

## BDD-based parity game solving

**Citation for published version (APA):**

Sanchez, L., Wesselink, J. W., & Willemse, T. A. C. (2018). BDD-based parity game solving: a comparison of Zielonka's recursive algorithm, priority promotion and fixpoint iteration. (Computer science reports; Vol. 1801). Eindhoven: Technische Universiteit Eindhoven.

**Document status and date:**

Published: 01/01/2018

**Document Version:**

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

**Please check the document version of this publication:**

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

**General rights**

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

[www.tue.nl/taverne](http://www.tue.nl/taverne)

**Take down policy**

If you believe that this document breaches copyright please contact us at:

[openaccess@tue.nl](mailto:openaccess@tue.nl)

providing details and we will investigate your claim.

Technische Universiteit Eindhoven  
Department of Mathematics and Computer Science

BDD-Based Parity Game Solving  
A comparison of Zielonka's Recursive Algorithm,  
Priority Promotion and Fixpoint Iteration

L. Sanchez, W. Wesselink and T.A.C. Willemse

18/01

ISSN 0926-4515

All rights reserved

editor: prof.dr.ir. J.J. van Wijk

Reports are available at:

<http://library.tue.nl/catalog/TUEPublication.csp?Language=dut&Type=ComputerScienceReports&Sort=Author&level=1> and

<http://library.tue.nl/catalog/TUEPublication.csp?Language=dut&Type=ComputerScienceReports&Sort=Year&Level=1>

Computer Science Reports 18-01  
Eindhoven, March 2018



# BDD-Based Parity Game Solving

## A Comparison of Zielonka’s Recursive Algorithm, Priority Promotion and Fixpoint Iteration

Lisette Sanchez, Wieger Wesselink, and Tim A.C. Willemse

Eindhoven University of Technology, The Netherlands

**Abstract.** Parity games are two player games with omega-winning conditions, played on finite graphs. Several algorithms for solving parity games have been proposed in the literature, and while the problem was recently shown to be solvable in quasi-polynomial time, so far, the question whether such games can be solved in polynomial time remains elusive. In practice, parity games play an important role in verification, satisfiability and synthesis. It is therefore important to identify algorithms that can efficiently deal with large and complex games that arise from such applications. In this paper, we describe our experiments with BDD-based implementations of Zielonka’s recursive algorithm, the more recent Priority Promotion algorithm and the Fixpoint-Iteration algorithm. We conclude that overall, Zielonka’s BDD-based algorithm beats the BDD-based Priority Promotion algorithm by a small margin for games that are characteristic of practical verification problems. For games with a large number of distinct priorities, Priority Promotion scales better. The Fixpoint-Iteration algorithm performs similar to Zielonka’s algorithm for games with at most 5 different priorities.

## 1 Introduction

Parity games [11, 19, 23] are infinite duration games played by two player on a finite directed graph. Each vertex in the game graph is owned by one of the two players and vertices are assigned a colour, or *priority*. The game is played by pushing a single token along the edges in the graph; the choice to which vertex the token is to move next is decided by the player owning the vertex currently holding the token. A parity winning condition determines the winner of this infinite play. A vertex in the game is won by the player that has a strategy for which, no matter how the opponent plays, every play from that vertex is won by her; the winner of each vertex is uniquely determined [19]. The parity game solving problem is to compute the set of vertices won by each player. From a practical viewpoint, parity games are interesting since they underlie typical verification, satisfiability and synthesis problems, see [8, 11, 1].

The simplicity of the game can be deceptive. In spite of a continued research effort, no polynomial algorithm for solving such games has been found. The problem of solving a parity game is known to be in  $UP \cap coUP$  [14], a class that neither precludes nor predicts the existence of a polynomial algorithm. In recent

years, several new algorithms have been devised for solving parity games, and only last year, the problem was shown to run in *quasi-polynomial time* [7].

Most parity game solving algorithms fall in one of two categories: ‘strategy identification’ (SI) algorithms and ‘dominion identification’ (DI) algorithms. The SI category of algorithms directly computes the winning strategies for both players; *e.g.* by means of policy iteration or by maintaining some statistics about the gameplays that can emerge from a vertex. The recent quasi-polynomial algorithms [7, 16] all fall in this category, but also classical algorithms such as Jurdziński’s *small progress measures* algorithm [15], and, to some extent, the Fixpoint-Iteration algorithm [6]. The DI category of algorithms all proceed by (recursively) decomposing the game graph in *dominions*: small subgraphs that are won by a single player and from which the opponent cannot escape. A classical exponent of the latter category is Zielonka’s *recursive algorithm* [23] but also the recently introduced Priority Promotion algorithm [5]. While DI algorithms typically have a worst-case running time complexity that is theoretically less attractive than that of SI algorithms, in practice, the SI are significantly outperformed by DI algorithms [21].

Since parity games that originate from practical verification problems can become quite large, it is natural to study techniques that help to solve such games, relying on compact, symbolic representations of the game graph such as Binary Decision Diagrams (BDDs). Unlike most SI algorithms, DI algorithms typically compute with subgraphs and as such are likely candidates admitting a symbolic implementation. Indeed, BDD-based implementations for parity game solving algorithm have been studied in the past [2].

In this paper, we describe our implementation of the PP algorithm using BDDs. Moreover, we compare its performance to a BDD-based implementation of Zielonka’s algorithm and a BDD-based implementation of the Fixpoint-Iteration algorithm. In particular, we reassess the conclusions from [21] in which Zielonka’s algorithm and the PP algorithm were shown to have similar performance when computing with a non-symbolic representation of the game graph. A few samples taken from a set of benchmarks for parity games encoding verification problems point at a much better scalability of our BDD-based solvers, compared to their explicit counterparts. Apart from comparing the performance on games originating from typical verification problems, we have assessed the performance of both BDD-based implementations by generating BDDs representing random game graphs. While we confirm that also in the symbolic setting the two algorithms perform similarly, our observations indicate that for games with a modest number of different priorities in the game, Zielonka beats the PP algorithm by a small margin, whereas for games with higher number of different priorities, the PP algorithm beats Zielonka. The Fixpoint-Iteration algorithm is competitive with Zielonka and PP only for games with at most 5 priorities; its shows a significant drop in performance for games with 6 priorities or more.

This paper is structured as follows. In Section 2, we introduce parity games and the relevant concepts. The three algorithms that we compare are then introduced and discussed in Section 3 and we describe how these can be implemented

using BDD techniques in Section 4. In Section 5, we describe our experimental evaluation of our implementations and we finish with conclusions in Section 6.

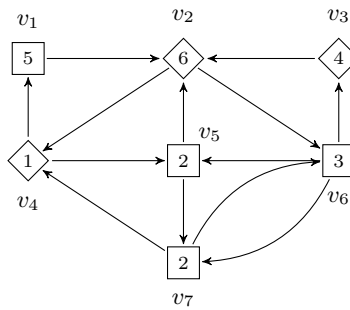
## 2 Parity Games

A parity game is an infinite duration game, played by players *odd*, denoted by 1 and *even*, denoted by 0, on a directed, finite graph. The game is formally defined as follows.

**Definition 1 (Parity game).** *A parity game is a tuple  $(V, E, p, (V_0, V_1))$ , where:*

- $V$  is a finite set of vertices, partitioned in a set  $V_0$  of vertices owned by player 0, and a set of vertices  $V_1$  owned by player 1,
- $E \subseteq V \times V$  is the edge relation; we assume that  $E$  is total, i.e. for all  $v \in V$ , there is some  $w \in V$  such that  $(v, w) \in E$ ,
- $p: V \rightarrow \mathbb{N}$  is a priority function that assigns priorities to vertices.

We depict parity games as graphs in which diamond-shaped vertices represent vertices owned by player 0 and box-shaped vertices represent vertices owned by player 1. Priorities, associated with vertices, are typically written inside vertices, see Figure 1.



**Fig. 1.** A parity game with 7 vertices, 6 different priorities, 3 vertices owned by player 0 and 4 vertices owned by player 1.

*Plays, Strategies and Winning.* We write  $v \rightarrow w$  whenever  $(v, w) \in E$ . Henceforth,  $\alpha \in \{0, 1\}$  denotes an arbitrary player. We write  $\bar{\alpha}$  for  $\alpha$ 's opponent; i.e.  $\bar{0} = 1$  and  $\bar{1} = 0$ . A sequence of vertices  $v_1, \dots, v_n$  is a *path* if  $v_m \rightarrow v_{m+1}$  for all  $1 \leq m < n$ . Infinite paths are defined in a similar manner. We write  $\pi_n$  to denote the  $n^{\text{th}}$  vertex in a path  $\pi$ .

