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Homogenized enriched continuum analysis of acoustic metamaterials with negative stiffness and double negative effects

A. Sridhar, L. Liu, V.G. Kouznetsova*, M.G.D. Geers
Department of Mechanical Engineering, Eindhoven University of Technology, Eindhoven P.O. Box 513, 5600 MB, The Netherlands

Abstract

This paper demonstrates the application of a recently developed enriched micro-inertial continuum based homogenization framework towards numerical dispersion and boundary value problem analyses of local resonance metamaterials exhibiting sub-wavelength negative stiffness and double negative effects (i.e. simultaneous negative effective mass density and stiffness). This is a novel development since homogenized structural dynamic analyses that specifically incorporate negative stiffness effects have not yet been extensively explored. The proposed methodology is successful in approximating the negative stiffness effect to a certain degree. Accordingly, an appropriate error estimation procedure based on dispersion analyses is proposed to identify the limits of the reliability of the homogenized model. The resulting methodology provides a highly efficient framework for the analysis of double negative metamaterial problems involving non-trivial macroscopic loading, the influence of the applied boundary conditions, and a complex unit cell design. This is illustrated through a case study involving the refraction analysis of a double negative metamaterial prism.

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1. Introduction

Acoustic metamaterials have attracted a wide-spread attention in recent years based on their extraordinary elastic wave manipulation capabilities, e.g. band-stop filtering, redirection, channeling and multiplexing, which are usually unreachable by conventional materials (Deymier, 2013; Hussein et al., 2014; Ma and Sheng, 2016). These specially designed composites initiate novel applications in wave transformation, noise attenuation, seismic wave isolation, acoustic cloaking and super-resolution imaging (Hu et al., 2011; Kaina et al., 2015; Mitchell et al., 2014; Wen et al., 2011; Zhang and Liu, 2008; Zhou et al., 2008; Zhu et al., 2011). Two phenomena are responsible for the above-mentioned extraordinary properties, namely Bragg scattering and local resonance. The former relies on the perfect periodicity of the structure and is dominant in the high-frequency (short-wavelength) regime, while the latter is also active in the low-frequency (long-wavelength) regime not requiring material periodicity. This paper focuses on the analysis of a subclass of acoustic metamaterials solely based on local resonance, henceforth termed local resonance acoustic metamaterials (LRAMs).

* Corresponding author.
E-mail address: v.g.kouznetsova@tue.nl (V.G. Kouznetsova).

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Due to the sub-wavelength characteristic of LRAMs, their governing elastodynamic behavior can be modeled as an effective homogeneous continuum with dynamic (frequency dependent) material coefficients that reproduce the dispersion effects due to local resonance (Ma and Sheng, 2016; Zhou et al., 2012). A stopband emerges when either the effective dynamic mass density or elastic material parameter becomes negative in a certain frequency range. Depending on the mechanism, the local resonance stopband can therefore be further classified as either a negative mass density or a negative stiffness based effect. A double negative metamaterial is the one which simultaneously exhibits negative mass and negative stiffness effects in a certain frequency range. The combined effect leads to a passband characterized by a negative effective group velocity and, consequently, a negative effective refractive index (Veselago, 1968). This effect may have potential applications in the field of subwavelength imaging towards an acoustic superlens, whose wavelength resolution surpasses the classical Rayleigh limit (Ma and Sheng, 2016). Relevant publications on the design and analysis of double negative LRAMs are Liu et al. (2011a,b), Lai et al. (2011), and Zhu et al. (2014).

In the literature, various analytical methods have been developed for deriving the effective medium description of simplistic micro-structures (Bückmann et al., 2015; Michelitsch et al., 2003; Sheng et al., 2007; Zhou et al., 2012). In recent years, formal multiscale frameworks for deriving the effective medium for a general metamaterial micro-structure design was proposed by several authors (Bacigalupo and Gambarotta, 2014; Boutin et al., 2014; Chesnais et al., 2012; Pham et al., 2013; Sridhar et al., 2016). Furthermore, exploiting numerical solution methods (e.g. FEM), a two-scale computational homogenization (FE2) framework suitable for simulating LRAM problems involving arbitrary, complex micro and macro-structure geometry, transient macro-scale loading, nonlinear material response etc. was proposed by Pham et al. (2013). In subsequent papers, this framework was modified (Sridhar et al., 2016) and combined with dynamic model reduction techniques (restricted to linear elasticity) resulting in an emergent enriched micro-inertial homogenized continuum of a micro-morphic type, as defined by Eringen (1999). The new framework provides a basis for constructing computationally efficient multiscale techniques for large scale LRAM boundary value problems and thereby a viable alternative to direct numerical simulations (DNS). Furthermore, in Sridhar et al. (2017), a plane wave based semi-analytical approach for the rapid computation of the dispersion and transmission spectra of LRAM structures exploiting the enriched micro-inertial continuum was developed. This approach provides a computationally cheaper alternative to the widely used wave based FE methods (Collet et al., 2011; Farzbod and Leamy, 2011; Kulpe et al., 2014; Mead, 1973) for steady state analysis of LRAMs. Finally, different approaches towards deriving micro-inertial continua of LRAMs presented in the literature (Sun and Huang, 2006; Wang and Sun, 2002) are to be mentioned.

The homogenized enriched continuum model proposed in Sridhar et al. (2016, 2017) has thus far only been used to analyze negative mass based metamaterials, while the negative stiffness effect has remained unexplored. This paper addresses this yet unstudied aspect of the homogenized enriched continuum and demonstrates the applicability of the model to the description of the negative stiffness/double negative effects, leading to an efficient computational multiscale framework for dispersion and structural dynamics analysis of acoustic metamaterials exhibiting such effects.

The structure of the paper is as follows. The essential aspects of the enriched micro-inertial continuum and the underlying homogenization framework are briefly recapitulated in Section 2 with a particular emphasis on the aspects related to the negative stiffness effects, not considered in previous publications. The dispersion eigenvalue problem of the enriched continuum is then derived by assuming a plane wave ansatz in Section 3. Next, the characteristics of the negative stiffness and the negative mass density stopbands are separately analyzed using a simplified model of the homogenized enriched continuum in Section 4. The results obtained here are used to gain useful insights on the negative stiffness approximation. In Section 5, an error estimation procedure assessing the reliability of the predicted negative stiffness effects is detailed. In Section 6, a computational multiscale analysis based on a negative refraction prism problem is presented, based on the homogenized enriched continuum. Finally, the conclusions are given in Section 7.

The following notations are used throughout the paper to represent different quantities and operations. Unless otherwise stated, scalars, vectors, second, third and fourth-order Cartesian tensors are generally denoted by a (or A), $\mathbf{a}$, $\mathbf{A}$, $\mathbf{A}^{(3)}$ and $\mathbf{A}^{(4)}$, respectively; $n_{\text{dim}}$ denotes the number of dimensions of the problem. The zero vector is denoted as $\mathbf{0}$. A right italic subscript is used to index the components of vectorial and tensorial quantities. The Einstein summation convention is used for all vector and tensor related operations represented in index notation. The standard operations are denoted as follows for a given basis $\hat{e}_p$, $p = 1, \ldots, n_{\text{dim}}$. dyadic product: $\hat{a} \otimes \hat{b} = a_p b_q \hat{e}_p \otimes \hat{e}_q$. dot product: $\mathbf{A} \cdot \hat{b} = A_{pq} b_q \hat{e}_p$ and double contraction: $\mathbf{A} : \mathbf{B} = A_{pq} B_{qp}$. Matrices of any type of quantity are in general denoted by $\mathbf{A}$ except for a column matrix, which is denoted by (•). The transpose of a second order tensor is defined as follows: for $\mathbf{A} = A_{pq} \hat{e}_p \otimes \hat{e}_q$, $\mathbf{A}^T = A_{qp} \hat{e}_q \otimes \hat{e}_p$. The transpose operation also simultaneously yields the transpose of the corresponding component matrix. The first and second time derivatives are denoted by (•) and (••), respectively.

