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Potential-Function Proofs for First-Order Methods

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Abstract

This note discusses proofs for convergence of first-order methods based on simple potential-function arguments. We cover methods like gradient descent (for both smooth and non-smooth settings), mirror descent, and some accelerated variants.

1 Introduction

Recall that a set \( K \subseteq \mathbb{R}^d \) is convex if for all \( x, y \in K \), the convex combination \( \lambda x + (1 - \lambda) y \in K \) for all \( \lambda \in [0, 1] \). A function \( f : \mathbb{R}^d \to \mathbb{R} \) over a convex set \( K \) is convex if

\[
f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y) \quad \forall x, y \in K, \forall \lambda \in [0, 1].
\]

This is called the zeroth-order definition. There are other equivalent notions: if the function is differentiable, the first-order definition is that \( f \) is convex over \( K \) if

\[
f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle \quad \forall x, y \in K.
\] (1.1)

(The second-order definition says that a twice-differentiable \( f \) is convex if its Hessian matrix \( \nabla^2 f \) is positive-semidefinite.) For this note, we assume our convex functions \( f \) are closed and differentiable. However, the proofs extend to non-differentiable functions in the natural way, using subgradients. (See, e.g., [HUL01] for more definitions and background on convexity and subgradients.)

The Problems. Given a convex function \( f : \mathbb{R}^d \to \mathbb{R} \), the (unconstrained) convex minimization problem wants to find a point \( \hat{x} \) such that

\[
f(\hat{x}) - \min_{x \in \mathbb{R}^d} f(x) \leq \epsilon.
\]

In the constrained version of the problem, we are also given a convex set \( K \), and the goal is to find a point \( \hat{x} \in K \) to achieve

\[
f(\hat{x}) - \min_{x \in K} f(x) \leq \epsilon.
\]

In either case, let \( x^* \) denote the minimizer for \( f(\cdot) \).

In online convex optimization over convex set \( K \), at each timestep \( t = 1, 2, \cdots \), the algorithm outputs a point \( x_t \in K \) and an adversary produces a convex function \( f_t \), so that the loss for the algorithm is \( f_t(x_t) \). Now the regret of the algorithm is

\[
\sum_{t=1}^T f_t(x_t) - \min_{x \in K} \sum_{t=1}^T f_t(x).
\]

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The Gradient Descent Framework. The so-called “gradient descent” framework is a class of iterative first-order methods for solving these convex minimization problems. Given any point \( x \), we assume that we can get the gradient \( \nabla f(x) \) of the function \( f \) at any point \( x \). Since the gradient gives the direction of steepest increase in function value, a natural approach is to move in the direction of the negative gradient. Variants of this general versatile approach have been central to convex optimization for many years.

In this note we give convergence arguments for many commonly studied versions of first-order methods using simple potential-function arguments. It is remarkable how far we can get using these extremely simple ideas. The intuition of viewing these methods as trying to minimize a potential has long been known to specialists; e.g., see the text of Nemirovski and Yudin [NY83, pp. 85–88] for a continuous perspective via Lyapunov functions, and more explicitly in recent papers [WWJ16, SBC16, KBB15, WRJ16, DO17] relating continuous and discrete updates. (E.g., Krichene et al. [KBB15] give the potential function we use in Section 5.2.) We feel this “potential-function” view gives a principled way to proving convergence bounds, since it imparts a clear structure and direction to proofs. We hope others will also find this perspective useful.

For the most part, the proofs here follow the same or similar steps as prior proofs, albeit sometimes in a different (and to our eyes, intuitive) order. Indeed, this note should be viewed as being a technical survey, giving a potential-based interpretation of commonly-known proofs. We omit most citations and instead refer readers to the excellent texts and lecture notes at the end of this section.

1.1 Results and Organization

All of the proofs use the same general potential:

\[
\Phi_t = a_t \cdot (f(x_t) - f(x^*)) + b_t \cdot \text{(distance from } x_t \text{ to } x^*). \tag{1.2}
\]

Naturally, different proofs use slightly different multipliers \( a_t, b_t \), and even the distance functions may vary. However, the general approach remains the same: we show that \( \Phi_{t+1} - \Phi_t \leq B_t \) (where \( B_t \) is often zero). Since the potential and distance terms remain non-negative, the telescoping sum gives

\[
\Phi_T \leq \Phi_0 + \sum_{t=1}^{T} B_t \implies f(x_T) - f(x^*) \leq \frac{\Phi_0 + \sum_{t} B_t}{a_T}.
\]

We begin in Section 2 with proofs of the basic (projected) gradient descent, for general and strongly convex functions; these even work in the online regret-minimization setting. Here the analysis is more along the lines of amortized analysis: we show that the amortized cost, namely the cost of the algorithm plus the increase in potential is at most the optimal cost (plus \( B \)). I.e., \( f_t(x_t) + (\Phi_{t+1} - \Phi_t) \leq f_t(x^*) + B \). This telescopes to imply the average regret is \( \frac{1}{T}(\sum_{t=1}^{T} (f(x_t) - f(x^*)) \leq B + \Phi_0/T \). The potential here is very simple: we set \( a_t = 0 \) and just use the distance of the current point \( x_t \) to the optimal point \( x^* \) (according to the “right” distance). E.g., for basic Gradient Descent, the potential is just a scaled version of \( \|x_t - x^*\|^2 \).

Next, we will see proofs for convergence of smooth convex functions in Section 3. In the simplest case we just set \( b_t = 0 \) and use \( a_t = t \) in (1.2) to prove \( B_t \approx 1/t \). This gives an error of \( \approx (\log T)/T \), which is in the right ballpark. (This can be optimized using better settings of the multipliers.) The proofs for projected smooth gradient descent, gradient descent for well-conditioned functions, and the Frank-Wolfe method, all follow this template.
We then proceed to mirror descent, which is a substantial generalization of gradient descent to more general norms. While the language necessarily becomes more technical (relying on dual norms and Bregman divergences), the ideas remain clean. Indeed, the structure of the potential-based proofs remains essentially the same as for basic gradient descent; the potential is now a Bregman divergence, a natural generalization of the squared distance. These proofs appear in Section 4.

Finally, we see potential-based proofs of Nesterov’s accelerated gradient descent method for smooth convex functions and well-conditioned convex functions. We can now use the full power of both terms in the potential function (1.2). The proofs are marginally more involved, but still rely on bounding the change in potential. In hindsight, the analysis for basic smooth optimization from Section 3 directly suggests how to get the acceleration.

References: There are many excellent sources for other proofs of these results, with comprehensive bibliographies; e.g., see the authoritative notes by Nesterov [Nes04], Ben-Tal and Nemirovski [BTN01], the monograph by Bubeck [Bub15], the textbooks by Cesa-Bianchi and Lugosi [CBL06], Hazan [Haz16], or lecture notes by Duchi [Duc16], Shalev-Shwartz [SS12], and Vishnoi [Vis15]. Recent work of Diakonikolas and Orecchia [DO17] gives alternate proofs of these methods viewing them as discretizing certain continuous dynamics, and controlling a certain “duality gap”.

Disclaimer: To avoid technical complications, we assume that our convex functions are closed and convex, and also differentiable; the convex sets $K$ are also closed with non-empty interior. We only work with the Euclidean norm $\| \cdot \|_2$ for the first few sections; general norms are discussed in Section 4.

2 Online Analyses

2.1 Basic Gradient Descent

The basic analysis works even for the online convex optimization case: at each step we are given a function $f_t$, we play $x_t$, and want to minimize the regret. In this case the update rule is:

$$x_{t+1} \leftarrow x_t - \eta_t \nabla f_t(x_t)$$

(2.3)

An equivalent form for this update, that is easily verified by taking derivatives with respect to $x$, is:

$$x_{t+1} \leftarrow \arg \min_x \left\{ \frac{1}{2} \| x - x_t \|^2 + \eta_t \langle x, \nabla f_t(x_t) \rangle \right\}$$

(2.4)

Intuitively, we want to move in the direction of the negative gradient, but don’t want to move too far.

Theorem 2.1 (Basic Gradient Descent). Let $x^* \in \mathbb{R}^n$. Gradient descent starting at point $x_0 \in \mathbb{R}^n$ and using updates (2.3) with step size $\eta_t = \frac{D}{G \sqrt{T}}$ for $T$ steps guarantees an average regret of

$$\frac{1}{T} \sum_{t=0}^{T-1} (f_t(x_t) - f_t(x^*)) \leq \frac{\eta G^2}{2} + \frac{D^2}{2 \eta T} \leq \frac{DG}{\sqrt{T}}.$$

Here $G$ is an upper bound on $\| \nabla f_t(x) \|$, and $D := \| x_0 - x^* \|$. 