A game starts by placing a token on vertex  $v \in V$ . Players move the token indefinitely according to a single simple rule: if the token is on some vertex  $v \in V_\alpha$ , player  $\alpha$  gets to move the token to an adjacent vertex. The *parity* of

the highest priority that occurs infinitely often on a thus constructed infinite sequence of vertices through the game defines the *winner* of the path: player 0 wins if, and only if this priority is even.

More formally, the moves of players 0 and 1 are determined by their respective *strategies*. Informally, a strategy for a player  $\alpha$  determines, for a vertex  $\pi_i \in V_\alpha$  the next vertex  $\pi_{i+1}$  that will be visited if the token visits  $\pi_i$ . In general, a strategy is a partial function  $\sigma: V^* \times V_\alpha \rightarrow V$  which, given a history of vertices visited by the token and a vertex on which the token currently resides, determines the next vertex. For simplicity and (due to the positional determinacy theorem for parity games [11]) without a loss of generality, we only consider *positional strategies*. A positional strategy (hereafter simply referred to as *strategy*) for a player  $\alpha$  is a partial function  $\sigma: V_\alpha \rightarrow V$  that is compatible with  $E$ , *i.e.* for all  $v \in V_\alpha$  it is the case that  $\sigma(v) \in \{w \in V \mid v \rightarrow w\}$ . An infinite path  $\pi$  is *compatible* with a given strategy  $\sigma$  if for all vertices  $\pi_i$  for which  $\sigma$  is defined, we have  $\pi_{i+1} = \sigma(\pi_i)$ . We refer to an infinite path through the graph that is compatible with a strategy for player 0 and a strategy for player 1 as a *play*. A strategy is *closed* on a set  $U$  iff every play that is compatible with that strategy remains within  $U$ . We say that a set of vertices  $U$  is  $\alpha$ -*closed* iff  $\alpha$  has a strategy that is closed on  $U$ .

A play is won by player 0 iff the *maximal* priority that occurs infinitely often along that play is *even*. More formally, given a play  $\pi$ , player 0 wins  $\pi$  iff:

$$\max\{k \in \mathbb{N} \mid \forall j \in \mathbb{N} : \exists i \in \mathbb{N} : i > j \wedge k = p(\pi_i)\} \text{ is even}$$

Strategy  $\sigma$  for player  $\alpha$  is *winning* from a vertex  $v$  if and only if  $\alpha$  is the winner of every play starting in  $v$  that is compatible with  $\sigma$ . A set of vertices  $W_\alpha$  is won by  $\alpha$  if for each vertex  $v \in W_\alpha$ , player  $\alpha$  has some winning strategy from  $v$ . Another consequence of the aforementioned positional determinacy theorem is that the set of vertices  $W_0$  won by player 0 and the set of vertices  $W_1$ , won by player 1, forms a partition of the set  $V$ : every vertex is won by exactly one player.

*Example 1.* Consider the parity game depicted in Figure 1. The vertices  $v_5, v_6$  and  $v_7$  are won by player 1 whereas vertices  $v_1, v_2, v_3$  and  $v_4$  are won by player 0. To see why vertex  $v_6$  and  $v_7$  are won by player 1, consider her strategy  $\sigma$  defined by  $\sigma(v_7) = v_6$  and  $\sigma(v_6) = v_5$ . Any play that is compatible with this strategy infinitely often visits vertex  $v_6$ , which has a dominating odd priority. Hence, such plays are won by player 1. Furthermore, note that this strategy is *closed* on  $\{v_6, v_7\}$ .  $\square$

*Subgames, Attractors and Dominions.* For a game  $G = (V, E, p, (V_0, V_1))$  and a set  $U \subseteq V$ , we define the *subgame* of  $G$ , denoted  $G \cap U$ , as the maximal substructure  $(V', E', p', (V'_0, V'_1))$  that is obtained by restricting the graph  $(V, E)$  to  $U$  and lifting the restriction to the other elements of  $G$ , *i.e.*  $V' = U$ ,  $E' = E \cap (U \times U)$ ,  $p'(v) = p(v)$  for all  $v \in U$ , and  $V'_0 = V_0 \cap U$  and  $V'_1 = V_1 \cap U$ . If the edge relation  $E'$  of  $G \cap U$  is again total, then the subgame is again a parity game. We furthermore use the abbreviation  $G \setminus A$  to denote the subgame  $G \cap (V \setminus A)$ .

Parity game solving can essentially be reduced to identifying or computing appropriate subgraphs that are entirely won by a single player and from which the opponent cannot escape. A subgame that has this property for a player  $\alpha$  is called an  $\alpha$ -dominion. Technically, an  $\alpha$ -dominion is a non-empty set of vertices  $D_\alpha \subseteq V$  such that player  $\alpha$  has a winning strategy from *all* vertices in the set  $D_\alpha$  that is closed on  $D_\alpha$ . We note that  $W_0$ , the set of vertices won by player 0, is a 0-dominion; likewise  $W_1$  is a 1-dominion.

*Example 2.* Reconsider the parity game from Figure 1. We already identified a closed strategy that is winning for player 1 on the set of vertices  $\{v_5, v_6, v_7\}$ . So this set is a 1-dominion. It is also a maximal 1-dominion. The set  $\{v_6, v_7\}$  is also a 1-dominion: a closed winning strategy  $\sigma$  for player 1 is given by  $\sigma(v_6) = v_7$  and  $\sigma(v_7) = v_6$ . This dominion is not maximal.  $\square$

For a non-empty set  $U \subseteq V$  and a player  $\alpha$ , the  $\alpha$ -attractor into  $U$ , denoted  $Attr_\alpha(U)$ , is the set of vertices for which player  $\alpha$  can force play into  $U$ . We define  $Attr_\alpha(U)$  as  $\bigcup_{i \geq 0} Attr_\alpha^i(U)$ , the limit of approximations of the sets  $Attr_\alpha^n(U)$ , which are inductively defined as follows:

$$\begin{aligned} Attr_\alpha^0(U) &= U \\ Attr_\alpha^{n+1}(U) &= Attr_\alpha^n(U) \\ &\quad \cup \{v \in V_\alpha \mid \exists u \in Attr_\alpha^n(U) : v \rightarrow u\} \\ &\quad \cup \{v \in V_{\bar{\alpha}} \mid \forall u \in V : v \rightarrow u \implies u \in Attr_\alpha^n(U)\} \end{aligned}$$

The *confined*  $\alpha$ -attractor, denoted  $Attr_\alpha(T, U)$ , is defined analogously by limiting the set of vertices to those from  $T$  in each approximation. That is, it represents the subset of vertices  $T' \subseteq T$  from which  $\alpha$  can force play to  $U$  while remaining in  $T$ . We note that for arbitrary set  $U \subseteq V$ , the subgame  $G \setminus Attr_\alpha(U)$  is again a parity game; that is, the edge relation of the subgame is again total. Furthermore, the  $\alpha$ -attractor into an  $\alpha$ -dominion  $D_\alpha$  is again an  $\alpha$ -dominion. Finally, observe that the set  $V \setminus Attr_\alpha(U)$  is  $\bar{\alpha}$ -closed.

*Example 3.* The 0-attractor into the set  $\{v_2\}$  contains the vertices  $v_1$  (which is owned by 1 but trivially attracted to  $v_6$  since it has but one successor vertex),  $v_3$  and  $v_4$ . Vertex  $v_5$  does not belong to the 0-attractor into  $\{v_2\}$  as player 1 can choose to move to  $v_7$ . For similar reasons, vertices  $v_6$  and  $v_7$  do not belong to the 0-attractor into  $\{v_2\}$ . Note that the set  $\{v_5, v_6, v_7\}$  is 1-closed.  $\square$

### 3 Three Parity Game Solving Algorithms

Algorithms for solving parity games essentially either use clever statistics for identifying winning strategies, or they employ a decomposition of the game play by identifying dominions in subgames. In this paper we focus on two algorithms that fit in the latter class. Moreover, we consider a third algorithm that, arguably, fits in the former class.