2. The homogenized enriched continuum of LRAMs

This section summarizes the main aspects of the enriched continuum model introduced in Pham et al. (2013) and Sridhar et al. (2016). The framework is based on the extension of the classical two-scale computational homogenization formalism (Kouznetsova et al., 2001; Nemat-Nasser and Hori, 1993) with added micro-inertial effects resulting from local resonance. The classical (non-enriched) approach postulates a scale separation principle which requires the characteristic size of heterogeneities to be much smaller than the (smallest) relevant wavelength of the macroscopic loading. This limit restricts the response of the micro-structure to be strictly quasi-static, thus excluding any micro-inertial effects, resulting
in non-dispersive homogenized governing equations. To circumvent this, a relaxed scale separation principle was proposed in Pham et al. (2013) where the classical scale separation was only required on the macroscopic wavelength in the matrix domain of the LRAM whereas the wavelength in the inclusion domain was allowed to scale freely, thereby allowing for local resonance effects. The extended framework based on the relaxed separation of scales requires the solution of the full balance of linear momentum equations at both the micro and macro-scales. The domain of the micro-scale problem, termed the Representative Volume Element (RVE), is given by the periodic unit of the LRAM. For randomly distributed inclusions, the RVE is selected such that it is statistically representative, provided that there is sufficient spacing between the inclusions to ensure the locality of the resonant behavior. The long wave assumption on the matrix domain of the LRAM justifies the use of periodic boundary conditions on the RVE to be applied to the micro-scale displacements (micro-fluctuations) resulting in a (spatially) local effective constitutive behavior. As such, the multiscale framework is valid in a nonlinear setting. A FE spatial discretization combined with an implicit Newmark time integration scheme was introduced in Pham et al. (2013) to solve the micro and macro-scales problems in a nested manner.

Restricting the analysis to linear elasticity, a model order reduction based on static-dynamic decomposition was first applied on the FE model of the RVE in Sridhar et al. (2016). The reduced basis consists of the quasi-static response of the RVE (under the applied macroscopic load) and a set of low frequency eigenmodes of the RVE under fixed boundaries that capture the local resonance modes of the LRAM. Applying the scale transition relations on the reduced model yields a compact set of homogenized enriched continuum equations where the modal amplitudes of the local resonance eigenmodes emerge as additional internal kinematic field variables (in addition to the displacements) at the macro-scale.

Details of the derivation of this framework can be found in Sridhar et al. (2016). For convenience, the important parts of this derivation have been summarized in Appendix A. The final equations describing the homogenized enriched continuum are expressed as follows:

**The macro balance of momentum**

\[
\nabla^\text{M} \cdot \sigma^\text{M} = \rho_M \ddot{u}^\text{M} = 0. 
\]

(1)

**Local resonance dynamics**

\[
\tilde{\omega}_\text{res}^2 \eta M + \tilde{\eta}_M = -\tilde{\sigma}^\text{M} \cdot \tilde{u}^\text{M} - \tilde{\mathbf{H}}^\text{M} : \nabla^\text{M} \tilde{u}^\text{M}, \ s \in Q. 
\]

(2)

**The homogenized constitutive relations**

\[
\sigma_M = \mathbf{C}_M^{(4)} : \nabla^\text{M} \tilde{u}^\text{M} + \mathbf{D}_M^{(4)} : \nabla^\text{M} \mathcal{H}^\text{M} + \sum_{s \in Q} \tilde{\mathbf{H}}^M \cdot \eta M. 
\]

(3a)

\[
\rho_M \ddot{u}^\text{M} = \sum_{s \in Q} \tilde{\mathbf{f}}^\text{M} \cdot \eta M. 
\]

(3b)

Here, \( \sigma_M \) represents the macro-scale Cauchy stress tensor, \( \rho_M \) the macro-scale momentum vector, \( \tilde{u}_M \) the macro-scale displacement vector and \( \eta_M \) the generalized (modal) amplitude of the \( s \)th local resonance eigenmode of the inclusion indexed by the set \( Q \). The subscript \( (\bullet)_M \) here denotes a macro-scale quantity. The coefficients \( \mathbf{C}_M^{(4)} \) and \( \rho_M \) represent the classical effective linear (static) elastic stiffness tensor and the effective (volume average) static mass density, respectively. The enriched material parameters are \( \tilde{\omega}_\text{res} \), \( \tilde{\mathbf{f}}^\text{M} \) and \( \tilde{\mathbf{H}}^\text{M} \) representing the eigenfrequency, the momentum coupling and the stress coupling, of the \( s \)th local resonance mode respectively. A nonzero \( \tilde{\mathbf{f}}^\text{M} \) coefficient indicates a dipolar local resonant mode (Zhou et al., 2012) while monopolar and quadrupolar modes are represented by the \( \tilde{\mathbf{H}}^\text{M} \) coefficient. Finally, \( \mathbf{D}_M^{(4)} \) is an emergent tensor that represents the inertia with respect to the elastic deformation of the whole RVE. Note that the terms \( \mathbf{D}_M^{(4)} \) and \( \tilde{\mathbf{H}}^\text{M} \) were neglected in previous works (Sridhar et al., 2016; 2017) where only the negative mass effects due to the dipolar local resonance were considered. Yet, these terms can play an important role in predicting narrow negative stiffness effects in specially designed LRAM unit cells, which is further exploited in this paper. To summarize, the (tensorial) coefficients \( \mathbf{C}_M^{(4)} \), \( \mathbf{D}_M^{(4)} \), \( \rho_M \), \( \tilde{\mathbf{f}}^\text{M} \), \( \tilde{\mathbf{H}}^\text{M} \) and \( \tilde{\omega}_\text{res} \) \( (\forall s \in Q) \) constitute the set of effective homogenized material parameters characterizing a given LRAM.

The number of the enriched local resonance degrees of freedom, given by \( N_Q \), depends on the minimum number of eigenmodes required to sufficiently accurately capture the dispersive behavior of the system in the desired frequency regime. A simple mode selection criterion for composing a compact reduction basis can be devised as follows. First, compute the set of low frequency eigenmodes up to the eigenfrequency that is several times higher than the highest desired frequency range of analysis. Let these modes be indexed by the set \( \hat{Q} \). For all \( s \in \hat{Q} \), compute the following terms,

\[
\mu_k = \frac{\sigma_k^\text{M}}{\rho_M}, \ 0 < k < n_{\text{dim}} \quad \text{with no summation on } k \text{ or } l. 
\]

(4)

\[
\mu_{Dkl} = \frac{\mathbf{D}_{Mklk}}{D_{Mklk}} \quad \forall k, l = 1, \ldots, n_{\text{dim}} \quad \text{(no summation on } k \text{ or } l). 
\]

(5)

where \( \sigma_k^\text{M} \), \( \mathbf{D}_{Mklk} \) and \( D_{Mklk} \) are the scalar components of their corresponding tensor counterparts, and \( \mu_k \) and \( \mu_{Dkl} \) represent the component modal mass and modal elastic inertia fractions of the \( s \)th mode, respectively. They give the percentage contribution of the local resonant mode to the total inertia of the RVE and hence, an estimate of the strength of
its macroscale coupling. The set $Q$ is composed by retaining only those modes from $Q$ that have at least one of the values of $\mu_{\rho}^k$ or $\mu_{Dk}^l$ ($k,l = 1, \ldots, n_{\dim}$) greater than a specified cutoff. Based on the stopband analysis presented in Section 4, a conservative heuristic estimate of the cutoff value on $\mu_{\rho}^k$ and $\mu_{Dk}^l$ can be made, for example, at 2.5% and 20%, respectively, to ensure an adequate coupling preserving accuracy of the model, without sacrificing model compactness.