3
Proof. Consider the potential function

\[ \Phi_t = \frac{1}{2\eta} \| x_t - x^* \|^2 \]  

which is positive for all \( t \). We show that, for some upper bound \( B \),

\[ f_t(x_t) - f_t(x^*) + \Phi_{t+1} - \Phi_t \leq B. \]  

(2.6)

Summing over all times \( t \), the average regret is

\[ \frac{1}{T} \sum_{t=0}^{T-1} (f_t(x_t) - f_t(x^*)) \leq B + \frac{1}{T} (\Phi_0 - \Phi_T) \leq B + \frac{\Phi_0}{T} = B + \frac{D^2}{2\eta T}. \]  

(2.7)

Now we can compute \( B \), and then balance the two terms. While the potential uses differences of the form \( x_t - x^* \), the key is to express as much as possible in terms of \( x_{t+1} - x_t \), because the update rule (2.3) implies

\[ x_{t+1} - x_t = -\eta \nabla f_t(x_t). \]  

(2.8)

**The Change in Potential.** Using that \( \| a + b \|^2 - \| a \|^2 = 2\langle a, b \rangle + \| b \|^2 \) for the Euclidean norm,

\[ \frac{1}{2}(\| x_{t+1} - x^* \|^2 - \| x_t - x^* \|^2) = \langle x_{t+1} - x_t, x_t - x^* \rangle + \frac{1}{2} \| x_{t+1} - x_t \|^2 \]

\[ = \eta_t \langle \nabla f_t(x_t), x^* - x_t \rangle + \frac{\eta_t^2}{2} \| \nabla f_t(x_t) \|^2 \]  

(2.9)

**The Amortized Cost:** Setting \( \eta_t = \eta \) for all steps,

\[ f_t(x_t) - f_t(x^*) + \Phi_{t+1} - \Phi_t \]

\[ = f_t(x_t) - f_t(x^*) + \langle \nabla f_t(x_t), x^* - x_t \rangle + \frac{\eta}{2} \| \nabla f_t(x_t) \|^2 \]

\[ \leq 0 + \frac{\eta}{2} \| \nabla f_t(x_t) \|^2 \leq \frac{\eta G^2}{2}. \]  

(by convexity, and the bound on gradients)

Substituting for \( B \) in (2.7) and simplifying with \( \eta = \frac{D}{G\sqrt{T}} \), we get the theorem. \( \square \)

The regret bound implies a convergence result for the offline case, i.e., for the case where \( f_t = f \) for all \( t \). Here, setting \( \tilde{x} := \frac{1}{T} \sum_{t=0}^{T-1} x_t \) shows

\[ f(\tilde{x}) - f(x^*) = f \left( \frac{1}{T} \sum_t x_t \right) - f(x^*) \leq \frac{1}{T} \sum_t (f(x_t) - f(x^*)) \leq \frac{DG}{\sqrt{T}} \leq \varepsilon, \]

as long as \( T \geq \left( \frac{DG}{\varepsilon} \right)^2 \) and \( \eta = \frac{\varepsilon}{GT} \).

If the time horizon \( T \) is unknown, setting a time-dependent step size of \( \eta_t = \frac{D}{G\sqrt{t}} \) works, with an identical proof. It is also well-known that the convergence bound above is the best possible in general, modulo constant factors (see, e.g., [Nes04, Thm 3.2.1] or [Bub15, Thm 3.13]).
2.1.1 Projected Gradient Descent

If we want to solve the constrained minimization problem for a convex body $K$, we update as follows:

\[
\begin{align*}
    x'_{t+1} &\leftarrow x_t - \eta_t \nabla f_t(x_t) \\
    x_{t+1} &\leftarrow \Pi_K(x'_{t+1}).
\end{align*}
\]  

(2.10)
(2.11)

where $\Pi_K(x') := \min_{x \in K} \|x - x'\|$ is the projection of $x'$ onto the convex body $K$. See Figure 2.1.

Proposition 2.2 (Pythagorean Property). Given convex body $K \subseteq \mathbb{R}^n$, let $a \in K$ and $b' \in \mathbb{R}^n$. Let $b = \Pi_K(b')$. Then $\langle a - b, b' - b \rangle \leq 0$. Hence $\|a - b\|^2 \leq \|a - b'\|^2$.

Proof. For the first part, the tangent plane at $b$ has $b'$ on one side, and all of $K$ (and hence $a$) on the other side. Hence the angle between $b' - b$ and $a - b$ must be obtuse, giving the negative inner product. For the second part, $\|a - b'\|^2 = \|a - b\|^2 + \|b - b'\|^2 + 2\langle a - b, b - b' \rangle$. But the latter two terms are positive, which proves the lemma.

Using this, we get that for any point $x^* \in K$,

\[
\|x_{t+1} - x^*\|^2 \leq \|x'_{t+1} - x^*\|^2
\]

Using the same potential function (2.5), this inequality implies:

\[
f_t(x_t) - f_t(x^*) + \Phi_{t+1} - \Phi_t \leq f_t(x_t) - f_t(x^*) + \frac{1}{2\eta_t}(\|x'_{t+1} - x^*\|^2 - \|x_t - x^*\|^2),
\]

or in other words, the projection only helps and we can follow the analysis from §2.1 starting at (2.9) to bound the amortized cost by $\frac{\eta G^2}{2}$. So this gives a regret bound identical to that of Theorem 2.1.

2.2 Strong Convexity Analysis

Let us a prove a better regret (and convergence) bound when the functions are “not too flat”. A function $f$ is $\alpha$-strongly convex if for all $x, y$

\[
f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\alpha}{2} \|y - x\|^2.
\]  

(2.12)

For $\alpha$-strongly convex functions $f_t$, we use the same update step but vary the step size $\eta_t$. Specifically,

\[
x_{t+1} \leftarrow x_t - \eta_t \nabla f_t(x_t),
\]  

(2.13)
where $\eta_t = \frac{1}{\alpha(t+1)}$. We only present a proof for the unconstrained case, the constrained case follows as in §2.1.1.

**Theorem 2.3** (GD: Strong Convexity). If the functions $f_t$ are $\alpha$-strongly convex and $G$ is an upper bound on $\|\nabla f_t(x)\|$, the update rule (2.13) with $\eta_t = \frac{1}{\alpha(t+1)}$ guarantees an average regret of

$$
\frac{1}{T} \sum_{t=0}^{T-1} \left( f_t(x_t) - f_t(x^*) \right) \leq \frac{G^2 \log T}{2T\alpha}.
$$

**Proof.** The potential function is now

$$
\Phi_t = \frac{1}{2} \eta_t - \frac{1}{2} \|x_t - x^*\|^2 = \frac{t\alpha}{2} \|x_t - x^*\|^2.
$$

The Change in Potential:

$$
\Phi_{t+1} - \Phi_t = \frac{\alpha t}{2} \|x_{t+1} - x^*\|^2 - \frac{\alpha(t - 1)}{2} \|x_t - x^*\|^2
$$

$$
= \frac{\alpha}{2} \|x_t - x^*\|^2 + \frac{1}{2\eta_t} \left( \|x_{t+1} - x^*\|^2 - \|x_t - x^*\|^2 \right)
$$

$$
= \frac{\alpha}{2} \|x_t - x^*\|^2 + \langle \nabla f_t(x_t), x^* - x_t \rangle + \frac{\eta_t}{2} \|\nabla f_t(x_t)\|^2
$$

(by 2.9)

The Amortized Cost:

$$
\left( f_t(x_t) - f_t(x^*) + \Phi_{t+1} - \Phi_t \right)
$$

$$
= f_t(x_t) - f_t(x^*) + \frac{\alpha}{2} \|x_t - x^*\|^2 + \langle \nabla f_t(x_t), x^* - x_t \rangle + \frac{\eta_t}{2} \|\nabla f_t(x_t)\|^2
$$

$$
\leq 0 \text{ by } \alpha\text{-strong convexity}
$$

$$
\leq \frac{\eta_t}{2} \|\nabla f_t(x_t)\|^2 \leq \frac{\eta_t G^2}{2} \quad \text{(by bound on gradients)}
$$

(2.15)

Now summing over all time steps $t$, the total regret is

$$
\sum_t \left( f_t(x_t) - f_t(x^*) \right) \leq \Phi_0 + \sum_t \frac{\eta_t}{2} G^2 \leq 0 + \frac{G^2 \log T}{2\alpha}.
$$

Hence total regret only increases logarithmically as $\log T$ with time if the $f_t$ are strongly convex, as opposed to $\sqrt{T}$ in Theorem 2.1.

Interestingly in the offline optimization setting where $f_t = f$, using the same analysis but a better averaging shows a convergence rate of $O(1/T)$ with respect to a convex combination of the points $x_t$.

**Theorem 2.4.** Let $f$ be $\alpha$-strongly convex with gradients satisfying $\|\nabla f(x)\| \leq G$, and $x_t$ be the iterates produced by applying the update rule (2.3) with $\eta_t = \frac{1}{\alpha t}$. For any $T \geq 1$, let $\bar{x}_T := \frac{1}{T} \sum_{t=1}^{T} \lambda_t x_t$ denote the convex combination of $x_t$ with $\lambda_t = \frac{2}{T(T+1)}$. Then,

$$
f(\bar{x}_T) - f(x^*) \leq \frac{G^2}{\alpha(T+1)}.
$$

**Proof.** Instead of summing up (2.15) directly over $t$ in the regret analysis above, we first multiply (2.15)
by $t$, and then sum over $t$ to obtain

$$
\sum_{t=1}^{T} t(f_t(x_t) - f_t(x^*)) \leq \frac{1}{2\alpha} TG^2.
$$

Using $f_t = f$ and dividing by $T(T + 1)/2$ throughout, and by the convexity of $f$ we obtain

$$
f(\bar{x}_T) - f(x^*) \leq \frac{G^2}{\alpha(T + 1)}.
$$

$\square$
3 Bounds for Smooth Functions

We now turn to the setting where the functions are smooth, i.e., not too curved. We know that in the online case, the average regret of $O(1/\sqrt{T})$ is tight even for linear functions. However we get better guarantees for the offline setting where the function $f_t = f$ for all time steps. The potential functions now look more like (1.2), and use the difference $(f(x_t) - f(x^*))$ in function value, not just in action space.

Define function $f$ to be $\beta$-smooth if for all $x, y$

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\beta}{2} \|y - x\|^2.$$  \hspace{1cm} (3.16)

Observe the inequality here is in the opposite direction from the definitions of convexity (1.1) and strong-convexity (2.12).

3.1 Smooth Gradient Descent

The update rule here is simple (where we use $\nabla_t := \nabla f(x_t)$ for brevity).

$$x_{t+1} \leftarrow x_t - \frac{1}{\beta} \nabla_t.$$  \hspace{1cm} (3.17)

We first show an analysis based on a very natural potential, that gives a slightly sub-optimal bound with an additional $\log T$ factor. Later we will see how to improve this by modifying potential slightly.