### 3.1 Zielonka's Recursive Algorithm

The algorithm by Zielonka is a divide and conquer algorithm that searches for dominions in subgames. The algorithm is essentially distilled from a constructive proof of the positional determinacy of parity games by, among others, Zielonka [23]. Despite the fact that the algorithm has a relatively bad theoretical worst-case complexity (it runs in  $\mathcal{O}(mn^d)$  where  $n$  is the number of vertices,  $m$  the number of edges and  $d$  is the number of different priorities in the game) and exponential worst-case examples are known for various classes of special games [13], the algorithm remains among the most successful solvers for parity games in practice, see [12] and the recent exploration [21].

Zielonka's algorithm (see Algorithm 1) constructs winning regions for both players out of the solution of subgames with fewer different priorities and fewer vertices. For this, the algorithm relies on the fact that higher priorities in the game dominate lower priorities, and that any forced revisit of these higher priorities is beneficial to the player aligning with the parity of the priority. For non-trivial games, the algorithm relies on attractor set computations (see line 8) to identify the set of vertices that can be forced to visit the maximal priority in the game; the remaining vertices are then solved recursively (see line 9).

The outcome of the recursion can be either that the entire subgame is won by the player (say  $\alpha$ ) that has the same parity as the maximal priority in the game *or* the opponent  $\bar{\alpha}$  wins a non-empty set of vertices. In case  $\alpha$  wins all vertices in the subgame, she wins the entire game (lines 11-12). This can be seen as follows: since the subgame is  $\bar{\alpha}$ -closed, the opponent can choose to stay within the subgame; choosing to do so means she will lose. So the only option she has is to escape. But the only escape she has leads to the maximal priority which has the parity of  $\alpha$ . Note that if  $\alpha$  is forced to leave this maximal priority she will again end up in the subgame; any play that does so *ad infinitum* visits the maximal priority infinitely often and is therefore won by  $\alpha$ . In case  $\alpha$  does *not* win the entire subgame, the set of vertices won by  $\bar{\alpha}$  in the subgame are also won by  $\bar{\alpha}$  in the larger game since the subgame was  $\bar{\alpha}$ -closed; in other words, it is a  $\bar{\alpha}$ -dominion and so is its  $\bar{\alpha}$ -attractor (the set  $B$  computed in line 13). Removing this dominion and recursively solving the remaining subgame (line 14) then leads to a solution of the entire game.

Several optimisations can be applied to Zielonka's algorithm. For instance, rather than solving the entire game at once, one can first decompose a game into strongly connected components and first solve the bottom components. The SCC decomposition can be integrated tightly in the algorithm so that in each recursive call first an SCC decomposition is performed. While this may sound expensive, in [13] it is shown that this actually allows the algorithm to run in polynomial time on many practically relevant classes of special games such as solitaire and dull games for which the original algorithm might require exponential time otherwise. Another observation, made in *e.g.* [18] is that in case  $W_{\bar{\alpha}} = Attr_{\bar{\alpha}}(W_{\bar{\alpha}})$  after the first recursive call, no second recursive call is needed.

---

**Algorithm 1** Zielonka's Algorithm

---

```
1: function ZIELONKA( $G$ )
2:   if  $V = \emptyset$  then
3:      $(W_0, W_1) \leftarrow (\emptyset, \emptyset)$ 
4:   else
5:      $m \leftarrow \max\{p(v) \mid v \in V\}$ 
6:      $\alpha \leftarrow m \bmod 2$ 
7:      $U \leftarrow \{v \in V \mid \Omega(v) = m\}$ 
8:      $A \leftarrow \text{Attr}_\alpha(U)$ 
9:      $(W'_0, W'_1) \leftarrow \text{ZIELONKA}(G \setminus A)$ 
10:    if  $W'_{\bar{\alpha}} = \emptyset$  then
11:       $(W_\alpha, W_{\bar{\alpha}}) \leftarrow (A \cup W'_\alpha, \emptyset)$ 
12:    else
13:       $B \leftarrow \text{Attr}_{\bar{\alpha}}(W'_{\bar{\alpha}})$ 
14:       $(W'_0, W'_1) \leftarrow \text{ZIELONKA}(G \setminus B)$ 
15:       $(W_\alpha, W_{\bar{\alpha}}) \leftarrow (W'_\alpha, W'_{\bar{\alpha}} \cup B)$ 
16:    end if
17:  end if
18:  return  $(W_0, W_1)$ 
19: end function
```

---

### 3.2 Priority Promotion

The recent *Priority Promotion* (PP) algorithm [5] starts, like Zielonka's recursive algorithm, with a game decomposition that aims at identifying dominions for a given player. In contrast to Zielonka's algorithm, however, the PP algorithm does not solve explicit subgames. Instead, within a fixed game, it uses a dominion searcher that maintains a set of vertices, along with an updated (promoted) priority mapping and zooms in on a dominion within that set of vertices. Once an  $\alpha$ -dominion is returned by the dominion searcher, its  $\alpha$ -attractor is removed from the game and the search for another dominion continues, see Algorithm 2, until the entire game is solved.

---

**Algorithm 2** Priority Promotion Solver

---

```
1: function SOLVEPP( $G$ )
2:    $(W_0, W_1) \leftarrow (\emptyset, \emptyset)$ 
3:   repeat
4:      $m \leftarrow \max\{p(v) \mid v \in V\}$ 
5:      $(W'_0, W'_1) \leftarrow \text{SEARCHDOMINION}(G, V, p, m)$ 
6:      $(W_0, W_1) \leftarrow (W_0 \cup W'_0, W_1 \cup W'_1)$ 
7:      $G \leftarrow G \setminus (W'_0 \cup W'_1)$ 
8:   until  $V = \emptyset$ 
9:   return  $(W_0, W_1)$ 
10: end function
```

---

The main complexity and novelty of the PP algorithm lies in the way it identifies a dominion. To this end, it relaxes the notion of a dominion to a *quasi*  $\alpha$ -dominion. A quasi  $\alpha$ -dominion is a set  $U$  of vertices for which  $\alpha$  has a strategy that guarantees that every play that remains within  $U$  is won by  $\alpha$ , or that exits  $U$  via the  $\bar{\alpha}$ -escape of  $U$ . The  $\alpha$ -escape of a set  $U$ , denoted  $esc_G^\alpha(U)$  is the set of vertices from which  $\alpha$  can force play out of  $U$  in a single move; *i.e.*  $v \in esc_G^\alpha(U)$ , for a game  $G$  holds iff  $v \in V_\alpha$  and for all  $w$  for which  $v \rightarrow w$  we have  $w \notin U$ , or  $v \in V_{\bar{\alpha}}$  and there is some  $w \notin U$  such that  $v \rightarrow w$ . A quasi  $\alpha$ -dominion  $U$  in a game  $G$  is said to be  $\alpha$ -closed iff  $esc_G^{\bar{\alpha}}(U) = \emptyset$ ; otherwise it is said to be  $\alpha$ -open. Note that an  $\alpha$ -closed quasi  $\alpha$ -dominion is an  $\alpha$ -dominion. When the  $\bar{\alpha}$ -escape set of a quasi  $\alpha$ -dominion only contains vertices with the highest priority in the game, the  $\alpha$  quasi-dominion is called a  $\alpha$ -region.

By searching for quasi-dominions, the algorithm avoids the ‘hard’ problem of identifying the (set of) priorities that lead to a player winning its dominion. Under certain conditions, these quasi-dominions offer just enough guarantees that when composed, the result is a dominion. Regions meet these conditions. The dominion searcher, see Algorithm 3, essentially traverses a partial order of *states* which contain information about the open regions the searcher has computed up to that point. Once the searcher finds a closed region it terminates and returns this dominion to the solver. Conceptually the searcher assigns to each  $\alpha$ -region it finds a priority that under-approximates the best value the opponent  $\bar{\alpha}$  must visit when escaping from the  $\alpha$ -region. A higher  $\alpha$ -region  $U_1$  can then be merged with a lower region  $U_2$ , yielding a new  $\alpha$ -region and improving the under-approximation of  $U_2$  by promoting its best escape value.

The searcher first checks whether the set of vertices  $U$  that dominate the current state induces a closed region  $A$  within the entire game; if so, the region is returned and the search is finished (lines 2-8). If the region is open, the opponent  $\bar{\alpha}$  may escape. When the region is open in the current state, she may try to escape to some inferior priority within the subgraph; if the region is closed in the current state, she can only escape to some region in the larger game and end up in a priority that dominates  $m_g$ .