### 3. The dispersion eigenvalue problem of the enriched continuum

In this section, the dispersion eigenvalue problem of the homogenized enriched continuum is derived by assuming a plane wave ansatz on the field variables, i.e. $\tilde{u}_M$ and $\eta_M$

$$\phi = (\phi) e^{i k \cdot \mathbf{x} - i \omega t},$$

where notation $(\phi)$ is used to denote the transformed variable. In (6), $\mathbf{k}$ and $\omega$ represent the wavevector and frequency of the plane wave, respectively and $i$ is the imaginary unit. The wave vector can be represented in terms of its magnitude and direction as $\mathbf{k} = k \mathbf{e}_\rho$, where $k = \frac{\omega}{\mathbf{c}_P}$ is the wave number, $\lambda$ the corresponding wavelength and $\mathbf{e}_\rho$ is the unit direction of propagation. Applying the plane wave transform (6) to Eqs. (1)-(3) yields,

$$ik \cdot \mathbf{\tilde{\sigma}}_M^T + i \omega \mathbf{\tilde{p}}_M = \mathbf{0},$$

(7a)

$$\omega^2 \mathbf{\tilde{\eta}}_M - \omega^2 \mathbf{\tilde{\eta}}_M = \omega^2 \mathbf{\tilde{\eta}}_M + i \mathbf{\tilde{\sigma}}_M : i \mathbf{k} \otimes \mathbf{\tilde{u}}_M. \ s \in Q$$

(7b)

$$\mathbf{\tilde{\sigma}}_M = (\omega^2 \mathbf{M}^{(4)} - \omega^2 \mathbf{M}^{(4)}) : i \mathbf{k} \otimes \mathbf{\tilde{u}}_M - \omega^2 \mathbf{\tilde{u}}_M.$$

(7c)

$$\mathbf{\tilde{p}}_M = -i \omega \left( \rho_M \mathbf{\tilde{u}}_M + \sum_{s \in Q} \eta_M ^{\tilde{s}} \mathbf{\tilde{\eta}}_M. \right).$$

(7d)

The terms $\mathbf{\tilde{\sigma}}_M$ and $\mathbf{\tilde{p}}_M$ are eliminated by substituting Eqs. (7c) and (7d) into Eq. (7a). Combining it with Eq. (7b) results in the following eigenvalue problem of the $k$ – $\omega$ form where $\omega$ is the (unknown) eigenvalue and $k$ and $\mathbf{e}_\rho$ act as parameters. **Homogenized dispersion problem: $k$ – $\omega$ form**

$$\left[ \begin{bmatrix} k^2 \mathbf{\tilde{\sigma}}_M : \mathbf{\tilde{\sigma}}_M \mathbf{Q}^{(4)} \mathbf{\tilde{\sigma}}_M + \omega^2 \left( \rho_M ^{\tilde{\sigma}} \mathbf{\tilde{u}}_M + \sum_{s \in Q} \eta_M ^{\tilde{s}} \mathbf{\tilde{\eta}}_M. \right) \end{bmatrix} \right] \left[ \mathbf{\tilde{u}}_M \right] = \left[ \mathbf{0} \right].$$

(8)

where $\omega_{res}^2$ denotes a diagonal matrix of size $N_Q$ containing the squares of eigenfrequencies and $\mathbf{\tilde{\sigma}}_M$ and $\mathbf{\tilde{\eta}}_M$ denote the column matrices of size $N_Q$ containing the corresponding quantities for each resonance mode; $\mathbf{Q}$ and $\mathbf{\tilde{\sigma}}_M$ are the column matrices of size $N_Q$ with all entries equal to the zero vector $\mathbf{0}$ and scalar $0$, respectively; and finally, $(\phi)^T$ stands for the left transpose of a tensor, i.e. $\mathbf{A}^{(4)T} = (A_{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l)^T = A_{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$. The dispersion problem can also be expressed in the $\omega$ – $k$ form, where $\omega$ is the (unknown) eigenvalue and $k$ and $\mathbf{e}_\rho$ are the parameters. To this end, eliminating $\mathbf{\tilde{u}}_M$ from Eqs. (8) and re-arranging the expression gives,

**Homogenized dispersion problem: $\omega$ – $k$ form**

$$\left[ k^2 \mathbf{\tilde{\sigma}}_M : \mathbf{\tilde{\sigma}}_M \mathbf{Q}^{(4)} \mathbf{\tilde{\sigma}}_M + \omega^2 \mathbf{\rho}_M ^{\tilde{\sigma}} \mathbf{\tilde{\sigma}}_M (\omega) \right] \cdot \mathbf{\tilde{e}}_\rho = \mathbf{0}.$$

(9)

where $\mathbf{Q}^{(4)} \mathbf{\tilde{\sigma}}_M ^{\tilde{e}} (\omega)$ and $\mathbf{\rho}_M ^{\tilde{\sigma}} (\omega)$ represent the effective dynamic stiffness and mass density tensors, respectively, expressed as,

$$\mathbf{C}^{(4)}_{M,\text{dyn}} (\omega) = \mathbf{C}^{(4)}_M - \omega^2 \left( \mathbf{D}^{(4)}_M + \sum_{s \in Q} \frac{\omega^2}{\omega_{res}^2} \eta_M ^{\tilde{s}} \mathbf{\tilde{\sigma}}_M ^{\tilde{s}} \otimes \mathbf{\tilde{\sigma}}_M ^{\tilde{s}}. \right).$$

(10)

$$\mathbf{\rho}_M ^{\tilde{\sigma}} (\omega) = \rho_M \mathbf{I} + \sum_{s \in Q} \frac{\omega^2}{\omega_{res}^2} \eta_M ^{\tilde{s}} \mathbf{\tilde{\sigma}}_M ^{\tilde{s}} \otimes \mathbf{\tilde{\sigma}}_M ^{\tilde{s}}.$$

(11)

Here, the subscript $(\cdot)_{\text{M, dyn}}$ distinguishes the dynamic macroscopic parameters from their quasi-static counterparts. The unit normalized eigenvectors of Eq. (9) are denoted by $\mathbf{\tilde{e}}_\phi$, where $p = 1, \ldots, n_{\dim}$, and called, the polarization mode. A polarization mode, $\mathbf{\tilde{e}}_\phi$, for some $p \in \{1, \ldots, n_{\dim}\}$ parallel to $\mathbf{\tilde{e}}_\rho$ (i.e. $\mathbf{\tilde{e}}_\phi \cdot \mathbf{\tilde{e}}_\rho = 1$), represents a pure compressive (P) wave whereas a polarization mode that is orthogonal to $\mathbf{\tilde{e}}_\rho$ (i.e. $\mathbf{\tilde{e}}_\phi \cdot \mathbf{\tilde{e}}_\rho = 0$) represents a pure shear (S) wave. All other orientations represent hybrid wave modes.