**Theorem 3.1 (Smooth Functions).** If $f$ is $\beta$-smooth and $D := \max_x \{\|x - x^*\|_2 \mid f(x) \leq f(x_0)\}$, the update rule (3.17) guarantees

$$f(x_T) - f(x^*) \leq \beta \frac{D^2(1 + \ln T)}{2T}.$$  \hspace{1cm} (3.19)

**Proof.** To show a convergence rate of $O(1/t)$, perhaps the most natural approach is to consider the potential

$$\Phi_t = t \cdot (f(x_t) - f(x^*))$$

and try to show that $\Phi_T = O(1)$. This works, but gives a weaker bound of $\Phi_T = O(\log T)$ (note that conveniently, $\Phi_0 = 0$) and hence $f(x_T) - f(x^*) = O(\log T)/T$. Later we get rid of the logarithmic term.

**The Potential Change:**

$$\Phi_{t+1} - \Phi_t = (t + 1)(f(x_{t+1}) - f(x^*)) - t(f(x_t) - f(x^*))$$

$$= (t + 1)(f(x_{t+1}) - f(x_t)) + (f(x_t) - f(x^*))$$  \hspace{1cm} (3.18)

To bound the first term, we use the smooth convexity of $f$ with $x = x_t$ and $y = x_{t+1} = x_t - \eta_t \nabla_t$:

$$f(x_{t+1}) \leq f(x_t) - \eta_t \cdot \|\nabla_t\|_2^2 + \frac{\beta}{2} \cdot \eta_t^2 \cdot \|\nabla_t\|_2^2.$$  \hspace{1cm} (3.19)

Our choice of $\eta_t = 1/\beta$ maximizes the reduction above to give

$$f(x_{t+1}) \leq f(x_t) - \frac{1}{2\beta} \|\nabla_t\|_2^2.$$  \hspace{1cm} (3.19)
For the second term in (3.18), just use convexity and Cauchy-Schwarz:

\[
f(x_t) - f(x^*) \leq \langle \nabla f(x_t), x_t - x^* \rangle \leq \|\nabla f(x_t)\|_2 \cdot \|x_t - x^*\|_2 \leq \|\nabla f(x_t)\|_2 \cdot D, \tag{3.20}
\]

where \( D := \max\{\|x - x^*\|_2 \mid f(x) \leq f(x_0)\} \). Note that (3.19) ensures that \( f(x_t) \leq f(x_{t-1}) \leq \ldots \leq f(x_0) \). So the potential change is

\[
\Phi_{t+1} - \Phi_t \leq (t + 1) \cdot \langle f(x_t), f(x^*) \rangle \leq (t + 1) \cdot (t + 2) \cdot \|\nabla f(x_t)\|_2 \cdot \|x_t - x^*\|_2 \leq 2D^2 \beta \cdot \frac{t + 1}{t + 2}.
\]

The RHS is maximized at \( \frac{D^2 \beta}{2(t + 1)} \) when \( \|\nabla f(x_t)\|_2 = \frac{D \beta}{t + 1} \), and hence

\[
f(x_T) - f(x^*) = \Phi_T = \frac{1}{T} \sum_{t=0}^{T-1} (\Phi_{t+1} - \Phi_t) \leq \frac{1}{T} \sum_{t=0}^{T-1} \frac{D^2 \beta}{2(t + 1)} \cdot \|\nabla f(x_t)\|_2 \cdot \|x_t - x^*\|_2 \leq \frac{D^2 \beta}{2T} \cdot (1 + \ln T).
\]

The intuition is evident from (3.19) and (3.20): we improve a lot by (3.19) when the gradients are large, or else we are close to the optimum by (3.20).

**A Tighter Analysis.** The logarithmic dependence in the above bound can be removed by a simple trick. We multiply the potential by a linear term in \( t \), which avoids the sum over \( 1/t \) in the analysis above.

**Theorem 3.2** (Smooth Functions: Take II). If \( f \) is \( \beta \)-smooth, and \( D := \max\{\|x - x^*\|_2 \mid f(x) \leq f(x_0)\} \), the update rule (3.17) guarantees

\[
f(x_T) - f(x^*) \leq \beta \cdot \frac{2D^2}{T + 1}.
\]

**Proof.** The potential now changes to

\[
\Phi_t = t(t + 1) \cdot (f(x_t) - f(x^*))
\]

The potential change is

\[
\Phi_{t+1} - \Phi_t = (t + 1)(t + 2) \cdot (f(x_{t+1}) - f(x_t)) + 2(t + 1) \cdot (f(x_t) - f(x^*))
\]

Plugging in (3.19) and (3.20) gives

\[
\leq (t + 1)(t + 2) \cdot \left(-\frac{1}{2\beta} \|\nabla f(x_t)\|^2 + 2(t + 1) \cdot \|\nabla f(x_t)\|_2 \cdot D \right) \leq 2D^2 \beta \cdot \frac{t + 1}{t + 2},
\]

where the last inequality is the maximum value of the preceding expression obtained at \( \|\nabla f(x_t)\| = \frac{2\beta D}{t + 2} \).

Summing over the time steps, \( \Phi_T \leq T \cdot 2D^2 \beta \), so

\[
f(x_T) - f(x^*) \leq \frac{2D^2 \beta \cdot T}{T(T + 1)} = \beta \cdot \frac{2D^2}{T + 1}.
\]

**3.1.1 Yet Another Proof**

Let’s see yet another proof that gets rid of the logarithmic term. Interestingly, the potential function now combines both the difference in the function value, and the distance in the “action” space.
Theorem 3.3 (Smooth Functions: Take III). If $f$ is $\beta$-smooth, the update rule (3.17) guarantees

$$f(x_T) - f(x^*) \leq \beta \frac{\|x_0 - x^*\|^2}{2T}.$$ 

Proof. Consider the potential of the form

$$\Phi_t = t(f(x_t) - f(x^*)) + \frac{a}{2} \|x_t - x^*\|^2$$

where $a$ will be chosen based on the analysis below. As $\Phi_0 = \frac{a}{2} \|x_0 - x^*\|^2$, if we show that $\Phi_t$ is non-increasing,

$$\frac{a}{2} \|x_0 - x^*\|^2 = \Phi_0 \geq \Phi_T = T(f(x_T) - f(x^*)) + \frac{a}{2} \|x_T - x^*\|^2$$

which gives $f(x_t) - f(x^*) \leq \frac{a}{2T} \|x_0 - x^*\|^2$ as desired.

The potential difference can be written as:

$$\Phi_{t+1} - \Phi_t = (t + 1)(f(x_{t+1}) - f(x_t)) + f(x_t) - f(x^*) + \frac{a}{2} (\|x_{t+1} - x^*\|^2 - \|x_t - x^*\|^2). \tag{3.22}$$

Using the bounds from the mentioned inequalities,

$$\leq (t + 1) \left( \frac{1}{2\eta_t} \|
abla f(x_t)\|^2 + (\nabla f(x_t), x_t - x^*) + \frac{a}{2} \left( 2\eta_t (\nabla f(x^*), x_t - x^*) + \eta_t^2 \|
abla f(x_t)\|^2 \right) \right) \tag{3.23}$$

where $\eta_t = 1/\beta$ in this case. Now, we set $a = 1/\eta_t = \beta$ to cancel the inner-product terms, which gives

$$\Phi_{t+1} - \Phi_t \leq -(t/2\beta) \|
abla f(x_t)\|^2 \leq 0. \tag*{\Box}$$

This guarantee is almost the same as in Theorem 3.2, with a slightly better definition of the distance term ($\|x_0 - x^*\|^2$ vs $D^2$). But we will revisit and build on this proof when we talk about Nesterov acceleration in Section 5.

3.1.2 Projected Smooth Gradient Descent

Let us consider the constrained minimization problem for a convex body $K$. As previously, the update becomes

$$x_{t+1}' \leftarrow x_t - (1/\beta) \nabla f(x_t), \quad x_{t+1} \leftarrow \Pi_K(x_{t+1}'). \tag{3.24}$$

However, the analysis is instructive; it is more involved than for the basic gradient descent setting previously as the potential involves terms that depend on $f$.

Theorem 3.4 (Constrained Smooth Optimization). If $f$ is $\beta$-smooth, the update rule (3.24) guarantees

$$f(x_T) - f(x^*) \leq \frac{2(f(x_0) - f(x^*)) + \beta \|x_0 - x^*\|^2}{T + 2}.$$
Proof. Consider the potential of the form

\[ \Phi_t = (t+2)(f(x_t) - f(x^*)) + \frac{\beta}{2} \|x_t - x^*\|^2. \]

The potential difference can be written as:

\[ \Phi_{t+1} - \Phi_t = (t+3) \left( f(x_{t+1}) - f(x_t) \right) + f(x_t) - f(x^*) + \frac{\beta}{2} \left( \|x_{t+1} - x^*\|^2 - \|x_t - x^*\|^2 \right). \]

(3.25)

Using the bounds from the mentioned inequalities,

\[ \leq (t+3) \left( \langle \nabla_t, x_{t+1} - x_t \rangle + \frac{\beta}{2} \|x_{t+1} - x_t\|^2 \right) + \langle \nabla_t, x_t - x^* \rangle + \frac{\beta}{2} (2 \|x_{t+1} - x_t, x_t - x^*\| + \|x_{t+1} - x_t\|^2) \]

\[ = (t+3) \langle \nabla_t, x_{t+1} - x_t \rangle + (t+4) \frac{\beta}{2} \|x_{t+1} - x_t\|^2 + \langle \nabla_t, x_t - x^* \rangle + \beta \langle x_{t+1} - x_t, x_t - x^* \rangle \]

As \( x_{t+1} \) is the projected value, we cannot directly use \( x_{t+1} - x_t = -\nabla_t / \beta \), but the following useful claim (that we prove later) still holds.

