepsilon (b_f(\tilde{u}_{f, t}), \phi_f(0))_{\Omega_f}.
\end{align*}
\]
for example
\[
\left( \partial_z u_f^\varepsilon(t), \partial_z \phi(t) \right)_\Omega_f = \int_0^1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \partial_z u_f^\varepsilon(t, x, y) \partial_z \phi(t, x, y) \, dx \, dy = \frac{1}{\varepsilon} \int_0^1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \partial_z u_f^\varepsilon(t, \varepsilon z, y) \partial_z \phi(t, \varepsilon z, y) \, dz \, dy
\]
(4.7)

\[= \frac{1}{\varepsilon} (\partial_z u_f^\varepsilon(t), \partial_z \phi(t))_\Omega_f.\]

### 4.2 Effective models

We will see that, depending on the scaling parameters for the porosity and the permeability \(\kappa\) and \(\lambda\), respectively, several different effective models occur in the limit \(\varepsilon \to 0\) (see Figure 4.2). We will consider the parameter range \(\kappa \in [-1, \infty)\) and \(\lambda \in [-1, \infty)\) for Problem \(P\). In addition, we will discuss the case \(\lambda \in (-\infty, -1)\) for Neumann no flow conditions at the fracture boundary.

The resulting effective models are presented in Figure 4.2 and brief descriptions of the models can be found in Table 4.1, where the abbreviations PDE and ODE stand for partial and ordinary differential equation, respectively. For the case \(\kappa = -1\) and \(\lambda > 1\), an equation for the fracture remains in the limit \(\varepsilon \to 0\), but the two matrix blocks decouple due to a no flow condition at the interface, for which reason the fracture equation plays no role. The Effective models VIII and IX arise only if no flow conditions are imposed at the fracture boundary. This is because they incorporate a spatially constant pressure in the fracture, which is prevented if the pressure head is fixed at the boundary or if an influx is given, entailing a non-zero pressure gradient.

Let us mention that if one allows the medium to be anisotropic by defining a permeability tensor and introducing different scaling powers with respect to different coordinate directions, even more models can arise in the limit \(\varepsilon \to 0\).
Fracture equation | Pressure continuity
---|---
Model I | Richards’ equation | ✓
Model II | elliptic equation | ✓
Model III | ODE in time | ✓
Model IV | flux continuity between matrix blocks | ✓
Model V | parabolic PDE determines pressure jump | —
Model VI | elliptic PDE determines pressure jump | —
Model VII | no flow between matrix blocks | —
Model VII\(^a\) | no flow between matrix blocks | —
Model VIII\(^b\) | ODE for spatially constant pressure | ✓
Model IX\(^b\) | spatially constant pressure | ✓

\(^a\) fracture equation remains without relevance for the decoupled model

\(^b\) no flow boundary conditions required in the fracture

Table 4.1: Summary of the effective models

### 4.2.1 Strong formulation

To begin with, we introduce the effective models in the strong form in physical variables. We subdivide the effective models in models with continuous pressure at the interface (Effective models I – IV), discontinuous pressure at the interface (Effective models V – VII\(^a\)), and in additional models which are only possible for Neumann no flow conditions at the fracture boundary (Effective models VIII and IX).

#### Models with continuous pressure

The first effective model consists of the Richards equation in the matrix block subdomains and the one-dimensional Richards equation in the fracture. It occurs for \(\kappa = \lambda = -1\).

\[
\begin{aligned}
\partial_t \theta_m(\psi_{m_j}) - \nabla \cdot \left( K_m(\theta_m(\psi_{m_j})) \nabla \psi_{m_j} \right) &= f_{m_j}, & \text{in } \Omega^T_{m_j}, \\
\partial_t \theta_f(\psi_f) - \partial_y \left( K_f(\theta_f(\psi_f)) \partial_y \psi_f \right) &= [\bar{q}_m]_\Gamma, & \text{on } \Gamma^T, \\
\psi_{m_j} &= \psi_f, & \text{on } \Gamma^T, \\
\psi_{m_j} &= 0, & \text{on } \partial \Omega_m \setminus \Gamma \times (0, T), \\
\psi_f &= 0, & \text{on } \partial \Gamma \times (0, T), \\
\psi_{m_j}(0) &= \psi_{m_j,1}, & \text{on } \Omega_m, \\
\psi_f(0) &= \psi_{f,1}, & \text{on } \Gamma,
\end{aligned}
\]

Effective model I:

where

\[
[\bar{q}_m]_\Gamma := (K_m(\theta_m(\psi_{m_j})) \nabla \psi_{m_1} \cdot \bar{n}_{m_1} + K_m(\theta_m(\psi_{m_2})) \nabla \psi_{m_2} \cdot \bar{n}_{m_2}) |_\Gamma
\]

is the flux difference between the two solid matrix subdomains acting as a source term for Richards’ equation in the fracture (note that \(\bar{n}_{m_1} = -\bar{n}_{m_2}\)).

If the porosity ratio does not change with vanishing fracture width and the permeability ratio is taken to be reciprocally proportional to the fracture width, i.e. \(\kappa > -1\) and \(\lambda = -1\), one ends up with an effective model consisting of the Richards equation in the matrix blocks and a stationary elliptic equation in the fracture:

\[
\begin{aligned}
\partial_t \theta_m(\psi_{m_j}) - \nabla \cdot \left( K_m(\theta_m(\psi_{m_j})) \nabla \psi_{m_j} \right) &= f_{m_j}, & \text{in } \Omega^T_{m_j}, \\
-\partial_y \left( K_f(\theta_f(\psi_f)) \partial_y \psi_f \right) &= [\bar{q}_m]_\Gamma, & \text{on } \Gamma^T, \\
\psi_{m_j} &= \psi_f, & \text{on } \Gamma^T, \\
\psi_{m_j} &= 0, & \text{on } \partial \Omega_m \setminus \Gamma \times (0, T), \\
\psi_f &= 0, & \text{on } \partial \Gamma \times (0, T), \\
\psi_{m_j}(0) &= \psi_{m_j,1}, & \text{on } \Omega_m.
\end{aligned}
\]

Effective model II:
On the other hand, only the flux term in the fracture vanishes in the limit for \( \kappa = -1 \) and \( \lambda \in (-1, 1) \), and an ordinary differential equation remains. This limit case corresponds to a fracture capable of storing and releasing water from the solid matrix, but which does not conduct fluid along the fracture:

\[
\begin{align*}
\frac{\partial \theta_m(\psi_{m,i})}{\partial t} - \nabla \cdot \left( K_m(\theta_m(\psi_{m,i})) \nabla \psi_{m,i} \right) &= f_{m,i}, \quad \text{in } \Omega_{m,i}^T, \\
\frac{\partial \theta_f(\psi_f)}{\partial t} &= \left[ \bar{q}_m \right]_\Gamma, \quad \text{on } \Gamma^T, \\
\psi_{m,i} &= \psi_f, \quad \text{on } \Gamma^T, \\
\psi_{m,i} &= 0, \quad \text{on } \partial \Omega_{m,i} \setminus \Gamma \times (0, T], \\
\psi_{m,i}(0) &= \psi_{m,i,1}, \quad \text{on } \Omega_{m,i}, \\
\psi_f(0) &= \psi_{f,1}, \quad \text{on } \Gamma,
\end{align*}
\]

Effective model III:

For \( \kappa > -1 \) and \( \lambda \in (-1, 1) \), an effective model results in which the fracture as a physical entity has disappeared. In this case, both the pressure and the flux are continuous on \( \Gamma \):

\[
\begin{align*}
\frac{\partial \theta_m(\psi_{m,i})}{\partial t} - \nabla \cdot \left( K_m(\theta_m(\psi_{m,i})) \nabla \psi_{m,i} \right) &= f_{m,i}, \quad \text{in } \Omega_{m,i}^T, \\
\left[ \bar{q}_m \right]_\Gamma &= 0, \quad \text{on } \Gamma^T, \\
\psi_{m,i} &= \psi_f, \quad \text{on } \Gamma^T, \\
\psi_{m,i} &= 0, \quad \text{on } \partial \Omega_{m,i} \setminus \Gamma \times (0, T], \\
\psi_{m,i}(0) &= \psi_{m,i,1}, \quad \text{on } \Omega_{m,i}, \\
\psi_f(0) &= \psi_{f,1},
\end{align*}
\]

Effective model IV:

Models with discontinuous pressure

If the permeability ratio is proportional to the fracture width, i.e. \( \lambda = 1 \), effective models result in which the pressure is discontinuous across the interface. For \( \kappa = -1 \), the pressure jump is determined by a parabolic differential equation on the spatial domain \((-\frac{1}{2}, \frac{1}{2})\), which does no longer represent a physical domain:

\[
\begin{align*}
\frac{\partial \theta_m(\psi_{m,i})}{\partial t} - \nabla \cdot \left( K_m(\theta_m(\psi_{m,i})) \nabla \psi_{m,i} \right) &= f_{m,i}, \quad \text{in } \Omega_{m,i}^T, \\
\psi_{m,2} - \psi_{m,1} &= \left[ \psi_f, \right], \quad \text{on } \Gamma^T, \\
\psi_{m,i} &= 0, \quad \text{on } \partial \Omega_{m,i} \setminus \Gamma \times (0, T], \\
\psi_{m,i}(0) &= \psi_{m,i,1}, \quad \text{on } \Omega_{m,i}, \\
\psi_f(0) &= \psi_{f,1}, \quad \text{for } z \in \left(-\frac{1}{2}, \frac{1}{2}\right), y \in \Gamma,
\end{align*}
\]

where \( \psi_f \) solves for each \( y \) a parabolic differential equation on a one-dimensional domain:

\[
\begin{align*}
\frac{\partial \theta_f(\psi_f)}{\partial t} - \frac{\partial}{\partial z} \left( K_f(\theta_f(\psi_f)) \frac{\partial \psi_f}{\partial z} \right) &= 0, \quad \text{for } z \in \left(-\frac{1}{2}, \frac{1}{2}\right), (t, y) \in \Gamma^T, \\
\psi_f(t, \frac{1}{2}, y) &= \psi_{m,1}(t, 0, y), \quad \text{for } (t, y) \in \Gamma^T, \\
K_f(\theta_f(\psi_f)) \frac{\partial \psi_f}{\partial z} \left(t, -\frac{1}{2}, y \right) \cdot \vec{n} &= \left( K_m(\theta_m(\psi_{m,1})) \nabla \psi_{m,1} \right) (t, 0, y) \cdot \vec{n}, \quad \text{for } (t, y) \in \Gamma^T, \\
\psi_f(t, \frac{1}{2}, y) &= \psi_{m,2}(t, 0, y), \quad \text{for } (t, y) \in \Gamma^T, \\
K_f(\theta_f(\psi_f)) \frac{\partial \psi_f}{\partial z} \left(t, \frac{1}{2}, y \right) \cdot \vec{n} &= \left( K_m(\theta_m(\psi_{m,2})) \nabla \psi_{m,2} \right) (t, 0, y) \cdot \vec{n}, \quad \text{for } (t, y) \in \Gamma^T,
\end{align*}
\]

and where we write

\[
\psi_f := \psi_f|_{z=\frac{1}{2}} - \psi_f|_{z=-\frac{1}{2}}.
\]
Similarly, in the effective model for \( \kappa > -1 \), the solution to a one-dimensional elliptic problem determines the pressure jump across the interface:

\[
\begin{align*}
\text{Effective model VI :} & \quad \begin{cases}
\frac{\partial t_m}{\partial t_m}(\psi_{m_1}) - \nabla \cdot \left( K_m(\theta_m(\psi_{m_1})) \nabla \psi_{m_1} \right) = f_{m_1}, & \text{in } \Omega_{m_1}^T, \\
\psi_{m_2} - \psi_{m_1} = \left[ \psi_f \right], & \text{on } \Gamma_T, \\
\psi_{m_1} = 0, & \text{on } \partial \Omega_{m_1} \setminus \Gamma \times (0, T], \\
\psi_{m_1}(0) = \psi_{m_1,1}, & \text{on } \Omega_{m_1},
\end{cases}
\end{align*}
\]

where \( \psi_f \) solves

\[
\begin{align*}
\frac{\partial z}{\partial z} \left( K_f(\theta_f(\psi_f)) \frac{\partial z}{\partial z} \psi_f \right) = 0, & \quad \text{for } z \in \left( -\frac{1}{2}, \frac{1}{2} \right), (t, y) \in \Gamma^T, \\
\psi_f(t, -\frac{1}{2}, y) = \psi_{m_1}(t, 0, y), & \quad \text{for } (t, y) \in \Gamma^T, \\
\left( K_f(\theta_f(\psi_f)) \nabla \psi_f \right) (t, -\frac{1}{2}, y) \cdot \vec{n} = (K_m(\theta_m(\psi_{m_1})) \nabla \psi_{m_1}) (t, 0, y) \cdot \vec{n}, & \quad \text{for } (t, y) \in \Gamma^T, \\
\psi_f(t, \frac{1}{2}, y) = \psi_{m_2}(t, 0, y), & \quad \text{for } (t, y) \in \Gamma^T, \\
\left( K_f(\theta_f(\psi_f)) \nabla \psi_f \right) (t, \frac{1}{2}, y) \cdot \vec{n} = (K_m(\theta_m(\psi_{m_2})) \nabla \psi_{m_2}) (t, 0, y) \cdot \vec{n}, & \quad \text{for } (t, y) \in \Gamma^T.
\end{align*}
\]

If the permeability ratio decreases even faster as \( \varepsilon \) vanishes, that is \( \lambda > 1 \), we obtain effective models in which the pressure is discontinuous across the interface, and the subsolutions in the two solid matrix blocks and the fracture are entirely decoupled from each other, separated by a no flow condition:

\[
\begin{align*}
\text{Effective model VII :} & \quad \begin{cases}
\frac{\partial t_m}{\partial t_m}(\psi_{m_1}) - \nabla \cdot \left( K_m(\theta_m(\psi_{m_1})) \nabla \psi_{m_1} \right) = f_{m_1}, & \text{in } \Omega_{m_1}^T, \\
\frac{\partial t_f}{\partial t_f}(\psi_f(t, z, y) = 0, & \text{for } z \in \left( -\frac{1}{2}, \frac{1}{2} \right), (t, y) \in \Gamma, \\
K_m(\theta_m(\psi_{m_1})) \nabla \psi_{m_1} \cdot \vec{n}_{m_1} = 0, & \text{on } \Gamma_T, \\
\psi_{m_1} = 0, & \text{on } \partial \Omega_{m_1} \setminus \Gamma \times (0, T], \\
\psi_{m_1}(0) = \psi_{m_1,1}, & \text{on } \Omega_{m_1}, \\
\psi_f(0) = \psi_{f,1}, & \text{for } z \in \left( -\frac{1}{2}, \frac{1}{2} \right), y \in \Gamma,
\end{cases}
\end{align*}
\]