For an arbitrary RVE geometry and wave direction, the dispersion problem (8) and (9) can be solved numerically. Due to the reduced nature of the enriched continuum problem, this is now computationally much cheaper in comparison to standard Bloch analysis techniques, especially in the 3D case. The achieved reduction is arguably also larger compared to the reduced order Bloch analysis techniques (Hussein, 2009; Krattiger and Hussein, 2014) due to the simplifying assumptions made in the homogenization.
4. Characteristics of the negative mass density and the negative stiffness stopbands

In this section, analytical closed form expressions of the stopband frequencies of the negative mass density and the negative stiffness stopbands are derived based on a simplified, non-dimensionalized version of the homogenized enriched continuum model. Note that comparable analyses have been performed in the literature (Bückmann et al., 2015; Milton and Willis, 2007; Sheng et al., 2007; Sugino et al., 2017) based on other models. The main purpose of this section is to exploit the analytical expressions to compare and contrast the two effects, and more importantly, to acquire insight into the nature of the negative stiffness effect as predicted by the homogenized enriched continuum.

The following assumptions are made to ease the analytical derivation. The macro-scale dimension is reduced to 1 ($n_{dim} = 1$), and only a single local resonance mode coupled to either the macro stress or momentum is assumed. The resulting approximation nevertheless retains the core aspects of the micro-inertial model. Applying those assumptions to the dispersion Eq. (9) and solving for $k$ gives,

$$k = \omega \sqrt{\frac{\rho_{M,\text{dyn}}}{\rho_{M,\text{dyn}}}},$$

where,

$$C_{M,\text{dyn}}(\omega) = C_{M} - \omega^2 D_{M} - \frac{H_{M}^2}{\omega_{\text{res}}^2} \omega^4 \left( \frac{\omega^2}{\omega_{\text{res}}^2} - \omega^2 \right)$$

$$\rho_{M,\text{dyn}}(\omega) = \rho_{M} + \frac{j_{M}^2}{\rho_{M}} \omega^2 \left( \frac{\omega^2}{\omega_{\text{res}}^2} - \omega^2 \right).$$

Here, $j_{M}, H_{M}, C_{M}, C_{M,\text{dyn}}$ and $\rho_{M,\text{dyn}}$ represent the scalar, one dimensional, versions of their corresponding tensor counterparts. Furthermore, in the subsequent analysis either $H_{M} = 0$ or $j_{M} = 0$ will be adopted in the above expression, focusing on either the negative mass or negative stiffness effect, respectively. Recasting Eq. (13) in terms of non-dimensionalized parameters gives,

$$C_{M,\text{dyn}}(\Omega) = \omega_{\text{res}}^2 D_{M} \left( \Omega_{\text{mat}}^2 - \Omega^2 - \mu_{D} \frac{\Omega^4}{1 - \Omega^2} \right),$$

$$\rho_{M,\text{dyn}}(\Omega) = \rho_{M} \left( 1 + \mu_{D} \frac{\Omega^2}{1 - \Omega^2} \right),$$

where the non-dimensionalized parameters are given by

$$\mu_{D} = \frac{H_{M}^2}{D_{M}},$$

$$\mu_{\rho} = \frac{j_{M}^2}{\rho_{M}},$$

$$\Omega = \frac{\omega}{\omega_{\text{res}}},$$

$$\Omega_{\text{mat}} = \frac{1}{\omega_{\text{res}}} \sqrt{\frac{C_{M}}{D_{M}}},$$

Here, $\mu_{\rho}$ and $\mu_{D}$ represent the modal mass and elastic inertia fraction, respectively, $\Omega$ the applied frequency normalized with respect to $\omega_{\text{res}}$ and $\Omega_{\text{mat}}$ the approximate resonance frequency of the matrix, also normalized with respect $\omega_{\text{res}}$. The modal mass fraction, $\mu_{\rho}$ directly gives the resonator mass as a fraction of the total mass of the RVE. The modal elastic inertia fraction, $\mu_{D}$, while proportional to the mass of the resonator, also depends on the inertial amplification provided by the coupling mechanism of the negative stiffness resonator (Frandsen et al., 2016). The frequency bounds of the local resonant stopband can be found by equating $k$ in Eq. (12) to 0 and $\infty$ and solving for $\Omega$.

The solution for the normalized frequency bounds (lower, $\Omega_{\text{low}}$ and upper, $\Omega_{\text{up}}$) and the bandwidth ($BW_{\text{norm}}$) due to the negative mass density effects only ($\mu_{\rho} \neq 0, \mu_{D} = 0$) is given as,

$$\Omega_{\text{low}} = 1,$$

$$\Omega_{\text{up}} = \frac{1}{\sqrt{1 - \mu_{\rho}}},$$

$$BW_{\text{norm}} = \frac{\Omega_{\text{up}} - \Omega_{\text{low}}}{\Omega_{\text{up}}},$$
\[ BW_{\text{norm}} = \Omega_{\text{up}} - \Omega_{\text{low}} = \frac{1}{\sqrt{1 - \mu_D}} - 1. \quad (20c) \]

Similarly, the solution for the normalized frequency bounds and the bandwidth due to negative stiffness effects \((\mu_D \neq 0, \mu_\rho = 0)\) is given as follows

\[ \Omega_{\text{low}} = \sqrt{\frac{1 + \Omega_{\text{mat}}^2 + \sqrt{(1 - \Omega_{\text{mat}}^2)^2 + 4\mu_D\Omega_{\text{mat}}^2}}{2(1 - \mu_D)}}, \quad (21a) \]

\[ \Omega_{\text{up}} = 1, \quad (21b) \]

\[ BW_{\text{norm}} = 1 - \sqrt{\frac{1 + \Omega_{\text{mat}}^2 + \sqrt{(1 - \Omega_{\text{mat}}^2)^2 + 4\mu_D\Omega_{\text{mat}}^2}}{2(1 - \mu_D)}}. \quad (21c) \]

Comparing Eqs. (20c) with (21c), it is clear that the normalized bandwidth due to negative mass effects is determined by one parameter only, namely \(\mu_\rho\), whereas the negative stiffness effects are governed by two parameters, \(\mu_D\) and \(\Omega_{\text{mat}}\). The plots of the \(BW_{\text{norm}}\) versus the normalized parameters are given in Fig. 1. The negative mass stopband (Fig. 1a) scales strongly with \(\mu_\rho\), especially for higher values of \(\mu_\rho\), whereas the negative stiffness stopband (Fig. 1b) only scales significantly with \(\mu_D\) for lower values of \(\Omega_{\text{mat}}\). For \(\Omega_{\text{mat}} >> 1\), the stopband essentially vanishes regardless of the value of \(\mu_D\), at which point the negative stiffness effect no longer manifests itself at the macro-scale.