Claim 3.5. \( \langle \nabla_t, x_{t+1} - x_t \rangle \leq -\beta \|x_{t+1} - x_t\|^2 \leq 0. \)

Using this, we have \((t+4/2)\beta \|x_{t+1} - x_t\|^2 \leq -(t+4/2) \langle \nabla_t, x_{t+1} - x_t \rangle \) and we can cancel all the \( \|x_{t+1} - x_t\|^2 \) terms above to get:

\[ \Phi_{t+1} - \Phi_t \leq \langle \nabla_t, x_{t+1} - x_t \rangle + \langle \nabla_t, x_t - x^* \rangle + \beta \langle x_{t+1} - x_t, x_t - x^* \rangle \]

\[ \leq \langle \nabla_t, x_{t+1} - x_t \rangle + \langle \nabla_t, x_t - x^* \rangle + \beta \langle x_{t+1} - x_t, x_t - x^* \rangle \]

where the second step uses that \((t+2)/2 \geq 1\) and that the first term gets more negative when \( t \) is larger. Now summing the first two terms,

\[ = \langle \nabla_t, x_{t+1} - x^* \rangle + \beta \langle x_{t+1} - x_t, x_t - x^* \rangle \]

There is a mismatch between the first inner product, which has a \( x_{t+1} - x^* \) term, and the second, which has \( x_t - x^* \). We can fix this by adding the positive term \( \beta \|x_{t+1} - x_t\|^2 \) to get:

\[ \leq \langle \nabla_t, x_{t+1} - x^* \rangle + \beta \langle x_{t+1} - x_t, x_{t+1} - x^* \rangle \]

\[ = \langle \nabla_t + \beta(x_{t+1} - x_t), x_{t+1} - x^* \rangle \]

\[ = \beta \langle x_{t+1} - x_{t+1}', x_{t+1} - x^* \rangle. \]

(by the update rule 3.24)

But this inner product is negative because of the Pythagorean property Proposition 2.2, so the potential decreases over time. Hence \( \Phi_t \leq \Phi_0 \), which completes the proof.

Proof of Claim 3.5. Subtracting the right side from the left, \( \langle \nabla_t, x_{t+1} - x_t \rangle + \beta \|x_{t+1} - x_t\|^2 = \beta ((x_t - x_{t+1}') + (x_{t+1} - x_t), x_{t+1} - x_t) = \beta (x_{t+1} - x_{t+1}', x_{t+1} - x_t) \leq 0 \) due to the Pythagorean property Proposition 2.2, and noting that \( x_t \in K. \)
3.1.3 The Frank-Wolfe Method

One drawback of projected gradient descent is the projection step: given a point \( x' \) and a body \( K \), finding the closest point \( \Pi_K(x') \) might be computationally expensive. Instead, we can use a different rule, the Frank-Wolfe method (also called conditional gradient descent), that implements each gradient step using linear optimization over the body \( K \). Loosely, at each timestep we find the point in \( K \) that is furthest from the current point in the direction of the negative gradient, and move a small distance towards it.

Formally, the update rule for Frank-Wolfe method is simple:

\[
\begin{align*}
y_t & \leftarrow \arg \min_{y \in K} \langle \nabla t, y \rangle \\
x_{t+1} & \leftarrow (1 - \eta_t)x_t + \eta_ty_t
\end{align*}
\] (3.26)

As we shall see below, setting \( \eta_t = 1/(t + 1) \) will give the following result.

![Figure 3.2: The Frank-Wolfe Update](image)

**Theorem 3.6** (Smooth Functions: Frank-Wolfe). If \( f \) is \( \beta \)-smooth, \( K \) is a convex body with \( D := \max_{x, y \in K} \|x - y\| \), then the update rule (3.26) guarantees

\[
f(x_T) - f(x^*) \leq \beta \frac{D^2 (1 + \ln T)}{2T}.
\]

**Proof.** The potential function remains unchanged:

\[
\Phi_t = t \cdot (f(x_t) - f(x^*)),
\]

and hence the change in potential is again:

\[
\Phi_{t+1} - \Phi_t = (t + 1)(f(x_{t+1}) - f(x_t)) + (f(x_t) - f(x^*)) \quad (3.27)
\]

To bound the change in potential (3.27), we observe that \( x_{t+1} - x_t = \eta_t(y_t - x_t) \).

\[
f(x_{t+1}) - f(x_t) \leq \langle \nabla t, x_{t+1} - x_t \rangle + \frac{\beta}{2}\|x_{t+1} - x_t\|^2 \quad \text{(by smoothness)}
\]

\[
= \eta_t \langle \nabla t, y_t - x_t \rangle + \frac{\beta \eta_t^2}{2}\|y_t - x_t\|^2.
\]

\[
\leq \eta_t \langle \nabla t, x^* - x_t \rangle + \frac{\beta \eta_t^2}{2}\|y_t - x_t\|^2. \quad \text{(by optimality of } y_t \text{)}
\]

\[
f(x_t) - f(x^*) \leq \langle \nabla t, x_t - x^* \rangle \quad \text{(by convexity)}
\]

Setting \( \eta_t := \frac{1}{t+1} \) cancels the linear terms and hence the potential change (3.27) is at most \( \beta \eta_t D^2/2 \).
Summing over $t$ and using $\Phi_0 = 0$, the final potential $\Phi_T \leq D^2 (1 + \ln T)$, and hence $f(x_T) - f(x^*) = \beta \frac{(1+\ln T)D^2}{2T}$.

We can remove the logarithmic dependence in the above bound, using the trick of multiplying the potential by $(t+1)$ as in Theorem 3.2. We omit the simple details of the following theorem.

**Theorem 3.7** (Smooth Functions: Frank-Wolfe, Take II). If $f$ is $\beta$-smooth, $K$ is a convex body with $D := \max_{x,y \in K} \|x - y\|$, then the update rule (3.26) with $\eta_t = 2/(t+1)$ guarantees

$$f(x_T) - f(x^*) \leq 2\beta \frac{D^2}{T+1}.$$ 

### 3.2 Well-Conditioned Functions

If a function is both $\alpha$-strongly convex and $\beta$-smooth, it must be that $\alpha \leq \beta$. The ratio $\kappa := \beta/\alpha$ is called the condition number of the convex function. We now show a much stronger convergence guarantee for “well-conditioned” functions, those for which $\kappa$ is small. The update rule is:

$$x_{t+1} \leftarrow x_t - \eta \nabla f_t$$

(3.28)

where $\eta = \frac{1}{\beta}$ just depends on the smoothness.

**Theorem 3.8** (GD: Well-Conditioned). Given a function $f$ that is both $\alpha$-strongly convex and $\beta$-smooth, the update rule (3.28) ensures

$$f(x_T) - f(x^*) \leq \exp(-T/\kappa) \cdot (f(x_0) - f(x^*)) \quad \text{for all } x^*.$$ 

**Proof.** We set $\gamma = 1/(\kappa - 1)$ for brevity\(^1\), and use the potential

$$\Phi_t = (1 + \gamma)^t \cdot (f(x_t) - f(x^*))$$

(3.29)

Note that this is a natural potential to use, as we wish to show that $f(x_T) - f(x_0)$ falls exponentially with $T$.

**The Potential Change:** A little rearrangement gives us

$$\Phi_{t+1} - \Phi_t = (1 + \gamma)^t \cdot \left( (1 + \gamma)(f(x_{t+1}) - f(x_t)) + \gamma(f(x_t) - f(x^*)) \right).$$

(3.30)

We bound the two terms separately. Using the smoothness analysis from (3.19):

$$f(x_{t+1}) - f(x_t) \leq -\frac{1}{2\beta} \|\nabla f_t\|^2.$$ 

And by the definition of strong convexity,

$$f(x_t) - f(x^*) \leq \langle \nabla f_t, x_t - x^* \rangle - \frac{\alpha}{2} \|x_t - x^*\|^2 \leq \frac{1}{2\alpha} \|\nabla f_t\|^2$$

\(^1\)Note that $\kappa = 1$ iff $f(x) = ax^2 + bx + c$ in which case it is easily checked that the optimum solution $x^*$ is reached in a single step.
where the second inequality uses \( \langle a, b \rangle - \|b\|^2 / 2 \leq \|a\|^2 / 2 \). Plugging this back into (3.30) gives

\[
(1 + \gamma)^t \left( -\frac{1 + \gamma}{2\beta} + \frac{\gamma}{2\alpha} \right) \|\nabla_t\|^2
\]

which is 0 by our choice of \( \gamma \). Hence, after \( T \) steps,

\[
f(x_T) - f(x^*) \leq (1 + \gamma)^{-T}(f(x_0) - f(x^*)) = (1 - 1/\kappa)^T(f(x_0) - f(x^*)) \leq e^{-T/\kappa}(f(x_0) - f(x^*)) \tag{3.31}
\]

Here we can show that \( x_T \) also gets rapidly closer to \( x^* \). If \( x^* \) is the optimal point, we know \( \nabla f(x^*) = 0 \). Now smoothness gives \( f(x_0) - f(x^*) \leq \frac{\beta}{2}\|x_0 - x^*\|^2 \), and strong convexity gives \( \frac{\alpha}{2}\|x_T - x^*\|^2 \leq f(x_T) - f(x^*) \). Plugging into (3.31) gives us that

\[
\|x_T - x^*\|^2 \leq \kappa e^{-T/\kappa} \cdot \|x_0 - x^*\|^2.
\]

To reduce the error by a factor of \( 1/2 \), it suffices to increase \( T \) additively by \( \kappa \ln 2 \). Hence if the condition number \( \kappa \) is constant, every constant number of rounds of gradient descent gives us one additional bit of accuracy! This behavior, where getting error bounded by \( \varepsilon \) requires \( O(\log \varepsilon^{-1}) \) steps, is called linear convergence in the numerical analysis literature.