In case the region is open in the current state (line 11-14), the priority function of the current state is updated to set all vertices in  $A$  to the currently dominating priority  $m_g$ . This is achieved by the update  $p_g^* \leftarrow p_g[U \mapsto m_g]$  which is defined as  $p_g^*(v) = m_g$  in case  $v \in U$  and  $p_g^*(v) = p_g(v)$  otherwise. The new priority for the subgraph that will be explored is set to the next largest priority in the graph (line 13) and  $A$  is removed from the subgraph (line 14). This new state is then explored recursively (see line 20). In case the region is closed in the current state (line 16-18), the opponent must escape (if she wants to) to vertices in the larger game. The *best escape priority* she can force is the minimal priority among the set of vertices that she can force play to in a single step. This is given by the function  $best_{p_g}^\alpha(A)$  which yields the minimum priority (according to  $p_g$ ) of the set  $\{w \in V \setminus A \mid \exists v \in A \cap V_\alpha : v \rightarrow w\}$ . The dominating priority  $m_g$  is updated to  $m_g^*$  to reflect this best escape priority (see line 16) after which  $p_g$  is updated to  $p_g^*$ . In this update, all vertices in  $A$  are set to priority  $m_g^*$  while

all vertices with priorities exceeding  $m_g^*$  remain unchanged. All vertices with priorities dominated by  $m_g^*$  are reset to their original value. This is achieved by the update in line 17. Here,  $p_g^{\geq m}$  yields the partial function that coincides with  $p_g$  on the (maximal) domain  $V' \subset V$  for which  $p_g(v) \geq m$  for all  $v \in V'$ , and is undefined elsewhere. The *update*  $p \uplus p_g$ , for a partial function  $p_g$  is then defined as  $(p \uplus p_g)(v) = p(v)$  in case  $v \notin \text{dom}(p_g)$  and  $p_g(v)$  otherwise. The subgraph that is explored next is set to all vertices with (promoted) priority no larger than  $m_g^*$ . This newly constructed state is then again recursively explored (see line 20). The PP algorithm resets all information in lower regions when promoting ver-

---

**Algorithm 3** Priority Promotion Dominion Searcher

---

```

1: function SEARCHDOMINION( $G, V_g, p_g, m_g$ )
2:    $\alpha \leftarrow m_g \bmod 2$ 
3:    $U \leftarrow \{v \in V_G \mid p_g(v) = m_g\}$ 
4:    $A \leftarrow \text{Attr}_\alpha(V_g, U)$ 
5:    $X \leftarrow \text{esc}_G^\alpha(A)$ 
6:   if  $X = \emptyset$  then
7:      $(W'_\alpha, W'_\alpha) \leftarrow (\text{Attr}_\alpha(A), \emptyset)$ 
8:     return  $(W'_0, W'_1)$ 
9:   else
10:     $X \leftarrow \text{esc}_{G \cap V_g}^\alpha(A)$ 
11:    if  $X \neq \emptyset$  then
12:       $p_g^* \leftarrow p_g[A \mapsto m_g]$ 
13:       $m_g^* \leftarrow \max\{k \mid k < m_g \wedge \exists v \in V_g : p_g^*(v) = k\}$ 
14:       $V_g^* \leftarrow V_g \setminus A$ 
15:    else
16:       $m_g^* \leftarrow \text{bep}_{p_g}^\alpha(A)$ 
17:       $p_g^* \leftarrow (p \uplus p_g^{\geq m_g^*})[A \mapsto m_g^*]$ 
18:       $V_g^* \leftarrow \{v \in V \mid p_g^*(v) \leq m_g^*\}$ 
19:    end if
20:    return SEARCHDOMINION( $G, V_g^*, p_g^*, m_g^*$ )
21:  end if
22: end function

```

---

tices. As observed in [3, 4], this can be improved by only resetting regions of the opponent, or even a subset of that. While this affects the performance for some artificial worst-case examples, these improvements do not lead to improvements on games stemming from practical applications, nor on random games [21].

### 3.3 The Fixpoint-Iteration Algorithm

The third algorithm we consider is the so-called *Fixpoint-Iteration* algorithm. It is essentially based on the linear reduction of the parity game solving problem to the modal  $\mu$ -calculus model checking problem. This transformation essentially views a parity game as a graph with state labels indicating the owner

(by associating a proposition **Even** or **Odd** to a state) and priority of a vertex (by associating a proposition  $\text{prio}_i$  to a state representing a vertex with priority  $i$ ), while the winning regions  $W_0$  and  $W_1$  of the game are characterised by a so-called Walukiewicz  $\mu$ -calculus formula.

Rather than using a general-purpose modal  $\mu$ -calculus model checker for model checking a parity game, we here recall the algorithm given by Bruse *et al.* [6], see Algorithm 4. This algorithm works directly on a (pre-processed) parity game, essentially checking the following formula<sup>1</sup> for games with least priority 0, and a similar formula for games with least priority 1:

$$\sigma X_{d-1} \dots \mu X_1. \nu X_0. ((\text{Even} \wedge \bigvee_{i=0}^{d-1} \diamond(\text{prio}_i \wedge X_i)) \vee (\text{Odd} \wedge \bigwedge_{i=0}^{d-1} \square(\overline{\text{prio}_i} \vee X_i))$$

The pre-processing guarantees that the parity game is *compressed*: *i.e.* it ensures that the set of priorities occurring in the game constitute an integer interval with least priority 0 or 1. Note that converting a parity game to a compressed parity game can be done without changing the solution to the game; the transformation is sometimes referred to as *priority compression*. The main loop of the algorithm

---

**Algorithm 4** Fixpoint-Iteration algorithm for compressed parity games with least priority 0 and maximal priority  $d - 1$ . The algorithm for compressed parity games with least priority 1 is analogous.

---

```

1: function FI( $G$ )
2:   function DIAMOND_BOX
3:      $U_0 \leftarrow \{v \in V_0 \mid \exists w \in V : v \rightarrow w \wedge w \in X_{p(w)}\}$ 
4:      $U_1 \leftarrow \{v \in V_1 \mid \forall w \in V : v \rightarrow w \implies w \in X_{p(w)}\}$ 
5:     return  $U_0 \cup U_1$ 
6:   end function
7:   for  $i \leftarrow d - 1, \dots, 0$  do
8:      $X_i \leftarrow$  if  $i$  is even then  $V$  else  $\emptyset$ 
9:   end for
10:  repeat
11:     $X'_0, X_0 \leftarrow X_0, \text{DIAMOND\_BOX}()$ 
12:     $i \leftarrow 0$ 
13:    while  $X_i = X'_i$  and  $i < d - 1$  do
14:       $i \leftarrow i + 1$ 
15:       $X'_i, X_i \leftarrow X_i, X_{i-1}$ 
16:       $X_{i-1} \leftarrow$  if  $i$  is odd then  $V$  else  $\emptyset$ 
17:    end while
18:  until  $i = d - 1$  and  $X_{d-1} = X'_{d-1}$ 
19:  return  $(X_{d-1}, V \setminus X_{d-1})$ 
20: end function

```

---

<sup>1</sup> Here,  $\mu$  and  $\nu$  are the least and greatest fixpoint, respectively,  $\diamond$  is the *may* modality,  $\square$  is the *must* modality, **Even**, **Odd** and  $\text{prio}_i$  are propositions and  $\overline{\text{prio}_i}$  is the negation of  $\text{prio}_i$ .

evaluates the fixpoint-free part of the Walukiewicz formula using the current values for the variables  $X_0 \dots X_{d-1}$ . The evaluation gives rise to new values for these variables until a fixpoint is reached for all variables; at that moment, variable  $X_{d-1}$  contains all vertices that satisfy the formula, and, by construction, are won by player 0.

Note that the re-initialisation of all variables  $X_0 \dots X_{d-1}$  can be optimised by exploiting monotonicity of the formula, reducing the overall complexity from exponential in  $d$  to exponential in  $\lceil d/2 \rceil$ , see *e.g.* [20]. This can be done by first computing the smallest index  $i$  for which  $X_i$  is not yet stable, and only resetting  $X_{i-1}, X_{i-3}, \dots$  afterwards, see also [6].

