Constrained order observer design for disturbance decoupled output estimation

Constrained Order Observer Design for Disturbance Decoupled Output Estimation

R.W.H. Merks, E.M.M. Kivits and S.Weiland

***Abstract*—From industry there is an increasing interest in applying model based control techniques to large-scale systems. Control designs like H_2 and H_∞ optimal control will typically contain a dynamic state observer in which the order will be equal to the model order. Consequently, real-time implementation of the controller may not be possible due to computational constraints. The problem of designing a constrained order observer for a large-scale system, with explicit guarantees on the output estimate is therefore a relevant problem in these applications.**

This paper addresses the problem of constructing (if it exists) an observer with a constrained order that decouples the output estimation errors from the disturbances that are acting on the system. In addition, stability of the output estimate, a characterization of the orders for which the observer can achieve disturbance decoupled estimation and the explicit construction of the observer are discussed.

***Index Terms*—Model/Controller reduction, Estimation, Observers for Linear systems, Algebraic/geometric methods**

I. Introduction

In various industrial applications, controllers are designed for systems that are described by partial differential equations [1][2]. Models of these systems are created by utilizing finite elements or finite volume methods, which lead to large-scale models that commonly contain well over $1,000$ states. For model based control designs like H_2 and H_∞ optimal control, the order of a dynamic output feedback controller is typically equal to the model order. For large-scale models, the synthesis of such a controller might therefore become numerically unreliable, or even infeasible. Furthermore, real-time implementation of the controller may not be possible due to computational constraints. The design of model based controllers with a constrained order is therefore a relevant problem in these applications.

Control designs like H_2 and H_∞ optimal control will typically contain a dynamic state observer. Consequently, the order of such a controller is determined by the observer order. For the design of a constrained order observer $\hat{\Sigma}_O$ for a system Σ, typically 3 approaches are considered as depicted in Fig. 1:

1) First system Σ is reduced in order, which leads to system $\hat{\Sigma}$. For $\hat{\Sigma}$ an observer $\hat{\Sigma}_O$ with the same order as $\hat{\Sigma}$ is created.
2) First an observer Σ_O is created for system Σ. Σ_O is then reduced in order, which leads to a low order observer $\hat{\Sigma}_O$.
3) The low order observer $\hat{\Sigma}_O$ is created directly for system Σ.

The design of a constrained order observer is typically performed by utilizing standard open-loop model order reduction techniques such as balanced truncation or moment matching. Control relevant model order reduction has been investigated extensively, which has led to various techniques as described in [3][4][5][6][7]. However, these techniques consider standard state-space systems and do not provide explicit guarantees on the interconnection of the constrained order observer $\hat{\Sigma}_O$ and full order system Σ.

The design of controllers or observers with a constrained order that *will* provide explicit guarantees on performance has, to the knowledge of the authors, received little attention. The problem of designing a low order observer that is optimal for a full order system has been addressed shortly in [8] in an H_2 optimal control context. The problem has also been investigated for disturbance decoupling problems in [9][10][11]. However, the range of orders for which the problem can be solved is not investigated.

Disturbance decoupling problems are closely related to H_2 optimal control problems, which is observed in [8][12]. Therefore, it is decided to further investigate the design of constrained order observers in a disturbance decoupling context, with the intention to extend the results to H_2 optimal observer design.

This paper addresses the problem of constructing (if it exists) an observer with a constrained order that decouples the output estimation errors from the disturbances that are acting on the system. In addition, stability of the output estimate, a characterization of the orders for which the observer can achieve disturbance decoupled output estimation and the explicit construction of the observer are discussed.

The paper is organized as follows. In section II the main problem and additional subproblems will formally be defined. The mathematical preliminaries are discussed in section III. The standard disturbance decoupled estimation problem is discussed in section IV. In section V the main results to the problems as defined in section II are presented. The paper is finalized with an academic example in section VI, followed by conclusions and future works in section VII.

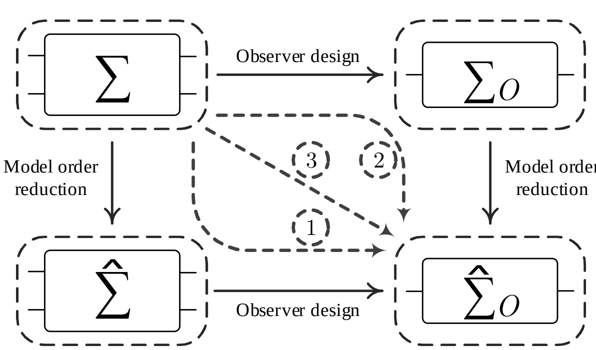

Fig. 1. Constrained order observer design approaches.

II. Problem Definition

Consider the generalized plant model of a system

$$\Sigma = \begin{cases} \dot{x}(t) = \ Ax(t) + B_u u(t) + B_w w(t) \\ y(t) = C_y x(t) \\ z(t) = C_z x(t) + D_{uz} u(t) \end{cases} \tag{1}$$

where $A : \mathcal{X} \to \mathcal{X}$, $B_u : \mathcal{U} \to \mathcal{X}$, $B_w : \mathcal{W} \to \mathcal{X}$, $C_y : \mathcal{X} \to \mathcal{Y}$, $C_z : \mathcal{X} \to \mathcal{Z}$ and $D_{uz} : \mathcal{U} \to \mathcal{Z}$ are linear mappings on finite dimensional vector spaces $\mathcal{X}$, $\mathcal{U}$, $\mathcal{W}$, $\mathcal{Y}$ and $\mathcal{Z}$. The vector signals $x(t)$, $u(t)$, $w(t)$, $y(t)$ and $z(t)$ of dimension n_x, n_u, n_w, n_y and n_z represent the state, known input, unknown disturbance, measured output and the control output, respectively.

Assumption 2.1: It is assumed that any state of Σ (1) can be reached using the pair $(u(t), w(t))$ and observed through the pair $(y(t), z(t))$. □

An observer provides a causal estimate $\hat{z}(t)$ of $z(t)$ on the basis of $u(t)$ and $y(t)$ according to

$$\Sigma_{Oh} = \begin{cases} \dot{h}(t) = Mh(t) + Nu(t) + Oy(t) \\ \hat{z}(t) = \ Ph(t) + Qu(t) + Ry(t) \end{cases} \tag{2}$$

with linear mappings $M : \mathcal{H} \to \mathcal{H}$, $N : \mathcal{U} \to \mathcal{H}$, $O : \mathcal{Y} \to \mathcal{H}$ $P : \mathcal{H} \to \mathcal{Z}$, $Q : \mathcal{U} \to \mathcal{Z}$, $R : \mathcal{Y} \to \mathcal{Z}$ and finite dimensional vector space $\mathcal{H}$. The vector signals $h(t)$ and $\hat{z}(t)$ of dimension n_h and n_z represent the state of the observer and the estimated control output, respectively. The order of the observer is given by $\dim(\mathcal{H})$. Connecting Σ_{Oh} (2) to Σ (1) leads to an error system $\Sigma_{\epsilon h}$ with inputs $w(t)$ and $u(t)$ and output estimation error $\epsilon_z(t) = z(t) - \hat{z}(t)$ as depicted in Fig. 2.