One may ask if the convergence for smooth, and for well-conditioned functions is optimal as a function of \( T \). The answer is no: a landmark result of Yu. Nesterov gives faster (and optimal) convergence rates. We will see a potential-function-based proof in Section 5.
4 The Mirror Descent Framework

The gradient descent algorithms in the previous sections work by adding some multiple of the gradient to current point. However, this should strike the reader as somewhat strange, since the point \( x_t \) and the gradient \( \nabla f(x_t) \) are objects that lie in different spaces and should be handled accordingly. In particular, if \( x_t \) lies in some vector space \( E \), the gradient \( \nabla f(x_t) \) lies in the dual vector space \( E^* \). (This did not matter earlier since \( \mathbb{R}^n \) equipped with the Euclidean norm is self-dual, but now we want to consider general norms and would like to be careful.)

A key insight of Nemirovski and Yudin was that substantially more general and powerful results can be obtained, without much additional work, by considering these spaces separately. For example, it is well-known (and we will see) that the classic multiplicative-weights update method can be obtained as a special case of this general approach.

4.1 Basic Mirror Descent

The key idea in mirror descent is to define a mapping between \( E \) to \( E^* \), called the mirror map. Given a point \( x_t \), we first map it to \( E^* \), make the gradient update there, and then use the inverse map back to obtain the point \( x_{t+1} \).

We start some basic concepts and notation. Consider some vector space \( E \) with an inner product \( \langle \cdot, \cdot \rangle \), and define a norm \( \| \cdot \| \) on \( E \). To measure distances in \( E^* \), we use the dual norm defined as

\[
\|y\|_* := \max_{x : \|x\|=1} \langle x, y \rangle.
\]

(4.32)

By definition we have

\[
\langle x, y \rangle \leq \|x\| \cdot \|y\|_*,
\]

(4.33)

which is often referred to as the generalized Cauchy-Schwarz inequality.

A function \( h \) is \( \alpha \)-strongly convex with respect to \( \| \cdot \| \) if

\[
h(y) \geq h(x) + \langle \nabla h(x), y - x \rangle + \frac{\alpha}{2} \|y - x\|^2
\]

Such a strongly convex function \( h \) defines a map from \( E \) to \( E^* \) via its gradient: indeed, the map \( x \mapsto \nabla h(x) \) takes the point \( x \in E \) into a point in the dual space \( E^* \). The strong convexity ensures that the map is 1-1 (i.e., \( \nabla h(x) \neq \nabla h(y) \) for \( x \neq y \)). Moreover, the map \( \nabla h(\cdot) \) is also surjective, so for any \( \theta \in E^* \) there is an inverse \( x \in E \) such that \( \nabla h(x) = \theta \). In fact, this inverse map is given by the gradient of the Fenchel dual for \( h \), i.e., \( \nabla h^*(\theta) = x \iff \nabla h(x) = \theta \). (For the reader not familiar with Fenchel duality, it suffices to interpret \( \nabla h^*(\theta) \) merely as \( (\nabla h)^{-1}(\theta) \).) Readers interested in the technical details can see, e.g., [BT03].
4.1.1 The Update Rules

Since $\nabla h : E \to E^*$ gives us a map from the primal space $E$ to the dual space $E^*$, we keep track of the image point $\theta_t = \nabla h(x_t)$ as well. Now, the updates are the natural ones, given by

$$
\begin{align*}
\theta'_{t+1} &= \theta_t - \eta_t \nabla f_t(x_t) \\
x'_{t+1} &= \nabla h^*(\theta'_{t+1}) \\
x_{t+1} &= \arg \min_{x \in K} D_h(x \parallel x'_{t+1}).
\end{align*}
$$

(4.34)

In other words, given $x_t \in E$, we add $\eta_t$ times the negative gradient to its image $\theta_t = \nabla h(x_t)$ in the dual space to get $\theta'_{t+1}$, pull the result back to $x'_{t+1} \in E$ (using the inverse mapping $x'_{t+1} = \nabla h^*(\theta'_{t+1})$), and project it back onto $K$ to get $x_t$. Of course, we may not want to use the Euclidean distance for the projection; the “right” distance in this case is the Bregman divergence, which we discuss shortly.

An equivalent way to present the mirror descent update is the following:

$$
x_{t+1} = \arg \min_{x \in K} \left\{ \langle \eta_t \nabla f_t(x_t), x - x_t \rangle + D_h(x \parallel x'_{t+1}) \right\}.
$$

(4.35)

This is generalization of (2.4). The equivalence is easy to see in the unconstrained case (just take derivatives), for the constrained case one uses the KKT conditions.

4.1.2 Bregman Divergences

Given a strictly convex function $h : \mathbb{R}^d \to \mathbb{R}$, define the Bregman divergence

$$
D_h(y \parallel x) := h(y) - h(x) - \langle \nabla h(x), y - x \rangle
$$

to be the “error” at $y$ between the actual function value and the value given by linearization at some point $x$. The convexity of $h$ means this quantity is non-negative; if $h$ is $\beta$-strongly convex with respect to the norm $\| \cdot \|$, then $D_h(y \parallel x) \geq \frac{\beta}{2} \| y - x \|^2$. Also, $D_h(y \parallel x)$ is a convex function of $y$ (for a fixed $x$, this is a convex function minus a linear term), and the gradient of the divergence with respect to the first argument is $\nabla_y (D_h(y \parallel x)) = \nabla h(y) - \nabla h(x)$.

For example, the function $h(x) := \frac{1}{2} \| x \|^2$ is 1-strongly convex with respect to $\ell_2$ (and hence strictly convex), and the associated Bregman divergence $D_h(y \parallel x) = \frac{1}{2} \| y - x \|^2$, half the squared $\ell_2$ distance.
This distance is not a metric, since it does not satisfy the triangle inequality. Or consider the negative entropy function \( h(x) := \sum_i x_i \ln x_i \) defined on the probability simplex \( \Delta_n := \{ x \in [0,1]^n \mid \sum_i x_i = 1 \} \). For \( x, y \in \Delta_n \), the associated Bregman divergence \( D_h(y \parallel x) = \sum_i y_i \ln y_i / x_i \), the relative entropy or Kullback-Leibler (KL) divergence from \( x \) to \( y \). This distance is not even symmetric in \( x \) and \( y \).

**Bregman projection.** Given a convex body \( K \) and a strictly convex function \( h \), we define the Bregman projection of a point \( x' \) on \( K \) as

\[
\Pi^h_K(x') = \arg \min_{x \in K} D_h(x \parallel x').
\]

If \( x' \in K \), then \( \Pi^h_K(x') = x' \) because \( D_h(x' \parallel x') = 0 \). For \( h(x) = \frac{1}{2} \| x \|^2 \), this corresponds to the usual Euclidean projection. A very useful feature of Bregman projections is that they satisfy a “Pythagorean inequality” with respect to the divergence, analogous to Fact 2.2.

**Proposition 4.1** (Generalized Pythagorean Property). *Given a convex body \( K \subseteq \mathbb{R}^n \), let \( a \in K \) and \( b' \in \mathbb{R}^n \). Let \( b = \Pi^h_K(b') \). Then

\[
(\nabla h(b') - \nabla h(b), a - b) \leq 0.
\]

In particular, \( D_h(a \parallel b') \geq D_h(a \parallel b) + D_h(b \parallel b') \) and hence that \( D_h(a \parallel b) \leq D_h(a \parallel b') \).

**Proof.** Recall that for any convex function \( g \) and convex body \( K \), if \( x^* = \arg \min_{x \in K} g(x) \) is the minimizer of \( g \) in \( K \), then \( \langle \nabla g(x^*), y - x^* \rangle \geq 0 \) for all \( y \in K \). Using \( g(x) = D_h(x \parallel b') \), and noting that \( g(x) \) is convex with \( \nabla g(x) = \nabla h(x) - \nabla h(b') \) and that the minimizer \( x^* = b \), we get \( \langle \nabla h(b) - \nabla h(b'), a - b \rangle \geq 0 \) for all \( a \in K \).

For the second part, expand the terms using the definition of \( D_h(a \parallel b) \) and cancel the common terms, the desired inequality turns out to be equivalent to \( \langle \nabla h(b') - \nabla h(b), a - b \rangle \leq 0 \). The last inequality uses that the divergences are non-negative.

### 4.1.3 The Analysis

We consider the more general online optimization setting, and prove the following regret bound.

**Theorem 4.2.** Let \( K \) be a convex body, \( f_1, \ldots, f_T \) be convex functions defined on \( K \), \( \| \cdot \| \) be a norm, and \( h \) be an \( \alpha_h \)-strongly convex function with respect to \( \| \cdot \| \). The mirror descent algorithm starting at \( x_0 \) and taking constant step size \( \eta_t = \eta \) in every iteration, produces \( x_1, \ldots, x_T \) such that

\[
\sum_{t=1}^T f_t(x_t) \leq \sum_{t=1}^n f_t(x^*) + \frac{D_h(x^* \parallel x_0)}{\eta} + \frac{\eta \sum_{t=1}^T \| \nabla f_t(x_t) \|^2}{2\alpha_h}, \text{ for all } x^* \in K \quad (4.36)
\]

**Proof.** Define the potential

\[
\Phi_t = \frac{D_h(x^* \parallel x_t)}{\eta}. \quad (4.37)
\]

Observe that plugging in \( h(x) = \frac{1}{2} \| x \|^2 \) gives us the potential function (2.5) for the Euclidean norm.