## 4 Implementing Parity Game Solvers Using BDDs

An explicit representation of a parity game  $G$  that encodes a verification problem of a (software or hardware) system quickly requires too much main memory. A symbolic representation of the game graph may then help to sidestep this problem. Binary Decision Diagrams (BDDs) are essentially a clever data structure that can be used to concisely and canonically represent propositional formulae, which in turn can be used to characterise sets. In our setting we have to represent the following sets: the set of vertices  $V$  of a game, the edges  $E$ , the partition  $(V_0, V_1)$  of  $V$  and the priority function  $p$ . For most parity games stemming from practical verification problems, the number of distinct priorities in the game is small. For that reason, it suffices to represent  $p$  as a collection of ordered sets.

We assume that the reader is familiar with BDDs and the algorithms manipulating these; for a comprehensive treatment we refer to [22, 10]. For the computations involved in the parity game solving algorithms of the previous section we provide the key ingredients below; each operation is described using an expression for the BDD-based implementation.

Henceforward we assume that  $\mathbf{x}$  is the vector of Boolean variables used to span the set of vertices  $V$ ; we assume  $\iota$  is a total injective mapping from  $V$  to truth-assignments for  $\mathbf{x}$ . The  $i$ -th Boolean variable of  $\mathbf{x}$  is denoted  $\mathbf{x}_i$ ; this notation extends to other vectors of Boolean variables. We represent a set  $U \subseteq V$  by a propositional formula encoded as a BDD  $\mathcal{U}(\mathbf{x})$  that ranges over  $\mathbf{x}$ ; as is standard, for a truth-assignment  $\mathbf{u}$  for  $\mathbf{x}$  we have  $\mathcal{U}(\mathbf{u}) = 1$  iff  $\iota(v) = \mathbf{u}$  for some  $v \in U$ . This way, we can represent, *e.g.* the sets  $V_0$  and  $V_1$  by BDDs  $\mathcal{V}_0(\mathbf{x})$  and  $\mathcal{V}_1(\mathbf{x})$ . Note that we only need representations of two of the three sets  $V, V_0$  and  $V_1$ ; the third can be constructed efficiently by computing the complement of the BDD. The set of edges is given by a BDD  $\mathcal{E}(\mathbf{x}, \mathbf{x}')$  that ranges over  $\mathbf{x}$  and  $\mathbf{x}'$ , where  $\mathbf{x}'$  is the vector of Boolean variables used to represent a successor vertex. Finally, the collection of BDDs  $\mathcal{P}_i(\mathbf{x})$  represents the set of vertices with priority  $i$ .

We first highlight the operations required for implementing Zielonka's algorithm using BDDs. Set comparison, intersection, union and complement are easily expressed as BDD operations using equivalence, conjunction, disjunction and negation. The main computational problem in Zielonka's algorithm is there-

fore the attractor set computation and computing a subgame. Since subgames are defined using set intersection, the operations involved when computing a subgame require conjunction of BDDs only; we omit further details. For the attractor set computation, which is also used in the PP algorithm, we first define the *pre-image* operation using BDDs; that is, given a set  $U \subseteq V$  and a player  $\alpha$ , we encode the operation  $pre_\alpha(U)$  which is defined as follows:

$$pre_\alpha(U) = \{v \in V_\alpha \mid \exists u \in U : v \rightarrow u\} \cup \{v \in V_{\bar{\alpha}} \mid \forall u \in V : v \rightarrow u \Rightarrow u \in U\}$$

Observe that we can restate this as follows:

$$pre_\alpha(U) = \{v \in V_\alpha \mid \exists u \in U : v \rightarrow u\} \cup \{v \in V_{\bar{\alpha}} \mid \neg \exists u \in V \setminus U : v \rightarrow u\}$$

Representing the above operation using standard BDD operations such as disjunction, conjunction, negation and existential quantification is then as follows:

$$\begin{aligned} & pre_\alpha(\mathcal{U}(\mathbf{x})) \\ = & (\mathcal{V}_\alpha(\mathbf{x}) \wedge \exists \mathbf{x}' . (\mathcal{E}(\mathbf{x}, \mathbf{x}') \wedge \mathcal{U}(\mathbf{x}')))) \vee (\mathcal{V}_{\bar{\alpha}}(\mathbf{x}) \wedge \neg \exists \mathbf{x}' . (\mathcal{E}(\mathbf{x}, \mathbf{x}') \wedge \neg \mathcal{U}(\mathbf{x}')))) \end{aligned}$$

Using the pre-image, the  $\alpha$ -attractor  $Attr_\alpha(\mathcal{U}(\mathbf{x}))$  (cf. Page 2) is then effectively implemented using BDDs. The confined  $\alpha$ -attractor  $Attr_\alpha(\mathcal{T}(\mathbf{x}), \mathcal{U}(\mathbf{x}))$ , used in the PP algorithm (see line 4) is defined analogously using an additional conjunction operator on BDDs. Note also that the operation  $esc_G^\alpha(A)$ , used in lines 5 and 10, can be implemented using the pre-image operation. The DIAMONDBOX() function of the FI algorithm can be implemented in a similar manner:

$$\begin{aligned} & (\mathcal{V}_0(\mathbf{x}) \wedge \bigvee_i (\exists \mathbf{x}' . (\mathcal{E}(\mathbf{x}, \mathbf{x}') \wedge \mathcal{P}_i(\mathbf{x}') \wedge \mathcal{X}_i(\mathbf{x}')))) \\ \vee & (\mathcal{V}_1(\mathbf{x}) \wedge \neg \bigvee_i (\exists \mathbf{x}' . (\mathcal{E}(\mathbf{x}, \mathbf{x}') \wedge \mathcal{P}_i(\mathbf{x}') \wedge \neg \mathcal{X}_i(\mathbf{x}')))) \end{aligned}$$

where, for each  $i$ ,  $\mathcal{X}_i(\mathbf{x})$  is the set of vertices represented by approximation  $X_i$  in Algorithm 4.

The operations in the PP algorithm that remain to be encoded involve the computation of the best escape priority and the computation of new priority mappings. For the best escape priority we require to compute minimal priority among a set of successor vertices of a given set of vertices  $U$ . We split this computation in two parts. First we identify the set of successor vertices using the operation  $post(U)$  defined as follows:

$$post(U) = \{v \in V \mid \exists u \in U : u \rightarrow v\}$$

This can effectively be encoded as follows:

$$post(\mathcal{U}(\mathbf{x})) = (\exists \mathbf{x} . \mathcal{E}(\mathbf{x}, \mathbf{x}') \wedge \mathcal{U}(\mathbf{x}))[\mathbf{x}' \mapsto \mathbf{x}]$$

Computing  $bep_{p_g}^\alpha(A)$ , see line 16, then can be implemented by searching for the least value  $m$  in the domain of mapping  $p_g$  for which the BDD  $\neg \mathcal{A}(\mathbf{x}) \wedge p_g(m) \wedge post(\mathcal{A}(\mathbf{x}) \wedge \mathcal{V}_\alpha(\mathbf{x}))$  is satisfiable. This can be done using a simple iteration.

Finally, the computations of  $p_g^*$  in lines 12 and 17 are simple updates of the function  $p_g$  which can be done by iterating over the domain of the mapping  $p$  (resp.  $p_g$ ) and computing new BDDs using conjunction and disjunction of the BDDs given by  $p$  and  $p_g$ . More specifically, for updates of the form  $(p \uplus p_g^{\geq m})[A \mapsto m]$ , we first compute the set of vertices in the domain of  $p_g$  that are above  $m$ , assuming we have BDDs  $\mathcal{P}_{g,i}(\mathbf{x})$  representing the set of vertices  $\{v \in V \mid p_g(v) = i\}$ :

$$\mathit{Above}_{\geq m}(\mathbf{x}) = \bigvee \{\mathcal{P}_{g,i}(\mathbf{x}) \mid i \geq m\}$$

The updated priority function  $p_g^*$  coinciding with  $(p \uplus p_g^{\geq m})[A \mapsto m]$ , where  $A$  is encoded by BDD  $\mathcal{A}(\mathbf{x})$  and the set of vertices  $\{v \in V \mid p(v) = i\}$  is encoded by  $\mathcal{P}_i(\mathbf{x})$  is then, for each value  $i \neq m$ , given by:

$$\neg \mathcal{A}(\mathbf{x}) \wedge ((\mathcal{P}_i(\mathbf{x}) \wedge \neg \mathit{Above}) \vee (\mathcal{P}_{g,i}(\mathbf{x}) \wedge \mathit{Above}))$$

whereas for value  $i = m$  we have:

$$\mathcal{A}(\mathbf{x}) \vee (\mathcal{P}_i(\mathbf{x}) \wedge \neg \mathit{Above}) \vee (\mathcal{P}_{g,i}(\mathbf{x}) \wedge \mathit{Above})$$

## 5 Experimental Evaluation

We have implemented Zielonka’s algorithm and the PP algorithm in Python, utilising a BDD package `dd2` which, next to a native Python BDD implementation, offers wrappers to C-based BDD implementations such as CUDD, Sylvan and BuDDy. For the timings we report on for our experiments we rely on the native Python implementation as our independent experiments indicate that its performance is comparable to that of CUDD. Note that the choice of programming language is secondary given that all essential time-consuming computations involve BDD creation and manipulation. We conducted all experiments on a Macbook Pro, 3.5 GHz Intel Core i7 (13-inch, 2017 model, *i.e.* dual core), with 16 Gb 2133 MHz LPDDR3 main memory, running macOS 10.13.2.