In the above model, which corresponds to \( \kappa = -1 \), physical quantities are assigned to the fracture, whereas the fracture is merely a geometrical entity that blocks the flow in the effective model for \( \kappa > -1 \):

\[
\begin{align*}
\text{Effective model VII' :} & \quad \begin{cases}
\frac{\partial t_m}{\partial t_m}(\psi_{m_1}) - \nabla \cdot \left( K_m(\theta_m(\psi_{m_1})) \nabla \psi_{m_1} \right) = f_{m_1}, & \text{in } \Omega_{m_1}^T, \\
K_m(\theta_m(\psi_{m_1})) \nabla \psi_{m_1} \cdot \vec{n}_{m_1} = 0, & \text{on } \Gamma_T, \\
\psi_{m_1} = 0, & \text{on } \partial \Omega_{m_1} \setminus \Gamma \times (0, T], \\
\psi_{m_1}(0) = \psi_{m_1,1}, & \text{on } \Omega_{m_1}.
\end{cases}
\end{align*}
\]
Additional models for no flow conditions

If one replaces the homogeneous Dirichlet boundary conditions in the fracture by Neumann no flow conditions, the following additional models can arise in the limit $\varepsilon \to 0$ for $\kappa = -1$ and $\lambda < -1$:

**Effective model VIII**:

$$
\begin{align*}
\partial_t \theta_m(\psi_{m_j}) - \nabla \cdot \left( K_m(\theta_m(\psi_{m_j})) \nabla \psi_{m_j} \right) &= f_{m_j}, & \text{in } \Omega_{m_j}, \\
\psi_f(t,y) &= \psi_f(t), & \text{on } \Gamma_T, \\
\partial_t \psi_f(t) &= \int_0^1 [\bar{\eta}_m]_I \, dy, & \text{on } \Gamma_T, \\
\psi_{m_j} &= \psi_f, & \text{on } \Gamma_T, \\
\psi_{m_j} &= 0, & \text{on } \partial \Omega_{m_j} \setminus \Gamma \times (0,T), \\
\psi_{m_j}(0) &= \psi_{m_j,1}, & \text{on } \Omega_{m_j}, \\
\psi_f(0) &= \psi_{f,1}, & \text{on } \Gamma.
\end{align*}
$$

(4.20)

**Effective model IX**:

$$
\begin{align*}
\partial_t \theta_m(\psi_{m_j}) - \nabla \cdot \left( K_m(\theta_m(\psi_{m_j})) \nabla \psi_{m_j} \right) &= f_{m_j}, & \text{in } \Omega_{m_j}, \\
\psi_f(t,y) &= \psi_f(t), & \text{on } \Gamma_T, \\
\int_0^1 [\bar{\eta}_m]_I \, dy &= 0, & \text{on } \Gamma_T, \\
\psi_{m_j} &= \psi_f, & \text{on } \Gamma_T, \\
\psi_{m_j} &= 0, & \text{on } \partial \Omega_{m_j} \setminus \Gamma \times (0,T), \\
\psi_{m_j}(0) &= \psi_{m_j,1}, & \text{on } \Omega_{m_j}.
\end{align*}
$$

(4.21)

Note that the pressure in the fracture is spatially constant here, and so is the pressure in the solid matrices at the interface due to the pressure continuity.

For $\kappa > -1$ and $\lambda < -1$, the pressure in the fracture takes a constant value at each time, in such a way that the total flux across the fracture is conserved:

$$
\begin{align*}
\partial_t \theta_m(\psi_{m_j}) - \nabla \cdot \left( K_m(\theta_m(\psi_{m_j})) \nabla \psi_{m_j} \right) &= f_{m_j}, & \text{in } \Omega_{m_j}, \\
\psi_f(t,y) &= \psi_f(t), & \text{on } \Gamma_T, \\
\psi_{m_j} &= \psi_f, & \text{on } \Gamma_T, \\
\psi_{m_j} &= 0, & \text{on } \partial \Omega_{m_j} \setminus \Gamma \times (0,T), \\
\psi_{m_j}(0) &= \psi_{m_j,1}, & \text{on } \Omega_{m_j}.
\end{align*}
$$

4.2.2 Weak formulation

We state the effective models in the weak formulation, which naturally arises when $\varepsilon$ goes to zero in Definition 4.1.1. For this, we introduce the function space for the fracture solution on the one-dimensional fracture:

$$
\mathcal{V}_f := \{ u \in L^2(\Gamma) : \partial_y u \in L^2(\Gamma), u = 0 \text{ on } \partial \Gamma \}.
$$

(4.22)

The pressure head in Effective models I – IV is continuous at the interface. Therefore, whenever the model converges towards one of these effective models, we expect the solution in the fracture to approach a constant value in $z$-direction as the fracture becomes narrower. For this reason, we define $z$-averaged quantities, which will prove useful in order to prove convergence towards Effective models I – IV:

**Definition 4.2.1 (Averaged quantities)** For each $\varepsilon > 0$, we define the horizontally averaged quantities

$$
\begin{align*}
\bar{u}_f^\varepsilon(t,y) &= \int_{-1/2}^{1/2} \bar{u}_f(t,z,y) \, dz, \\
\bar{b}_f(\bar{u}_f^\varepsilon)(t,y) &= \int_{-1/2}^{1/2} b_f(\bar{u}_f^\varepsilon)(t,z,y) \, dz, \\
\bar{b}_f(\bar{u}_{f,1}^\varepsilon)(y) &= \int_{-1/2}^{1/2} b_f(\bar{u}_{f,1}^\varepsilon)(z,y) \, dz, \\
f_f(t,y) &= \int_{-1/2}^{1/2} f(t,z,y) \, dz.
\end{align*}
$$

(4.23)

In Effective models VIII and IX, the pressure head is spatially constant along the entire fracture. Therefore, we define the following averages, which will be used for Problem $P_N$ in Section 4.5:

41
**Definition 4.2.2** (Averaged quantities for Problem $\mathcal{P}_N$) For each $\varepsilon > 0$, we define

\[
\begin{align*}
\bar{u}^\varepsilon_f(t) &:= \int_0^1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{u}^\varepsilon_f(t, z, y) \, dz \, dy = \int_0^1 \bar{u}^\varepsilon_f(t, y) \, dy, \\
\bar{b}^\varepsilon_f(t) &:= \int_0^1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{b}^\varepsilon_f(t, z, y) \, dz \, dy = \int_0^1 \bar{b}^\varepsilon_f(t, y) \, dy, \\
\bar{b}^\varepsilon_{f,i} &:= \int_0^1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{b}^\varepsilon_{f,i}(z, y) \, dz \, dy = \int_0^1 \bar{b}^\varepsilon_{f,i}(y) \, dy, \\
f^\varepsilon_f(t) &:= \int_0^1 \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t, z, y) \, dz \, dy = \int_0^1 f(t, y) \, dy.
\end{align*}
\] (4.24)

From the regularity of $\bar{u}^\varepsilon_f$, which will be shown later, we obtain $\bar{u}^\varepsilon_f \in L^2(0, T; \bar{\mathcal{V}}_f)$ and $\bar{u}^\varepsilon_f \in L^2(0, T)$.

**Models with continuous pressure**

**Definition 4.2.3** (Effective model I) The triple $(U_{m_1}, U_{m_2}, \bar{U}_f) \in L^2(0, T; \mathcal{V}_{m_1}) \times L^2(0, T; \mathcal{V}_{m_2}) \times L^2(0, T; \bar{\mathcal{V}}_f)$ is called a solution to **Effective model I** if

\[
K^{-1}_{m_1}(U_{m_1})(t, z, y) = K^{-1}_f(\bar{U}_f)(t, y) = K^{-1}_{m_2}(U_{m_2})(t, z, y) \text{ on } \Gamma \text{ for a.e. } t \in [0, T],
\] (4.25)
in the sense of traces, and

\[
- (b_m(U_{m_1}), \partial_t \phi_{m_1})_{\Omega_{m_1}} - (b_m(U_{m_2}), \partial_t \phi_{m_2})_{\Omega_{m_2}} - (b_f(\bar{U}_f), \partial_t \phi_f)_{\Gamma^T} \\
+ (\nabla U_{m_1}, \nabla \phi_{m_1})_{\Omega_{m_1}} + (\nabla U_{m_2}, \nabla \phi_{m_2})_{\Omega_{m_2}} + (\partial_y \bar{U}_f, \partial_y \phi_f)_{\Gamma^T} \\
= (f_{m_1}, \phi_{m_1})_{\Omega_{m_1}} + (f_{m_2}, \phi_{m_2})_{\Omega_{m_2}} \\
+ (b_m(K_m(\psi_{m_1, i})), \phi_{m_1}(0))_{\Omega_{m_1}} + (b_m(K_m(\psi_{m_2, i})), \phi_{m_2}(0))_{\Omega_{m_2}} + (\bar{b}_f(K_f(\psi_{f,i})), \phi_f(0))_{\Gamma} \tag{4.26}
\]

for all $(\phi_{m_1}, \phi_{m_2}, \phi_f) \in W^{1,2}(0, T; \mathcal{V}_{m_1}) \times W^{1,2}(0, T; \mathcal{V}_{m_2}) \times W^{1,2}(0, T; \bar{\mathcal{V}}_f)$ satisfying

\[
\phi_{m_1}(t, z, y) = \phi_{m_2}(t, z, y) = \phi_f(t, y) \text{ on } \Gamma \text{ for a.e. } t \in [0, T],
\] (4.27)

and

\[
\phi_i(T) = 0 \quad \text{for } i \in \{m_1, m_2, f\}. \tag{4.28}
\]

**Definition 4.2.4** (Effective model II) The triple $(U_{m_1}, U_{m_2}, \bar{U}_f) \in L^2(0, T; \mathcal{V}_{m_1}) \times L^2(0, T; \mathcal{V}_{m_2}) \times L^2(0, T; \bar{\mathcal{V}}_f)$ is called a solution to **Effective model II** if

\[
K^{-1}_{m_1}(U_{m_1})(t, z, y) = K^{-1}_f(\bar{U}_f)(t, y) = K^{-1}_{m_2}(U_{m_2})(t, z, y) \text{ on } \Gamma \text{ for a.e. } t \in [0, T],
\] (4.29)
in the sense of traces, and

\[
- (b_m(U_{m_1}), \partial_t \phi_{m_1})_{\Omega_{m_1}} - (b_m(U_{m_2}), \partial_t \phi_{m_2})_{\Omega_{m_2}} \\
+ (\nabla U_{m_1}, \nabla \phi_{m_1})_{\Omega_{m_1}} + (\nabla U_{m_2}, \nabla \phi_{m_2})_{\Omega_{m_2}} + (\partial_y \bar{U}_f, \partial_y \phi_f)_{\Gamma^T} \tag{4.30}
\]

for all $(\phi_{m_1}, \phi_{m_2}, \phi_f) \in W^{1,2}(0, T; \mathcal{V}_{m_1}) \times W^{1,2}(0, T; \mathcal{V}_{m_2}) \times W^{1,2}(0, T; \bar{\mathcal{V}}_f)$ satisfying

\[
\phi_{m_1}(t, z, y) = \phi_{m_2}(t, z, y) = \phi_f(t, y) \text{ on } \Gamma \text{ for a.e. } t \in [0, T],
\] (4.31)

and

\[
\phi_i(T) = 0 \quad \text{for } i \in \{m_1, m_2, f\}. \tag{4.32}
\]
Definition 4.2.5 (Effective model III) The triple \((U_{m1}, U_{m2}, \overline{U}_f) \in L^2(0, T; \nu_{m1}) \times L^2(0, T; \nu_{m2}) \times L^2(0, T; \overline{\nu}_f)\) is called a solution to Effective model III if
\[
K_m^{-1}(U_{m1})(t, z, y) = K_f^{-1}(\overline{U}_f)(t, y) = K_m^{-1}(U_{m2})(t, z, y) \text{ on } \Gamma \text{ for a.e. } t \in [0, T],
\]
in the sense of traces, and
\[
\begin{align*}
- (b_m(U_{m1}), \partial_t \phi_{m1})_{\Omega_{m1}} - (b_m(U_{m2}), \partial_t \phi_{m2})_{\Omega_{m2}} &- \left( b_f(\overline{U}_f), \partial_t \phi_f \right)_{\Gamma_T} \\
+ (\nabla U_{m1}, \nabla \phi_{m1})_{\Omega_{m1}} + (\nabla U_{m2}, \nabla \phi_{m2})_{\Omega_{m2}} &+ (f_m, \phi_{m1})_{\Omega_{m1}} + (b_m(K_m(\psi_{m2}, 1)), \phi_{m2}(0))_{\Omega_{m2}} + (b_f(K_f(\bar{\psi}_f, 1)), \phi_f(0))_{\Gamma_T},
\end{align*}
\]
for all \((\phi_{m1}, \phi_{m2}, \phi_f) \in W^{1,2}(0, T; \nu_{m1}) \times W^{1,2}(0, T; \nu_{m2}) \times W^{1,2}(0, T; \overline{\nu}_f)\) satisfying
\[
\phi_{m1}(t, z, y) = \phi_{m2}(t, z, y) = \phi_f(t, y) \text{ on } \Gamma \text{ for a.e. } t \in [0, T],
\]
and
\[
\phi_i(T) = 0 \quad \text{for } i \in \{m_1, m_2, f\}.
\]

Definition 4.2.6 (Effective model IV) The triple \((U_{m1}, U_{m2}, \overline{U}_f) \in L^2(0, T; \nu_{m1}) \times L^2(0, T; \nu_{m2}) \times L^2(0, T; \overline{\nu}_f)\) is called a solution to Effective model IV if
\[
K_m^{-1}(U_{m1})(t, z, y) = K_f^{-1}(\overline{U}_f)(t, y) = K_m^{-1}(U_{m2})(t, z, y) \text{ on } \Gamma \text{ for a.e. } t \in [0, T],
\]
in the sense of traces, and
\[
\begin{align*}
- (b_m(U_{m1}), \partial_t \phi_{m1})_{\Omega_{m1}} - (b_m(U_{m2}), \partial_t \phi_{m2})_{\Omega_{m2}} &+ (\nabla U_{m1}, \nabla \phi_{m1})_{\Omega_{m1}} + (\nabla U_{m2}, \nabla \phi_{m2})_{\Omega_{m2}} \\
= (f_m, \phi_{m1})_{\Omega_{m1}} + (f_m, \phi_{m2})_{\Omega_{m2}} + (b_m(K_m(\psi_{m1}, 1)), \phi_{m1}(0))_{\Omega_{m1}} + (b_m(K_m(\psi_{m2}, 1)), \phi_{m2}(0))_{\Omega_{m2}},
\end{align*}
\]
for all \((\phi_{m1}, \phi_{m2}, \phi_f) \in W^{1,2}(0, T; \nu_{m1}) \times W^{1,2}(0, T; \nu_{m2}) \times W^{1,2}(0, T; \overline{\nu}_f)\) satisfying
\[
\phi_{m1}(t, z, y) = \phi_{m2}(t, z, y) = \phi_f(t, y) \text{ on } \Gamma \text{ for a.e. } t \in [0, T],
\]
and
\[
\phi_i(T) = 0 \quad \text{for } i \in \{m_1, m_2, f\}.
\]

Models with discontinuous pressure

The interfaces \(\Gamma_1\) and \(\Gamma_2\) in Effective models V and VI do no longer represent physical entities, but describe the boundary of the virtual domain along which the pressure jump is computed. Models with discontinuities in the capillary pressure are considered in e.g. [50, 51].