**The Potential Change:** For brevity, use \( \nabla_t := \nabla f_t(x_t) \).

\[
D_h(x^* \parallel x_{t+1}) - D_h(x^* \parallel x_t)
\]

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implies that Kullback-Liebler divergence. Moreover, Pinsker’s inequality says that

\[ KL(D_{\theta_t}) = h(x^*) - h(x_{t+1}^*) - \mathbb{E}_{\theta_{t+1}}\left[h(x_{t+1}^*) - h(x^*) - (x_{t+1}^* - x_{t+1})\right] \]

\[ \leq D_h(x^* \parallel x_{t+1}) - D_h(x^* \parallel x_t) \]  

(generalized Pythagorean ppty.)

\[ = h(x^*) - h(x_{t+1}^*) - \langle \nabla h(x_{t+1}^*), x^* - x_{t+1} \rangle - h(x^*) + h(x_t) + \langle \nabla h(x_t), x^* - x_t \rangle \]

\[ = h(x_t) - h(x_{t+1}^*) - \langle \theta_{t+1}^*, x_t - x_{t+1} ^* \rangle - \langle \theta_{t+1} - \theta_t, x^* - x_t \rangle \]

\[ = h(x_t) - h(x_{t+1}^*) - \langle \theta_{t+1}^*, x_t - x_{t+1} ^* \rangle + \langle \eta_t \nabla_t, x_t - x_{t+1}^* \rangle + \langle \eta_t \nabla_t, x^* - x_t \rangle \]

strong convexity

\[ \leq -\frac{\alpha_h}{2} \| x_{t+1}^* - x_t \|^2 + \eta_t \langle \nabla_t, x_t - x_{t+1}^* \rangle + \eta_t \langle \nabla_t, x^* - x_t \rangle \]

\[ \leq \frac{\eta_t^2}{2\alpha_h} \| \nabla_t \|^2 + \eta_t \langle \nabla_t, x^* - x_t \rangle. \]  

(4.38)

The last inequality uses generalized Cauchy-Schwarz to get \( (a, b) \leq \|b\| \|a\| \leq \|b\|^2 / 2 + \|a\|^2 / 2 \). Observe that (4.38) precisely maps to (2.9) when we consider the Euclidean norm.

**The Amortized Cost:** Recall that we set \( \eta_t = \eta \) for all steps. Hence, dividing (4.38) and substituting,

\[ f_t(x_t) - f_t(x^*) + (\Phi_{t+1} - \Phi_t) \leq \frac{f_t(x_t) - f_t(x^*) + \langle \nabla_t, x^* - x_t \rangle + \eta_t \langle \nabla_t, x^* - x_t \rangle}{2\alpha_h} \]

\[ \leq 0 \text{ by convexity of } f_t \]

The total regret then becomes

\[ \sum_t (f_t(x_t) - f_t(x^*)) \leq \Phi_0 + \sum_t \frac{\eta_t}{2\alpha_h} \| \nabla_t \|^2 \leq \frac{D_h(x^* \parallel x_0)}{\eta} + \frac{\eta \sum_{t=1}^T \| \nabla_t \|^2}{2\alpha_h}. \]

Hence the proof.

### 4.1.4 Special Cases

To get some intuition, let us look at some well-known special cases. If we use the \( \ell_2 \) norm, and \( h(x) := \frac{1}{2} \|x\|_2^2 \) which is clearly 1-strongly convex with respect to \( \ell_2 \), the associated Bregman divergence \( D_h(x^* \parallel x) = \frac{1}{2} \|x^* - x\|_2^2 \). Moreover, the Euclidean norm is self-dual, so if we bound \( \|\nabla f_t\|_2 \) by \( G \), the total regret bound above is \( \frac{1}{2\eta} \|x^* - x_0\|_2^2 + \eta TG^2 / 2 \). This is the same result for projected gradient descent we derived in Theorem 2.1—and in fact the algorithm is also precisely the same.

Now consider the \( \ell_1 \) norm, with \( K \) being the probability simplex \( \Delta_n := \{x \in [0, 1]^n \mid \sum_i x_i = 1\} \). If we choose the negative entropy function \( h(x) := \sum_i x_i \ln x_i \), then \( D_h(x^* \parallel x) \) is just the well-known Kullback-Liebler divergence. Moreover, Pinsker’s inequality says that \( KL(p \parallel q) \geq \|p - q\|_1^2 \), which implies that \( h \) is 1-strongly convex with respect to \( \ell_1 \). Applying Theorem 4.2 now gives a regret bound of

\[ \frac{KL(x^* \parallel x_0)}{\eta} + \frac{\eta}{2} \sum_t \| \nabla_t \|_\infty. \]

Let’s also see what the mirror-descent algorithm does in this case. The mirror map takes the point \( x \) to \( \nabla h(x) = (1 + \log x_j)_j \), and the inverse map takes \( \theta \) to \( \nabla h^*(\theta) = (e^{\theta_j} - 1)_j \). This point may be outside the probability simplex, so we do a Bregman projection, which in this case corresponds to just a rescaling \( x \mapsto x / \|x\|_1 \). Unrolling the process, one can get a closed-form expression for the point \( x_T \):

\[ (x_T)_i = \frac{(x_0)_i \exp \{ \sum_t (\nabla_t)_i \}}{\sum_j (x_0)_j \exp \{ \sum_t (\nabla_t)_j \}}. \]
E.g., if we specialize even further to online linear optimization, where each function \( f_t(x) = \langle \ell_t, x \rangle \) for some \( \ell_t \in [0, 1]^n \), the gradient is \( \ell_t \) and its \( \ell_\infty \)-norm is \( ||\ell_t||_\infty \leq 1 \), giving us the familiar regret bound of \( KL(x^*, x_0) + \frac{\eta T}{2} \) that we get from the multiplicative weights/Hedge algorithms. Which is not surprising, since this algorithm is precisely the Hedge algorithm!

4.2 An Aside: Smooth Functions and General Norms

Let us consider minimizing a function that is smooth with respect to non-Euclidean norms, in the unconstrained case. When we consider an arbitrary norm \( \| \cdot \| \), the definition of a smooth function (3.16) extends seamlessly. Now we can define an update rule by naturally extending (2.4):

\[
\begin{align*}
    x_{t+1} \leftarrow \arg \min_x \left\{ \frac{1}{2} \| x - x_t \|^2 + \eta_t \langle \nabla_t, x - x_t \rangle \right\},
\end{align*}
\]

(4.39)

where the norm is no longer the Euclidean norm, but the norm in question. To evaluate the minimum on the right side, we can use basic Fenchel duality: given a function \( g \), its Fenchel dual is defined as

\[
g^*(\theta) := \max_z \{ \langle \theta, z \rangle - g(z) \}.
\]

If we define \( g(z) = \frac{1}{2} \| z \|^2 \), it is known that \( g^*(\theta) = \frac{1}{2} \| z \|^2_\ast \) (see [BV04, Example 3.27]). Hence

\[
\begin{align*}
    \min_x \left\{ \frac{1}{2} \| x - x_t \|^2 + \eta_t \langle \nabla_t, x - x_t \rangle \right\} &= -\max_x \left\{ \eta_t \langle \nabla_t, x_t - x \rangle - \frac{1}{2} \| x_t - x \|^2 \right\} \\
    &= -\max_z \left\{ \langle \eta_t \nabla_t, z \rangle - \frac{1}{2} \| z \|^2 \right\} = -\frac{1}{2} \| \eta_t \nabla_t \|^2_\ast
\end{align*}
\]

(4.40)

If a function \( f \) is \( \beta \)-smooth with respect to the norm, then setting \( \eta_t = \frac{1}{\beta} \) gives:

\[
\begin{align*}
    f(x_{t+1}) \leq f(x_t) + \langle \nabla_t, x_{t+1} - x_t \rangle + \frac{\beta}{2} \| x_{t+1} - x_t \|^2 \\
    = f(x_t) + \beta \left( \langle \eta_t \nabla_t, x_{t+1} - x_t \rangle + \frac{1}{2} \| x_{t+1} - x_t \|^2 \right) = f(x_t) + \beta \cdot \left( -\frac{1}{2} \| \eta_t \nabla_t \|^2_\ast \right),
\end{align*}
\]

where the last equality uses that \( x_{t+1} \) is the minimizer of the expression in (4.40). Summarizing, we get

\[
f(x_{t+1}) \leq f(x_t) - \frac{1}{2\beta} \| \nabla_t \|^2_\ast.
\]

(4.41)

This is analogous to the expression (3.17). Now we can continue the proof as in §3.1, again defining \( D := \max \{ \| x - x^* \| \mid f(x) \leq f(x_0) \} \), and using the generalized Cauchy-Schwarz inequality to get the general-norm analog of Theorem 3.2:

**Theorem 4.3 (GD: Smooth Functions for General Norms).**  Given a function \( f \) that is \( \beta \)-smooth with respect to the norm \( \| \cdot \| \), the update rule (4.39) ensures

\[
f(x_T) - f(x^*) \leq \beta \frac{2D^2}{T+1}.
\]
5 Nesterov Acceleration: A Potential Function Proof

In Section 3, we proved a convergence rate of $O(1/T)$ for smooth functions, using both projected gradient descent and the Frank-Wolfe method. But the lower bound is only $\Omega(1/T^2)$. In this case, the algorithm can be improved: Yurii Nesterov showed how to do it using his "accelerated gradient descent" methods. Recently there has been much interest in gaining a deeper understanding of this process, with proofs using "momentum" methods and continuous-time updates [WWJ16, SBC16, KBB15, WRJ16, DO17].

Let us now see potential-based proofs for his theorem, both for the smooth case, and for the well-conditioned case. We consider only the unconstrained case (i.e., when $K = \mathbb{R}^n$) and the Euclidean norm; the extension to general norms is sketched in §5.3.

5.1 An Illustrative Failed Attempt

One way to motivate Nesterov’s accelerated algorithm is to revisit the proof for smooth functions in Section 3.1.1. Let us recall the essential facts. The potential was

$$\Phi_t = t(f(x_t) - f(x^*)) + \frac{a}{2} \|x_t - x^*\|^2$$

for some $a > 0$. Hence the potential difference was:

$$\Phi_{t+1} - \Phi_t = (t + 1) (f(x_{t+1}) - f(x_t)) + f(x_t) - f(x^*) + \frac{a}{2} \left( \|x_{t+1} - x^*\|^2 - \|x_t - x^*\|^2 \right).$$

$(3.19)$ (convexity)

$$\leq (t + 1) \cdot \left( -\frac{1}{2\beta} \|\nabla f\|_2^2 + \langle \nabla f, x_t - x^* \rangle + a \eta_t \langle \nabla f, x_t - x^* \rangle + \frac{a^2}{2} \|\nabla f\|_2^2 \right) = -t/2\beta \|\nabla f\|_2^2 \leq 0.$$

In that last expression we set $\eta_t = 1/\beta$ and $a = 1/\eta_t = \beta$ to cancel the inner-product terms.