External signals $u(t)$ and $w(t)$ are both acting as inputs to the error system $\Sigma_{\epsilon h}$. From an estimation perspective, both signals are viewed as disturbances and a distinction is made by assuming that the signal $u(t)$ is known and $w(t)$ unknown.

The *main problem* that will be addressed in this paper is the *disturbance decoupled estimation problem with a constrained order* (DDEP-CO). The DDEP-CO cannot be solved for all systems and all observer orders. Therefore, in addition to solving the DDEP-CO, it will be investigated under which conditions the DDEP-CO can be solved.

Definition 2.2: Σ_{Oh} (2) *solves* the DDEP-CO for (Σ (1), n_o) if $\dim(\mathcal{H}) = n_o$ and the output estimation error $\epsilon_z(t) = z(t) - \hat{z}(t)$ is independent of input $u(t)$ and disturbance $w(t)$. The DDEP-CO *can be solved* for (Σ (1), n_o) if such an observer exists. □

In addition, the following problems will be investigated:

- Adding stability to the estimation error in the DDEP-CO.
- The characterization of all integers n_o and a lower bound $n_o^\star$ for which the DDEP-CO can be solved.
- The explicit construction of an observer that will solve the DDEP-CO for a given value n_o.

III. Mathematical Preliminaries

A. Operators

For a matrix A, its transpose, kernel, image, spectrum and pseudo inverse are denoted by $A^\top$, $\ker(A)$, $\mathrm{im}(A)$, $\sigma(A)$ and $A^\dagger$, respectively.

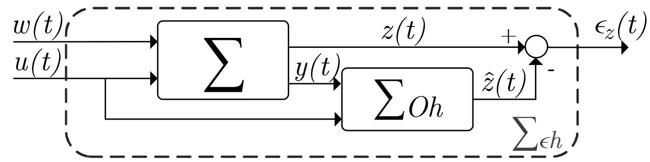

Fig. 2. Error system $\Sigma_{\epsilon h}$.

Let $\mathcal{R}$ and $\mathcal{S}$ be subspaces of a finite-dimensional vector space $\mathcal{X}$. If $\mathcal{R} \cap \mathcal{S} = 0$, the sum of subspaces becomes a direct sum, which is denoted by $\mathcal{R} \oplus \mathcal{S}$.

The oblique projection onto $\mathcal{R}$, along $\mathcal{S}$ is denoted by $\Pi(\mathcal{R}, \mathcal{S}) : \mathcal{X} \to \mathcal{R}$. This projection is unique if $\mathcal{R} \oplus \mathcal{S} = \mathcal{X}$ and orthogonal if $\Pi(\mathcal{R}, \mathcal{R}^\perp)$ is used.

Introduce a linear map $A : \mathcal{X} \to \mathcal{Y}$ and a subspace $\mathcal{R} \subseteq \mathcal{X}$, then the map $\underline{A} : \mathcal{R} \to \mathcal{Y}$ defined as $\underline{A}x = Ax$ for $x \in \mathcal{R}$ is called the domain restriction of A to $\mathcal{R}$ and is denoted $A|\mathcal{R}$.

A subspace $\mathcal{R} \subseteq \mathcal{X}$ leads to an equivalence relation between vectors $x_1, x_2 \in \mathcal{X}$ in the sense that $x_1 \simeq x_2$ if $x_1 - x_2 \in \mathcal{R}$. The set of all such equivalence classes is called the quotient space $\mathcal{X}$ modulo $\mathcal{R}$ and is denoted $\mathcal{X} \bmod \mathcal{R}$. For any $x \in \mathcal{X}$ the equivalence class of which x is an element is denoted by $\bar{x} = \{y : x - y \in \mathcal{R}\}$.

Subspace algebra is discussed in [11], equivalence classes and modulo relations are discussed in [12] and more information about oblique projections can be found in [14][15].

B. Disturbance Decoupling

Definition 3.1: Consider Σ (1) with $u(t) = 0$. Then Σ (1) is *disturbance decoupled* if the control output $z(t)$ is independent of disturbance $w(t)$. □

Disturbance decoupling is closely related to geometric control theory, which makes use of so-called invariant subspaces. This relation is shown using the following lemma.

Lemma 3.2: Consider Σ (1) with $u(t) = 0$. Then Σ (1) is disturbance decoupled if and only if there exists a subspace $\mathcal{S}_A \subseteq \mathcal{X}$ such that $\mathrm{im}(B_w) \subseteq \mathcal{S}_A \subseteq \ker(C_z)$, $A\mathcal{S}_A \subseteq \mathcal{S}_A$. □

Proof: The proof can be found in [12, Thm. 4.6]. ■

Remark 3.3: Lemma 3.2 can be interpreted as follows. State trajectories starting in $\mathcal{S}_A$ will remain in $\mathcal{S}_A$ when $w(t) = 0$, because $A\mathcal{S}_A \subseteq \mathcal{S}_A$. The effects of a disturbance $w(t)$ on state trajectories are restricted to $\mathcal{S}_A$ by additionally requiring that $\mathrm{im}(B_w) \subseteq \mathcal{S}_A$. Finally, any effect on state trajectories within $\mathcal{S}_A$ will not be visible on $z(t)$, when $\mathcal{S}_A \subseteq \ker(C_z)$. □

C. Conditioned Invariant Subspaces

Conditioned invariant subspaces rely on concepts of geometric control theory as presented in [11][12][13].

A specific type of observer Σ_{Oh} (2) is the so-called Luenberger state observer with estimated output

$$\Sigma_O = \begin{cases} \dot{\hat{x}}(t) = \left(A + LC_y\right)\hat{x}(t) + B_u u(t) - Ly(t) \\ \hat{z}(t) = \quad P\hat{x}(t) + Qu(t) + Ry(t) \end{cases} \tag{3}$$

where observer gain $L : \mathcal{Y} \to \mathcal{X}$ is a linear mapping and vector signal $\hat{x}(t)$ of dimension n_x represents an estimate for the state $x(t)$ of Σ (1). Consider Σ (1) connected to Σ_O (3), define the state estimation error $e(t) = x(t) - \hat{x}(t)$ and consider the output estimation error $\epsilon_z(t) = z(t) - \hat{z}(t)$. This leads to the error system

$$\Sigma_\epsilon = \begin{cases} \left(\frac{\dot{x}(t)}{\dot{e}(t)}\right) = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & A+LC_y \end{array}\right)\left(\frac{x(t)}{e(t)}\right) \\ \qquad + \left(\frac{B_u}{0}\right)u(t) + \left(\frac{B_w}{B_w}\right)w(t) \\ \epsilon_z(t) = \left(\begin{array}{c|c} C_z - RC_y - P & P \end{array}\right)\left(\frac{x(t)}{e(t)}\right) \\ \qquad + \left(D_{uz} - Q \right)u(t) \end{cases} \tag{4}$$

which has an extended state $x^\epsilon(t) = \begin{pmatrix} x(t)^\top & e(t)^\top \end{pmatrix}^\top$ and extended system matrices A^ϵ, B_u^ϵ, B_w^ϵ, C_z^ϵ and D_{uz}^ϵ. The extended state-space is characterized by the vector space $\mathcal{X}^\epsilon = \mathcal{X}^x \oplus \mathcal{X}^e$, which consists of subspaces $\mathcal{X}^x = \mathcal{X}$ for the state $x(t)$ and $\mathcal{X}^e = \mathcal{X}$ for the estimation error $e(t)$.