Definition 4.2.7 (Effective model V) The triple \((U_{m1}, U_{m2}, U_f) \in L^2(0, T; \nu_{m1}) \times L^2(0, T; \nu_{m2}) \times L^2(0, T; \nu_f)\) is called a solution to Effective model V if
\[
\begin{align*}
K_m^{-1}(U_{m1})(t, z, y) &= K_f^{-1}(U_f)(t, z, y) \text{ on } \Gamma_1, \\
K_m^{-1}(U_{m2})(t, z, y) &= K_f^{-1}(U_f)(t, z, y) \text{ on } \Gamma_2,
\end{align*}
\]
for a.e. \(t \in [0, T]\).
in the sense of traces, and
\[
\begin{align*}
- (b_m(U_m), \partial_t^2 y_m)_{\Omega_{t_1}^m} - (b_m(U_{m_2}), \partial_t^2 y_{m_2})_{\Omega_{t_2}^m} - \left( b_f(U_f), \partial_t^2 y_f \right)_{\Omega_f^m} \\
+ \left( \nabla U_m, \nabla y_m \right)_{\Omega_{t_1}^m} + \left( \nabla U_{m_2}, \nabla y_{m_2} \right)_{\Omega_{t_2}^m} + \left( \partial_t^2 U_f, \partial_t^2 y_f \right)_{\Omega_f^m} \\
= (f_{m_1}, \Phi_{m_1})_{\Omega_{t_1}^m} + (f_{m_2}, \Phi_{m_2})_{\Omega_{t_2}^m} \\
+ (b_m(K_m(\psi_{m_1})), \Phi_{m_1}(0))_{\Omega_{t_1}^m} + (b_m(K_m(\psi_{m_2})), \Phi_{m_2}(0))_{\Omega_{t_2}^m} + \left( b_f(K_f(\psi_f)), \Phi_f(0) \right)_{\Omega_f^m},
\end{align*}
\]
for all \((\phi_{m_1}, \phi_{m_2}, \phi_f) \in W^{1,2}(0, T; \nu_{m_1}) \times W^{1,2}(0, T; \nu_{m_2}) \times W^{1,2}(0, T; \nu_f)\) satisfying
\[
\phi_{m_1} = \phi_f \text{ on } \Gamma_1 \quad \text{and} \quad \phi_{m_2} = \phi_f \text{ on } \Gamma_2 \quad \text{for a.e. } t \in [0, T],
\]
and
\[
\phi_1(t) = 0 \quad \text{for } i \in \{m_1, m_2, f\}.
\]

**Definition 4.2.8 (Effective model VI)** The triple \((U_{m_1}, U_{m_2}, U_f) \in L^2(0, T; \nu_{m_1}) \times L^2(0, T; \nu_{m_2}) \times L^2(0, T; \nu_f)\) is called a solution to **Effective model VI** if
\[
\begin{align*}
K_m^{-1}(U_{m_1})(t, z, y) = K_f^{-1}(U_f)(t, z, y) & \quad \text{on } \Gamma_1, \\
K_m^{-1}(U_{m_2})(t, z, y) = K_f^{-1}(U_f)(t, z, y) & \quad \text{on } \Gamma_2,
\end{align*}
\]
for a.e. \(t \in [0, T]\),
\[
\phi_{m_1} = \phi_f \text{ on } \Gamma_1 \quad \text{and} \quad \phi_{m_2} = \phi_f \text{ on } \Gamma_2 \quad \text{for a.e. } t \in [0, T],
\]
and
\[
\phi_1(T) = 0 \quad \text{for } i \in \{m_1, m_2, f\}.
\]

**Definition 4.2.9 (Effective model VII)** The triple \((U_{m_1}, U_{m_2}, U_f) \in L^2(0, T; \nu_{m_1}) \times L^2(0, T; \nu_{m_2}) \times L^2(0, T; \nu_f)\) is called a solution to **Effective model VII** if
\[
\begin{align*}
- (b_m(U_m), \partial_t^2 y_m)_{\Omega_{t_1}^m} - (b_m(U_{m_2}), \partial_t^2 y_{m_2})_{\Omega_{t_2}^m} - \left( b_f(U_f), \partial_t^2 y_f \right)_{\Omega_f^m} \\
+ \left( \nabla U_m, \nabla y_m \right)_{\Omega_{t_1}^m} + \left( \nabla U_{m_2}, \nabla y_{m_2} \right)_{\Omega_{t_2}^m} + \left( \partial_t^2 U_f, \partial_t^2 y_f \right)_{\Omega_f^m} \\
= (f_{m_1}, \Phi_{m_1})_{\Omega_{t_1}^m} + (f_{m_2}, \Phi_{m_2})_{\Omega_{t_2}^m} \\
+ (b_m(K_m(\psi_{m_1})), \Phi_{m_1}(0))_{\Omega_{t_1}^m} + (b_m(K_m(\psi_{m_2})), \Phi_{m_2}(0))_{\Omega_{t_2}^m} + \left( b_f(K_f(\psi_f)), \Phi_f(0) \right)_{\Omega_f^m},
\end{align*}
\]
for all \((\phi_{m_1}, \phi_{m_2}, \phi_f) \in W^{1,2}(0, T; \nu_{m_1}) \times W^{1,2}(0, T; \nu_{m_2}) \times W^{1,2}(0, T; \nu_f)\) satisfying
\[
\phi_{m_1} = \phi_f \text{ on } \Gamma_1 \quad \text{and} \quad \phi_{m_2} = \phi_f \text{ on } \Gamma_2 \quad \text{for a.e. } t \in [0, T],
\]
and
\[
\phi_1(t) = 0 \quad \text{for } i \in \{m_1, m_2, f\}.
\]
**Definition 4.2.10 (Effective model VII***) The pair \((U_{m_1}, U_{m_2}) \in L^2(0, T; \mathcal{V}_{m_1}) \times L^2(0, T; \mathcal{V}_{m_2})\) is called a solution to Effective model VII** if
\[
\begin{align*}
- (b_m(U_{m_1}), \partial_t \phi_{m_1})_{\Omega_{m_1}^3} - (b_m(U_{m_2}), \partial_t \phi_{m_2})_{\Omega_{m_2}^3} + (\nabla U_{m_1}, \nabla \phi_{m_1})_{\Omega_{m_1}^3} + (\nabla U_{m_2}, \nabla \phi_{m_2})_{\Omega_{m_2}^3} \\
= (f_{m_1}, \phi_{m_1})_{\Omega_{m_1}^3} + (f_{m_2}, \phi_{m_2})_{\Omega_{m_2}^3} + (b_m(K_m(\psi_{m_1, 1})), \phi_{m_1}(0))_{\Omega_{m_1}^3} + (b_m(K_m(\psi_{m_2, 1})), \phi_{m_2}(0))_{\Omega_{m_2}^3},
\end{align*}
\]
for all \((\phi_{m_1}, \phi_{m_2}) \in W^{1,2}(0, T; \mathcal{V}_{m_1}) \times W^{1,2}(0, T; \mathcal{V}_{m_2})\) satisfying
\[
\phi_{m_1}(T) = \phi_{m_2}(T) = 0.
\]

**Additional models for no flow conditions**

Models for Neumann no flow boundary conditions in the fracture subdomain in which the fracture solution is spatially constant and the pressure is continuous across the interface read as follows:

**Definition 4.2.11 (Effective model VIII)** The triple \((U_{m_1}, U_{m_2}, \bar{U}_f) \in L^2(0, T; \mathcal{V}_{m_1}) \times L^2(0, T; \mathcal{V}_{m_2}) \times L^2(0, T)\) is called a solution to Effective model VIII if
\[
\begin{align*}
K_m^{-1}(U_{m_1})(t, z, y) = K_f^{-1}(\bar{U}_f)(t) = K_m^{-1}(U_{m_2})(t, z, y) \text{ on } \Gamma \text{ for a.e. } t \in [0, T],
\end{align*}
\]
in the sense of traces, and
\[
\begin{align*}
- (b_m(U_{m_1}), \partial_t \phi_{m_1})_{\Omega_{m_1}^3} - (b_m(U_{m_2}), \partial_t \phi_{m_2})_{\Omega_{m_2}^3} - \int_0^T b_f(U_f) \partial_t \phi_f dt \\
+ (\nabla U_{m_1}, \nabla \phi_{m_1})_{\Omega_{m_1}^3} + (\nabla U_{m_2}, \nabla \phi_{m_2})_{\Omega_{m_2}^3} \\
= (f_{m_1}, \phi_{m_1})_{\Omega_{m_1}^3} + (f_{m_2}, \phi_{m_2})_{\Omega_{m_2}^3} + (b_m(K_m(\psi_{m_1, 1})), \phi_{m_1}(0))_{\Omega_{m_1}^3} + (b_m(K_m(\psi_{m_2, 1})), \phi_{m_2}(0))_{\Omega_{m_2}^3} + b_f(K_f(\bar{\psi}_{f, 1})) \phi_f(0),
\end{align*}
\]
for all \((\phi_{m_1}, \phi_{m_2}, \phi_f) \in W^{1,2}(0, T; \mathcal{V}_{m_1}) \times W^{1,2}(0, T; \mathcal{V}_{m_2}) \times W^{1,2}(0, T)\) satisfying
\[
\phi_{m_1}(t, z, y) = \phi_{m_2}(t, z, y) = \phi_f(t) \text{ on } \Gamma \text{ for a.e. } t \in [0, T],
\]
and
\[
\phi_i(t) = 0 \quad \text{for } i \in \{m_1, m_2, f\}.
\]

**Definition 4.2.12 (Effective model IX)** The triple \((U_{m_1}, U_{m_2}, \bar{U}_f) \in L^2(0, T; \mathcal{V}_{m_1}) \times L^2(0, T; \mathcal{V}_{m_2}) \times L^2(0, T)\) is called a solution to Effective model IX if
\[
\begin{align*}
K_m^{-1}(U_{m_1})(t, z, y) = K_f^{-1}(\bar{U}_f)(t) = K_m^{-1}(U_{m_2})(t, z, y) \text{ on } \Gamma \text{ for a.e. } t \in [0, T],
\end{align*}
\]
in the sense of traces, and
\[
\begin{align*}
- (b_m(U_{m_1}), \partial_t \phi_{m_1})_{\Omega_{m_1}^3} - (b_m(U_{m_2}), \partial_t \phi_{m_2})_{\Omega_{m_2}^3} + (\nabla U_{m_1}, \nabla \phi_{m_1})_{\Omega_{m_1}^3} + (\nabla U_{m_2}, \nabla \phi_{m_2})_{\Omega_{m_2}^3} \\
= (f_{m_1}, \phi_{m_1})_{\Omega_{m_1}^3} + (f_{m_2}, \phi_{m_2})_{\Omega_{m_2}^3} + (b_m(K_m(\psi_{m_1, 1})), \phi_{m_1}(0))_{\Omega_{m_1}^3} + (b_m(K_m(\psi_{m_2, 1})), \phi_{m_2}(0))_{\Omega_{m_2}^3},
\end{align*}
\]
for all \((\phi_{m_1}, \phi_{m_2}, \phi_f) \in W^{1,2}(0, T; \mathcal{V}_{m_1}) \times W^{1,2}(0, T; \mathcal{V}_{m_2}) \times W^{1,2}(0, T)\) satisfying
\[
\phi_{m_1}(t, z, y) = \phi_{m_2}(t, z, y) = \phi_f(t) \text{ on } \Gamma \text{ for a.e. } t \in [0, T],
\]
and
\[
\phi_i(t) = 0 \quad \text{for } i \in \{m_1, m_2, f\}.
\]
4.3 Uniform estimates with respect to $\varepsilon$

In order to prove the convergence of Problem $\mathcal{P}$ towards one of the effective models introduced in the previous section, we establish estimates for the solution and its derivatives which do not depend on $\varepsilon$, similar to the uniform estimates with respect to $\Delta t$ in Chapter 3.

Testing with $\varepsilon$-independent functions $\psi_t(t,z,y) = \psi_f(t,y)$ in the fracture (and ergo $\psi_{m_1}$ and $\psi_{m_2}$ fulfilling $\psi_{m_1}(t,z,y)|_{\Gamma_1} = \psi_{m_2}(t,z,y)|_{\Gamma_2} = \psi_f(t,y)$ for a.e. $t \in [0, T]$) in Definition 4.1.1 gives

$$
-(b_m(u_{m_1}), \partial_t \psi_{m_1})_{\Omega_{m_1}} - (b_m(u_{m_2}), \partial_t \psi_{m_2})_{\Omega_{m_2}} - \varepsilon^{k+1} (b_{\tilde{f}}(\tilde{u}_{f}), \partial_t \psi_f)_{\Gamma_T} \\
+ (\nabla u_{m_1}, \nabla \psi_{m_1})_{\Omega_{m_1}} + (\nabla u_{m_2}, \nabla \psi_{m_2})_{\Omega_{m_2}} + \varepsilon^{k+1}(\partial_t \tilde{u}_{f}, \partial_y \psi_f)_{\Gamma_T} \\
= (f_{m_1}, \psi_{m_1})_{\Omega_{m_1}} + (f_{m_2}, \psi_{m_2})_{\Omega_{m_2}} + \varepsilon(f_{\tilde{f}}, \psi_f)_{\Gamma_T},
$$

(4.62)

for all $(\psi_{m_1}, \psi_{m_2}, \psi_f) \in W^{1,2}(0,T; \mathcal{V}_{m_1}) \times W^{1,2}(0,T; \mathcal{V}_{m_2}) \times W^{1,2}(0,T; \bar{\mathcal{V}}_f)$ satisfying

$$
\psi_{m_1}(t,z,y)|_{\Gamma_1} = \psi_{m_2}(t,z,y)|_{\Gamma_2} = \psi_f(t,y) \quad \text{for a.e. } t \in [0, T],
$$

and

$$
\psi_f(T) = 0 \quad \text{for } i \in \{m_1, m_2, f\}.
$$

(4.63)

(4.64)

The proof of the following lemma provides us with a first restriction on the powers of $\varepsilon$, required to establish uniform bounds with respect to $\varepsilon$:

**Lemma 4.3.1** Let $\kappa \geq -1$ and Assumption $(A_\lambda)$ be satisfied. Then, there exists a $C > 0$ independent of $\varepsilon$ such that

$$
\|\eta_{m_1}^{\varepsilon}\|_{L^2(0,T; \mathcal{V}_{m_1})}^2 + \|\eta_{m_2}^{\varepsilon}\|_{L^2(0,T; \mathcal{V}_{m_2})}^2 + \varepsilon^{k+1}\|\tilde{u}_{f}^{\varepsilon}\|_{L^2(0,T)}^2 + \varepsilon^{k+1}\|\nabla \tilde{u}_{f}^{\varepsilon}\|_{L^2(0,T)}^2 + \varepsilon^{k-1}\|\partial_y \tilde{u}_{f}^{\varepsilon}\|_{L^2(0,T)}^2 \leq C.
$$

(4.65)