The parameter \(\Omega_{\text{mat}}\) is strongly connected to the homogenizability of the multiscale framework. Since \(\Omega_{\text{mat}}\) represents the factor of separation between the local resonance and the matrix resonance frequency, the condition \(\Omega_{\text{mat}} >> 1\) needs to be satisfied in order to ensure that the response of the matrix domain is predominantly quasi-static in the local resonance frequency regime \((\omega \sim \omega_{\text{res}} / \Omega \sim 1)\). The parameter \(\Omega_{\text{mat}}\) can therefore be considered as a measure quantifying the separation of scales within the matrix domain at \(\omega \sim \omega_{\text{res}}\). Higher order asymptotic corrections with respect to \(\Omega_{\text{mat}}\) are required to adequately model the negative stiffness effect for \(\Omega_{\text{mat}}\) approaching 1. Accordingly, the present methodology only approximates narrow negative stiffness effects for \(\Omega_{\text{mat}} \geq 1\) with restricted accuracy. This justifies the need for an error estimation procedure to identify the applicability and reliability of the homogenized model, which is presented in the next section.

5. Assessment of the reliability limits of the homogenized model

This section presents a heuristic approach for determining the limits of applicability and reliability of the homogenization model. As discussed in Section 4, the negative stiffness effect is only significant in low scale separation regimes where the homogenizability of the problem is not necessarily guaranteed. Hence, it is required to introduce an appropriate accuracy measure to identify its reliability limits. The best and the most straightforward error measure for any given macroscopic problem would be to compare the homogenized solution with the DNS. This is naturally infeasible as it would downgrade the prime purpose of a multiscale analysis. Therefore, the dispersion analysis is proposed here as the benchmark for assessing the accuracy of the enriched homogenized model in a relatively cheap and reliable way. The reference solution is
determined using Bloch analysis on a metamaterial unit cell which accounts for all higher order scattering effects. Here, the finite element based Bloch analysis (Farzbod and Leamy, 2011) is used as it is directly compatible with the present numerical multiscale framework, where the periodic boundary conditions on the RVE are replaced by the Bloch boundary conditions. The additional computational problem now takes the form of a $k - \omega$ eigenvalue problem on the RVE for a given set of $k$ points and wave directions. Since the problem is limited to a single RVE, the overall cost remains significantly lower than DNS. By restricting the analysis to a few important wavevector points, a relatively cheap approach for evaluating the reliability of the model results. This evaluation has to be performed only once for a given RVE and therefore contributes to the offline costs only.

Since the underlying enriched continuum homogenization framework retains a periodic boundary condition on the RVE for all wavenumbers (as required by the relaxed scale separation principle), the dispersion model derived from it (Eqs. (9) and (8)) corresponds to a Bloch analysis at $k = 0$ (provided that sufficient number of the periodic eigenmodes are incorporated into the model). The model therefore deviates for wavenumbers further from zero in regimes where the relaxed scale separation does not hold. Accordingly, a first measure quantifying the reliability of the model is obtained by computing the errors in the predicted eigenfrequencies at the symmetry points of the edge of the Brillouin zone. The frequency range of the applicability of the enriched continuum model is then determined by the lowest eigenfrequency within the computed set that does not satisfy a required error tolerance. Subsequent evaluations can be performed in a similar fashion at selected points inside the Brillouin zone. The number of wavevector points selected ultimately depends on the desired accuracy, to be balanced with computational cost limitations of the overall analysis.

Note that since the stopbands/passbands due to the negative stiffness/double negative effect are typically very narrow (1% of the local resonance frequency), even a small error in the eigenfrequency might be significant relative to the stopband/passband. Thus, in order to obtain a more sensitive estimate, the bandwidth error of the predicted stopbands/passbands can be computed instead. This can be determined by evaluating the difference between the eigenfrequencies computed at the edge of the Brillouin zone and at $k = 0$.

6. Dispersion and macro-scale boundary value analysis of a double-negative example problem

This section presents a case study to demonstrate the applicability of the enriched homogenized continuum model for dispersion and macro-scale boundary value problem analyses of metamaterials with double negative effects. First, the dispersion spectrum of the considered LRAM unit cell is computed using the enriched continuum and validated against Bloch analyses. Next, inspired by the works of Zhu et al. (2014), an example steady state macro-scale boundary value problem of a negatively refracting metamaterial prism is solved using the enriched homogenized continuum.

The considered double negative LRAM unit cell is shown in Fig. 2. The inclusion within the metamaterial unit cell is modeled using discrete spring-mass mechanisms. It is embedded in a continuum matrix modeled in 2D with a plane strain assumption. The discrete inclusion model is justified since the present work focuses on the emerging effects and not on the actual manufacturing of the metamaterial. Nevertheless, the spring mass model can be easily converted to practice using compliant mechanisms. The inclusion consists of a resonator directly coupled to the matrix and a pair of outer resonators, connected via a herringbone shaped, two-bar mechanism as shown in the figure. The negative stiffness and mass effects necessary for the double negative passband are generated separately by the outer and central resonators, respectively. The parameters of the unit cell given in Table 1, are selected such that a double negative passband emerges within the homogenizable regime. The inclusion constitutes about 75% of the total mass of the RVE, which can be achieved in a realistic setting. The initial angle of the herringbone mechanism as shown in Fig. 2 is kept very low at 15° in order to generate a high inertial amplification (Frandsen et al., 2016), thereby greatly enhancing the negative stiffness effect.

The matrix of the unit cell is discretized using a FE mesh with plane strain quadrilateral elements with a maximum element size of 2 mm. Convergence of the results with respect to the mesh size has been verified. The spring-mass inclusion model is coupled to the FE model of the matrix at the respective nodes. The total number of degrees of freedom of the resulting model is 272. The homogenized enriched continuum properties are derived using the procedure given in
Table 1  
Geometric and material properties of the considered unit cell.

<table>
<thead>
<tr>
<th>Property</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of unit cell</td>
<td>ℓ</td>
<td>20 mm</td>
</tr>
<tr>
<td>Diameter of inclusion</td>
<td>D</td>
<td>10 mm</td>
</tr>
<tr>
<td>Initial herringbone angle</td>
<td>α</td>
<td>15°</td>
</tr>
<tr>
<td>Central resonator stiffness</td>
<td>β₁</td>
<td>62.5 MPa</td>
</tr>
<tr>
<td>Central resonator mass</td>
<td>m₁</td>
<td>0.628 kg/m</td>
</tr>
<tr>
<td>Herringbone resonator stiffness</td>
<td>β₂</td>
<td>125 MPa</td>
</tr>
<tr>
<td>Herringbone resonator mass</td>
<td>m₂</td>
<td>0.314 kg/m</td>
</tr>
<tr>
<td>Young’s modulus of matrix</td>
<td>E&lt;sub&gt;mat&lt;/sub&gt;</td>
<td>25 GPa</td>
</tr>
<tr>
<td>Poisson’s ratio of matrix</td>
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</tr>
<tr>
<td>Mass density of matrix</td>
<td>ρ&lt;sub&gt;mat&lt;/sub&gt;</td>
<td>2000 kg/m³</td>
</tr>
</tbody>
</table>

