MASTER

Decidability of bisimilarity and axiomatisation for sequential processes in the presence of intermediate termination

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Master Thesis

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Abstract

An alternative semantics for sequential composition in a setting with intermediate termination was proposed in a recent article by Baeten, Luttik and Yang. This thesis considers several open questions regarding sequential processes with intermediate termination that use the revised semantics for sequential composition ($\mathrm{TSP}^{;}(A)$).

The first question concerns the axiomatisation of $\mathrm{TSP}^{;}(A)$ with respect to bisimilarity. A ground-complete axiomatisation is proposed for the theory $\mathrm{TSP}^{;}(A)$ extended with an auxiliary operator. This auxiliary operator is used to remove intermediate termination from terms. Additionally, it is shown that $\mathrm{TSP}^{;}(A)$ does not afford a ground-complete axiomatisation with respect to bisimilarity without an auxiliary operator. Finally, it is argued that with this specific auxiliary operator, bisimilarity does not afford an ω-complete axiomatisation for $\mathrm{TSP}^{;}(A)$.

The second question concerns the decidability of bisimilarity for processes definable by finite guarded recursive specification over $\mathrm{TSP}^{;}(A)$. It is shown that every recursive $\mathrm{TSP}^{;}(A)$-specification can be transformed to a normal form, which allows us to eliminate redundant intermediate termination from processes. Using this normal form, the existing decidability proofs for context-free processes without intermediate termination, can be adapted for $\mathrm{TSP}^{;}(A)$. Finally, it is argued that the technique used to transform $\mathrm{TSP}^{;}(A)$-specifications to a normal form for which decidability can be proven, cannot be applied to $\mathrm{TSP}^{\cdot}(A)$-specifications.

Preface

This master thesis is the result of my graduation project for the Computer Science and Engineering master at Eindhoven University of Technology (TU/e). The project was carried out within the Formal System Analysis research group of the Mathematics and Computer Science department of TU/e. Prior to this graduation project, I carried out a seminar project within the Formal System Analysis research group, investigating the axiomatisation of $\mathrm{TSP}^{;}(A)$. The results in Section 4.1 were obtained during this seminar project. The foundations for the result in Section 4.2 were made during this seminar project as well, and this result was finalized during my graduation project.

I would like to thank Bas Luttik, for providing me with the opportunity to perform this graduation project. I would like to thank him for his guidance during this project, and for reviewing my work many times, providing me with valuable feedback.

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Astrid Belder
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Chapter 1

Introduction

This chapter introduces the research context and research problems for this thesis. Moreover, the results obtained during the graduation project are summarized and the structure of this report is provided.

1.1 Research context

In the last decade, several studies have been conducted into the integration of concurrency theory with the theory of formal languages [4] [17]. An important difference between these two theories, is the different notion of equivalence that is used. In formal language theory, languages are specified by a grammar or automaton. Grammars are sometimes referred to as language generators and automata as language acceptors. Two grammars are considered to be equal if they generate the same language, i.e., if they are *language-equivalent*, and similarly automata are considered equal if they accept the same language. This interpretation of behaviour does not take moments of choice into account. For example, an automaton that accepts an a and then has a choice to either accept a b or a c, accepts the same language as an automaton that has the choice to either accept ab or ac. This notion of expressiveness is therefore only suitable if an automaton is considered to be a stand-alone machine that automatically follows a predetermined sequence of operations. In concurrency theory, however, the interaction between a system and its environment is taken into consideration. An environment may be a user or another system interacting with the system. A system is also often considered to be a number of components interacting with each other. As some choices within a system need to be resolved by interaction, the moment of choice may be relevant in an interactive system. Therefore, to preserve all relevant moments of choice, a finer notion of equivalence is needed. One of the finest behavioural equivalences used in the concurrency theory is *bisimilarity*.