We compare the performance of our implementation of Zielonka’s algorithm and the PP algorithm as described in Sections 3 and 4. To analyse the performance of both algorithms on parity games beyond those from a fixed benchmark set, we analyse their running times on randomly generated games. Such games, however, may be poor predictors for the performance of the algorithms on practical problems. Since one of our aims is to assess the speed at which BDD-based solvers can solve parity games that encode practical verification problems, we also compare their performance on some of the cases of the Keiren benchmark set [17].

---

<sup>2</sup> See <https://github.com/johnyf/dd>, version 0.5.4, by Ioannis Filippidis



## 5.1 Random Parity Games

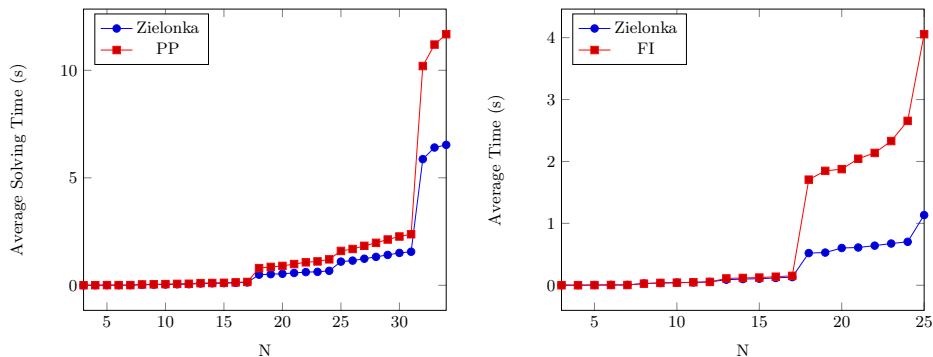
The PGSolver [12] is a collection of parity game solving algorithms programmed in OCaml, which includes tools to generate self-loop-free random parity games with a fixed number of vertices. Since the tool generates explicit graphs in the PGSolver format, we automatically converted these games to BDDs using a binary encoding for the vertices.

Unfortunately (but not unexpected), our experiments indicate that for games that consist of more than 2000 vertices, the BDD approach starts to fail dramatically; the BDD representations of the games become overly complex and unstructured, leading to excessive running times of both our Zielonka and PP implementations. The main reason for the poor performance is the lack of structure provided by the binary encoding of vertices: typically, BDDs remain concise if two adjacent vertices (*i.e.* vertices related by  $E$ ) only differ in their representation by a relatively small number of bits. The binary encoding, however, does not guarantee this.

In view of the above observation, we resort to another approach for generating random games and testing the scalability of both implementations. Instead of generating explicit games, we generate random BDDs representing games of a fixed size but with different characteristics. We do this as follows. We generate games consisting of exactly  $2^N$  vertices using a vector  $\mathbf{x}$  of Boolean variables of length  $N$ . The sets  $V_0$  and  $V_1$ , the priority function  $p$  and the edges  $E$  are then generated by constructing a ‘random’ clause. For  $V_0$  we generate a conjunctive clause by uniformly choosing, for each variable  $\mathbf{x}_i$  whether the variable occurs positively, negatively, or not at all. The set  $V_1$  is set to be the complement of  $V_0$ . For the priority mapping we use a similar approach ensuring that the priority mapping partitions the set of vertices  $V$ ; we set an upper bound of  $K$  priorities, meaning that we generate  $K$  clauses. Since it may happen that some clauses become unsatisfiable, there may occasionally be fewer than  $K$  priorities in the games we generate. Finally, we construct the edge relation  $E$ . To this end, we first randomly select a subset  $\mathbf{u}$  of the Boolean variables  $\mathbf{x}$ . Next, we generate  $M$  clauses where we again uniformly generate a clause based on a positive, negative or no occurrence of a variable in  $\mathbf{u}$ , and, similarly for their copies  $\mathbf{u}'$ . For the remaining variables we require that the primed copies  $\mathbf{x}'_i$  are equivalent to the unprimed ones, *i.e.*  $\mathbf{x}'_i \leftrightarrow \mathbf{x}_i$ .

We measured the performance of all three algorithms on a collection of randomly generated parity games where we varied  $N$  and  $K$ ; we set  $M$  to 4. For each  $N$ , for  $3 \leq N \leq 34$ , we measured the total time to solve 5000 games. Figure 2 (left) plots the size of the randomly generated parity games (each consisting of  $2^N$  vertices) on the  $x$ -axis against the time (in seconds) on the  $y$ -axis for both PP and Zielonka. The jumps in our plot are due to an increase in the number of priorities: for  $N < 18$ , games have at most 5 priorities, for  $N < 25$ , our games have at most 6 priorities, for  $N < 32$ , we generated games with at most 7 priorities, whereas for  $N \geq 32$  we generated games with at most 8 priorities. Figure 2 (right) plots the size of randomly generated parity games (using the

same setup, but with a sample size of 500 instead of 5000) against the average time (in seconds) on the  $y$ -axis for Zielonka and the FI algorithm.



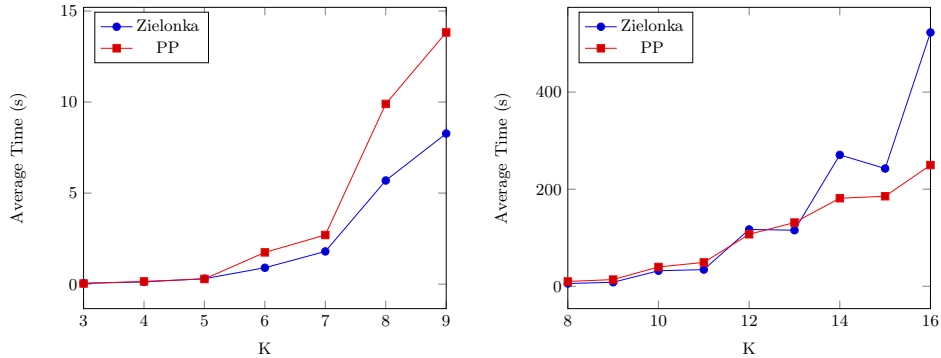
**Fig. 2.** Left: Average solving time (in seconds) per game per instance of  $N$ , consisting of  $2^N$  vertices each using Zielonka and the PP algorithm (sample size: 5000 games per instance of  $N$ ). Right: Average solving time (in seconds) per game per instance of  $N$ , consisting of  $2^N$  vertices each using Zielonka and the FI algorithm (sample size: 500 games per instance of  $N$ ).

We observe that the FI algorithm is no match for Zielonka as soon as the number of different priorities in the game grows beyond 5 (*i.e.* for  $N \geq 18$ ). Furthermore, Zielonka’s algorithm consistently outperforms PP. While for small games the difference is not very significant, Zielonka seems to scale better as games get more complex, as witnessed by the larger values for  $N$ . We remark that the performance of both algorithms is still rather good: solving games consisting of  $2^{34}$  vertices with 8 priorities takes, on average 6.4 seconds using Zielonka and 11.7 seconds using PP. We repeat the experiment, fixing  $N$  to 32,  $M$  again to 4 and varying  $K$  from 4 to 16, and generate 500 games for each configuration. The results for  $K \leq 9$  are depicted in Figure 3 (left), whereas the results for  $K \geq 8$  are depicted in Figure 3 (right).