Table 2  
The computed homogenized enriched continuum parameters of the considered double negative LRAM. Quantities not displayed in the table are equal to zero.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>ρ&lt;sub&gt;mat&lt;/sub&gt; [kg/m³]</td>
<td>3180</td>
</tr>
<tr>
<td>C&lt;sup&gt;111&lt;/sup&gt; [GPa]</td>
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</tr>
<tr>
<td>C&lt;sub&gt;111&lt;/sub&gt; = C&lt;sub&gt;2222&lt;/sub&gt;</td>
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<tr>
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<td>5.09</td>
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<tr>
<td>D&lt;sub&gt;1111&lt;/sub&gt; = 2.269</td>
<td></td>
</tr>
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<tr>
<td>D&lt;sub&gt;1122&lt;/sub&gt; = 0.093</td>
<td></td>
</tr>
<tr>
<td>D&lt;sub&gt;1122&lt;/sub&gt; = D&lt;sub&gt;2212&lt;/sub&gt;</td>
<td>0.029</td>
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<td>ω&lt;sub&gt;0&lt;/sub&gt; [Hz]</td>
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<tr>
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<td>ρ&lt;sub&gt;mat&lt;/sub&gt; [kg m⁻³]</td>
<td>39.94</td>
</tr>
</tbody>
</table>

Fig. 3. Plot of (a) the dispersion spectrum of the double negative LRAM computed using Bloch analysis (blue) and the enriched homogenized model (red) along with the corresponding local resonance mode shapes and (b) the frequency response of relevant components of the effective dynamic mass density and stiffness tensor. The double negative passband region is shaded in cyan. Note, that within the double negative passband, the homogenized dispersion spectrum appears to not predict any solutions at higher wavenumbers. This is just a consequence of the ω – k approach, which requires densely spaced frequency points in order to capture the spectrum at near zero group velocities. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Sridhar et al. (2016) (summarized in Appendix A). The computed enriched material parameters are given in Table 2. Three local resonance modes are identified and included in the description of the enriched continuum. The first mode is the oscillation of the central spring mass, the second and third modes are, respectively, the out of phase and in-phase oscillations of the outer resonators (see Fig. 3 for reference). The dispersion spectrum of the homogenized model is computed using Eq. (9), for a wave propagating in the ℓ₁ direction and plotted in Fig. 3. Both the real and imaginary components of the wavenumber are shown. Three distinct bands are observed: a negative mass stopband (2233–3146 Hz) along the compressive wave branch due to the action of the central resonator, a narrow negative stiffness band (2936–3017 Hz) also along
the compressive wave branch due to the out of phase motion of the outer resonators and finally, a negative mass stopband (3144–4445 Hz) along the shear wave branch due to the in-phase oscillations of the outer resonators. The overlap of the negative stiffness and mass stopband along the compressive branch yields a double negative passband from 2936 to 3017 Hz.

The dispersion spectrum predicted by the homogenized model is validated against the spectrum computed using the standard FE Bloch analysis (Farzbod and Leamy, 2011). A very good match exists between the homogenized model and the Bloch spectrum in the frequency regime of interest. The bandwidth error equals 12.9% in the double negative passband predicted by the homogenized model. A closer look at the double negative passband reveals an excellent agreement between the two spectra at low wavenumbers with a gradual divergence at higher wavenumbers. This provides further evidence that the bandwidth error results from higher order scale effects not accounted for in the homogenization framework. Furthermore, the group velocity in the passband is nearly zero at high wavenumbers. This implies that a 12.9% error in the bandwidth is still acceptable since most of the error stems from less critical regions within the passband. The computational cost of the homogenized dispersion analysis mainly consists of evaluating the enriched material parameters and the reliability assessment via Bloch analysis, both performed in an “offline” stage only once for a given unit cell design. The cost of solving the dispersion problem Eq. (9) is insignificant as it only involves a few unknowns. Thus, the total cost is of the same order as a Bloch analysis for a few wavenumbers only and is therefore significantly lower compared to a full Bloch analysis. It however should be emphasized that the main added value of the proposed model is its direct applicability to the solution of (initial) boundary value problems in finite domains as will be illustrated in the following.

A macro-scale steady state boundary value analysis of the double negative prism is carried out next. The setup of the problem, inspired by the works of Zhu et al. (2014), is illustrated in Fig. 4. It consists of a rectangular domain made up of a homogeneous medium, with an embedded metamaterial prism. The relevant dimensions of the macro-scale problem are indicated in Fig. 4 and given in Table 3. The size of the metamaterial region spans a domain effectively composed of 7020 unit cells. The homogeneous region is made up of the same material as the matrix of the metamaterial unit cell. A horizontal harmonic traction of 1.67 MPa is applied on a part of the domain boundary to simulate an incident compressive wave, orthogonal to one of the sides of the prism (see Fig. 4 for reference). An absorbing boundary condition via matched impedance with respect to the homogeneous medium is applied on the rest of the boundary to prevent multiple reflections within the homogeneous domain, aiding the visualization of the refracted wave. However, the reflections within the LRAM prism domain and at the traction boundary are not mitigated, serving the aim of illustrating the ability of the developed homogenization framework to simulate finite domain boundary value problems with general boundary conditions. Fully resolving the metamaterial domain by directly modeling all the unit cells would lead to a computationally prohibitive simulation. Instead, here the metamaterial domain is modeled using the developed homogenized enriched continuum model, leading to a significant reduction in computational costs, as will be discussed later. Note, that the enriched continuum app-

| Table 3 |
| Geometric properties of the macro-scale problem. |
| Property | Symbol | Value |
| Base length of domain | \( B \) | 6 m |
| Height of domain | \( H \) | 12 m |
| Base length of prism | \( b \) | 1.8 m |
| Height of prism | \( h \) | 3.12 m |
| Height of excitation region | \( a \) | 1.74 m |

Fig. 4. The setup of the macroscopic problem: a rectangular domain of a homogeneous medium with an embedded double negative LRAM prism (dark shaded). A horizontal harmonic traction is uniformly applied on the marked area of the left boundary. Absorbing boundary conditions are applied on all other external boundaries.
Fig. 5. The steady state response of the double negative metamaterial prism system for a normal incident compressive wave at an applied frequency of 2985 Hz (a–c), 3000 Hz (d–f) and 3600 Hz (g–i). The LRAM exhibits negative refraction at 2985 Hz and 3000 Hz and ordinary (positive) refraction at 3600 Hz. Plots (a), (d) and (g) give the total real displacement; (b), (e) and (h) the real P polarized displacement gradient; and (c), (f) and (i) the real S polarized displacement gradient. The colorbars in the left column are associated to the displacements while the colorbars in the right column are associated to the polarized displacement gradients, corresponding to the given applied frequency simulation. The values of the refraction angles ($\theta_{rP}$ and $\theta_{rS}$) determined using Snell’s law are also displayed and plotted using solid black vector arrows. The dashed black line denotes normal to the surface.

The problem has been validated against direct numerical simulations (DNS) on other boundary value problems in Sridhar et al. (2016, 2017) and therefore this validation step is not repeated here.

The problem is discretized using a uniform linear quadrilateral FE mesh with element size 60 mm $\times$ 60 mm. This element size accurately resolves wavelengths up to 300 mm without noticeable dispersion effects for the parameters used in the present analysis. The same, linear, approximation is applied to both the displacement and micro-inertial degrees of freedom. Each node in the mesh contains 5 degrees of freedom: 2 displacements and 3 enriched degrees of freedom. The total number of degrees of freedom of the resulting model is about 42,000. The steady state simulation in the frequency domain is carried out at 2985 Hz, 3000 Hz (both in the negative refraction regime) and 3600 Hz (in the ordinary (positive) refraction...
regime). Due to Poisson effects, a compressive wave yields both a compressive and a shear wave upon refraction. To resolve the individual waves, (i.e. P and S polarized waves) a Helmholtz decomposition is employed where the irrotational (curl-free) and divergence free components of the macroscopic displacement field are computed, which is detailed in Appendix B.