The estimation error $e(t)$ of Σ_ϵ (4) can be influenced by the mapping L. Similar to lemma 3.2, there exists invariant subspaces related to the estimation error $e(t)$.

Definition 3.4: Given a subspace $\mathcal{K} \subseteq \mathcal{X}$, a *conditioned invariant subspace* $\mathcal{S}(\mathcal{K})$ is a subspace $\mathcal{S}(\mathcal{K}) \subseteq \mathcal{X}$ that contains $\mathcal{K}$ and for which there exists a mapping $L : \mathcal{Y} \to \mathcal{X}$ such that $(A + LC_y)\mathcal{S}(\mathcal{K}) \subseteq \mathcal{S}(\mathcal{K})$. □

Proposition 3.5: Given a subspace $\mathcal{K} \subseteq \mathcal{X}$, there exists a unique smallest conditioned invariant subspace that contains $\mathcal{K}$, which is denoted by $\mathcal{S}^\star(\mathcal{K})$. This subspace satisfies $\mathcal{S}^\star(\mathcal{K}) \subseteq \mathcal{S}(\mathcal{K})$, for any conditioned invariant subspace $\mathcal{S}(\mathcal{K})$. □

Proof: The proof can be found in [12, Thm. 5.5, 5.7]. ■

IV. Disturbance Decoupled Estimation Problem (DDEP)

A. Problem Definition

For the DDEP, the aim is to find mappings L, P, Q and R such that disturbance decoupling is achieved on Σ_ϵ (4). Similar to remark 3.3, the effects of $w(t)$ on the estimation error $e(t)$ of Σ_ϵ (4) should be restricted to some subspace. Therefore we will consider conditioned invariant subspaces that contain $\text{im}(B_w)$. For notational convenience we will define $\mathcal{S} = \mathcal{S}(\text{im}(B_w))$ and $\mathcal{S}^\star = \mathcal{S}^\star(\text{im}(B_w))$.

Remark 4.1: By using lemma 3.2 and by interpreting $u(t)$ as a disturbance, it follows that Σ_ϵ (4) is disturbance decoupled if and only if there exists a subspace $\mathcal{S}_{A^\epsilon} \subseteq \mathcal{X}^\epsilon$ such that $\text{im}(\begin{pmatrix} B_u^\epsilon & B_w^\epsilon \end{pmatrix}) \subseteq \mathcal{S}_{A^\epsilon} \subseteq \ker(C_z^\epsilon)$ and $A^\epsilon \mathcal{S}_{A^\epsilon} \subseteq \mathcal{S}_{A^\epsilon}$, for $Q = D_{uz}$ and for some L, P and R.

Using assumption 2.1, observe that $\mathcal{S}_{A^\epsilon} = \mathcal{X}^x \oplus \mathcal{S}^\star$ (with $\mathcal{S}^\star \subseteq \mathcal{X}^e$) is the *smallest* subspace that will satisfy $\text{im}(\begin{pmatrix} B_u^\epsilon & B_w^\epsilon \end{pmatrix}) \subseteq \mathcal{S}_{A^\epsilon}$ and $A^\epsilon \mathcal{S}_{A^\epsilon} \subseteq \mathcal{S}_{A^\epsilon}$ for some L. Finally, Σ_ϵ (4) can be disturbance decoupled when there exist mappings P and R such that $\mathcal{S}_{A^\epsilon} \subseteq \ker(C_z^\epsilon)$ is satisfied. □

The following lemma shows when this last inclusion can be satisfied for some P and R.

Lemma 4.2: Σ_ϵ (4) *can be* disturbance decoupled if and only if $\mathcal{S}^\star \cap \ker(C_y) \subseteq \ker(C_z)$. □

Proof: The proof can be found in [12, Cor. 5.22] for an observer Σ_{Oh} (2). In theorem 4.6 it will be shown that the use of a Luenberger observer is non-restrictive. ■

Remark 4.3: Lemma 4.2 can be interpreted as follows. Assumption 2.1 implies that $\epsilon_z(t)$ should be independent of $x(t)$. The subspace $\mathcal{S}^\star$ is the smallest subspace to which the effect of $w(t)$ on estimation error $e(t)$ can be restricted. Measurements $y(t)$ are not directly affected by $w(t)$, which implies that certain estimation errors $e(t)$ are directly visible on $y(t)$. Σ_ϵ (4) can therefore be disturbance decoupled if and only if $\epsilon_z(t)$ is independent of all estimation errors $e(t)$ that are affected by $w(t)$, but which cannot directly be compensated by $y(t)$. □

Whether the DDEP can be solved for Σ (1) is strongly related to $\mathcal{S}^\star$, as shown in lemma 4.2. This is why in literature the subspace $\mathcal{S}^\star$ is considered [11][12]. However, it will be shown in section V that the values n_o for which the DDEP-CO can be solved are related to the dimension of conditioned invariant subspaces. The DDEP will therefore be related to a set of conditioned invariant subspaces $\mathbb{S}$ as defined in (5) below.

Definition 4.4: Σ_O (3) *solves* the DDEP for (Σ (1), $\mathcal{S}$) if the output estimation error $\epsilon_z(t)$ of Σ_ϵ (4) is independent of input $u(t)$ and disturbance $w(t)$. Furthermore, for Σ_ϵ (4) it holds that $e(t) \in \mathcal{S}$ for all $t \geq t_0$ and for any $w(t)$, provided that $e(t_0) = e_0 \in \mathcal{S}$. □

B. Design of a Full Order Luenberger Observer

Consider a set of conditioned invariant subspaces

$$\mathbb{S} = \{\mathcal{S} : \mathcal{S} = \mathcal{S}(\text{im}(B_w)),\ \mathcal{S} \cap \ker(C_y) \subseteq \ker(C_z)\}\,. \tag{5}$$

Note that $\mathcal{S}^\star(\text{im}(B_w)) \in \mathbb{S}$, whenever $\mathbb{S} \neq \emptyset$.