These second set of experiments sketch a slightly different picture, showing that both Zielonka and PP have similar performance on average for games with few priorities. However, as the number of different priorities in the games increase, the performance of both algorithms degrades, with Zielonka suffering a significantly worse performance degradation than PP. This suggests that these algorithms will have more difficulty handling symbolic games that derive from, *e.g.* synthesis problems which typically have a large number of priorities, and that PP is likely to perform better than Zielonka for such use cases.

## 5.2 The Keiren Parity Game Solving Benchmark Set

In [17], Keiren describes and provides a set of parity games that originate from over 300 model checking problems and more than 200 equivalence checking prob-



**Fig. 3.** Average solving time (in seconds) per game with  $2^{32}$  vertices per instance of  $K$  using Zielonka and the PP algorithm (sample size: 500 games per instance of  $K$ .)

lems. The main obstacle in reusing the data set is that the games are encoded as explicit graphs in the PGSolver format. This means that most of the structure that a BDD solver can typically exploit for compactly representing the game graph is lost. As we concluded from the random games generated by PGSolver, a binary encoding of the sets involved leads to severe performance degradation of the solvers, and given the size of the graphs, there is little hope the running times of our algorithms on such encoded BDDs provide meaningful information. We have coped with this by generating BDDs from the original specifications for a few of the model checking games included in the Keiren benchmark set.<sup>3</sup> Note that the conversion of the original specifications to BDDs is not straightforward: most games stem from model checking problems for system models that employ (unbounded) lists, natural numbers, *etcetera*, for which no trivial bounds can be established. The results can be found in Table 1.

We note that solving the explicit game for Onebit<sub>8</sub> (the onebit sliding window protocol with 8 different messages) requires over 140 seconds using the `pbespgsolve` tool of the mCRL2 [9] tool set and consumes 16Gb of memory. In contrast, our BDD algorithms both solve the resulting game in under a second and require a negligible amount of memory. We were unable to solve the Onebit<sub>32</sub> game using the `pbespgsolve` tool. Our experiments suggest that PP and Zielonka will perform comparably on games originating from verification problems. Moreover, the Fixpoint-Iteration algorithm is in several cases slightly faster than PP and Zielonka; this aligns with our observations that on random games with very few priorities, the Fixpoint-Iteration algorithm is competitive with Zielonka’s algorithm.

<sup>3</sup> We developed a special-purpose `pbes2bdd` tool for converting (a fragment of) parameterised Boolean equation systems in Standard Recursive Form, containing only Boolean data parameters, to BDDs. The tool currently does not guarantee totality of the generated parity game; instead, it relies on the user to guarantee that this is the case.

Model	Property	Zielonka PP		FI	$\#x$
Chatbox	Absence of Deadlock	0.04	0.05	0.01	18
	It is possible to JOIN infinitely often	0.02	0.02	0.00	18
	Invariantly LEAVE can follow JOIN	0.04	0.05	0.01	18
Onebit <sub>2</sub>	Infinitely often send and receive	0.61	0.77	0.67	26
	Absence of Deadlock	0.16	0.32	0.08	26
Onebit <sub>8</sub>	Infinitely often send and receive	3.30	3.28	3.36	38
	Absence of Deadlock	0.86	1.81	0.37	38
Onebit <sub>32</sub>	Infinitely often send and receive	10.20	9.99	9.48	50
	Absence of Deadlock	2.39	5.19	1.09	50

**Table 1.** Running times (in seconds) for selected model checking problems;  $\#x$  indicates the number of Boolean variables needed to encode the set of vertices.

### 5.3 Discussion and Threats to Validity

The correctness of the tested algorithms was established in the respective papers in which they were published. To gain confidence in the correctness of our implementation we have made sure that our code closely matches the pseudocode given in the preceding sections. Moreover, we cross-checked the answers of all solvers, comparing the computed winning sets for both players for each game we solved. Alternatively, one might compute winning strategies during game solving; such strategies provide the certificates that can be checked in polynomial time. While computing winning strategies can be done straightforwardly in an explicit setting, it is not clear how to do the same in a symbolic setting. Finally, we base some of our conclusions on randomly generated games; such games may not have the structure of games that originate from practical verification problems or synthesis problems. Moreover, we have reported on the average time spent on solving games; these solving times are no indication for solving a given game. In fact, while running our experiments we have come across games that require a factor 10 or more time to solve, and it is hard to predict which of the algorithms we studied will solve a given game quickest.

Our results indicate that, overall, the performance of Zielonka’s algorithm and the PP algorithm is comparable in most cases. For games with a small number of priorities, Zielonka seems to perform slightly better than PP, suggesting that in practice, the former is the solver of first choice. However, given that both solvers require at most seconds for even the largest models, containing up to  $2^{34}$  vertices, either algorithm would work in practice. The Fixpoint-Iteration algorithm is only able to keep up with Zielonka for games with at most 5 priorities. Comparing the performance of our symbolic solvers to the explicit solvers, however, indicates that there the BDD-based solvers offer huge performance gains.

## 6 Conclusions

We studied three algorithms for solving parity games, *viz.* Zielonka’s recursive algorithm, the recently introduced Priority Promotion algorithm, and the

Fixpoint-Iteration algorithm. Both Zielonka and PP work by repeatedly identifying dominions by decomposing a parity game, but both have a different way of doing so; the Fixpoint-Iteration algorithm essentially solves a model checking problem on a parity game. We have described how these algorithms can be implemented using Binary Decision Diagrams (BDDs) and subsequently assessed the performance of our implementations. Since there are no available benchmark sets for experimenting on symbolically encoded BDDs, we have selected a few cases from the Keiren benchmark set for parity game solvers [17] and manually modified these so that their PBES representations could be converted to BDDs using the mCRL2 tool `pbes2bdd` we developed for this purpose. In addition, we have assessed our implementations on random games of varying sizes and complexities. Our results show that both Zielonka and PP perform remarkably similar on average, with the classic Zielonka algorithm slightly outperforming the PP solver for games with few priorities. The Fixpoint-Iteration algorithm keeps up with Zielonka for games with at most 5 priorities but performs poorly for games with a larger number of priorities. As soon as the number of priorities increases beyond the number typically found in games encoding verification problems, also the performance of Zielonka and PP is seriously compromised, with Zielonka being affected worse. The performance degradation, however, is unlikely to be specific to the use of BDDs.

## References

1. A. Arnold, A. Vincent, and I. Walukiewicz. Games for synthesis of controllers with partial observation. *TCS*, 303(1):7–34, 2003.
2. M. Bakera, S. Edelkamp, P. Kissmann, and C.D. Renner. Solving  $\mu$ -calculus parity games by symbolic planning. In *MoChArt*, volume 5348 of *Lecture Notes in Computer Science*, pages 15–33. Springer, 2008.
3. M. Benerecetti, D. Dell’Erba, and F. Mogavero. A delayed promotion policy for parity games. In *GandALF*, volume 226 of *EPTCS*, pages 30–45, 2016.
4. M. Benerecetti, D. Dell’Erba, and F. Mogavero. Improving priority promotion for parity games. In *Haifa Verification Conference*, volume 10028 of *Lecture Notes in Computer Science*, pages 117–133, 2016.
5. M. Benerecetti, D. Dell’Erba, and F. Mogavero. Solving parity games via priority promotion. In *CAV (2)*, volume 9780 of *Lecture Notes in Computer Science*, pages 270–290. Springer, 2016.
6. F. Bruse, M. Falk, and M. Lange. The fixpoint-iteration algorithm for parity games. In *GandALF*, volume 161 of *EPTCS*, pages 116–130, 2014.
7. C.S. Calude, S. Jain, B. Khossainov, W. Li, and F. Stephan. Deciding parity games in quasipolynomial time. In *STOC*, pages 252–263. ACM, 2017.
8. T. Chen, B. Ploeger, J. van de Pol, and T.A.C. Willemse. Equivalence Checking for Infinite Systems Using Parameterized Boolean Equation Systems. In *CONCUR’07*, pages 120–135, 2007.
9. S. Cranen, J.F. Groote, J.J.A. Keiren, F.P.M. Stappers, E.P. de Vink, W. Weselink, and T.A.C. Willemse. An overview of the mCRL2 toolset and its recent advances. In *TACAS*, volume 7795 of *Lecture Notes in Computer Science*, pages 199–213. Springer, 2013.