The plots of the real component of the total displacement and the P and S polarized displacement gradients at each of the applied frequencies are given in Fig. 5. The negative refraction effects occurring in the double negative zone are apparent, Fig. 5(a)–(f) especially in contrast to the ordinary (positive) refraction effect Fig. 5(g)–(i). Multiple refracted waves of each polarization can be observed, mostly resulting from multiple internal reflections within the LRAM prism. It is interesting to note that in the negative refraction regime, the energy is mostly channeled through the S wave upon refraction, thus presenting an example of mode conversion, a well known feature of double negative LRAMs (Wu et al., 2011; Zhu et al., 2014). For comparison, the values of refraction angles computed using Snell’s law (see Appendix C) are also given in Fig. 5 and plotted there as solid black arrows. A good match results between the prediction of Snell’s law and the primary refracted waves computed through the multiscale simulation.

The script for the entire FE simulation was implemented in MATLAB on a standard desktop computer with an Intel(R) Core(TM) i5-4200H CPU, 2.80GHz processor and 4GB RAM, without parallelization. Due to the relatively small RVE model, the runtime for the offline computations of the homogenized coefficients was negligible. The direct Bloch analysis (for the estimation of the accuracy of the homogenized model) took 4.5 s for 100 wavenumber points. Finally, the runtime of the macro-scale enriched continuum boundary value problem simulation (online) was about 20 s for a given applied frequency. The gain in terms of computational costs by using the homogenized enriched continuum based framework instead of the fully resolved DNS can be indirectly quantified by comparing the number of degrees of freedom involved. In the DNS case, the LRAM prism domain would entail a periodic repetition of the discretized RVE model while the mesh in the homogeneous part would need to be refined in the vicinity of the prism to ensure nodal connectivity. The total number of degrees of freedom of the DNS model can therefore be estimated to be in the order of $10^6$, which exceeds the homogenized model by a factor of 100. Thus, a significant reduction in the overall computational cost can be expected.

7. Conclusions

This paper demonstrated the application of the homogenized enriched continuum approach, by solving a computationally efficient macroscopic boundary value problem that enables the analysis of negative stiffness and double negative LRAMs. Unlike negative mass, the negative stiffness effect is only approximated by the homogenized enriched continuum, limiting it to narrow stopbands. Simplified analytical expressions for the stopband bandwidths generated by each effect were derived and it was shown that a low scale separation is required in order to observe significant negative stiffness effects. To ensure the reliability of the model in this regime, an assessment criteria based on the relative error in the eigen-frequencies between the enriched continuum and the reference Bloch spectra at relevant wavevector points was introduced. Special attention was also given to the relative error in the predicted bandwidth of the stopbands/double negative passbands as it gives a more sensitive estimate of the accuracy of the predicted negative stiffness effect.

The methodology was demonstrated on a 2D case study of a negatively refracting LRAM prism. The finite element method was used to discretize the micro and macro problems. The steady state response to an incident compressive wave on the LRAM prism was simulated in the propagating zone and within the negative refraction zone. A clear illustration of the positive and negative refraction effects was shown. The high computational gain using the enriched continuum as opposed to the DNS was discussed, accompanied by a reduction in the total number of degrees of freedom by a factor of 100. A detailed dispersion analysis of the considered LRAM was also carried out where the spectrum of the enriched continuum model was validated against the full Bloch spectrum. Close inspection of the spectra revealed that the error between the two was only significant at higher wavenumbers within the passband where the group velocities are very low (and therefore the negative refraction effect is weak). This implies that a relatively high bandwidth error can be tolerated for practical purposes. The exact value of a generally applicable limit, for any given unit cell, still remains an open question.

Despite the reduction in accuracy, the present methodology offers an approach that is fairly easy to implement using standard FE methods under arbitrary transient loading and boundary conditions and complex LRAM unit cell geometries, at a fraction of the cost of a full scale direct numerical simulation.

Acknowledgments

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Appendix A. Homogenized enriched continuum parameters

The final expressions for the homogenized parameters to be computed based on the discretized RVE model are given here for convenience, omitting the details of the derivation. Interested readers are referred to Sridhar et al. (2016) for further details.
The point of departure is the discrete dynamic model of the RVE, obtained via the standard Galerkin FE discretization applied to the linear governing micro-scale equations. It is assumed that the periodic boundary conditions have already been implemented on the model. The discrete force balance is expressed as,

$$\mathbf{K} \cdot \ddot{\mathbf{u}} + \mathbf{M} \cdot \dot{\mathbf{u}} = \mathbf{f},$$  \hspace{1cm} (A.1)

where $\mathbf{K}$ and $\mathbf{M}$ represent the stiffness and mass matrices respectively, $\mathbf{u}$ and $\mathbf{f}$ are the column matrices of the nodal displacements and applied traction forces, respectively. The coupling between the micro-scale RVE and the macro-scale is achieved through a prescribed displacement condition at the vertex nodes of the RVE. Hence, the degrees of freedom are partitioned into the prescribed vertex degrees of freedom denoted by $(\bullet)_p$ and the remaining, free degrees of freedom, denoted by $(\bullet)_M$. Next, a reduction based on static-dynamic decomposition is applied to the above model in which the reduced basis is composed of the quasi-static response to the prescribed displacements and a compact set of periodic eigenmodes of the system with the prescribed degrees of freedom set to zero. Without going into the details, the obtained reduced order model is expressed as follows,

$$\mathbf{K}^{qs}_{pp} \cdot \ddot{\mathbf{u}}_p + \mathbf{M}^{qs}_{pp} \cdot \dot{\mathbf{u}}_p + \mathbf{M}_{\text{coup}} \dot{\mathbf{u}}_M = \mathbf{f}_p,$$ \hspace{1cm} (A.2)

$$\omega^2 \mathbf{M} \cdot \ddot{\mathbf{u}}_M + \mathbf{K} \cdot \mathbf{u}_M + \frac{1}{\sqrt[3]{\mathbf{M}}_{\text{coup}}} \cdot \mathbf{u}_p = \mathbf{0},$$ \hspace{1cm} (A.3)

where,

$$\mathbf{K}^{qs}_{pp} = \mathbf{K}_{pp} - \mathbf{K}_{pf} \cdot \mathbf{K}_{ff}^{-1} \cdot \mathbf{K}_{fp},$$ \hspace{1cm} (A.4)

$$\mathbf{M}^{qs}_{pp} = \mathbf{S}^T \cdot \mathbf{M}_f \cdot \mathbf{S} + 2 \mathbf{M}_f \cdot \mathbf{S} + \mathbf{M}_{pp},$$ \hspace{1cm} (A.5)

$$\mathbf{M}_{\text{coup}} = \mathbf{S}^T \cdot \mathbf{M}_f \cdot \mathbf{S}.$$ \hspace{1cm} (A.6)