Theorem 4.5: Σ_O (3) *can solve* the DDEP for (Σ (1), $\mathcal{S}$) if and only if $\mathcal{S} \in \mathbb{S}$. □

Proof: ($\Leftarrow$) If $\mathcal{S} \in \mathbb{S}$, the subspace $S_{A^\epsilon} = \mathcal{X}^x \oplus \mathcal{S}$ must be used in remark 4.1. Then $\mathcal{S}_{A^\epsilon} \subseteq \ker(C_z^\epsilon)$ can be satisfied for some P and R if, according to lemma 4.2, $\mathcal{S} \cap \ker(C_y) \subseteq \ker(C_z)$ is satisfied. This is true for any $\mathcal{S} \in \mathbb{S}$.

($\Rightarrow$) $\mathcal{S}$ must be a conditioned invariant subspace in order to satisfy the requirement that $e(t) \in \mathcal{S}$ for all $t \geq t_0$ and for any $w(t)$, provided that $e(t_0) = e_0 \in \mathcal{S}$. Again consider the subspace $S_{A^\epsilon} = \mathcal{X}^x \oplus \mathcal{S}$ in remark 4.1 and observe that $\mathcal{S}_{A^\epsilon} \subseteq \ker(C_z^\epsilon)$ requires that $\mathcal{S} \cap \ker(C_y) \subseteq \ker(C_z)$, which implies that $\mathcal{S} \in \mathbb{S}$. ■

Theorem 4.6: Let $\mathcal{S} \in \mathbb{S}$. Then the following statements are equivalent:

(i) Σ_O (3) *solves* the DDEP for (Σ (1), $\mathcal{S}$),
(ii) For Σ_ϵ (4) it holds that $e(t) \in \mathcal{S}$ for all $t \geq t_0$ and for any $w(t)$, provided that $e(t_0) = e_0 \in \mathcal{S}$. $\epsilon_z(t)$ is independent of $u(t)$. In addition, $\epsilon_z(t_0) = 0$ for any $x(t_0)$, $e(t_0) \in \mathcal{S}$,
(iii) $(A + LC_y)\mathcal{S} \subseteq \mathcal{S}$, $Q = D_{uz}$, $C_z - RC_y - P = 0$, $P\mathcal{S} = 0$. □

Proof: (i) $\Rightarrow$ (ii): By definition, the first 2 requirements in (ii) are satisfied. Then observe that "$\epsilon_z(t)$ is independent of $u(t)$ and $w(t)$", will imply that $\epsilon_z(t)$ is independent of $x(t)$ (by assumption 2.1) and estimation errors $e(t)$ that are affected by $w(t)$, which are $e(t) \in \mathcal{S}$. This implies that $\epsilon_z(t_0) = 0$ for any $x(t_0)$, $e(t_0) \in \mathcal{S}$.

(ii) $\Rightarrow$ (iii): $e(t) \in \mathcal{S}$ for all $t \geq t_0$ and for any $w(t)$, provided that $e(t_0) = e_0 \in \mathcal{S}$ implies that $(A + LC_y)\mathcal{S} \subseteq \mathcal{S}$.

$\epsilon_z(t_0) = 0$ for any $x(t_0)$, $e(t_0) \in \mathcal{S}$, implies that $C_z - RC_y - P = 0$, $P\mathcal{S} = 0$. $\epsilon_z(t)$ is independent of $u(t)$ by additionally imposing $Q = D_{uz}$.

(iii) $\Rightarrow$ (i): $(A + LC_y)\mathcal{S} \subseteq \mathcal{S}$ implies that $e(t) \in \mathcal{S}$ for all $t \geq t_0$ and for any $w(t)$, provided that $e(t_0) = e_0 \in \mathcal{S}$. Then, by adding $C_z - RC_y - P = 0$, $P\mathcal{S} = 0$ it can be concluded that $\epsilon_z(t)$ is independent of $w(t)$. Finally, by imposing $Q = D_{uz}$ it can be concluded that $\epsilon_z(t)$ is independent of $u(t)$. ■

V. Main Results

A. Disturbance Decoupled Estimation Problem with a Constrained Order (DDEP-CO)

Using theorem 4.6, observer Σ_O (3) is designed such that $(A + LC_y)\mathcal{S} \subseteq \mathcal{S}$ and $P\mathcal{S} = 0$. This implies that the states $\hat{x}(t) \in \mathcal{S}$ are unobservable in $\hat{z}(t)$. Σ_O (3) therefore solves the DDEP, when the states $x(t)$ are estimated modulo $\mathcal{S}$.

A low order observer can be created by defining a subspace $\mathcal{S}_m \subseteq \mathcal{X}$, which is an arbitrary complement of $\mathcal{S}$ in the sense that $\mathcal{X} = \mathcal{S} \oplus \mathcal{S}_m$. Using $\mathcal{S}_m$, one element is taken from the equivalence class that is defined by $\mathcal{X}$ mod $\mathcal{S}$.

An observer of the form Σ_{Oh} (2), with order $\dim(\mathcal{S}_m) = \dim \mathcal{X} - \dim \mathcal{S}$, is now given by

$$\hat{\Sigma}_O \quad = \quad \begin{cases} \dot{\hat{x}}_m(t) = \hat{M}\hat{x}_m(t) + \hat{N}u(t) + \hat{O}y(t) \\ \hat{z}(t) \ = \hat{P}\hat{x}_m(t) \ + \hat{Q}u(t) + \hat{R}y(t) \end{cases} \tag{6}$$

with $\hat{M} = \Pi(\mathcal{S}_m, \mathcal{S})A|\mathcal{S}_m + L_m C_y|\mathcal{S}_m,\ \hat{N} = \Pi(\mathcal{S}_m, \mathcal{S})B_u,$

$$\hat{O} = -L_m, \quad \hat{P} = (C_z - (C_z|\mathcal{T})(C_y|\mathcal{T})^{\dagger}C_y)|\mathcal{S}_m,$$
$$\hat{Q} = D_{uz}, \quad \hat{R} = (C_z|\mathcal{T})(C_y|\mathcal{T})^{\dagger},$$

and where $\mathcal{T}$ satisfies $(\mathcal{S} \cap \ker(C_y)) \oplus \mathcal{T} = \mathcal{S}$. Linear mapping $L_m : \mathcal{Y} \to \mathcal{S}_m$ satisfies $\Pi(\mathcal{S}_m, \mathcal{S})A|\mathcal{S} + L_m C_y|\mathcal{S} = 0$, which relates to theorem 4.6 by observing that $(A + LC_y)\mathcal{S} \subseteq \mathcal{S}$ holds if and only if $\Pi(\mathcal{S}_m, \mathcal{S})A|\mathcal{S} + \Pi(\mathcal{S}_m, \mathcal{S})LC_y|\mathcal{S} = 0$.