10. R. Drechsler and B. Becker. *Binary Decision Diagrams - Theory and Implementation*. Springer, 1998.
11. E.A. Emerson and C.S. Jutla. Tree automata, mu-calculus and determinacy. In *FOCS'91*, pages 368–377, Washington, DC, USA, 1991. IEEE Computer Society.
12. O. Friedmann and M. Lange. Solving parity games in practice. In *ATVA*, volume 5799 of *Lecture Notes in Computer Science*, pages 182–196. Springer, 2009.
13. M. Gazda and T.A.C. Willemse. Zielonka’s recursive algorithm: dull, weak and solitaire games and tighter bounds. In *GandALF*, volume 119 of *EPTCS*, pages 7–20, 2013.
14. M. Jurdziński. Deciding the winner in parity games is in  $UP \cap co-UP$ . *IPL*, 68(3):119–124, 1998.
15. M. Jurdziński. Small progress measures for solving parity games. In *STACS'00*, volume 1770 of *LNCS*, pages 290–301. Springer, 2000.
16. M. Jurdzinski and R. Lazic. Succinct progress measures for solving parity games. In *LICS*, pages 1–9. IEEE Computer Society, 2017.
17. J.J.A. Keiren. Benchmarks for parity games. In *FSEN*, volume 9392 of *Lecture Notes in Computer Science*, pages 127–142. Springer, 2015.
18. Y. Liu, Z. Duan, and C. Tian. An improved recursive algorithm for parity games. In *TASE*, pages 154–161. IEEE Computer Society, 2014.
19. R. McNaughton. Infinite games played on finite graphs. *APAL*, 65(2):149–184, 1993.
20. H. Seidl. Fast and simple nested fixpoints. *Inf. Process. Lett.*, 59(6):303–308, 1996.
21. T. van Dijk. Oink: an implementation and evaluation of modern parity game solvers. In *TACAS*, 2018. To appear; see <https://arxiv.org/abs/1801.03859>.
22. I. Wegener. *Branching Programs and Binary Decision Diagrams*. SIAM, 2000.
23. W. Zielonka. Infinite games on finitely coloured graphs with applications to automata on infinite trees. *TCS*, 200(1-2):135 – 183, 1998.



If you want to receive reports, send an email to: [wsinsan@tue.nl](mailto:wsinsan@tue.nl) (we cannot guarantee the availability of the requested reports).

*In this series appeared (from 2012):*

12/01	S. Cranen	Model checking the FlexRay startup phase
12/02	U. Khadim and P.J.L. Cuijpers	Appendix C / G of the paper: Repairing Time-Determinism in the Process Algebra for Hybrid Systems ACP
12/03	M.M.H.P. van den Heuvel, P.J.L. Cuijpers, J.J. Lukkien and N.W. Fisher	Revised budget allocations for fixed-priority-scheduled periodic resources
12/04	Ammar Osaiweran, Tom Fransen, Jan Friso Groote and Bart van Rijnsoever	Experience Report on Designing and Developing Control Components using Formal Methods
12/05	Sjoerd Cranen, Jeroen J.A. Keiren and Tim A.C. Willemse	A cure for stuttering parity games
12/06	A.P. van der Meer	CIF MSOS type system
12/07	Dirk Fahland and Robert Prüfer	Data and Abstraction for Scenario-Based Modeling with Petri Nets
12/08	Luc Engelen and Anton Wijs	Checking Property Preservation of Refining Transformations for Model-Driven Development
12/09	M.M.H.P. van den Heuvel, M. Behnam, R.J. Bril, J.J. Lukkien and T. Nolte	Opaque analysis for resource-sharing components in hierarchical real-time systems - extended version -
12/10	Milosh Stolikj, Pieter J. L. Cuijpers and Johan J. Lukkien	Efficient reprogramming of sensor networks using incremental updates and data compression
12/11	John Businge, Alexander Serebrenik and Mark van den Brand	Survival of Eclipse Third-party Plug-ins
12/12	Jeroen J.A. Keiren and Martijn D. Klabbers	Modelling and verifying IEEE Std 11073-20601 session setup using mCRL2
12/13	Ammar Osaiweran, Jan Friso Groote, Mathijs Schuts, Jozef Hooman and Bart van Rijnsoever	Evaluating the Effect of Formal Techniques in Industry
12/14	Ammar Osaiweran, Mathijs Schuts, and Jozef Hooman	Incorporating Formal Techniques into Industrial Practice
13/01	S. Cranen, M.W. Gazda, J.W. Wesselink and T.A.C. Willemse	Abstraction in Parameterised Boolean Equation Systems
13/02	Neda Noroozi, Mohammad Reza Mousavi and Tim A.C. Willemse	Decomposability in Formal Conformance Testing
13/03	D. Bera, K.M. van Hee and N. Sidorova	Discrete Timed Petri nets
13/04	A. Kota Gopalakrishna, T. Ozcelebi, A. Liotta and J.J. Lukkien	Relevance as a Metric for Evaluating Machine Learning Algorithms
13/05	T. Ozcelebi, A. Weffers-Albu and J.J. Lukkien	Proceedings of the 2012 Workshop on Ambient Intelligence Infrastructures (WAmI)
13/06	Lotfi ben Othmane, Pelin Angin, Harold Weffers and Bharat Bhargava	Extending the Agile Development Process to Develop Acceptably Secure Software
13/07	R.H. Mak	Resource-aware Life Cycle Models for Service-oriented Applications managed by a Component Framework
13/08	Mark van den Brand and Jan Friso Groote	Software Engineering: Redundancy is Key
13/09	P.J.L. Cuijpers	Prefix Orders as a General Model of Dynamics



14/01	Jan Friso Groote, Remco van der Hofstad and Matthias Raffelsieper	On the Random Structure of Behavioural Transition Systems
14/02	Maurice H. ter Beek and Erik P. de Vink	Using mCRL2 for the analysis of software product lines
14/03	Frank Peeters, Ion Barosan, Tao Yue and Alexander Serebrenik	A Modeling Environment Supporting the Co-evolution of User Requirements and Design
14/04	Jan Friso Groote and Hans Zantema	A probabilistic analysis of the Game of the Goose
14/05	Hrishikesh Salunkhe, Orlando Moreira and Kees van Berkel	Buffer Allocation for Real-Time Streaming on a Multi-Processor without Back-Pressure
14/06	D. Bera, K.M. van Hee and H. Nijmeijer	Relationship between Simulink and Petri nets
14/07	Reinder J. Bril and Jinkyu Lee	CRTS 2014 - Proceedings of the 7th International Workshop on Compositional Theory and Technology for Real-Time Embedded Systems
14/08	Fatih Turkmen, Jerry den Hartog, Silvio Ranise and Nicola Zannone	Analysis of XACML Policies with SMT
14/09	Ana-Maria Şutii, Tom Verhoeff and M.G.J. van den Brand	Ontologies in domain specific languages – A systematic literature review
14/10	M. Stolikj, T.M.M. Meyfroyt, P.J.L. Cuijpers and J.J. Lukkien	Improving the Performance of Trickle-Based Data Dissemination in Low-Power Networks
15/01	Önder Babur, Tom Verhoeff and Mark van den Brand	Multiphysics and Multiscale Software Frameworks: An Annotated Bibliography
15/02	Various	Proceedings of the First International Workshop on Investigating Dataflow In Embedded computing Architectures (IDEA 2015)
15/03	Hrishikesh Salunkhe, Alok Lele, Orlando Moreira and Kees van Berkel	Buffer Allocation for Realtime Streaming Applications Running on a Multi-processor without Back-pressure
15/04	J.G.M. Mengerink, R.R.H. Schiffelers, A. Serebrenik, M.G.J. van den Brand	Evolution Specification Evaluation in Industrial MDSE Ecosystems
15/05	Sarmen Keshishzadeh and Jan Friso Groote	Exact Real Arithmetic with Perturbation Analysis and Proof of Correctness
15/06	Jan Friso Groote and Anton Wijs	An $O(m \log n)$ Algorithm for Stuttering Equivalence and Branching Bisimulation
17/01	Ammar Osaiweran, Jelena Marincic and Jan Friso Groote	Assessing the quality of tabular state machines through metrics
17/02	J.F. Groote and e.P. de Vink	Problem solving using process algebra considered insightful
18/01	L. Sanchez, W. Wesselink and T.A.C. Willemse	BDD-Based Parity Game Solving: A comparison of Zielonka's Recursive Algorithm, Priority Promotion and Fixpoint Iteration