**Proof.** Recalling the estimate in Lemma 3.4.2, we find that in view of Remark 3.4.1 for any $l \in \{0, \ldots, N\}$ the solution to the (not Kirchhoff transformed) time-discrete problem satisfies

$$
\frac{m_\theta}{2} \sum_{l \in \{1, \ldots, N\}} \sum_{j=1}^{2} \|\psi_{m,l,j}^{\varepsilon}\|_{\Omega_{m,l,j}}^2 + \varepsilon^{k-1} \frac{M_\theta}{2} \sum_{l \in \{1, \ldots, N\}} \|\psi_{f,l}^{\varepsilon}\|_{\Omega_{f,l}}^2 + \Delta t \frac{m_K}{2} \sum_{l=1}^{2} \sum_{j=1}^{N} \|
abla \psi_{m,l,j}^{\varepsilon}\|_{\Omega_{m,l,j}}^2 + \varepsilon^{k-1} \frac{M_\theta}{2} \sum_{l \in \{1, \ldots, N\}} \|\psi_{f,l}^{\varepsilon}\|_{\Omega_{f,l}}^2 \\
\leq M_\theta \sum_{j=1}^{2} \|\psi_{m,l,j}^{\varepsilon}\|_{\Omega_{m,l,j}}^2 + \varepsilon^{k-1} \frac{M_\theta}{2} \|\psi_{f,l}^{\varepsilon}\|_{\Omega_{f,l}}^2 + \Delta t \frac{C_m}{2m_K} \sum_{j=1}^{2} \sum_{k=1}^{N} \|f_{m,l,j}^{k}\|_{\Omega_{m,l,j}}^2 + \varepsilon^{k-1} \frac{M_\theta}{2} \|\psi_{f,l}^{\varepsilon}\|_{\Omega_{f,l}}^2.
$$

Note that $\varepsilon$ was kept fixed in Chapter 3, for which reason it was omitted in the notation. Let $\tilde{f}_j^{\varepsilon}(z,y) = f_j^{\varepsilon}(z,y)$ and $\tilde{u}_{f,l}^{\varepsilon}(z,y) = u_{f,l}^{\varepsilon}(z,y)$. The Poincaré constants are bounded uniformly in $\varepsilon$ due to the rectangular geometry. The initial data are assumed to be $\varepsilon$-independent. Using $\kappa \geq -1$ and Assumption $(A_\lambda)$, the right hand side of the above equation can be bounded uniformly in $\varepsilon$ for all $\Delta t > 0$ by

$$
\frac{m_\theta}{2} \sum_{j=1}^{2} \|\psi_{m,l,j}^{\varepsilon}\|_{\Omega_{m,l,j}}^2 + \varepsilon^{k-1} \frac{M_\theta}{2} \|\psi_{f,l}^{\varepsilon}\|_{\Omega_{f,l}}^2 + \Delta t \frac{C_m}{2m_K} \sum_{j=1}^{2} \sum_{k=1}^{N} \|f_{m,l,j}^{k}\|_{\Omega_{m,l,j}}^2 + \varepsilon^{k-1} \frac{M_\theta}{2} \|\psi_{f,l}^{\varepsilon}\|_{\Omega_{f,l}}^2 \\
= \frac{M_\theta}{2} \sum_{j=1}^{2} \|\psi_{m,l,j}^{\varepsilon}\|_{\Omega_{m,l,j}}^2 + \varepsilon^{k+1} \frac{M_\theta}{2} \|\psi_{f,l}^{\varepsilon}\|_{\Omega_{f,l}}^2 + \Delta t \frac{C_m}{2m_K} \sum_{j=1}^{2} \sum_{k=1}^{N} \|f_{m,l,j}^{k}\|_{\Omega_{m,l,j}}^2 + \varepsilon^{k-1} \frac{M_\theta}{2} \|\psi_{f,l}^{\varepsilon}\|_{\Omega_{f,l}}^2 \\
\leq C,
$$

where the constant $C \geq 0$ is independent of $\varepsilon$ and $\Delta t$. Therefore, this estimate translates to the time-interpolated functions $\Psi_{\Delta t}$ by Lemma 3.5.1. Consequently, the weak convergence in equation (3.21) and the weak lower semicontinuity of the norm yield

$$
\sum_{j=1}^{2} \|\psi_{m,l,j}^{\varepsilon}\|_{L^2(0,T; \mathcal{V}_{m,l,j})}^2 + \varepsilon^{k+1}\|\psi_{f,l}^{\varepsilon}\|_{L^2(0,T; \bar{\mathcal{V}}_f)}^2 + \varepsilon^{k+1}\|\nabla \psi_{f,l}^{\varepsilon}\|_{L^2(0,T)}^2 \leq C.
$$
By the properties of the Kirchhoff transformation from Lemma 3.2.1, analogous estimates for \( u^f_j \) follow immediately:

\[
\sum_{j=1}^{2} \| u^f_j \|^2_{L^2(0,T;\mathcal{V}_{m_j})} + \| \nabla u^f_j \|^2_{\mathcal{V}'_{T}} \leq C.
\]

Finally, the identities \( \| u^f_j \|^2_{\mathcal{V}'_{T}} = \varepsilon \| \nabla u^f_j \|^2_{\mathcal{V}'_{T}} + \varepsilon \| \nabla u^f_j \|^2_{\mathcal{V}'_{T}} \) give

\[
\sum_{j=1}^{2} \| u^f_j \|^2_{L^2(0,T;\mathcal{V}_{m_j})} + \varepsilon \| \nabla u^f_j \|^2_{\mathcal{V}'_{T}} + \varepsilon \| \nabla u^f_j \|^2_{\mathcal{V}'_{T}} \leq C,
\]

which concludes the proof.

This leads us to similar estimates for \( \bar{a}^f_j \):

**Lemma 4.3.2** Let \( \kappa \geq -1 \) and Assumption \((A_{\lambda})\) be satisfied. Then there exists a \( C > 0 \) independent of \( \varepsilon \) such that

\[
\varepsilon^{k+1} \| \bar{a}^f_j \|^2_{\mathcal{V}'_{T}} + \varepsilon^{k+1} \| \partial_y \bar{a}^f_j \|^2_{\mathcal{V}'_{T}} \leq C.
\]  

**Proof.** By Jensen’s inequality (see Lemma 2.3.4), we have

\[
\| \bar{a}^f_j \|^2_{\mathcal{V}'_{T}} = \int_{0}^{T} \left( \frac{1}{2} \bar{a}^f_j(t, z, y) \right) dz \left( \frac{1}{2} \bar{a}^f_j(t, z, y) \right) dy dt \leq \int_{0}^{T} \left( \frac{1}{2} \bar{a}^f_j(t, z, y) \right)^2 dz dy dt = \| \bar{a}^f_j \|^2_{\mathcal{V}'_{T}}.
\]

The estimate for the second term follows with the same argument, and Lemma 4.3.1 gives the result.

Moreover, the \( L^\infty \) estimate from Lemma 3.6.2 allows us to bound \( \bar{a}^f_j \) in \( L^2(\Omega^f_T) \) and \( \bar{a}^f_j \) in \( L^2(\Gamma^T) \) uniformly with respect to \( \varepsilon \) as the following lemma shows:

**Lemma 4.3.3** There exists a constant \( C > 0 \) such that for all \( \kappa, \lambda \in \mathbb{R} \)

\[
\| \bar{a}^f_j \|_{L^\infty(\Omega^f_T)} + \| \bar{a}^f_j \|_{L^\infty(\Gamma^T)} + \| \bar{a}^f_j \|_{\Omega^f_T} + \| \bar{a}^f_j \|_{\Gamma^T} \leq C.
\]  

**Proof.** The constant \( M_{\Phi} := \max\{ M_{\Psi}, M_{\Gamma_T^\infty} \} \) in the estimate in Lemma 3.6.2 is independent of \( \varepsilon \). Therefore, \( \bar{a}^f_j \) and \( \bar{a}^f_j \) are uniformly bounded in \( L^\infty(\Omega^f_T) \) and \( L^\infty(\Gamma^T) \), respectively, with respect to \( \varepsilon \). Since the time interval and the domain have finite measure, this implies uniform estimates for \( \bar{a}^f_j \) in \( L^2(0, T; L^2(\Omega^f)) \) and \( \bar{a}^f_j \) in \( L^2(0, T; L^2(\Gamma)) \) as well.

The estimates in Lemmas 4.3.1 – 4.3.3 give rise to a subsequence of \( \varepsilon \to 0 \) along which we get

\[
\begin{align*}
\bar{u}^f_j \to U_{m_j}, & \quad \text{weakly in } L^2(0, T; \mathcal{V}_{m_j}), \\
\bar{a}^f_j \to \bar{U}_f, & \quad \text{weakly in } L^2(0, T; L^2(\Omega^f)), \\
\bar{a}^f_j \to \bar{U}_f, & \quad \text{weakly in } L^2(0, T; L^2(\Gamma)),
\end{align*}
\]

by the Eberlein–Šmulyan theorem. Moreover, if \( \lambda = -1 \) we obtain a uniform bound for \( \| \partial_y \bar{a}^f_j \|^2_{\Gamma^T} \) in Lemma 4.3.2, implying that for a subsequence

\[
\bar{a}^f_j \to \bar{U}_f, \quad \text{weakly in } L^2(0, T; \mathcal{V}_{\dot{Y}_f}).
\]

The next lemma furnishes the strong convergence of \( \bar{u}^f_j \to U_{m_j} \) in \( L^2(0, T; L^2(\Omega_{m_j})) \) along a subsequence of \( \varepsilon \to 0 \). This will prove essential for the convergence of the non-linear expression \( b_m(u^f_j) \) and for the convergence on the boundary.

**Lemma 4.3.4** Let \( \kappa \geq -1 \) and Assumption \((A_{\lambda})\) be satisfied. Then, there exists a subsequence of \( \varepsilon \to 0 \) along which

\[
\bar{u}^f_j \to U_{m_j}, \quad \text{strongly in } L^2(0, T; L^2(\Omega_{m_j})).
\]  

\(47\)
Proof. We will show this statement for the untransformed variable \( \Psi_{m_j}^\varepsilon = K^{-1}(u_{m_j}^\varepsilon) \); the result for \( u_{m_j}^\varepsilon \) follows immediately from the Lipschitz continuity of the Kirchhoff transformation. The capital letter \( \Psi \) is used here to conform to the notation in the previous chapter.

First, note that the right hand side in the a priori estimate in Lemma 3.4.2 is bounded uniformly with respect to \( \varepsilon \) as \( \varepsilon \) decreases to zero due to \( \kappa \geq -1 \) and Assumption \((A_\lambda)\). Therefore, the estimate for the time derivative \( \partial_t \Theta_{m_j}^\varepsilon \) in Lemma 3.5.2 is independent of \( \varepsilon \). From this, we infer in the limit \( \Delta t \to 0 \) that

\[
\| \partial_t \Theta_{m_j}^\varepsilon \|^2_{L^2(0,\Delta t;W^{-1,2}(\Omega_{m_j}))} \leq C,
\]

where the constant \( C \) is independent of \( \varepsilon \). From the strong convergence in equation (3.26) and the Lipschitz continuity of \( \theta_{m^{-1}} \), we know that

\[
\Psi_{m_j}^\varepsilon = \theta_{m^{-1}}(\Theta_{m_j}^\varepsilon).
\]

Hence, we obtain using the properties of the Kirchhoff transformation (see Proposition 3.2.1) together with Lemma 4.3.1 and the Lipschitz continuity of \( \theta_{m^{-1}} \)

\[
\| \theta_m(\Psi_{m_j}^\varepsilon) \|^2_{L^2(0,T;V_{m_j})} + \| \partial_t \theta_m(\Psi_{m_j}^\varepsilon) \|^2_{L^2(0,T;W^{-1,2}(\Omega_{m_j}))} \leq C.
\]

Thus, the existence of a strongly convergent subsequence \( \{ \theta_m(\Psi_{m_j}^\varepsilon) \}_\varepsilon \) in \( L^2(0,T;L^2(\Omega_{m_j})) \) in the limit \( \varepsilon \to 0 \) follows from the Aubin–Lions–Simon theorem along the lines of the proof of Lemma 3.5.5, in formulæ

\[
\theta_m(\Psi_{m_j}^\varepsilon) \to \theta_m(\Psi_{m_j}), \quad \text{strongly in } L^2(0,T;L^2(\Omega_{m_j})).
\]

The Lipschitz continuity of \( \theta_{m^{-1}} \) gives

\[
\Psi_{m_j}^\varepsilon \to \Psi_{m_j}, \quad \text{strongly in } L^2(0,T;L^2(\Omega_{m_j})),
\]

and the statement of the lemma follows from the Lipschitz continuity of \( K_m \). \( \square \)

In order to show that the difference between the traces of \( \bar{\alpha}_f^\varepsilon \) and the averaged quantity \( \bar{\alpha}_f^\varepsilon \) on the interface converges to zero as \( \varepsilon \to 0 \), we will make use of Proposition 4.3 in [53]:

**Proposition 4.3.1**

Let \( \Omega = (-\frac{1}{2}, \frac{1}{2}) \times (0, L) \), \( f \in W^{1,2}(\Omega) \), and let \( f : [0, L] \to \mathbb{R} \) be defined as \( f(y) = \int_{\frac{1}{2}}^{\frac{1}{2}} f(\xi, y) \, d\xi \). Then, in the sense of traces

\[
\| f(\xi_0, \cdot) - f \|_{L^1(O_L)} \leq \| \partial_\xi f \|_{L^1(O_L)}
\]

for each \( \xi_0 \in [-\frac{1}{2}, \frac{1}{2}] \).

This proposition together with Lemma 4.3.1 give the following estimate:

**Lemma 4.3.5**

There exists a \( C > 0 \) independent of \( \varepsilon \) such that for any \( z_0 \in [-\frac{1}{2}, \frac{1}{2}] \) it holds

\[
\| \bar{\alpha}_f^\varepsilon (\cdot, z_0, \cdot) - \bar{\alpha}_f^\varepsilon \|_{L^\infty(O_f)} \leq C.
\]

**Remark 4.3.1** \((\lambda \geq 1)\) Lemma 4.3.1 shows that the requirement \( \lambda \leq 1 \) is necessary in order to keep the left hand side in equation 4.72 bounded when \( \varepsilon \) vanishes (and convergence is only achieved for \( \lambda < 1 \)). However, in the case of \( \lambda \geq 1 \), all estimates in this section remain valid due to Assumption \((A_\lambda)\). Therefore, the solution to the effective model cannot be expected to be continuous across the interface then, but the existence of convergent subsequences is still guaranteed.

For \( \lambda \leq -1 \), the gradient of the fracture solution \( \nabla \bar{u}_f^\varepsilon \) is bounded in \( L^2(0,T;L^2(\Omega_f)) \) according to Lemma 4.3.1. From this, we infer just as for the matrix block solutions in Lemma 4.3.4 that there exists a strongly convergent subsequence of \( \varepsilon \to 0 \) along which

\[
\bar{u}_f^\varepsilon \to u_f, \quad \text{strongly in } L^2(0,T;L^2(\Omega_f)),
\]

making use of the Aubin–Lions–Simon theorem 2.2.3. The case \( \lambda \leq -1 \) is the most interesting one, since it means that the fracture permeability grows at least as fast as the fracture width \( \varepsilon \) decreases. We analyse the other values of \( \lambda \) as well, but under additional assumptions.
For $\lambda \in (-1, 1]$, the estimate in Lemma 4.3.1 does not provide a uniform estimate for $\partial_y \tilde{u}_f^\epsilon$ any longer. For this reason, we have to assume that the $y$-derivative of $\tilde{u}_f^\epsilon$ remains bounded when $\epsilon$ approaches zero in order to obtain a strongly convergent subsequence.