Remark 5.1: This observer can intuitively be interpreted by choosing $\mathcal{S}_m = \mathcal{S}^{\perp}$ and $\mathcal{T} = \mathcal{S} \cap (\ker(C_y)^{\perp})$. With mapping L_m, disturbance $w(t)$ leads to estimation errors $e(t) \in \mathcal{S}$, while $w(t)$ will have no effect on $e(t) \in \mathcal{S}^{\perp}$. The mappings $\hat{P}$, $\hat{Q}$ and $\hat{R}$ satisfy (iii) in theorem 4.6 for $e(t) \in \mathcal{S}^{\perp}$. □

Theorem 5.2: Let $\mathcal{S} \in \mathbb{S}$ satisfy $\dim(\mathcal{S}) = \dim(\mathcal{X}) - n_o$ and let $\mathcal{S}_m$ be such that $\mathcal{S} \oplus \mathcal{S}_m = \mathcal{X}$. Then $\hat{\Sigma}_O$ (6) with mapping L_m satisfying $\Pi(\mathcal{S}_m, \mathcal{S})A|\mathcal{S} + L_m C_y|\mathcal{S} = 0$ *solves* the DDEP-CO for (Σ (1), n_o). □

Proof: Statement (iii) in theorem 4.6 introduces requirements on Σ_O (3) to solve the DDEP.

Observe that $P\mathcal{S} = 0$ requires that $P|(\mathcal{S} \cap \ker(C_y)) = 0$ and $P|\mathcal{T} = 0$, where $\mathcal{T}$ satisfies $(\mathcal{S} \cap \ker(C_y)) \oplus \mathcal{T} = \mathcal{S}$. To additionally satisfy $C_z - RC_y - P = 0$, we separately consider the restriction of this matrix on $\mathcal{S} \cap \ker(C_y)$, $\mathcal{T}$ and $\mathcal{S}_m$.

- $(C_z - RC_y - P)|(\mathcal{S} \cap \ker(C_y)) = C_z|(\mathcal{S} \cap \ker(C_y)) = 0$, which holds by definition for any $\mathcal{S} \in \mathbb{S}$.
- $(C_z - RC_y - P)|\mathcal{T} = (C_z - RC_y)|\mathcal{T} = 0$, which requires that $R = (C_z|\mathcal{T})(C_y|\mathcal{T})^{\dagger}$.
- $(C_z - RC_y - P)|\mathcal{S}_m = 0$, which requires that $P|\mathcal{S}_m = (C_z - RC_y)|\mathcal{S}_m$.

Then observe that $(A + LC_y)\mathcal{S} \subseteq \mathcal{S}$ and $P\mathcal{S} = 0$ imply that states $\hat{x}(t) \in \mathcal{S}$ are unobservable in $\hat{z}(t)$. Removing the unobservable states of Σ_O (3) and keeping the states $\hat{x}(t) \in \mathcal{S}_m$ will lead to $\hat{\Sigma}_O$ (6).

$(A + LC_y)\mathcal{S} \subseteq \mathcal{S}$ holds if and only if $\Pi(\mathcal{S}_m, \mathcal{S})A|\mathcal{S} + \Pi(\mathcal{S}_m, \mathcal{S})LC_y|\mathcal{S} = 0$. Observe that $L_m = \Pi(\mathcal{S}_m, \mathcal{S})L$, to conclude that $\hat{\Sigma}_O$ (6) solves the DDEP-CO for (Σ (1), n_o). ■

B. Stability

In the previous sections, the output estimation error $\epsilon_z(t)$ is made independent of $w(t)$ and $u(t)$. An additional requirement is that $\epsilon_z(t) \to 0$ as $t \to \infty$ for any initial estimation error $e(t)$. This is related to stability of $e(t)$.

Definition 5.3: A subset $\mathbb{C}_g \subseteq \mathbb{C}$ is a *stability domain*, if $\mathbb{C}_g \cap \mathbb{R}$ is nonempty and $\mathbb{C}_g$ is symmetric with respect to the real axis. □

In the absence of disturbances and inputs, the estimation error satisfies $\dot{e}(t) = (A + LC_y)e(t)$. The requirement that $(A + LC_y)\mathcal{S} \subseteq \mathcal{S}$, in combination with the fact that $\hat{\Sigma}_O$ (6) does not estimate the states in $\mathcal{S}$, implies that $\sigma(A + LC_y) = \sigma(A|\mathcal{S}) \cup \sigma((A + LC_y)|(\mathcal{X} \bmod \mathcal{S}))$. This can be seen by partitioning the error dynamics as $e(t) = \begin{pmatrix} e_1(t)^{\top} & e_2(t)^{\top} \end{pmatrix}$, where $e_1(t)$ describes the estimation error in $\mathcal{S}$ and $e_2(t)$ the estimation error in $\mathcal{X}$ mod $\mathcal{S}$. This leads to

$$A + LC_y = \begin{pmatrix} A|\mathcal{S} & * \\ 0 & (A+LC_y)|(\mathcal{X} \bmod \mathcal{S}) \end{pmatrix}.$$

For this reason, the observer gain L only determines $\sigma((A + LC_y)|(\mathcal{X} \bmod \mathcal{S}))$. This gives rise to a result that is similar to definition 3.4 and proposition 3.5.

Definition 5.4: Given a subspace $\mathcal{K} \subseteq \mathcal{X}$ and a stability domain $\mathbb{C}_g$, a *detectability subspace* $\mathcal{S}_g(\mathcal{K})$ is a subspace $\mathcal{S}_g(\mathcal{K}) \subseteq \mathcal{X}$ that contains $\mathcal{K}$ and for which there exists a mapping $L : \mathcal{Y} \to \mathcal{X}$ such that $(A + LC_y)\mathcal{S}_g(\mathcal{K}) \subseteq \mathcal{S}_g(\mathcal{K})$ and $\sigma((A + LC_y)|(\mathcal{X} \bmod \mathcal{S}_g(\mathcal{K}))) \subseteq \mathbb{C}_g$. □

Proposition 5.5: Given a subspace $\mathcal{K} \subseteq \mathcal{X}$ and a stability domain $\mathbb{C}_g$, there exists a unique smallest detectability subspace that contains $\mathcal{K}$, which is denoted by $\mathcal{S}_g^{\star}(\mathcal{K})$. This subspace satisfies $\mathcal{S}_g^{\star}(\mathcal{K}) \subseteq \mathcal{S}_g(\mathcal{K})$, for any detectability subspace $\mathcal{S}_g(\mathcal{K})$. □

Proof: The proof can be found in [12, Thm. 5.11, 5.14]. ■

Stability is added to $\epsilon_z(t)$ by utilizing the set

$$\mathbb{S}_g = \{\mathcal{S}_g : \mathcal{S}_g = \mathcal{S}_g(\mathrm{im}(B_w)),\ \mathcal{S}_g \cap \ker(C_y) \subseteq \ker(C_z)\}$$

in the observer design according to theorem 5.2, with L_m satisfying $\Pi(\mathcal{S}_m, \mathcal{S}_g)A|\mathcal{S}_g + L_m C_y|\mathcal{S}_g = 0$ and $\sigma(\Pi(\mathcal{S}_m, \mathcal{S}_g)A|\mathcal{S}_m + L_m C_y|\mathcal{S}_m) \subseteq \mathbb{C}_g$ for some stability domain $\mathbb{C}_g$.