Observe that the potential is actually decreasing quite considerably by $-t/2\beta \|\nabla f\|_2^2$, but we are ignoring this large decrease. A first (incorrect) attempt to harness this extra term would be to change the potential to

$$\Phi_t = t(t + 1) (f(x_t) - f(x^*)) + \frac{a}{2} \|x_t - x^*\|^2.$$

At first glance, the potential change $\Phi_{t+1} - \Phi_t$ would be

$$(t + 2)(t + 1) \cdot \left( -\frac{1}{2\beta} \|\nabla f\|_2^2 + 2(t + 1) \cdot \langle \nabla f, x_t - x^* \rangle + a \eta_t \langle \nabla f, x_t - x^* \rangle + \frac{a^2}{2} \|\nabla f\|_2^2 \right) \quad (5.42)$$

Now if we change the step length $\eta_t$ from $1/\beta$ to something more aggressive, say $\eta_t = \frac{t+1}{2\beta}$, and choose $a = 4\beta$, the inner-product terms cancel, and the potential reduction is at most $\|\nabla f\|_2^2 (-\frac{(t+1)(t+2)}{2\beta} + \frac{(t+1)^2}{2\beta^2}) \leq 0$. This would seem to give us an $O(1/T^2)$ convergence.

So where’s the mistake? It’s in our erroneous use of $(3.19)$: we used the cautious update with $\eta_t = 1/\beta$ to get the first term of $-\frac{1}{2\beta} \|\nabla f\|_2^2$, but the aggressive update with $\eta_t = \frac{t+1}{2\beta}$ elsewhere. To fix this, how about running two processes, one cautious and one aggressive, and then combining them together linearly (with decreasing weight on the aggressive term) to get the new point $x_t$? This is precisely what Nesterov’s Accelerated Method does; let’s see it now. (This is another way to arrive at the elegant linear-coupling view that Allen-Zhu and Orecchia present in [AO14].)
### 5.2 Getting the Acceleration

As we just sketched, one way to view the accelerated gradient descent method is to run two iterative processes for $y_t$ and $z_t$, and then combine them together to get the actual point $x_t$. The proof is almost the same as before.

**The Update Steps:** Start with $x_0 = y_0 = z_0$. At time $t$, play $x_t$. For brevity, define $\nabla_t := \nabla f(x_t)$.

Now consider the update rules, where the color is used to emphasize the subtle differences:

\[
\begin{align*}
y_{t+1} &\xlongrightarrow{} x_t - \frac{1}{\lambda} \nabla f(x_t) \quad (5.43) \\
z_{t+1} &\xlongrightarrow{} z_t - \eta_t \nabla f(x_t) \quad (5.44) \\
x_{t+1} &\xlongrightarrow{} (1 - \tau_{t+1})y_{t+1} + \tau_{t+1}z_{t+1}. \quad (5.45)
\end{align*}
\]

In (5.44), we will choose the “aggressive” step size $\eta_t = \frac{\lambda + 1}{2\beta}$ as we did in the above failed attempt. In (5.45) the mixing weight is $\tau_t = \frac{2}{1+2\beta}$, but this will arise organically below.

**The Potential:** This is the same one from the failed attempt:

\[
\Phi(t) = t(t+1) \cdot (f(y_t) - f(x^*)) + 2\beta \cdot \|z_t - x^*\|^2 \quad (5.46)
\]

**The Potential Change:** Define $\Delta \Phi_t = \Phi(t+1) - \Phi(t)$. By the standard GD analysis in (2.9),

\[
\frac{1}{2} (\|z_{t+1} - x^*\|^2 - \|z_t - x^*\|^2) = \frac{\eta_t^2}{2} \|\nabla_t\|^2 + \eta_t \langle \nabla_t, x^* - z_t \rangle \quad (5.47)
\]

\[
\implies \Delta \Phi_t = t(t+1) \cdot (f(y_{t+1}) - f(y_t)) + 2(t+1) \cdot (f(y_{t+1}) - f(x^*)) + 4\beta \left( \frac{\eta_t^2}{2} \|\nabla_t\|^2 + \eta_t \langle \nabla_t, x^* - z_t \rangle \right)
\]

By smoothness and the update rule for $y_{t+1}$, (3.19) implies $f(y_{t+1}) \leq f(x_t) - \frac{1}{\lambda} \|\nabla_t\|^2$. Substituting, and dropping the resulting (negative) squared-norm term,

\[
\Delta \Phi_t \leq t(t+1) \cdot (f(x_t) - f(y_t)) + 2(t+1) \cdot (f(x_t) - f(x^*)) + 4\beta \eta_t \langle \nabla_t, x^* - z_t \rangle \\
\leq t(t+1) \cdot \langle \nabla_t, x_t - y_t \rangle + 2(t+1) \cdot \langle \nabla_t, x_t - x^* \rangle + 2(t+1) \cdot \langle \nabla_t, x^* - z_t \rangle
\]

using convexity for the first two terms, and $\eta_t = \frac{\lambda + 1}{2\beta}$ for the last one. Collecting like terms,

\[
\Delta \Phi_t \leq (t+1) \cdot \langle \nabla_t, (t+2)x_t - ty_t - 2z_t \rangle = 0, \quad (5.48)
\]

by using (5.45) and $\tau_t = \frac{2}{1+2\beta}$. Hence $\Phi_t \leq \Phi_0$ for all $t \geq 0$. This proves:

**Theorem 5.1 (Accelerated GD).** Given a $\beta$-smooth function $f$, the update rules (5.43)-(5.45) ensure

\[
f(y_t) - f(x^*) \leq 2\beta \frac{\|z_0 - x^*\|^2}{t(t+1)}.
\]

#### 5.2.1 An Aside: Optimizing Parameters and Other Connections

Suppose we choose the generic potential $\Phi(t) = \lambda^2_{t-1} (f(y_t) - f(x^*)) + \frac{\beta}{2} \|z_t - x^*\|^2$, where $\lambda_t = O(t^2)$, and try to optimize the calculation above. Having $\lambda^2_t - \lambda^2_{t-1} = \lambda_t$ and $\tau_t = 1/\lambda_t$ makes the calculations
work out very cleanly. Solving this recurrence leads to the (somewhat exotic-looking) weights

$$\lambda_0 = 0, \quad \lambda_t = \frac{1 + \sqrt{1 + 4\lambda_{t-1}^2}}{2}$$  \hspace{1cm} (5.49)

used in the standard exposition of AGM2.

The update rules (5.43)–(5.45) have sometimes been called AGM2 (Nesterov’s second accelerated method) in the literature. A different set of update rules (called AGM1) are the following: for the optimized choice of $\lambda_t$ from (5.49), define:

\[
\begin{align*}
y_{t+1} &\leftarrow x_t - \frac{1}{\beta} \nabla f(x_t) \\
x_{t+1} &\leftarrow \left(1 - \frac{1 - \lambda_t}{\lambda_{t+1}}\right)y_{t+1} + \frac{1 - \lambda_t}{\lambda_{t+1}} y_t
\end{align*}
\]  \hspace{1cm} (5.50)

Let us show the simple equivalence (also found in, e.g., [DT14, KF16]).

**Lemma 5.2.** Using updates (5.50–5.51) and setting $z_t := \lambda_t x_t - (\lambda_t - 1)y_t = \lambda_t(x_t - y_t) + y_t$, and $\tau_t := 1/\lambda_t$ leads to the updates (5.43–5.45).

**Proof.** Clearly $y_t$ is the same as above, so it suffices to show that $z_t$ and $x_t$ behave identically. Indeed, rewriting the definition of $z_t$ and substituting $\tau_t = 1/\lambda_t$ gives

\[
x_t = (1 - \tau_t)y_t + \tau_t z_t.
\]

\[
z_{t+1} - z_t = (\lambda_{t+1}(x_{t+1} - y_{t+1}) + y_{t+1}) - (\lambda_t(x_t - y_t) + y_t)
\]  \hspace{1cm} (5.52)

Moreover, rewriting (5.51) gives

\[
\lambda_{t+1}(x_{t+1} - y_{t+1}) - (1 - \lambda_t)(y_t - y_{t+1}) = 0
\]  \hspace{1cm} (5.53)

Subtracting (5.53) from (5.52) gives

\[
z_{t+1} - z_t = \lambda_t y_{t+1} - \lambda_t x_t = -\frac{\lambda_t}{\beta} \nabla f(x_t)
\]  \hspace{1cm} (5.54)

Recalling that $\lambda_t = 1/\tau_t = (\beta \eta_t)$, this is precisely the update rule $z_{t+1} \leftarrow z_t - \eta_t \nabla f(x_t)$. This shows the equivalence of the two update rules. \hfill \Box

### 5.3 The Extension to Arbitrary Norms

Given an arbitrary norm $\|\cdot\|$, the update rules now use the gradient descent update (4.39) for smooth functions for the $y$ variables, and the mirror descent update rules (4.34) for the $z$ variables:

\[
\begin{align*}
y_{t+1} &\leftarrow \arg\min_y \left\{ \frac{\beta}{2} \|y - x_t\|^2 + \langle \nabla f(x_t), y - x_t \rangle \right\} \\
z_{t+1} &\leftarrow \arg\min_z \left\{ \langle \eta_t \nabla f(x_t), z \rangle + D_h(z \parallel z_t) \right\} \\
x_{t+1} &\leftarrow (1 - \tau_{t+1}) y_{t+1} + \tau_{t+1} z_{t+1}.
\end{align*}
\]  \hspace{1cm} (5.55)
Given the discussion in the preceding sections, the update rules are the natural ones: the first is the update (4.39) for smooth functions, and the second is the usual mirror descent update rule (4.35) for the strongly-convex function $h$. The step size is now set to $\eta_t = \frac{(t+1)\alpha_h}{2\beta}$, and the potential function becomes:

$$\Phi(t) = t(t+1) \cdot (f(y_t) - f(x^*)) + \frac{4\beta}{\alpha_h} \cdot D_h(x^* \| z_t),$$

which on substituting $D_h(x^* \| z_t) = \frac{1}{2} \| x^* - z_t \|^2$ and $\alpha_h = 1$ gives (5.46).