In the case $\lambda > 1$, we cannot bound any of the derivatives, and we need to assume the uniform boundedness of the gradient of $\tilde{u}_f^\epsilon$ additionally. This is summarised in the following assumption:

**Assumption 4.3.1** (Assumption for $\lambda > -1$) There exists a constant $C > 0$ such that for all $\epsilon > 0$ it holds

\[
(A_{\lambda}) \quad \begin{cases} 
\| \partial_y \tilde{u}_f^\epsilon \|_{\Omega_f^\epsilon} \leq C, & -1 < \lambda \leq 1, \\
\| \partial_y \tilde{u}_f^\epsilon \|_{\Omega_f^\epsilon} + \| \partial_y \tilde{u}_f^\epsilon \|_{\Omega_f^\epsilon} \leq C, & \lambda > 1.
\end{cases}
\]

With the additional Assumption $(A_{\lambda})$, we get a strongly convergent subsequence as in equation (4.73) for all values of $\lambda \in \mathbb{R}$ by the Aubin–Lions–Simon theorem. Similarly, Lemma 4.3.3 together with Assumption $(A_{\lambda})$ yields a strongly convergent subsequence

\[
\tilde{u}_f^\epsilon \rightharpoonup \bar{U}_f, \quad \text{strongly in } L^2(0, T; L^2(\Gamma)).
\]  

### 4.4 Upscaling theorem

With the tools of the previous chapter at hand, we are able to prove the convergence of Problem $\mathcal{P}$ towards one of the effective models in Section 4.2 for different values of the scaling powers $\kappa$ and $\lambda$:

**Theorem 4.4.1** (Upscaling theorem for Problem $\mathcal{P}$) For the following ranges of $\kappa$ and $\lambda$, these tuples are a solution to the following effective models:

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$\lambda$</th>
<th>Effective model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>$-1$</td>
<td>$\left( U_{m_1} , U_{m_2} , \bar{U}_f \right)$ Effective model I,</td>
</tr>
<tr>
<td>$\in (-1, \infty)$</td>
<td>$-1$</td>
<td>$\left( U_{m_1} , U_{m_2} , \bar{U}_f \right)$ Effective model II,</td>
</tr>
<tr>
<td>$-1$</td>
<td>$\in (-1, 1)$</td>
<td>$\left( U_{m_1} , U_{m_2} , \bar{U}_f \right)$ Effective model III,</td>
</tr>
<tr>
<td>$\in (-1, \infty)$</td>
<td>$\in (-1, 1)$</td>
<td>$\left( U_{m_1} , U_{m_2} , \bar{U}_f \right)$ Effective model IV,</td>
</tr>
<tr>
<td>$-1$</td>
<td>$1$</td>
<td>$\left( U_{m_1} , U_{m_2} , U_f \right)$ Effective model V,</td>
</tr>
<tr>
<td>$\in (-1, \infty)$</td>
<td>$1$</td>
<td>$\left( U_{m_1} , U_{m_2} , U_f \right)$ Effective model VI,</td>
</tr>
<tr>
<td>$\in (-1, \infty)$</td>
<td>$\in (1, \infty)$</td>
<td>$\left( U_{m_1} , U_{m_2} , U_f \right)$ Effective model VII,</td>
</tr>
</tbody>
</table>

### Proof

- **$\lambda < 1$:**

Choose arbitrary test functions $(\phi_{m_1}, \phi_{m_2}, \phi_f) \in W^{1,2}(0, T; V_{m_1}) \times W^{1,2}(0, T; V_{m_2}) \times W^{1,2}(0, T; W_0^{1,2}(\Gamma))$ satisfying equations (4.63) and (4.64), and denote the terms in equation (4.62) by $I_1, \ldots, I_{12}$. For all values of $\kappa$ and $\lambda$, the term $I_6$ vanishes in the limit $\epsilon \to 0$ due to

\[
|I_6| = \epsilon \left| \left( f_f^\epsilon , \phi_f \right)_{\Gamma_T} \right| \leq \epsilon \| f_f^\epsilon \|_{\Gamma_T} \| \phi_f \|_{\Gamma_T} \to 0.
\]

The terms $I_7, I_8, I_{10}, I_{11}$ do not depend on $\epsilon$ and remain unchanged as $\epsilon$ approaches zero. The strong $L^2$ convergence from Lemma 4.3.4 and the Lipschitz continuity of $b_1$ give

\[
I_1 = - \langle b_m(u_{m_1}'), \partial_t \phi_{m_1} \rangle_{\Omega_{m_1}^\epsilon} \to - \langle b_m(U_{m_1}) , \partial_t \phi_{m_1} \rangle_{\Omega_{m_1}^\epsilon},
\]

and the convergence of $I_2$ follows with the same argument. Making use of the weak convergence from equation (4.68)\_1, we obtain

\[
I_4 = \langle \nabla u_{m_1}', \nabla \phi_{m_1} \rangle_{\Omega_{m_1}^\epsilon} \to \langle \nabla U_{m_1} , \nabla \phi_{m_1} \rangle_{\Omega_{m_1}^\epsilon},
\]

and similarly for $I_5$. As regards the fracture solution, we distinguish the cases $\lambda = -1$ and $\lambda > -1$: in case of $\lambda = -1$, the weak convergence from equation (4.69) yields

\[
I_6 = \left( \partial_y u_f^\epsilon , \partial_y \phi_f \right)_{\Gamma_T} \to \left( \partial_y U_f , \partial_y \phi_f \right)_{\Gamma_T},
\]

49
whereas for \( \lambda > -1 \), we get
\[
|l_0| = \epsilon^{\lambda+1} \left| \left( \partial_y \tilde{a}_f, \partial_y \varphi_f \right)_{\Gamma_T} \right| \leq \epsilon^{\lambda+1} \| \partial_y \tilde{a}_f \|_{\Gamma_T} \| \partial_y \varphi_f \|_{\Gamma_T} \leq \epsilon^{\lambda+1} C \| \partial_y \varphi_f \|_{\Gamma_T} \to 0,
\]
where we made use of the estimate for \( \partial_y \tilde{a}_f \) in Lemma 4.3.2.

It remains to consider the terms \( I_3 \) and \( I_{12} \). Here, we make a distinction between the cases \( \kappa = -1 \) and \( \kappa > -1 \).

First, consider the case \( \kappa = -1 \), where \( I_3 = - \left( \tilde{b}_f (\tilde{a}_f^\kappa), \partial_t \varphi_f \right)_{\Gamma_T} \) and \( I_{12} \) is independent of \( \epsilon \).

For the term \( I_3 \), we start by estimating
\[
\left| \left( \tilde{b}_f (\tilde{a}_f^\kappa) - \tilde{b}_f (\bar{U}_f), \partial_t \varphi_f \right)_{\Gamma_T} \right| \leq \left| \left( \tilde{b}_f (\tilde{a}_f^\kappa) - \tilde{b}_f (\bar{U}_f), \partial_t \varphi_f \right)_{\Gamma_T} \right| + \left| \left( \tilde{b}_f (\tilde{a}_f^\kappa), \partial_t \varphi_f \right)_{\Gamma_T} \right|.
\]

For the first term on the right hand side we obtain using the Lipschitz continuity of \( b_f \) and Lemma 4.3.5
\[
\left| \left( \tilde{b}_f (\tilde{a}_f^\kappa) - \tilde{b}_f (\bar{U}_f), \partial_t \varphi_f \right)_{\Gamma_T} \right| \leq \| \tilde{b}_f (\tilde{a}_f^\kappa) - \tilde{b}_f (\bar{U}_f) \|_{\Gamma_T} \| \partial_t \varphi_f \|_{\Gamma_T}
\]
\[
= \left\| \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \tilde{b}_f (\tilde{a}_f^\kappa) - \tilde{b}_f (\bar{U}_f) \right) dz \right\|_{\Gamma_T} \| \partial_t \varphi_f \|_{\Gamma_T}
\]
\[
\leq M_\theta \left\| \int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{U}_f - \bar{U}_f \right\| dz \| \partial_t \varphi_f \|_{\Gamma_T}
\]
\[
\leq M_\theta C \epsilon^{\frac{1}{1+2}} \| \partial_t \varphi_f \|_{\Gamma_T},
\]
which goes to zero as \( \epsilon \to 0 \).

The second term vanishes due to
\[
\left| \left( \tilde{b}_f (\tilde{a}_f^\kappa) - \tilde{b}_f (\bar{U}_f), \partial_t \varphi_f \right)_{\Gamma_T} \right| \leq M_\theta \| \tilde{a}_f^\kappa - \bar{U}_f \|_{\Gamma_T} \| \partial_t \varphi_f \|_{\Gamma_T},
\]
and the strong convergence in equation (4.74). This shows that
\[
I_3 = - \left( \tilde{b}_f (\tilde{a}_f^\kappa), \partial_t \varphi_f \right)_{\Gamma_T} \to - \left( \tilde{b}_f (\bar{U}_f), \partial_t \varphi_f \right)_{\Gamma_T}.
\]

For \( \kappa > -1 \), we estimate
\[
|I_3| = \epsilon^{\kappa+1} \left| \left( \tilde{b}_f (\tilde{a}_f^\kappa), \partial_t \varphi_f \right)_{\Gamma_T} \right| \leq \epsilon^{\kappa+1} M_\theta \| \tilde{a}_f^\kappa \|_{\Omega_T} \| \partial_t \varphi_f \|_{\Gamma_T} \leq \epsilon^{\kappa+1} M_\theta C \| \partial_t \varphi_f \|_{\Gamma_T} \to 0,
\]
in view of Lemma 4.3.3, and similarly, we get
\[
|I_{12}| = \epsilon^{\kappa+1} \left| \left( \tilde{b}_f (\tilde{a}_f^\kappa), \varphi_f (0) \right)_{\Gamma_T} \right| \leq \epsilon^{\kappa+1} M_\theta \| \tilde{a}_f^\kappa \|_{\Omega_T} \| \varphi_f (0) \|_{\Gamma_T} \leq \epsilon^{\kappa+1} M_\theta C \| \varphi_f (0) \|_{\Gamma_T} \to 0.
\]

Finally, the Dirichlet interface condition for the pressure head has to be proven. It turns out that a weakly convergent subsequence in \( L^2 (0, T; L^2 (\Gamma)) \) in the fracture suffices for this purpose, and Assumption \( (A_\gamma) \) is not necessary at this point. We present the proof for the interface \( \Gamma_1 \), the proof for the interface \( \Gamma_2 \) follows along the same lines.

As the weak convergence of \( \tilde{a}_f^\kappa \) towards \( \bar{U}_f \) does not directly imply the weak convergence of \( K_f^{-1} (\tilde{a}_f^\kappa) \) towards \( K_f^{-1} (\bar{U}_f) \), we define the function \( R (u_{m_1}) := (K_f \circ K_m^{-1}) (u_{m_1}) \) in order to transform the interface condition \( K_m^{-1} (u_{m_1}) = K_f^{-1} (\bar{U}_f) \) on \( \Gamma_1 \) into a linear expression in \( \bar{U}_f \), namely
\[
R (u_{m_1}) = \bar{U}_f \quad \text{on} \ \Gamma_1.
\]
Now we take an arbitrary test function \( \phi \in L^2(0, T; L^2(\Gamma_1)) \) and estimate
\[
\left| \left( R(U_{m_1}) - R_F, \phi \right) \right|_{L^2(T)} \leq \left| \left( R(U_{m_1}) - R(u_{m_1}^\varepsilon), \phi \right) \right|_{L^2(T)} + \left| \left( R(u_{m_1}^\varepsilon) - \bar{u}_F^\varepsilon, \phi \right) \right|_{L^2(T)}.
\]

Let us denote the terms on the right hand side by \( J_1, \ldots, J_4 \). As \( u_{m_1}^\varepsilon \) and \( \bar{u}_F^\varepsilon \) satisfy the interface condition, we immediately get \( J_2 = 0 \). Note that Assumption (A_k) implies the Lipschitz continuity of \( R \) with Lipschitz constant \( \frac{M_K}{m_K} \). Making use of this and the Cauchy–Schwarz inequality yields
\[
J_1 \leq \frac{M_K}{m_K} \| U_{m_1} - u_{m_1}^\varepsilon \|_{L^2(T)} \| \phi \|_{L^2(T)},
\]
and one shows as in the proof of Theorem 3.5.1 that \( J_1 \to 0 \) in the limit \( \varepsilon \to 0 \) using the trace inequality, the strong convergence in Lemma (4.3.4) and the boundedness of the gradient due to the weak convergence in equation (4.69). For the term \( J_3 \), we obtain
\[
J_3 \leq \| \bar{u}_F^\varepsilon - \bar{u}_F \|_{L^2(T)} \| \phi \|_{H^1(T)},
\]
and from Lemma 4.3.5 we infer that \( J_3 \to 0 \). Finally, the weak convergence in equation (4.69) yields \( J_4 \to 0 \).

Since \( \phi \in L^2(0, T; L^2(\Gamma_1)) \) was arbitrary, one has \( R(U_{m_1}) = \bar{U}_f \) on \( \Gamma_1 \) and therefore \( K_{m_1}^{-1}(U_{m_1}) = K_f^{-1}(\bar{U}_f) \) on \( \Gamma_1 \) in the sense of traces, which concludes the proof in the case \( \lambda < 1 \).