Finally, observe that $\mathbb{S}_{g_1} \subseteq \mathbb{S}_{g_2}$, whenever $\mathbb{C}_{g_1} \subseteq \mathbb{C}_{g_2}$. This implies that $\mathbb{S}_g \subseteq \mathbb{S}$ for any $\mathbb{C}_g$ and that $\mathbb{S}_g = \mathbb{S}$, when $\mathbb{C}_g = \mathbb{C}$.

C. Conditions on the Observer Order

Given any stability domain $\mathbb{C}_g$, the values n_o for which the DDEP-CO can be solved are determined by the dimension of elements in $\mathbb{S}_g$. This leads to the following main result.

Theorem 5.6: Let $\mathcal{S}_m$ be such that $\mathcal{S}_g \oplus \mathcal{S}_m = \mathcal{X}$. Then $\hat{\Sigma}_O$ (6) *can solve* the DDEP-CO for (Σ (1), n_o), with $\sigma(\Pi(\mathcal{S}_m, \mathcal{S}_g)A|\mathcal{S}_m + L_m C_y|\mathcal{S}_m) \subseteq \mathbb{C}_g$ if and only if there exists an $\mathcal{S}_g \in \mathbb{S}_g$ with $\dim(\mathcal{S}_g) = \dim(\mathcal{X}) - n_o$. □

Proof: This follows directly from theorem 5.2. ■

Let $n_o^\star$ be the smallest value for which the DDEP-CO can be solved, which is interesting from a computational perspective.

Theorem 5.7: Let, without loss of generality, C_y and C_z be of full row rank. Then $n_o^\star \geq \dim(\mathcal{Z}) - \dim(\mathcal{Y})$. □

Proof: Observe that $\mathbb{S}_g \subseteq \mathbb{S}$, for any $\mathbb{C}_g$. For this reason we will consider elements $\mathcal{S}$ from the largest set $\mathbb{S}$. Theorem 5.6 indicates that the DDEP-CO can be solved if there exists a subspace $\mathcal{S}$ for which it must hold that $\dim(\mathcal{S}) - \dim(\mathcal{Y}) \leq \dim(\mathcal{S} \cap \ker(C_y)) \leq \dim(\ker(C_z)) = \dim(\mathcal{X}) - \dim(\mathcal{Z})$. This imposes an upper bound on $\dim(\mathcal{S}) \leq \dim(\mathcal{X}) - \dim(\mathcal{Z}) + \dim(\mathcal{Y})$, which combined with $n_o = \dim(\mathcal{X}) - \dim(\mathcal{S})$ implies that $n_o^\star \geq \dim(\mathcal{Z}) - \dim(\mathcal{Y})$. ■

D. Construction of the Constrained Order Observer

There exist various numerical algorithms to construct a basis for $\mathcal{S}^\star$. Subspace operations, in combination with an iterative procedure to construct a basis for $\mathcal{S}^\star$ are implemented in [16]. The so-called *special coordinate basis* of a system, as introduced in [8] and which is numerically implemented in [17], can be used to determine a basis for both $\mathcal{S}^\star$ and $\mathcal{S}_g^\star$.

These algorithms can be used to numerically create an observer that will solve the DDEP-CO. An observer of small order is created by finding elements $\mathcal{S}_g \in \mathbb{S}_g$ of large dimension. However, a characterization of elements $\mathcal{S}_g \in \mathbb{S}_g$ and a numerical algorithm to find largest dimensional elements remains, to the knowledge of the authors, an open problem. For this reason, it is assumed that a basis for $\mathcal{S}_g \in \mathbb{S}_g$ is found.

Observe that a state transformation, given by a nonsingular mapping T such that $\bar{x}(t) = Tx(t)$, will not change the input to output behavior of a system. Therefore an observer can be designed for Σ (1), using any equivalent system that is created by applying a state transformation.

For $\mathcal{S}_g \in \mathbb{S}_g$, let $\mathcal{S}_m$ be such that $\mathcal{S}_g \oplus \mathcal{S}_m = \mathcal{X}$. Furthermore introduce $\mathcal{S}_1 = \mathcal{S}_g \cap \ker(C_y)$ and let $\mathcal{S}_2$ be such that $\mathcal{S}_1 \oplus \mathcal{S}_2 = \mathcal{S}_g$. For example, one could choose $\mathcal{S}_m = \mathcal{S}_g^\perp$ and $\mathcal{S}_2 = \mathcal{S}_g \cap (\ker(C_y)^\perp)$. Define

$$T_\mathcal{S} = \begin{pmatrix} \Pi(\mathcal{S}_1, \mathcal{S}_2 \oplus \mathcal{S}_m)^\top & \Pi(\mathcal{S}_2, \mathcal{S}_1 \oplus \mathcal{S}_m)^\top & \Pi(\mathcal{S}_m, \mathcal{S}_g)^\top \end{pmatrix}^\top$$

and observe that $T_\mathcal{S} : \mathcal{X} \to \mathcal{X}$ is nonsingular.

Proposition 5.8: Let the columns of $K_1 \in \mathbb{R}^{n_x \times k_1}$ and $K_2 \in \mathbb{R}^{n_x \times k_2}$ be orthonormal bases for subspaces $\mathcal{K}_1 \subseteq \mathcal{X}$ and $\mathcal{K}_2 \subseteq \mathcal{X}$ of dimension k_1 and k_2, respectively. Let these subspaces satisfy $\mathcal{K}_1 \oplus \mathcal{K}_2 = \mathcal{X}$, which implies that $n_x = k_1 + k_2$. Furthermore, let $K_{2p} \in \mathbb{R}^{n_x \times k_1}$ be an orthonormal basis for $\mathcal{K}_2^\perp$. Then the projection onto $\mathcal{K}_1$ and along $\mathcal{K}_2$ is numerically given by $\Pi(\mathcal{K}_1, \mathcal{K}_2) = K_1^\top K_1 (K_{2p}^\top K_1)^{-1} K_{2p}^\top$ as discussed in [15, Sec. 1.12]. □

State transformation $T_\mathcal{S}$ applied to Σ (1) creates the system

$$\Sigma^T = \begin{cases} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \\ \qquad + \begin{pmatrix} B_{u1} \\ B_{u2} \\ B_{u3} \end{pmatrix} u(t) + \begin{pmatrix} B_{w1} \\ B_{w2} \\ 0 \end{pmatrix} w(t) \\ y(t) = \begin{pmatrix} 0 & C_{y2} & C_{y3} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \\ z(t) = \begin{pmatrix} C_{z1} & C_{z2} & C_{z3} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + D_{uz} u(t) \end{cases} \tag{7}$$

where the state-space of Σ (1) is partitioned in such a way that $\mathcal{X}_1 = T_\mathcal{S}\mathcal{S}_1$, $\mathcal{X}_2 = T_\mathcal{S}\mathcal{S}_2$ and $\mathcal{X}_3 = T_\mathcal{S}\mathcal{S}_m$. Therefore $\mathcal{X}_1 \oplus \mathcal{X}_2$ is a detectability subspace for Σ^T (7).