1.1.1 Context-free processes

One of the notions of formal language theory that has been extensively studied in the setting of concurrent systems, is the correspondence between context-free grammars (CFG) and so called context-free processes, i.e., a process that has a set of traces that is equal to a context-free language. A process theory that is used to define context-free processes and that allows a natural translation from a context-free grammar to a context-free process, is the theory of sequential processes ($\mathrm{TSP}^{\cdot}(A)$). The syntax of this process theory consists of the constants for deadlock ($\mathbf{0}$) and intermediate termination ($\mathbf{1}$), an action-prefix operator for each action in $a \in A$ ($a.$), and the operators for alternative composition ($+$) and sequential composition ($\cdot$). There is a natural translation from a CFG to a $\mathrm{TSP}^{\cdot}(A)$-specification. For every non-terminal character in a CFG, a recursion variable is introduced in $\mathrm{TSP}^{\cdot}(A)$, and for every terminal an action is introduced. The empty string in a CFG, often denoted with ε, corresponds to the termination constant ($\mathbf{1}$) of $\mathrm{TSP}^{\cdot}(A)$. The rewrite rules for a non-terminal are translated to a defining equation for a recursion variable. For example, consider the CFG with non-terminals S and A, terminals a, b and c and production rules $S \to aSA \mid b$ and $A \to c \mid \varepsilon$. This can be straightforwardly translated to a $\mathrm{TSP}^{\cdot}(A)$-specification with recursion variables S and A, actions a, b and c and defining equations $S = a.SA + b.\mathbf{1}$ and $A = c.\mathbf{1} + \mathbf{1}$.

1.2 Problem description

The most obvious semantics for sequential composition, causes some issues modulo bisimilarity. One of these issues is the transparency caused by intermediate termination in a sequential context. For example, considering the specification from the previous paragraph, the sequence AA contains transparency, as A can terminate and A can execute a c-action. As a solution to these issues, Baeten et al. [6] have proposed a revised operational semantics for the sequential composition operator. This revised semantics does not allow processes in a sequential context to terminate if they are able to execute an action, and thus eliminates the problem of transparency. The theory of sequential processes that uses these revised operational semantics is denoted with $\mathrm{TSP}^{;}(A)$. With this new process theory several open research questions arise.

First of all, there are no published results regarding an axiomatisation for $\mathrm{TSP}^{;}(A)$ with respect to bisimilarity. An axiomatisation is ground-complete with respect to bisimilarity, if for all closed process-terms (terms without free variables) it holds that if they are bisimilar, then they can be derived to be equal from the axiomatisation. While there is a well known ground-complete axiomatisation for $\mathrm{TSP}^{\cdot}(A)$, this axiomatisation is not sound with respect to bisimilarity according to the new operational semantics. This is caused by the fact that, in the new semantics, sequential composition does not distribute from the right over alternative composition. Hence, a ground-complete axiomatisation for $\mathrm{TSP}^{;}(A)$ cannot straightforwardly be adapted from this existing axiomatisation, which poses the following research question:

Does bisimilarity afford a ground-complete axiomatisation for $\mathrm{TSP}^{;}$*?*

A well known result in formal language theory, is that it is undecidable whether two context-free grammars produce the same language. In contrast to the undecidability of the equivalence problem for context-free grammars, it turns out that bisimilarity is decidable for context-free processes. The first positive result regarding decidability of context-free processes was published by Baeten, Bergstra and Klop [3] and addressed the decidability of bisimilarity for normed BPA-processes. Note that BPA is $\mathrm{TSP}^{\cdot}(A)$, with action constants instead of the action prefix operator, and without intermediate termination and deadlock. Also, normed processes are processes that can eventually succesfully terminate. Alternative, and arguably simpler, decidability proofs for normed BPA-processes were published by Caucal [10], Hüttel and Stirling [14] and Groote [13]. In 1992, this result was extended by Christensen, Hüttel and