- \( \lambda \geq 1 \):

We choose arbitrary test functions \((\phi_{m_1}, \phi_{m_2}, \phi_f) \in W^{1,2}(0, T; V_{m_1}) \times W^{1,2}(0, T; V_{m_2}) \times W^{1,2}(0, T; V_f)\) satisfying the conditions in Definition 4.1.1, and denote the terms in equation (4.3) by \( I_{1}, \ldots, I_{13} \). Note that the test function \( \phi_f \) may depend on \( z \) now. The term \( I_{10} \) vanishes in the limit \( \varepsilon \to 0 \) as before because
\[
|I_{10}| = \varepsilon \left| \left( \bar{f}_F, \phi_f \right) \right|_{L^2(T)} \leq \varepsilon \| \bar{f}_F \|_{L^2(T)} \| \phi_f \|_{L^2(T)} \to 0.
\]
The terms \( I_8, I_9, I_{11}, I_{12} \) are \( \varepsilon \)-independent, and the convergence of the terms \( I_1, I_2, I_4, I_5 \) follows as in the case \( \lambda < 1 \). The term \( I_7 \) vanishes thanks to the estimate in Lemma 4.3.1:
\[
|I_7| = \varepsilon^{\lambda + 1} \left| \left( \partial_y \bar{u}_F^\varepsilon, \partial_y \phi_f \right) \right|_{L^2(T)} \leq \varepsilon^{\lambda + 1} \| \partial_y \bar{u}_F^\varepsilon \|_{L^2(T)} \| \partial_y \phi_f \|_{L^2(T)} \leq \varepsilon^{\lambda + 1} C \| \partial_y \phi_f \|_{L^2(T)} \to 0.
\]
For the term \( I_6 \), we first consider first the case \( \lambda = 1 \): then, \( \| \partial_z \bar{u}_F^\varepsilon \|_{L^2(T)} \) is uniformly bounded with respect to \( \varepsilon \) due to Lemma 4.3.1. By the Eberlein–Šmulian theorem 2.2.1, there exists a subsequence of \( \varepsilon \to 0 \) along which
\[
\partial_z \bar{u}_F^\varepsilon \rightharpoonup \partial_z U_f \quad \text{weakly in } L^2(0, T; L^2(\Omega_f)).
\]
This gives immediately
\[
I_6 = \left( \partial_z \bar{u}_F^\varepsilon, \partial_z \phi_f \right)_{L^2(T)} \rightarrow \left( \partial_z U_f, \partial_z \phi_f \right)_{L^2(T)}.
\]
If \( \lambda > 1 \), we estimate in view of Lemma 4.3.1
\[
|I_6| = \varepsilon^{\lambda - 1} \left| \left( \partial_z \bar{u}_F^\varepsilon, \partial_z \phi_f \right) \right|_{L^2(T)} \leq \varepsilon^{\lambda - 1} \| \partial_z \bar{u}_F^\varepsilon \|_{L^2(T)} \| \partial_z \phi_f \|_{L^2(T)} \leq \varepsilon^{\lambda - 1} C \| \partial_z \phi_f \|_{L^2(T)} \to 0.
\]
For the terms \( I_3 \) and \( I_{13} \), we distinguish the cases \( \kappa = -1 \) and \( \kappa > -1 \). For \( \kappa = -1 \), we use the strong convergence in equation (4.73) and the Lipschitz continuity of \( b_f \) to obtain
\[
I_5 = \left( b_f(\bar{u}_F^\varepsilon), \partial_t \phi_f \right)_{L^2(T)} \rightarrow \left( b_f(U_f), \partial_t \phi_f \right)_{L^2(T)}.
\]
and $I_3$ is independent of $\varepsilon$. For $\kappa > -1$, both terms vanish due to

$$|I_3| = \varepsilon^{k+1} \left| \left( b_f(\hat{u}_f), \partial_t \psi_f \right) \right|_{\Omega_f} \leq \varepsilon^{k+1} \|b_f(\hat{u}_f)\|_{\Omega_f} \|\partial_t \psi_f\|_{\Omega_f} \leq \varepsilon^{k+1} M_{\theta} \|\hat{u}_f\|_{\Omega_f} \|\partial_t \psi_f\|_{\Omega_f} \to 0,$$

and similarly

$$|I_3| = \varepsilon^{k+1} \left| \left( b_f(\hat{u}_f), \psi_f(0) \right) \right|_{\Omega_f} \leq \varepsilon^{k+1} \|b_f(\hat{u}_f)\|_{\Omega_f} \|\psi_f(0)\|_{\Omega_f} \leq \varepsilon^{k+1} M_{\theta} \|\hat{u}_f\|_{\Omega_f} \|\psi_f(0)\|_{\Omega_f} \to 0.$$

It remains to show the continuity of the pressure head at the interfaces $\Gamma_1$ and $\Gamma_2$ for $\lambda = 1$, which imposes the boundary conditions for the partial differential equation that determines the pressure jump across the fracture. Again, we show this only for $\Gamma_1$, for the case of $\Gamma_2$ we get analogous results. In contrast to the case $\lambda < 1$, we need to make use of the trace inequality in Theorem 2.1.1, for which reason we explicitly exploit Assumption \((A_{\gamma})\) in order to get a strongly convergent subsequence. We choose an arbitrary test function $\phi \in L^2(0; T; L^2(\Gamma_1))$ and estimate

$$\left| \left( R(U_{m_1}) - U_f, \phi \right)_{\Gamma_{1,T}} \right| \leq \left| \left( R(U_{m_1}) - R(u'_{m_1}), \phi \right)_{\Gamma_{1,T}} \right| + \left| \left( R(u'_{m_1}) - \tilde{u}_f, \phi \right)_{\Gamma_{1,T}} \right| + \left| \left( \tilde{u}_f - U_f, \phi \right)_{\Gamma_{1,T}} \right|.$$

We denote the terms on the right hand side by $I_1, I_2$ and $I_3$. The term $I_2$ equals zero as before due to the interface condition, and the term $I_1$ can be treated as in the case $\lambda < 1$. Finally, the term $I_3$ converges to zero using the trace inequality, the strong convergence in equation (4.73) and the boundedness of the gradients due to Assumption \((A_{\gamma})\).

Hence, we are left with the respective weak formulations of the effective models in the limit $\varepsilon \to 0$. □

### 4.5 The case $\lambda < -1$

Consider the case when the permeability ratio between the fracture and the matrix blocks rises faster than the fracture width decreases, that is $\lambda < -1$. Recall that in order to derive Essential models I – IV, we tested equation (4.3) with $z$-independent functions. Now, we expect the permeability to increase so rapidly that the fracture solution becomes spatially constant in the limit $\varepsilon \to 0$. For showing this, we will use test functions which are constant in space, i.e. $\phi_f(t, z, y) = \phi_f(t)$. Since we have chosen homogeneous Dirichlet boundary conditions in Problem $\mathcal{P}$, the only constant function in the test function space $\mathcal{V}_f$ is the zero function.

For this reason, we consider the problem now with Neumann no flow boundary conditions in the fracture. The strong formulation writes as

\[
\begin{align*}
\partial_t \theta_m(\psi_{m_j}^\varepsilon) + \nabla \cdot \psi_{m_j}^\varepsilon &= f_{m_j} \quad &\text{in } \Omega_{m_j}^\varepsilon, \\
\psi_{m_j}^\varepsilon &= -K_m(\theta_m(\psi_{m_j}^\varepsilon))\nabla \psi_{m_j}^\varepsilon \quad &\text{in } \Omega_{m_j}^\varepsilon, \\
\partial_t (\varepsilon^k \psi_{f_j}^\varepsilon) + \nabla \cdot \psi_{f_j}^\varepsilon &= f_{f_j} \quad &\text{in } \Omega_{f_j}^\varepsilon, \\
\psi_{f_j}^\varepsilon &= -\varepsilon^k K_f(\theta_f(\psi_{f_j}^\varepsilon))\nabla \psi_{f_j}^\varepsilon \quad &\text{in } \Omega_{f_j}^\varepsilon, \\
\psi_{m_j}^\varepsilon &= 0 \quad &\text{on } \partial \Omega_{m_j} \setminus \Gamma_j \times (0, T), \quad \\
\psi_{f_j}^\varepsilon &= 0 \quad &\text{on } \partial \Omega_f \setminus \Gamma_j \times (0, T), \quad \\
\psi_{m_j}^\varepsilon \cdot \vec{n} &= \psi_{f_j}^\varepsilon \cdot \vec{n} \quad &\text{on } \Gamma_j \times (0, T), \quad \\
\psi_{m_j}^\varepsilon &= \psi_{f_j}^\varepsilon \quad &\text{on } \Gamma_j \times (0, T), \quad \\
\psi_{f_j}^\varepsilon(0) &= \psi_{f_j} \quad &\text{in } \Omega_i.
\end{align*}
\]

Problem $\mathcal{P}_N$:

The altered boundary condition is expressed in the weak formulation by modifying the function spaces for the fracture solution and the fracture test function:

**Definition 4.5.1** (Weak solution to Problem $\mathcal{P}_N$) A triple $(u_{m_1}, u_{m_2}, u_f) \in L^2(0; T; \mathcal{V}_{m_1}) \times L^2(0; T; \mathcal{V}_{m_2}) \times L^2(0; T; W^{1,2}(\Omega_f))$ is called a weak solution to the Kirchhoff transformed Problem $\mathcal{P}_N$ if

\[
\mathcal{K}_m^{-1}(u_{m_1}) = \mathcal{K}_f^{-1}(u_f) \quad \text{on } \Gamma_1 \quad \text{and} \quad \mathcal{K}_m^{-1}(u_{m_2}) = \mathcal{K}_f^{-1}(u_f) \quad \text{on } \Gamma_2 \quad \text{for a.e. } t \in [0, T],
\]

(4.76)
in the sense of traces, and
\[-(b_m(u_m^1), \partial_t \phi_m^1)_{\Omega_t^{m_1}} \leq -(b_m(u_m^2), \partial_t \phi_m^2)_{\Omega_t^{m_2}} - \varepsilon^k \left(b_f(u_f), \partial_t \phi_f \right)_{\Omega_f^T} \]
\[+ \left(\nabla u_{m^1}, \nabla \phi_m^1\right)_{\Omega_t^{m_1}} + \left(\nabla u_{m^2}, \nabla \phi_m^2\right)_{\Omega_t^{m_2}} + \varepsilon \left(\nabla u_f, \nabla \phi_f \right)_{\Omega_f^T} \]
\[\leq (f_m, \phi_m^1)_{\Omega_t^{m_1}} + (f_m, \phi_m^2)_{\Omega_t^{m_2}} + \left( f_f, \phi_f \right)_{\Omega_f^T} \]
\[+ (b_m(u_m^1), \phi_m^1(0))_{\Omega_t^{m_1}} + (b_m(u_m^2), \phi_m^2(0))_{\Omega_t^{m_2}} + \varepsilon \left( b_f(u_f), \phi_f(0) \right)_{\Omega_f^T} \] (4.77)

for all \((\phi_{m_1}, \phi_{m_2}, \phi_f) \in W^{1,2}(0, T; \Omega) \times W^{1,2}(0, T; \Omega) \times W^{1,2}(0, T; W^{1,2}(\Omega_f))\) satisfying
\[\phi_{m_1} = \phi_f \text{ on } \Gamma_1 \quad \text{and} \quad \phi_{m_2} = \phi_f \text{ on } \Gamma_2 \text{ for a.e. } t \in [0, T],\] (4.78)
and
\[\phi_i(T) = 0 \quad \text{for } i \in \{m_1, m_2, f\}.\] (4.79)

Testing with functions \((\phi_{m_1}, \phi_{m_2}, \phi_f)\) such that \(\phi_f(t, z, y) = \phi_f(t)\) and \(\phi_{m_1}(t, z, y)|_{\Gamma_1} = \phi_{m_2}(t, z, y)|_{\Gamma_2} = \phi_f(t)\) for a.e. \(t \in [0, T]\) in Definition 4.1.1 yields
\[-(b_m(u_m^1), \partial_t \phi_m^1)_{\Omega_t^{m_1}} \leq -(b_m(u_m^2), \partial_t \phi_m^2)_{\Omega_t^{m_2}} - \varepsilon^{k+1} \int_0^T b_f(\bar{u}_f^\varepsilon) \partial_t \phi_f dt \]
\[+ \left(\nabla u_{m^1}, \nabla \phi_m^1\right)_{\Omega_t^{m_1}} + \left(\nabla u_{m^2}, \nabla \phi_m^2\right)_{\Omega_t^{m_2}} \]
\[\leq (f_m, \phi_m^1)_{\Omega_t^{m_1}} + (f_m, \phi_m^2)_{\Omega_t^{m_2}} + \varepsilon \int_0^T f_f \phi_f dt \]
\[+ (b_m(u_m^1), \phi_m^1(0))_{\Omega_t^{m_1}} + (b_m(u_m^2), \phi_m^2(0))_{\Omega_t^{m_2}} + \varepsilon^{k+1} b_f(\bar{u}_f^\varepsilon) \phi_f(0), \] (4.80)

for all \((\phi_{m_1}, \phi_{m_2}, \phi_f) \in W^{1,2}(0, T; \Omega) \times W^{1,2}(0, T; \Omega) \times W^{1,2}(0, T)\) satisfying
\[\phi_{m_1}(t, z, y)|_{\Gamma_1} = \phi_{m_2}(t, z, y)|_{\Gamma_2} = \phi_f(t) \text{ for a.e. } t \in [0, T],\] (4.81)
and
\[\phi_i(T) = 0 \quad \text{for } i \in \{m_1, m_2, f\}.\] (4.82)

Now, one mimics the previous procedure to obtain the same a priori estimate as in Lemma 3.4.2. Note that the space for the time-discrete fracture solution \(W^{1,2}(\Omega_f)\) equals again the space for the time-discrete test function \(\phi_f\), for which reason we are allowed to employ the time-discrete solutions as test functions in the time-discrete weak formulation. Thus, the estimates in Lemmas 4.3.1 and Lemma 4.3.3 can be carried over to the current situation.

Jensen’s inequality immediately gives
\[\|\bar{u}_f^\varepsilon\|_{W^{1,2}(0,T)}^2 \leq \|u_f^\varepsilon\|^2_{\Omega_f^T} \leq \|\bar{u}_f^\varepsilon\|^2_{\Omega_f^T},\] (4.83)
just as in Lemma 4.3.2.

Therefore, the Eberlein–Šmulian theorem 2.2.1 provides a subsequence \(\varepsilon \rightarrow 0\) along which
\[u_m^1 \rightharpoonup u_{m^1} \quad \text{weakly in } L^2(0, T; \Omega),\]
\[u_f^\varepsilon \rightharpoonup u_f \quad \text{weakly in } L^2(0, T).\] (4.84)

Due to \(\lambda < -1\), all derivatives of \(u_m^1\) and \(u_f^\varepsilon\) are bounded uniformly in \(\varepsilon\) by the estimate in Lemma 4.3.1, and we obtain in addition by the Aubin–Lions–Simon theorem 2.2.3
\[u_m^1 \rightarrow u_{m^1} \quad \text{strongly in } L^2(0, T; L^2(\Omega)),\]
\[u_f^\varepsilon \rightarrow u_f \quad \text{strongly in } L^2(0, T).\] (4.85)

We state the upscaling theorem for the case \(\lambda < -1\).
Theorem 4.5.1 (Upscaling theorem for Problem $\mathcal{P}_N$) For the following ranges of $\kappa$ and $\lambda$, these tuples are a solution to the following effective models:

\[
\begin{align*}
\kappa = -1, & \quad \lambda \in (-\infty, -1) : & \quad (U_{m_1}, U_{m_2}, \hat{U}_f) & \quad \text{Effective model VIII,} \\
\kappa \in (-1, \infty), & \quad \lambda \in (-\infty, -1) : & \quad (U_{m_1}, U_{m_2}, \hat{U}_f) & \quad \text{Effective model IX.}
\end{align*}
\]

Proof. We choose arbitrary test functions $(\phi_{m_1}, \phi_{m_2}, \phi_f) \in W^{1,2}(0,T; \mathcal{V}_{m_1}) \times W^{1,2}(0,T; \mathcal{V}_{m_2}) \times W^{1,2}(0,T)$ satisfying equations (4.81) and (4.82) and denote the resulting terms in equation (4.80) by $I_1, \ldots, I_{11}$. The terms $I_1, I_2, I_3, I_4, I_5, I_5, I_7, I_9$, and $I_{10}$ are treated exactly as in the upscaling theorem for Problem $\mathcal{P}$, Theorem 4.4.1. For the term $I_8$, we calculate

\[
|I_8| = \varepsilon \left| \int_0^T \dot{f}_f \phi_f \, dt \right| = \varepsilon \left| \left( \dot{f}_f, \phi_f \right)_{1,T} \right| \to 0,
\]

as in Theorem 4.4.1. For the remaining terms, consider first the case $\kappa > -1$: then we have

\[
|I_3| = \varepsilon^{k+1} \left| \int_0^T \dot{b}(\ddot{\alpha}_f) \partial_t \phi_f \, dt \right| \leq \varepsilon^{k+1} M_\theta \| \ddot{\alpha}_f \|_{\Omega_T^2} \| \partial_t \phi_f \|_{(0,T)} \to 0,
\]

and

\[
|I_{11}| = \varepsilon^{k+1} |\ddot{b}(\phi_{f,1})| \| \phi_f(0) \| \leq \varepsilon^{k+1} M_\theta \| \ddot{b}(\phi_{f,1}) \|_{\Omega_f} \| \phi_f(0) \| \to 0.
\]