Observe that $\mathrm{im}(B_w) \subseteq \mathcal{S}_g$ is satisfied for Σ (1), which implies for Σ^T (7) that $B_{w3} = 0$. $\mathcal{S}_1 \subseteq \ker(C_y)$ for Σ (1) implies that $C_{y1} = 0$ for Σ^T (7). Finally, there exists a mapping L such that $(A + LC_y)\mathcal{S}_g \subseteq \mathcal{S}_g$ for Σ (1), which implies for Σ^T (7) that there exists a mapping L_3 such that $\begin{pmatrix} A_{31} & A_{32} + L_3 C_{y2} \end{pmatrix} = 0$ and therefore $A_{31} = 0$.

An implementation of $\hat{\Sigma}_O$ (6) is created by using $\mathcal{X}_3$ of Σ^T (7), which is related to $\mathcal{S}_m$ of Σ (1), to the construct the observer

$$\hat{\Sigma}_O^T = \begin{cases} \dot{\hat{x}}_3(t) = (\, A_{33} + L_3 C_{y3} \,)\hat{x}_3(t) + B_{u3} u(t) - L_3 y(t) \\ \hat{z}(t) = (\, C_{z3} - C_{z2} C_{y2}^\dagger C_{y3} \,)\hat{x}_3(t) \\ \qquad + D_{uz} u(t) + (\, C_{z2} C_{y2}^\dagger \,) y(t) \end{cases} \tag{8}$$

Applying $\hat{\Sigma}_O^T$ (8) to Σ^T (7) will result in the error system $\hat{\Sigma}_\epsilon^T$ (9) by observing that $e_1(t) = x_1(t)$ and $e_2(t) = x_2(t)$. For stability domain $\mathbb{C}_g$ and $\mathcal{S}_g \in \mathbb{S}_g$ there exists, by definition, a mapping L_3 such that $A_{32} + L_3 C_{y2} = 0$ and $\sigma(A_{33} + L_3 C_{y3}) \subseteq \mathbb{C}_g$. $\hat{\Sigma}_\epsilon^T$ (9) can therefore be disturbance decoupled when $C_{z1} = 0$. This requires that $\mathcal{S}_g \cap \ker(C_y) \subseteq \ker(C_z)$ holds Σ (1), which is satisfied for any $\mathcal{S}_g \in \mathbb{S}_g$.

VI. Example

As addressed in section V-D, numerical algorithms to characterize large dimensional elements $\mathcal{S} \in \mathbb{S}$ and $\mathcal{S}_g \in \mathbb{S}_g$ do not exist yet. At this moment it is therefore unclear what orders of $\hat{\Sigma}_O$ (6) can be achieved for large-scale systems. An example is therefore presented in order to illustrate the construction of constrained order observers. Consider a system in a form similar to Σ^T (7), as described by

$$\Sigma_1 = \begin{cases} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0.1 & -2 & 0 & 1 \\ 0 & -0.5 & -0.4 & 0 \\ 0 & 0.2 & 0.2 & -2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} w(t) \\ y(t) = \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix} x(t), \qquad z(t) = \begin{pmatrix} 0 & 1 & 1 & 2 \end{pmatrix} x(t) \end{cases} \tag{10}$$

where $\mathcal{S}^\star = \mathcal{X}_1 \oplus \mathcal{X}_2$ and $\mathcal{S}^{\star\perp} = \mathcal{X}_3 \oplus \mathcal{X}_4$. Furthermore, observe that for stability domain $\mathbb{C}_g = \mathbb{C}_-$ it holds that $\mathcal{S}_g^\star = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ and $\mathcal{S}_g^{\star\perp} = \mathcal{X}_4$.

There exists an observer $\hat{\Sigma}_{O1}$ that solves the DDEP-CO for (Σ_1 (10), n_o), with $n_o = 2$. The inclusion $\mathcal{S}^\star \subseteq \mathcal{S}_g^\star$ leads to the conclusion that $\mathcal{S}^\star \notin \mathbb{S}_g$, which by theorem 5.6 implies that the observer will be unstable. This can also be seen from the fact that $L_3 = 0.5$ must hold in order to decouple state 3 from state 2, which will render the dynamics for $\hat{x}_3$ unstable.

The observer that solves the DDEP-CO for (Σ_1 (10), n_0), with $n_o = 1$ and for stability domain $\mathbb{C}_g = \mathbb{C}_-$, will guarantee that $\epsilon_z(t)$ is independent of $w(t)$ and that $\epsilon_z(t) \to 0$ as $t \to \infty$ for any initial estimation error. This observer is given by

$$\hat{\Sigma}_{O2} = \begin{cases} \dot{\hat{x}}_4(t) = -2\hat{x}_4(t) + 0.2y(t) \\ \hat{z}(t) = \;\; 2\hat{x}_4(t) + 1y(t) \end{cases} \tag{11}$$

Let disturbances be of the form $w_i(t) = c_{i0} + c_{i1}\sin(\omega_i t) + c_{i2} n(t)$, where $n(t)$ is unitary white noise. Assume that the disturbances are characterized by $c_{10} = -c_{20} = 10$, $c_{11} =$

$$\hat{\Sigma}_{\epsilon}^{T} \quad = \quad \left\{ \begin{array}{rl} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \hline \dot{e}_3(t) \end{pmatrix} & = \left(\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & 0 \\ 0 & A_{32} & A_{33} & 0 \\ \hline 0 & A_{32}+L_3C_{y2} & 0 & A_{33}+L_3C_{y3} \end{array} \right) \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \hline e_3(t) \end{pmatrix} + \begin{pmatrix} B_{u1} \\ B_{u2} \\ B_{u3} \\ \hline 0 \end{pmatrix} u(t) + \begin{pmatrix} B_{w1} \\ B_{w2} \\ 0 \\ \hline 0 \end{pmatrix} w(t) \\ \epsilon_z(t) & = \left(\begin{array}{ccc|c} C_{z1} & 0 & 0 & C_{z3} - C_{z2}C_{y2}^{\dagger}C_{y3} \end{array} \right) \begin{pmatrix} x(t) \\ \hline e_3(t) \end{pmatrix} \end{array} \right. \tag{9}$$

$c_{21} = 20$, $\omega_1 = 1$, $\omega_2 = 3$ and $c_{12} = c_{22} = 100$. Then Fig. 3 shows $\hat{\Sigma}_{O2}$ (11) applied to Σ_1 (10) in a simulation, with initial states $x(0) = \begin{pmatrix} 10 & 10 & 10 & 10 \end{pmatrix}^{\top}$ and $\hat{x}_4(0) = -30$. The simulation shows that $\epsilon_z(t)$ is not affected by $w(t)$ and it will converge to 0 as time progresses.