with $\mathbf{S} = -\mathbf{K}_{pf}^{-1} \cdot \mathbf{K}_{fp}$, the Schur complement of the stiffness matrix, representing the quasi-static response, $\mathbf{\phi} = [\mathbf{\phi}_1, \ldots, \mathbf{\phi}_n]^T$ the set of periodic eigenvectors corresponding to the modal amplitudes $\bar{\mathbf{u}}_M = [\bar{u}_{M1}, \ldots, \bar{u}_{Mn}]^T$, computed via the solution of the following eigen-value problem with the volume average mass normalization condition ($s \in \mathcal{Q}$),

$$\left( \mathbf{K}_{ff} - s \omega_s^2 \mathbf{M}_f \right) \cdot \mathbf{\phi} = \mathbf{0},$$ \hspace{1cm} (A.7)

$$\frac{1}{V} \mathbf{S}^T \cdot \mathbf{M}_f \cdot \mathbf{S} \cdot \mathbf{\phi} = 1,$$ \hspace{1cm} (A.8)

where $V$ is the volume of the RVE. The discretized upscaling relations that relate the macroscopic stress $\mathbf{\sigma}_M$ and momentum rate, $\mathbf{\dot{p}}_M$ to the prescribed nodal forces, $\mathbf{f}_p$ is given as,

$$\mathbf{\sigma}_M = \frac{1}{V} \mathbf{f}_p^T \odot \Delta \mathbf{x},$$ \hspace{1cm} (A.9a)

$$\mathbf{\dot{p}}_M = \frac{1}{V} \mathbf{f}_p \mathbf{\dot{f}}_p,$$ \hspace{1cm} (A.9b)

where $\Delta \mathbf{x} = \mathbf{x}_p - \mathbf{x}_C$ is the column of relative position vectors of the prescribed (vertex) nodes with respect to the geometric center of the RVE, $\mathbf{x}_C$ and $\mathbf{f}_p$ is a column matrix of size $n_{dim} + 1$ where each entry is 1. Similarly the downscaling relations that relate the prescribed nodal displacements to the macroscopic displacement $\mathbf{u}_M$ and its gradient $\nabla M \mathbf{u}_M$ is given as,

$$\dot{\mathbf{u}}_p = \mathbf{L}_p \mathbf{u}_M + (\nabla M \mathbf{u}_M)^T \cdot \Delta \mathbf{x},$$ \hspace{1cm} (A.10)

The homogenized macroscopic constitutive relations (3) are obtained by substituting Eqs. (A.2) and (A.10) into Eq. (A.9). The micro-dynamics Eq. (2) is obtained by substituting Eq. (A.10) into Eq. (A.3). Note that, in these substitutions, the terms involving the product of $\mathbf{K}^{qs}_{pp}$ and $\mathbf{L}_p$ vanish since $\mathbf{L}_p$ is the rigid body mode of $\mathbf{K}^{qs}_{pp}$. The resulting expressions for the enriched continuum parameters in terms of the reduced order model quantities are given as,

$$\mathbf{C}^{(4)}_M = \frac{1}{V} \left( \Delta \mathbf{x}^T \otimes \mathbf{K}_{qs} \otimes \Delta \mathbf{x} \right)^{LT},$$ \hspace{1cm} (A.11)

$$\mathbf{D}^{(4)}_M = \frac{1}{V} \left( \Delta \mathbf{x}^T \otimes \mathbf{M}_{qs} \otimes \Delta \mathbf{x} \right)^{LT},$$ \hspace{1cm} (A.12)
\[ \rho_M = \frac{1}{V} (i^T \mathbf{M}_{QD}) : \mathbf{1}, \]  
(A.13)

\[ s^* \mathbf{H}_M = \frac{1}{V} \mathbf{M}_{\text{coup}} \otimes \Delta \mathbf{x}, \]  
(A.14)

\[ s^* \mathbf{J}_M = \frac{1}{V} \mathbf{J}^T \cdot \mathbf{M}_{\text{coup}}. \]  
(A.15)

**Appendix B. Wave field decomposition**

The total displacement solution is a mixture of P and S wave fields. One of the well known decomposition techniques is the Helmholtz decomposition (Aki and Richards, 2002). In this method, the total displacement field, \( \mathbf{U} \) is expressed as

\[ \mathbf{U} = \nabla A + \mathbf{v} \times \mathbf{B}, \]  
(B.1)

where \( A \) and \( B \) denote the scalar and vector Lamé potentials, respectively. The first and second terms of in (B.1) represent P and S wave components of \( \mathbf{U} \), respectively. Since the P wave is irrotational and the S wave divergence free in an isotropic medium, one can apply the divergence and curl operators on Eq. (B.1), thereby yielding

\[ P = \nabla \cdot \nabla A, \]  
(B.2a)

\[ S = \nabla \times (\nabla \times \mathbf{B}), \]  
(B.2b)

which clearly indicates that the P and S wave components are carried by the scalar P and vector S respectively. In a 2D case, Eq. (B.2) can be further simplified as,

\[ P = \partial U_x / \partial x + \partial U_y / \partial y, \]  
(B.3a)

\[ S = \partial U_x / \partial y - \partial U_y / \partial x. \]  
(B.3b)

It can be seen from Eqs. (B.3) that P and S physically represent strains and hence exhibit a phase shift of \( \pi/2 \) compared to the actual displacement field. Although such a decomposition is less accurate in the metamaterial zone due to its anisotropy, it is still efficient and convenient to separate P and S components of the refracted waves in the isotropic homogeneous material zone, which is the main interest here.

**Appendix C. Determination of refraction angles using Snell’s law**

The classical Snell’s law for the refraction angle of the P and S wave resulting from a single incident (P or S) wave at the interface of two isotropic media is given as follows (Kolsky, 1964)

\[ \frac{\sin \theta_i}{v_1} = \frac{\sin \theta_{SP}}{v_{2P}} = \frac{\sin \theta_{SS}}{v_{2S}}, \]  
(C.1)

where \( \theta_i \) is the incidence angle, \( \theta_{SP} \) and \( \theta_{SS} \) the refracted angles of the P and S waves, respectively, \( v_1 \) the (effective) phase velocity in the first medium and, \( v_{2P} \) and \( v_{2S} \) the (effective) phase velocity of the P and S waves in the second medium, respectively. Note, that the above expression always predicts a positive refraction angle (for positive incidence), even in the negative refraction regime. Hence, a negative sign is “manually” added in this regime.

Strictly speaking, the above law cannot be applied to the LRAM prism problem discussed in Section 6, since the LRAM is anisotropic and the assumption of a single incident wave is not valid due to the multiple internal reflections within the prism domain. Therefore, it only serves here as an approximation provided for reference.

The parameters in Eq. (C.1) are determined as follows. The incident angle is computed as \( \theta_i = \arctan(b/h) \). The wave speeds are estimated as,

\[ v_1 = \sqrt{\rho_{\text{Mdyn1111}}(\omega) / \rho_{\text{Mdyn11}}(\omega)}, \]  
(C.2a)

\[ v_{2P} = \sqrt{E_{\text{mat}} / \rho_{\text{mat}}}, \]  
(C.2b)

\[ v_{2S} = \sqrt{G_{\text{mat}} / \rho_{\text{mat}}}, \]  
(C.2c)
where,
\[ E_{\text{mat}} = \frac{1 - v_{\text{mat}}}{(1 + v_{\text{mat}})(1 - 2v_{\text{mat}})} E_{\text{mat}}. \]  \hfill (C.3a)

\[ G_{\text{mat}} = \frac{E_{\text{mat}}}{2(1 + v_{\text{mat}})}. \]  \hfill (C.3b)

References