For $\kappa = -1$, the term $I_{11}$ is independent of $\varepsilon$. To tackle the term $I_3$, we write

\[
\int_0^T \left( b_f(\ddot{\alpha}_f) - b_f(\bar{U}_f) \right) \partial_t \phi_f \, dt = \int_0^T \left( b_f(\ddot{\alpha}_f) - b_f(\bar{\alpha}_f) \right) \partial_t \phi_f \, dt + \int_0^T \left( b_f(\ddot{\alpha}_f) - b_f(\bar{\alpha}_f) \right) \partial_t \phi_f \, dt.
\]

The second term on the right hand side converges to zero due to the strong convergence in equation (4.85) and the Lipschitz continuity of $b_f$. For the first term on the right hand side, we compute

\[
\left| \int_0^T \left( b_f(\ddot{\alpha}_f) - b_f(\bar{\alpha}_f) \right) \partial_t \phi_f \, dt \right| \leq \| b_f(\ddot{\alpha}_f) - b_f(\bar{\alpha}_f) \|_{(0,T)} \| \partial_t \phi_f \|_{(0,T)}
\]

\[
= \left. \| b_f(\ddot{\alpha}_f) - b(\bar{\alpha}_f) \| \right|_{(0,T)} \| \partial_t \phi_f \|_{(0,T)}
\]

\[
\leq M_\theta \left. \| \ddot{\alpha}_f - \bar{\alpha}_f \| \right|_{(0,T)} \| \partial_t \phi_f \|_{(0,T)}
\]

\[
\leq M_\theta C_{p_f} \| \nabla \ddot{\alpha}_{f,1} \|_{\Omega_T^2} \| \partial_t \phi_f \|_{(0,T)},
\]

where we made use of Poincaré’s inequality. Since we have $\lambda < -1$, Lemma 4.3.1 gives

\[
\| \nabla \ddot{\alpha}_{f,1} \|_{\Omega_T^2} \leq C \varepsilon^{-\frac{1}{2}} \to 0
\]

in the limit $\varepsilon \to 0$. It follows that

\[
I_3 = \int_0^T \ddot{b}(\alpha_f) \partial_t \phi_f \, dt \to \int_0^T \ddot{b}(\bar{U}_f) \partial_t \phi_f \, dt.
\]

The pressure continuity at the interface is shown along the lines of the proof of Theorem 4.4.1 for the case $\lambda < 1$, which concludes the proof. \[\square\]
Chapter 5

Numerical simulations

This chapter is dedicated to numerical studies aiming at the numerical validation of the theoretical upscaling results from the previous chapter. The simulations are carried out using a standard finite volume scheme implemented in MATLAB. The code solves the model in physical variables as stated in Problem $\mathcal{P}$. We use a matching grid, composed of uniform rectangular cells for partitioning the two-dimensional subdomains, and intervals of equal size for the one-dimensional fracture in the effective models. The flux is computed with a two-point flux approximation (TPFA) scheme. We use an implicit Euler discretisation in time with fixed time step, and the modified Picard scheme for the linearisation (see [31, 35] for a comparison of the classical linearisation schemes for Richards’ equation). We employ a monolithic approach and solve the system of equations for the entire domain at once.

Our numerical example deals with the injection of water into a groundwater aquifer, which is crossed by a fracture featuring a higher permeability. Boundary and initial conditions are illustrated in Figure 5.1. Although our analysis was limited to homogeneous Dirichlet conditions, we expect the theoretical results to hold for the more interesting boundary conditions in our numerical example as well. All geometric dimensions (including the pressure heads) are given in meters. In the simulations with a two-dimensional fracture, the fracture width takes the values $\varepsilon \in \{1, 0.1, 0.01, 0.001, 0.0001\}$. We impose no flow conditions on the boundary except for the inflow region in the lower edge of the left matrix block subdomain and the right upper boundary, where a Dirichlet condition allows for outflow. Thus, the water must enter or cross the fracture to leave the domain. The parameters of the van Genuchten parametrisation of the water content and the hydraulic conductivity are listed in Figure 5.1. These parameters are taken from [54], and correspond to silt loam and Touched silt loam in the matrix blocks and in the fracture, respectively. We take the end time of the simulation to be $T = 4 \text{ d}$ and the time step is chosen as $\Delta t = 3 \text{ h}$. The grid size is chosen as $\Delta y = \Delta z = 0.025$ in the solid matrix and $\Delta y = 0.025$ and $\Delta z \in \{0.025, 0.005, 0.001, 0.0002\}$ corresponding to the different fracture widths $\varepsilon$.

![Figure 5.1: Simulation parameters](image)
5.1 Models with continuous pressure

Convergence towards Effective model I: \( \kappa = \lambda = -1 \)

For this choice of \( \kappa \) and \( \lambda \), we expect the solution to converge towards Effective model I featuring the one-dimensional Richards equation in the fracture. Figure 5.2 depicts the pressure head and the saturation of Effective model I at time \( t = 4 \) d. One observes that the pressure head in the lower left matrix block and in the lower fracture has risen, whereas the pressure head in the right matrix block has little increased. The lower left corner of the domain is fully saturated.

Figure 5.3 shows that the averages of the pressure head across the fracture width do almost not differ from each other for different fracture widths with this choice of \( \kappa \) and \( \lambda \). Still, one notices that the averages converge towards the solution to the effective model as \( \epsilon \) approaches zero.

Convergence towards Effective model II: \( \kappa = 0, \lambda = -1 \)

When \( \kappa = 0 \) and \( \lambda = -1 \), the time derivative term is expected to vanish in the limit \( \epsilon \to 0 \), resulting in Effective model II. The solution to Effective model II, plotted in Figure 5.4, is similar to the solution to Effective model I, but the pressure height in the fracture and the surrounding matrix blocks is slightly higher. The difference is more noticeable in Figure 5.5. The averaged pressure heads in the fracture increase monotonically for decreasing \( \epsilon \). This is because the fracture gradually loses its ability to store water as the fracture width becomes smaller. For a fracture width of \( \epsilon = 0.1 \), the averaged pressure head is already close to the pressure head of Effective model II.

Convergence towards Effective model III: \( \kappa = -1, \lambda = 0 \)

For \( \kappa = -1 \) and \( \lambda = 0 \), we expect the flux term in the fracture to vanish and hence the solution to converge towards Effective model III, which incorporates an ordinary differential equation in time in the fracture. Figures 5.6 and 5.7 show that the solution to Effective model III significantly deviates from the solutions to Effective models I and II. Whereas the maximal pressure head in Effective models I and II was less than \(-1.65 \) m in the fracture and the entire fracture is thus clearly unsaturated, the pressure head in Effective model III is positive in the vicinity of the lower domain boundary implying that the fracture is saturated in this region. Since the flux term does not scale with the fracture width in this case, the fracture cannot distribute the water flowing in from the lower left corner fast enough within the fracture and the water is stored in the lower part of the fracture, while the upper fracture half is barely affected by the inflow for small fracture widths. Instead, water crosses the fracture leading to high saturations in the lower right matrix block in contrast to Effective models I and II.

Notice that although our mathematical analysis is limited to the unsaturated case, the averaged pressure heads evidently converge towards the solution to the effective model. In comparison to Effective models I and II, the convergence is considerably slower and for a fracture width of \( \epsilon = 0.01 \), the fracture is still completely unsaturated after 4 days.

Convergence towards Effective model IV: \( \kappa = \lambda = 0 \)

In the case where the fracture equation does not scale with respect to the fracture width, i.e. \( \kappa = \lambda = 0 \), our theoretical results predict convergence towards Effective model IV, in which the pressure and the flux are continuous across the fracture. The solution to the effective model is depicted in Figure 5.8. Large parts of the domain are fully saturated, in all three subdomains.

Figure 5.9 shows that the pressure heads in the fracture are much higher than in Effective model III, due to the fact that the time derivative term (which is sometimes called storage term) disappears in the fracture, and the fracture can neither store nor conduct water along the fracture in the effective model. For this reason, the pressure head at the lower boundary of the fracture is more than \( 1 \) m higher than in Effective model III. Besides, the influence of the inflow at the lower left matrix block reaches out further and a larger part of the fracture is saturated. The convergence speed towards the effective solution is similarly slow as in Effective model III.
Figure 5.2: Solution to Effective model I at $t = 4 \, \text{d}$

Figure 5.3: $z$-averaged pressure profile in the fracture for different $\varepsilon$ at $t = 4 \, \text{d}$ for $\kappa = \lambda = -1$
Figure 5.4: Solution to Effective model II at $t = 4 \text{ d}$

Figure 5.5: $z$-averaged pressure profile in the fracture for different $\varepsilon$ at $t = 4 \text{ d}$ for $\kappa = 0, \lambda = -1$
Figure 5.6: Solution to Effective model III at $t = 4 \text{ d}$

Figure 5.7: $z$-averaged pressure profile in the fracture for different $\varepsilon$ at $t = 4 \text{ d}$ for $\kappa = -1, \lambda = 0$
Figure 5.8: Solution to Effective model IV at \( t = 4 \, \text{d} \)

Figure 5.9: \( z \)-averaged pressure profile in the fracture for different \( \varepsilon \) at \( t = 4 \, \text{d} \) for \( \kappa = \lambda = 0 \)
5.2 Models with discontinuous pressure

Convergence towards Model V: $\kappa = -1$, $\lambda = 1$

We consider now the solutions for the scaling parameters $\kappa = -1$ and $\lambda = 1$. The choice $\lambda = 1$ is a critical one in that it is the smallest parameter for which we can expect the solution to the effective model no longer to be continuous across the fracture since the permeability of the fracture tends to zero too fast. The jump in the fracture is determined by a partial differential equation.

Figure 5.11 shows the solution to the expected effective model in the limit $\epsilon \to 0$, namely Effective model V. It stands out that the solution closely resembles the solution to Effective model III, where $\lambda = 0$ was chosen, and the pressure jump across the fracture is very small. This is due to the high permeability in the fracture that enters the Richards equation in one “virtual” space dimension, which determines the pressure jump (see equation (4.14)), because the high permeability yields a low pressure gradient along the virtual dimension. In the case of a confining fracture with low permeability, one would expect the pressure jump to be considerably higher for this choice $\kappa$ and $\lambda$.

The jump of the pressure head across the interface is presented in Figure 5.12 in logarithmic scaling. As for the solutions corresponding to two-dimensional fractures, the pressure is continuous at the interface by definition. Therefore, the value of $\psi_{m_1}|_{\Gamma_1} - \psi_{m_2}|_{\Gamma_2}$ is evaluated at the centres of the cells adjacent to the fracture. Note that the pressure jump is not monotonic in $\mu$-direction, but exhibits a bump at the saturation front.

Convergence towards Model VI: $\kappa = 0$, $\lambda = 1$

For the choice $\kappa = 0$, $\lambda = 1$, the model is expected to converge towards Effective model VI. The only difference compared to Effective model V consists in the partial differential equation from which the pressure jump is computed, which is now elliptic. Figure 5.13 shows that also in this case, the jump in the pressure head is
small, and the solution to Effective model VI differs only slightly from the solution to Effective model IV (reminiscent of the similarity between the solutions of Effective model III and V). Large parts of the domain are saturated after 4 d. As in the previous case, a bump in the pressure jump occurs at the saturation front, which has advanced further, as Figure 5.14 exhibits.

Convergence towards Model VII: $\kappa = -1, \lambda = 2$

For the scaling parameter $\kappa = -1$ and $\lambda = 2$, the permeability in the fracture depends quadratically on the fracture width and hence decreases rapidly as the fracture width goes to zero. This results in the decoupling of the matrix blocks with a no flow condition at the interface in the limit, expressed by Effective model VII. Since the water can thus not exit the left matrix block, we stop the simulation at an end time of $t = 2$ d – afterwards, the pressure rockets upwards because the injection of more water in the already saturated domain requires huge pressures which are no longer physical.

Figure 5.15 shows the solution to Effective model VII at $t = 2$ d. The fracture saturation is averaged over $z$ in the right subplot. As expected, the water remains in the left matrix block and does not enter or cross the fracture due to the no flow condition. The pressure head at the injection boundary is already higher than 3.5 m at this time.

For this large value of $\lambda$, the convergence occurs remarkably slowly as Figure 5.16 demonstrates: for a fracture width of 0.001, the maximum pressure head in the fracture is not even half as high as in the effective model, and even for a fracture width of 0.0001, the deviation with respect to the effective model is more than 0.5 m. Figure 5.17 reveals that the lower pressure heads at the interface $\Gamma_1$ for larger fracture widths come with (slightly) higher pressure heads at the interface $\Gamma_2$. Hence, as long as the fracture still possesses a positive width, the high pressures in the left matrix block are relieved by pumping water into the barely permeable fracture. This can also be seen in Figure 5.18, which depicts the absolute values of the normal fluxes along the fracture, summed over both interfaces. The narrower the fracture becomes, the smaller become the fluxes near the lower boundary of the interfaces, converging to the no flow condition. The flux intensity across the interfaces is not monotonic along the fracture, but has bumps for $\varepsilon$ between 0.1 and 0.001, which move in positive $y$-direction as the fracture width decreases.

Convergence towards Model VII*: $\kappa = 0, \lambda = 2$

For the choice of parameters $\kappa = 0, \lambda = 2$, convergence towards the same effective model as in the previous case is expected – the equation for the fracture which remains in the convergence analysis is meaningless for the effective model as it decouples anyway in the limit owing to the no flow interface condition.

Now, the storage term does not scale with respect to the fracture width, leading to higher pressures on both interfaces as the capacity of the fracture to store water gradually decreases (see Figures 5.19 and 5.20). On one hand, this means that the pressure $\psi_{m_1}$ on $\Gamma_1$ approaches the solution to the effective model more rapidly from below, whereas on the other hand, the rise in pressure $\psi_{m_2}$ on $\Gamma_2$ over time is noticeably higher than in the previous case considered, yielding slower convergence towards the effective model on $\Gamma_2$.

Figure 5.21 shows that the flux profile along the fracture is smoother than for $\kappa = -1$, but the magnitude of the normal fluxes across the interfaces near the lower boundary are similar in both cases for $\varepsilon \geq 0.001$. For the smallest positive fracture width considered, that is $\varepsilon = 0.0001$, the fluxes across the interfaces are lower for $\kappa = 0$ than for $\kappa = -1$. 
Figure 5.11: Solution to Effective model V at $t = 4\, d$.