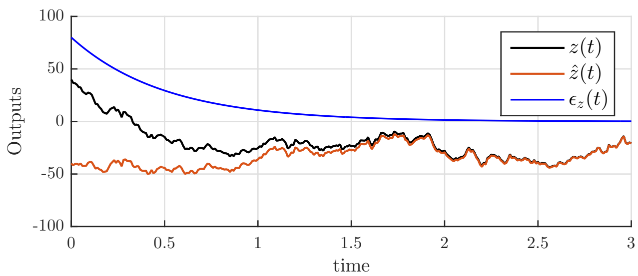

Fig. 3. A simulation of $z(t)$, $\hat{z}(t)$ and $\epsilon_z(t)$.

VII. Conclusions & Future Works

This paper addresses the characterization and synthesis of an observer for a linear and time-invariant generalized plant model of a system. The observer renders the output estimation error $\epsilon_z(t)$ of Σ_ϵ (4) independent of input $u(t)$ and disturbance $w(t)$, while in addition imposing a constraint on the state dimension of the observer. The constrained order observer is designed directly for the full order system, which is corresponding to approach 3 in Fig. 1.

A novel insight into the direct relation between the state dimension of the constrained order observer and the dimension of conditioned invariant subspaces $\mathcal{S}_g \in \mathbb{S}_g$ is presented. An observer of small order is created by finding elements $\mathcal{S}_g \in \mathbb{S}_g$ with a large dimension.

There exist algorithms that can construct a basis for $\mathcal{S}^\star$ [16] and $\mathcal{S}_g^\star$ [17]. A numerical procedure to construct bases for elements $\mathcal{S}_g \in \mathbb{S}_g$ of larger dimension does, to the knowledge of the authors, not exist. A lower bound on the observer order is constructed. However, the actual smallest order of the observer has analytically not been characterized yet. Consequently, both of these problems must be investigated further in order to make the presented approach applicable in practice.

The absence of a direct feed-through term from $w(t)$ to $y(t)$ has been assumed. If this term is present, the design of an observer will be similar when the definition of conditioned invariant subspaces is extended as described in [11, Sec. 4.5][18, Def. 2.1]. The design of observer gain L will remain unchanged, while matrices P and R must be adjusted. However, these adjustments are not required when the observer is used in a dynamic output feedback controller, for which these matrices are chosen with a different objective.

The design of a dynamic output feedback controller with a constrained order that achieves disturbance decoupling will be considered next. In order to design such a controller, the geometric control concepts of controlled invariant subspaces $\mathcal{V}$ and $(\mathcal{S}, \mathcal{V})$-pairs are used.

Disturbance decoupling problems play an important role in H_2 optimal control problems. It has been shown in [8][12] that a proper transformation of a generalized plant establishes that a controller (or observer) is H_2 optimal if and only if it achieves disturbance decoupling for the transformed (error) system. The results of this paper are therefore relevant for constrained order H_2 optimal observers as well.

References

[1] M. Topor, S. Deaconu, and L. Tutelea, "Homo-Heteropolar Synchronous Machine for Low Power Variable Speed Wind or Hydro Applications: Design, 3D FEM Validation and Control", *16th Eur. Conf. on Power Electronics and Applications*, Lappeenranta, 2014, pp. 1-10.

[2] Y. Zhao, H. Wang, H. Zhang and L. Xiao, "Position-Sensorless Control of DC + AC Stator Fed Doubly Salient Electromagnetic Motor Covered Full Speed Range", *IEEE Transactions on Industrial Electronics,* vol. 62, no. 12, pp. 7412-7423, 2015.

[3] E. Jonckheere and L. Silverman, "A New Set of Invariants for Linear Systems – Application to Reduced Order Compensator Design", *IEEE Transactions on Automatic Control,* vol. 28, no. 10, pp. 953-964, 1983.

[4] P. Wortelboer, "Frequency-Weighted Balanced Reduction of Closed-Loop Mechanical Servo-Systems: Theory and Tools", Ph.D. thesis, Delft University of Technology, 1994.

[5] P. Goddard, "Performance-Preserving Controller Approximation", Ph.D. thesis, Trinity College, Cambridge, 1995.

[6] G. Obinata and B. Anderson, *Model Reduction for Control System Design.* London: Springer London, 2013.

[7] A. Varga and B. Anderson, "Accuracy-Enhancing Methods for Balancing-Related Frequency-Weighted Model and Controller Reduction", *Automatica,* vol. 39, no. 5, pp. 919-927, 2003.

[8] A. Saberi, P. Sannuti and B. Chen, *H_2 Optimal Control.* London: Prentice Hall, 1995.

[9] M. Mutsaers, "Control Relevant Model Reduction and Controller Synthesis for Complex Dynamical Systems", Ph.D. thesis, Eindhoven University of Technology, 2012.

[10] M. Mutsaers and S. Weiland, "A Model Reduction Scheme with Preserved Optimal Performance", *50th IEEE Conference on Decision and Control and European Control Conference,* Orlando, FL, 2011, pp. 7176-7181.

[11] G. Basile and G. Marro, *Controlled and Conditioned Invariants in Linear System Theory.* Englewood Cliffs, N.J.: Prentice Hall, 1992.

[12] H. Trentelman, A. Stoorvogel and M. Hautus, *Control Theory for Linear Systems.* London: Springer London, 2001.

[13] W. Wonham, *Linear Multivariable Control.* New York: Springer, 1979.

[14] Y. Saad, *Numerical Methods for Large Eigenvalue Problems.* Philadelphia, PA: Society for Industrial and Applied Mathematics, 2011.

[15] Y. Saad, *Iterative Methods for Sparse Linear Systems.* Philadelphia, PA: Society for Industrial and Applied Mathematics, 2003.

[16] G. Marro. "Geometric Approach" Internet: http://www3.deis.unibo.it/Staff/FullProf/GiovanniMarro/geometric.htm [09- May- 2018].

[17] B. Chen, "Linear Systems Toolkit" Internet: http://uav.ece.nus.edu.sg/~bmchen/linsyskit/index.html [09- May- 2018].

[18] A. Stoorvogel and J. van der Woude, "The Disturbance Decoupling Problem with Measurement Feedback and Stability for Systems with Direct Feedthrough Matrices", *Systems & Control Letters*, vol. 17, no. 3, pp. 217-226, 1991.