Stirling [11], who proved decidability for arbitrary BPA-processes. Later, it has been shown independently by Bosscher [8] and Srba [16], that the proofs for decidability of bisimilarity of BPA-processes can easily be extended to include deadlock behaviour (BPA_δ). However, as seen in the previous section, for a full correspondence with the formal language theory the notion of intermediate termination is needed as well, as it facilitates a direct translation of ε-productions of a CFG. It turns out that the decidability result is considerably more difficult to obtain for the full set of $\mathrm{TSP}^{\cdot}(A)$-processes (including intermediate termination). In [17] it is shown that bisimilarity is decidable for all transparency-restricted $\mathrm{TSP}^{\cdot}(A)$-processes, but the question of decidability for the full set of $\mathrm{TSP}^{\cdot}(A)$-processes remains open. As the semantics of $\mathrm{TSP}^{;}(A)$ eliminate the notion of transparency, it is interesting to consider the question of decidability for processes defined by $\mathrm{TSP}^{;}(A)$. One may argue, that the semantics of $\mathrm{TSP}^{;}(A)$ makes the translation from CFG to context-free process less direct. For example, consider the CFG with production rules $S \to aSA \mid b$, $A \to cB \mid \varepsilon$ and $B \to d \mid \varepsilon$. A straightforward translation would yield the $\mathrm{TSP}^{;}(A)$-specification with defining equations $S = a.SA + b.\mathbf{1}$, $A = c.B + \mathbf{1}$ and $B = d.\mathbf{1} + \mathbf{1}$. This specification is not language-equivalent to the CFG, as the CFG may produce the word *aabcc* from the start-symbol S, and the $\mathrm{TSP}^{;}(A)$-specification may not execute *aabcc* starting from S. Instead, a language-equivalent $\mathrm{TSP}^{;}(A)$-specification could, for example, consist of the defining equations $S = a.SA + b.\mathbf{1}$ and $A = c.\mathbf{1} + c.d.\mathbf{1} + \mathbf{1}$. Although, the translation from CFG to context-free process is less direct, it is still interesting to consider the question of decidability for processes defined by $\mathrm{TSP}^{;}(A)$, as this set of context-free processes does include the notion of intermediate termination. This leads to the following research question:

Is bisimilarity decidable for processes definable in $\mathrm{TSP}^{;}(A)$ *with finite guarded recursive specifications?*

Finally, looking into the decidability problem for $\mathrm{TSP}^{;}(A)$ may result in new insights that can be used to solve the decidability problem for context-free processes defined by $\mathrm{TSP}^{\cdot}(A)$. Hence, we look into the following research question:

Is bisimilarity decidable for processes definable in $\mathrm{TSP}^{\cdot}(A)$ *with finite guarded recursive specifications?*

1.3 Results and outline

In this thesis, several results are established with respect to $\mathrm{TSP}^{;}(A)$. First, it is shown that by adding an auxiliary operator to this theory, there exists an axiomatisation that is sound and ground-complete with respect to bisimilarity. This auxiliary operator is the unary operator NT that expresses the non-terminating part of a term, i.e. it removes intermediate termination. This operator is used to express the distribution of the sequential composition over the alternative composition in a finite number of axioms. Without this auxilary operator, it is not possible to specify a finite ground-complete axiomatisation for $\mathrm{TSP}^{;}(A)$. This result is proven, by showing that for any finite set of equations E, an equation exists that is sound with respect to bisimilarity but cannot be derived from E. While adding the NT-operator to $\mathrm{TSP}^{;}(A)$ facilitates a ground-complete axiomatisation, the resulting theory does not afford a finite ω-complete axiomatisation with respect to bisimilarity, i.e., complete for all processes definable in $\mathrm{TSP}^{;}(A)$ (including free variables). The proof for this result is similar to the proof that bisimilarity does not afford a finite axiomatisation for $\mathrm{TSP}^{;}(A)$ without NT.

The second result presented in this thesis states that bisimilarity is decidable for processes definable in $\mathrm{TSP}^{;}(A)$ with finite guarded recursive specifications. It is shown that every guarded recursive $\mathrm{TSP}^{;}(A)$-specification can be transformed to a special Greibach Normal Form. For specifications in this normal form it holds that for all sequences definable by this specification, there exists a bisimilar sequence definable by this specification that does not contain any reachable redundant intermediate termination. This property of sequences allows us to reuse the standard decidability proof for BPA processes [11, 9]. The main part of this proof consists of showing that for any pair of bisimilar processes, there exists a finite relation containing the pair that is a *base* for a bisimulation relation. This means that the congruence closure of this finite relations is a bisimulation.

Finally, it is argued that the properties used in the decidability proof of bisimilarity for $\mathrm{TSP}^{;}(A)$ do not apply to $\mathrm{TSP}^{\cdot}(A)$-terms, which complicates the proof of decidablity for $\mathrm{TSP}^{\cdot}(A)$. One of these properties is that the size of a $\mathrm{TSP}^{;}(A)$-term can be bounded by its norm, i.e. the minimum number of actions needed before it can terminate. This property does not hold for terms including transparency, and thus does not hold for all $\mathrm{TSP}^{\cdot}(A)$-terms. Moreover, it is shown that the transparency of $\mathrm{TSP}^{\cdot}(A)$-terms does not allow us to use the cancellation property, which is used in the proof for $\mathrm{TSP}^{;}(A)$.