We already have all the pieces to bound the change in potential. Use the mirror descent analysis (4.38) to get

$$D_h(x^* \| z_{t+1}) - D_h(x^* \| z_t) \leq \frac{\eta_t^2}{2\alpha_h} \| \nabla \|_w^2 + \eta_t \langle \nabla, x^* - x_t \rangle.$$

which replaces (5.47). Infer $f(y_{t+1}) \leq f(x_t) - \frac{1}{2\gamma} \| \nabla \|^2$ from (4.41) in the smooth case. Substitute these into the analysis from §5.2 (with minor changes for the $\alpha_h$ term) to get the following theorem:

**Theorem 5.3** (Accelerated GD: General Norms). Given a $\beta$-smooth function $f$ with respect to norm $\| \cdot \|$, the update rules (5.55) ensure

$$f(y_t) - f(x^*) \leq \frac{4\beta}{\alpha_h} \cdot \frac{D_h(x^* \| z_0) - D_h(x^* \| z_t)}{t(t+1)}.$$

### 5.4 Strongly Convex with Acceleration

Now consider the case when the function $f$ is well-conditioned with condition number $\kappa = \beta/\alpha$. We now use the following updates (which look very much like the AGM1 updates):

$$y_{t+1} \leftarrow x_t - \frac{1}{\beta} \nabla f(x_t) \tag{5.57}$$

$$x_{t+1} \leftarrow \left(1 + \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)y_{t+1} - \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}y_t. \tag{5.58}$$

For the analysis, it will be convenient to define $\tau = \frac{1}{\sqrt{\kappa} + 1}$ and set

$$z_{t+1} := \frac{1}{\tau} x_{t+1} - \frac{1-\tau}{\tau} y_{t+1}. \tag{5.59}$$

We now show that that the error after $t$ steps is

$$f(y_t) - f(x^*) \leq (1 + \gamma)^{-t} \left(\frac{\alpha + \beta}{2} \| x_0 - x^* \|^2\right), \tag{5.60}$$

where $\gamma = \frac{1}{\sqrt{\kappa} - 1}$ (as in Section 3.2, for $\kappa = 1$ the algorithm reaches optimum in a single step and $y_1 = x^*$, and hence we assume that $\kappa > 1$). This improves on the error of $(1 + 1/\kappa)^{-t} \frac{\beta}{2} \| x_0 - x^* \|^2$ we get from Section 3.2.

**The Potential:** Consider the potential

$$\Phi(t) = (1 + \gamma)^t \left(f(y_t) - f(x^*) + \frac{\alpha}{2} \| z_t - x^* \|^2\right).$$
Observe that $\Phi_0 = f(y_0) - f(x^*) + \frac{\alpha}{2}\|z_0 - x^*\|^2$. As $x_0 = y_0 = z_0$, and by $\beta$-smoothness of $f$,

$$\Phi_0 \leq \frac{\alpha + \beta}{2}\|x_0 - x^*\|^2.$$

**Change in Potential:** To show the error bound (5.60), it suffices to show that $\Delta \Phi(t) = \Phi(t+1) - \Phi(t) \leq 0$ for each $t$. This is equivalent to showing

$$\left(1 + \gamma\right)(f(y_{t+1}) - f(x^*)) - (f(y_t) - f(x^*)) + \frac{\alpha}{2} \left(1 + \gamma\right)\|z_{t+1} - x^*\|^2 - \|z_t - x^*\|^2 \leq 0$$

We first bound the terms involving $f$ in the most obvious way. As above, we use $\nabla_t$ as short-hand for $\nabla f(x_t)$. By $\beta$-smoothness and the update rule, again $f(y_{t+1}) \leq f(x_t) - \frac{1}{2\beta}\|\nabla t\|^2$. So,

$$\begin{align*}
\left(1 + \gamma\right)(f(y_{t+1}) - f(x^*)) - (f(y_t) - f(x^*)) \\
\leq f(x_t) - f(y_t) + \gamma(f(x_t) - f(x^*)) - (1 + \gamma)\frac{1}{2\beta}\|\nabla t\|^2 \\
\leq \langle \nabla_t, x_t - y_t \rangle + \gamma \left(\langle \nabla_t, x_t - x^* \rangle - \frac{\alpha}{2}\|x_t - x^*\|^2 \right) - \frac{1 + \gamma}{2\beta}\|\nabla t\|^2,
\end{align*}$$

where the last inequality used convexity and strong convexity respectively.

We now want to remove references to $y_t$. By definition (5.59), $z_t = \left(\frac{1}{\gamma} - 1\right)(x_t - y_t) + x_t = \sqrt{\gamma}(x_t - y_t) + x_t$, so we infer $\gamma(z_t - x^*) = \sqrt{\gamma}(x_t - y_t) + \gamma(x_t - x^*)$. Using $\sqrt{\gamma} = 1 + \gamma$, simple algebra gives $\gamma(z_t - x^*) = \frac{1}{1 + \gamma}(\gamma(z_t - x^*) + \gamma^2(x_t - x^*))$.

For brevity we use $X_t := x_t - x^*$, $Z_t := z_t - x^*$, and substitute the above expression into (5.61) to get

$$\frac{1}{1 + \gamma}\langle \nabla_t, \gamma Z_t + \gamma^2 X_t \rangle - \frac{\alpha\gamma}{2}\|X_t\|^2 - \frac{1 + \gamma}{2\beta}\|\nabla t\|^2.$$  

Now, let us upper bound the terms in $\Delta \Phi(t)$ involving $z$. Conveniently, we can relate $z_{t+1}$ and $z_t$ using a simple calculation that we defer for the moment.

**Claim 5.4.** $z_{t+1} = (1 - \frac{1}{\sqrt{\gamma}})z_t + \frac{1}{\sqrt{\gamma}}x_t - \frac{1}{\alpha\sqrt{\gamma}}\nabla t$ and so $z_{t+1} - x^* = \frac{1}{1 + \gamma} Z_t + \frac{\gamma}{1 + \gamma} X_t - \frac{\gamma}{\alpha(1 + \gamma)}\nabla t$.

Now use Claim 5.4 and expand using $\|a + b + c\|^2 = \|a\|^2 + \|b\|^2 + \|c\|^2 + 2\langle a, b \rangle + 2\langle b, c \rangle + 2\langle a, c \rangle$:

$$\begin{align*}
(1 + \gamma)\|z_{t+1} - x^*\|^2 - \|z_t - x^*\|^2 \\
= \frac{1}{1 + \gamma} \left(\|Z_t\|^2 + \gamma^2\|X_t\|^2 + \frac{\gamma^2}{\alpha^2}\|\nabla t\|^2 + 2\gamma\langle Z_t, X_t \rangle - \frac{2\gamma^2}{\alpha}\langle \nabla_t, Z_t \rangle - \frac{2\gamma^2}{\alpha}\langle \nabla_t, X_t \rangle\right) - \|Z_t\|^2.
\end{align*}$$

Now sum (5.62) and $\alpha/2$ times (5.63). The terms involving $\|\nabla t\|^2$ cancel since $\frac{1 + \gamma}{2\beta} = \frac{\alpha\gamma^2}{2\alpha^2(1 + \gamma)}$ (by the definition of $\gamma$). Moreover, the inner-product terms involving $\nabla_t$ also cancel. Hence the potential change is at most

$$\Delta \Phi(t) \leq \frac{\alpha\gamma}{2}\|X_t\|^2 \left(1 - \frac{\gamma}{1 + \gamma}\right) + \frac{\alpha}{2}\|Z_t\|^2 \left(\frac{1}{1 + \gamma} - 1\right) + \frac{\alpha\gamma}{1 + \gamma}\langle Z_t, X_t \rangle$$

$$= -\frac{\alpha\gamma}{2(1 + \gamma)} \left(\|X_t\|^2 + \|Z_t\|^2 - 2\langle Z_t, X_t \rangle\right)$$

$$= -\frac{\alpha\gamma}{2(1 + \gamma)}\|Z_t - X_t\|^2 \leq 0.$$  

(5.64)
Hence the potential does not increase, as claimed. It only remains to prove Claim 5.4.

**Proof of Claim 5.4.** The expression of $x_{t+1}$ from (5.58) can be written as $(2 - 2\tau)y_{t+1} - (1 - 2\tau)y_t$. Plugging into the expression for $z_{t+1}$ from (5.59) gives

$$z_{t+1} = \frac{1}{\tau} \left( (2 - 2\tau)y_{t+1} - (1 - 2\tau)y_t - (1 - \tau)y_{t+1} \right)$$

$$= \frac{1}{\tau} \left( (1 - \tau)y_{t+1} - (1 - 2\tau)y_t \right).$$

Using the update rule (5.57) for $y_{t+1}$, and the relation $x_t = (1 - \tau)y_t + \tau z_t$ to eliminate $y_t$

$$= \frac{1}{\tau} \left( (1 - \tau) \left( x_t - \frac{1}{\beta} \nabla_t \right) - \frac{1 - 2\tau}{1 - \tau} (x_t - \tau z_t) \right)$$

$$= \frac{1 - 2\tau}{1 - \tau} z_t + \frac{\tau}{1 - \tau} x_t - \frac{1 - \tau}{\tau \beta} \nabla_t.$$

Using $\tau = 1/(\sqrt{\kappa} + 1)$ and $\beta = \kappa \alpha$ now gives the claim. \hfill \Box

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**References**