Figure 5.12: Pressure jump across the fracture in logarithmic scaling for different $\epsilon$ at $t = 4\, d$ for $\kappa = -1$, $\lambda = 1$. 
Figure 5.13: Solution to Effective model VI at $t = 4 \ d$

Figure 5.14: Pressure jump across the fracture in logarithmic scaling for different $\varepsilon$ at $t = 4 \ d$ for $\kappa = 0, \lambda = 1$
Figure 5.15: Solution to Effective model VII at \( t = 2 \, \text{d} \)

Figure 5.16: Pressure head \( \psi_{m_1} \) at \( \Gamma_1 \) for different \( \varepsilon \) at \( t = 2 \, \text{d} \) for \( \kappa = -1, \lambda = 2 \)
Figure 5.17: Pressure head $\psi_{m2}$ at $\Gamma_2$ for different $\varepsilon$ at $t = 2\,d$ for $\kappa = -1, \lambda = 2$

Figure 5.18: Sum of absolute values of normal fluxes at $\Gamma_1$ and $\Gamma_2$ for different $\varepsilon$ at $t = 2\,d$ for $\kappa = -1, \lambda = 2$
Figure 5.19: Pressure head $\psi_{m_1}$ at $\Gamma_1$ for different $\varepsilon$ at $t = 2$ d for $\kappa = 0, \lambda = 2$

Figure 5.20: Pressure head $\psi_{m_2}$ at $\Gamma_2$ for different $\varepsilon$ at $t = 2$ d for $\kappa = 0, \lambda = 2$
Figure 5.21: Sum of absolute values of normal fluxes at \( \Gamma_1 \) and \( \Gamma_2 \) for different \( \epsilon \) at \( t = 2 \) d for \( \kappa = 0, \lambda = 2 \)

### 5.3 Additional models for no flow conditions in the fracture

**Convergence towards Model VIII: \( \kappa = -1, \lambda = -2 \)**

For the choice \( \kappa = -1 \) and \( \lambda = -2 \), theory predicts convergence towards a spatially constant solution in the fracture as described by Effective model VIII. This can only happen if Neumann no flow boundary conditions are imposed because Dirichlet conditions fix the pressure head inadmissibly and non-zero Neumann conditions imply a pressure gradient contradicting the spatially constant pressure. We hence carry out the simulation for Problem \( P_N \). Since the fracture permeability increases quadratically with respect to decreasing fracture widths, we execute simulations only for \( \epsilon \geq 0.001 \) – for smaller fracture widths, the linearisation scheme failed to converge.

Figure 5.22 shows the solution to the effective model at \( t = 4 \) d. Due to the “infinite” permeability in the fracture, the inflowing water at the lower part of the fracture is immediately distributed along the fracture and is discharged into both matrix blocks at the upper part of the fracture, observable by the pressure gradient pointing towards the fracture in this region.

The \( z \)-averaged pressure head profiles in the fracture are plotted in Figure 5.23. While the pressure head slightly decreases along the fracture for \( \epsilon \geq 0.1 \), it is virtually constant for \( \epsilon \leq 0.01 \). However, a small deviation of 0.003 [m] between the value towards which the solutions of Problem \( P_N \) seem to converge and the solution to Effective model VIII is noticeable. This flaw occurred for other choices of \( \kappa \) and \( \lambda \) in our simulations as well when Neumann conditions were imposed on both sides of the fracture, for which reason we suspect that this is due to numerical errors.

**Convergence towards Model IX: \( \kappa = 0, \lambda = -2 \)**

Finally, we consider the parameters \( \kappa = 0 \) and \( \lambda = -2 \) for Problem \( P_N \). This means that the storage capacity of the fracture vanishes in the limit \( \epsilon \to 0 \), but the pressure within the fracture is instantly equalised when water flows in or out, as modelled by Effective model IX. For this case, the linearisation scheme only converged for \( \epsilon \geq 0.01 \) owing to the fast increase of the fracture permeability.

Figure 5.24 shows that the pressure head in the fracture has risen much more than for the case \( \kappa = -1 \) and the matrix blocks are more saturated, too. Figure 5.25 exhibits the increase in pressure in the fracture as the fracture width becomes smaller due to the decreasing storage capacity.
Figure 5.22: Solution to Effective model VIII at $t = 4 \text{ d}$

Figure 5.23: $z$-averaged pressure profile in the fracture for different $\varepsilon$ at $t = 4 \text{ d}$ for $\kappa = -1, \lambda = -2$
5.4 Discussion of the numerical results

In summary, our simulations show the convergence towards the correct effective models as derived by rigorous upscaling for all choices of $\kappa$ and $\lambda$ under consideration. This is despite the fact that fully saturated zones occurred in the simulations, which was excluded in the analysis, and despite the more complex boundary conditions as compared to the analysis. It turns out that the parameters $\kappa$ and $\lambda$ are crucial as regards the convergence speed: for $\lambda = -1$, the effective models are an accurate approximation of the original model even for wide fractures, whereas the effective models are only suitable to approximate very thin fractures for larger values of $\lambda$. This behaviour is in accordance with the estimate in Lemma 4.3.5, which gives sharper bounds for the error between the fracture solution and its $z$-average as $\lambda$ becomes smaller. In addition, the choice $\kappa = -1$ resulted in faster convergence compared with $\kappa = 0$ in our simulations.
Discussion and outlook

In this thesis, we considered a two-dimensional model for subsurface flow through a porous medium containing a fracture, consisting of Richards’ equation for the description of the flow in both the fracture and the surrounding matrix blocks. The parametrisation of the hydraulic properties in the fracture and the matrix blocks may differ from one another. The model is supplemented with physical interface conditions expressing the conservation of mass and momentum.

First, we fixed the fracture width and applied Rothe’s method, i.e. an implicit Euler time discretisation, in order to prove that the coupled model possesses a solution, provided that the time-discrete system has a solution. Proving the latter lies out of the scope of this thesis, but existence results for coupled elliptic problems can be found in e.g. [27]. For the existence proof, we established a priori bounds for the time-discrete solution, independent of the choice of the time step. In a next step, time-interpolated functions were introduced. By virtue of the a priori bounds, compactness arguments, namely the Eberlein–Šmulian theorem and the Aubin–Lions–Simon theorem, could be applied in order to get convergent subsequences as the time step approaches zero. The limits of these subsequences were shown to be a solution to the model.

Then, the fracture width was viewed as a model parameter and by means of rigorous upscaling, we proved that one obtains different effective models in the limit of vanishing fracture width. Depending on the scaling of the permeability and porosity with respect to the fracture width, a variety of models can arise in the limit. In these effective models, the pressure can be continuous or discontinuous at the fracture, and the flux in the fracture can be zero, given by a differential equation, or described as an interface condition between the surrounding matrix blocks. The upscaling was carried out by rescaling the fracture and the variables related to it, by proving uniform a priori bounds with respect to the fracture width, and again by compactness arguments. Finally, the convergence towards the effective models was examined numerically, and the simulation results were consistent with our theoretical findings.

Many extensions of this work are possible: additional effective models could be derived by allowing a permeability tensor and introducing scaling parameters with respect to different coordinate directions. Moreover, the model could be supplemented by (reactive) transport, or gravity could be incorporated. The analysis in this thesis was carried out only for the case of a fully unsaturated porous medium – with more elaborate analysis, one could extent the results to the degenerate elliptic-parabolic case. For the upscaling, one could try to avoid the additional boundedness assumption for the derivatives of the fracture solution in the case \( \lambda > -1 \).

As regards the numerics, domain decomposition schemes in the spirit of [6] could be applied in order to accelerate the simulations. Besides, it would be interesting to consider the upscaling for more complex fracture geometries or entire fracture networks.
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List of Figures

1.1 Definition of the REV [5, 25] ................................................. 5
1.2 Schematic representation of flow through a porous medium and the REV scale ............................................. 6
1.3 Wetting angle $\gamma$ at the interface of two fluid phases and one solid phase ................................................. 8
1.4 Typical profiles of $\theta(\psi)$ and $K(\theta(\psi))$ as given by the van Genuchten–Mualem model ..................................... 10
1.5 Geometry of the fracture and the surrounding matrix blocks .......................................................... 11
1.6 Dimensionless geometry of the fracture and the surrounding matrix blocks ................................................ 12

4.1 Geometry in rescaled variables and upscaled geometry ......................... 35
4.2 Overview of the resulting effective models for different values of the scaling parameters $\kappa$ and $\lambda$ .............................. 37

5.1 Simulation parameters .......................................................... 55
5.2 Effective model I: Solution at $t = 4\,d$ ........................................ 57
5.3 Effective model I ($\kappa = \lambda = -1$): $z$-averaged pressure profile in the fracture at $t = 4\,d$ ......................... 57
5.4 Effective model II: Solution at $t = 4\,d$ ........................................ 58
5.5 Effective model II ($\kappa = 0, \lambda = -1$): $z$-averaged pressure profile in the fracture at $t = 4\,d$ ....................... 58
5.6 Effective model III: Solution at $t = 4\,d$ ........................................ 59
5.7 Effective model III ($\kappa = -1, \lambda = 0$): $z$-averaged pressure profile in the fracture at $t = 4\,d$ ..................... 59
5.8 Effective model IV: Solution at $t = 4\,d$ ........................................ 60
5.9 Effective model IV ($\kappa = \lambda = 0$): $z$-averaged pressure profile in the fracture at $t = 4\,d$ ....................... 60
5.10 Convergence towards Effective models I – IV: $L^2$ error of $z$-averaged pressure head at $t = 4\,d$ ...................... 61
5.11 Effective model V: Solution at $t = 4\,d$ ........................................ 63
5.12 Effective model V ($\kappa = -1, \lambda = 1$): pressure jump across the fracture at $t = 4\,d$ ................................. 63
5.13 Effective model VI: Solution at $t = 4\,d$ ........................................ 64
5.14 Effective model VI ($\kappa = 0, \lambda = 1$): pressure jump across the fracture at $t = 4\,d$ ................................. 64
5.15 Effective model VII: Solution at $t = 2\,d$ ........................................ 65
5.16 Effective model VII ($\kappa = -1, \lambda = 2$): pressure head $\psi_m$ at $t = 2\,d$ .................................................. 65
5.17 Effective model VII ($\kappa = -1, \lambda = 2$): pressure head $\psi_{m_2}$ at $t = 2\,d$ .................................................. 66
5.18 Effective model VII ($\kappa = -1, \lambda = 2$): fluxes across the interfaces at $t = 2\,d$ ........................................ 66
5.19 Effective model VII* ($\kappa = 0, \lambda = 2$): pressure head $\psi_m$ at $t = 2\,d$ .................................................. 67
5.20 Effective model VII* ($\kappa = 0, \lambda = 2$): pressure head $\psi_{m_2}$ at $t = 2\,d$ .................................................. 67
5.21 Effective model VII* ($\kappa = 0, \lambda = 2$): fluxes across the interfaces at $t = 2\,d$ ........................................ 68
5.22 Effective model VIII: Solution at $t = 4\,d$ ........................................ 69
5.23 Effective model VIII ($\kappa = -1, \lambda = -2$): $z$-averaged pressure profile in the fracture at $t = 4\,d$ ............ 69
5.24 Effective model IX: Solution at $t = 4\,d$ ........................................ 70
5.25 Effective model IX ($\kappa = 0, \lambda = -2$): $z$-averaged pressure profile in the fracture at $t = 4\,d$ ............ 70
List of physical quantities

Greek letters
\( \alpha \)  Van Genuchten parameter  \( \mathrm{m}^{-1} \)
\( \Gamma \)  Interface
\( \gamma \)  Wetting angle
\( \varepsilon \)  Dimensionless fracture width
\( \theta \)  Volume content
\( \theta_R \)  Residual water content
\( \theta_S \)  Saturated water content
\( \kappa \)  Scaling parameter for porosity
\( \lambda \)  Scaling parameter for permeability
\( \mu \)  Dynamic viscosity  \( \mathrm{kg \ m}^{-1} \mathrm{s}^{-2} \)
\( \rho \)  Mass density  \( \mathrm{kg \ m}^{-3} \)
\( \sigma \)  Interfacial tension  \( \mathrm{kg \ s}^{-2} \)
\( \tau \)  Non-equilibrium coefficient  \( \mathrm{kg \ m}^{-1} \mathrm{s}^{-2} \)
\( \phi \)  Porosity
\( \psi \)  Pressure head  \( \mathrm{m} \)
\( \Omega \)  Domain, bounded subset of \( \mathbb{R}^d \)
\( \partial \Omega \)  Boundary of domain \( \Omega \)

Roman letters
\( A \)  Area  \( \mathrm{m}^2 \)
\( b \)  Water content as a function of \( u \)
\( d \)  Capillary tube diameter  \( \mathrm{m} \)
\( f \)  Source / sink term  \( \mathrm{s}^{-1} \)
\( g \)  Gravitational constant  \( \mathrm{m \ s}^{-2} \)
\( K \)  Hydraulic conductivity  \( \mathrm{m \ s}^{-1} \)
\( K_S \)  Saturated hydraulic conductivity  \( \mathrm{m \ s}^{-1} \)
\( k \)  Intrinsic permeability  \( \mathrm{m}^2 \)
\( k_r \)  Relative permeability
\( K \)  Kirchhoff transformation
\( L \)  Edge length of a solid block  \( \mathrm{m} \)
\( l \)  Fracture width  \( \mathrm{m} \)
\( m \)  Mass  \( \mathrm{kg} \)
\( N \)  Number of discrete times
\( n \)  Number of phases
\( n \)  Van Genuchten parameter
\( \vec{n} \)  Normal vector
\( p \)  Pressure  \( \mathrm{kg \ m}^{-1} \mathrm{s}^{-2} \)
\( p_c \)  Capillary pressure  \( \mathrm{kg \ m}^{-1} \mathrm{s}^{-2} \)
\( Q \)  Volume discharge  \( \mathrm{m}^3 \mathrm{s}^{-1} \)
\( q \)  Source / sink density  \( \mathrm{kg \ m}^{-3} \mathrm{s}^{-1} \)
\( S \)  Saturation
\( T \)  Final time  \( \mathrm{s} \)
\( t \)  Time  \( \mathrm{s} \)
\( \Delta t \)  Time step  \( \mathrm{s} \)
$u$  Kirchhoff variable  \( m^2 \text{s}^{-1} \)

$V$  Volume  \( m^3 \)

$v$  Darcy velocity  \( m \text{s}^{-1} \)

$v_{av}$  Average velocity  \( m \text{s}^{-1} \)

$W$  Energy functional  \( m \)

**Subscripts**

- $f$: Subscript for fracture
- $I$: Subscript for initial condition
- $i$: Subscript for subdomain $m_1, m_2, \text{or} f$
- $j$: Subscript for matrix block 1 or 2
- $k$: Subscript for discrete time $t_k$
- $m$: Subscript for matrix block
- $m_1$: Subscript for left matrix block
- $m_2$: Subscript for right matrix block
- $n$: Subscript for non-wetting phase
- $\Delta t$: Subscript for the time step
- $w$: Subscript for wetting phase
- $\alpha$: Subscript for phase

**Superscripts**

- $d$: Superscript for spatial dimension, here 1, 2, or 3
- $i$: Superscript for subdomain $m_1, m_2, \text{or} f$
- $k$: Superscript for discrete time $t_k$
Bibliography