The report is structured as follows. In Chapter 2, preliminaries are presented regarding the theory of sequential processes and bisimilarity. Chapter 3 illustrates the notion of transparency and other unwanted features caused by intermediate termination, and introduces the revised semantics for sequential composition as a solution to these unwanted features. Chapter 4 describes the results regarding the axiomatisation of $\mathrm{TSP}^{;}(A)$, and in Chapter 5 it is proven that bisimilarity is decidable for $\mathrm{TSP}^{;}(A)$. In Chapter 6, some insights are presented regarding the decidability of bisimilarity for $\mathrm{TSP}^{\cdot}(A)$. Finally, in Chapter 7 conclusions are drawn and some future work is proposed.

Chapter 2

Preliminaries

In this chapter preliminaries on the formal description of the theory of sequential processes including deadlock and intermediate termination are introduced.

2.1 Syntax and semantics

Let V be a countably infinite set of *free* variables. Let A be a finite set of actions containing typical elements a, b and c, and let the theory of sequential processes over A be denoted with $\text{TSP}^{\cdot}(A)$. The set of $\text{TSP}^{\cdot}(A)$-terms is generated by the following syntax:

$$p ::= \mathbf{0} \mid \mathbf{1} \mid a.p \mid p + p \mid p \cdot p \mid x \ \ .$$

In this syntax the constant symbol $\mathbf{0}$ denotes a deadlock, an unsuccessfully terminated process. The constant symbol $\mathbf{1}$ denotes a successfully terminated process. For each a in the set of actions A, the unary operator $a.$ denotes the action prefix, which means that the process $a.p$ can execute an action a and transform into process p. Furthermore, the binary operator $+$ denotes the alternative composition of two processes. The $\sum$-symbol is often used to abbreviate the alternative composition of processes. For $n \in \mathbb{N}$, the process $\sum_{i=1}^{n} p_i$ represents the alternative composition $p_1 + ... + p_n$. If $n = 0$, then $\sum_{i=1}^{n} p_i = \mathbf{0}$ by convention. The binary operator $\cdot$ denotes the sequential composition of processes. Lastly, x ranges over the set of free variables V.

A term that does not contain free variables is called *closed*. To emphasize that an arbitrary term may contain free variables, it is often called *open*. Typically, symbols p, q and r are used to range over closed terms, and symbols s, t and u are used to range over open terms. For terms s and t we use $s \equiv t$ to denote that s and t are syntactically equal.

The operational semantics of closed $\text{TSP}^{\cdot}(A)$-terms consist of transition relations $\xrightarrow{a}$ for $a \in A$ and a termination predicate $\downarrow$. The predicate $p \xrightarrow{a} p'$ is used to denote that process p can execute an a action and transform into process p', and $\downarrow$ is used to indicate that a process can successfully terminate. The complete set of operational rules for $\text{TSP}^{\cdot}(A)$-terms can be found in Figure 2.1.

If $w = a_1 a_2 \ldots a_n$ is a sequence of n actions, then $p \xrightarrow{w} p_n$ is used to denote that $p \xrightarrow{a_1} p_1 \xrightarrow{a_2} p_2 \ldots p_{n-1} \xrightarrow{a_n} p_n$ for some processes $p_1 \ldots p_{n-1}$. We say that p_n is *reachable* from p if

$$\frac{}{a.p \xrightarrow{a} p} \qquad \frac{p \xrightarrow{a} p'}{p + q \xrightarrow{a} p'} \qquad \frac{q \xrightarrow{a} q'}{p + q \xrightarrow{a} q'}$$

$$\frac{}{\mathbf{1}\downarrow} \qquad \frac{p\downarrow}{(p+q)\downarrow} \qquad \frac{q\downarrow}{(p+q)\downarrow}$$

$$\frac{p\downarrow \quad q\downarrow}{p \cdot q\downarrow} \qquad \frac{p \xrightarrow{a} p'}{p \cdot q \xrightarrow{a} p' \cdot q} \qquad \frac{p\downarrow \quad q \xrightarrow{a} q'}{p \cdot q \xrightarrow{a} q'}$$

Figure 2.1: Operational semantics for $\mathrm{TSP}^{\cdot}(A)$.

and only if there exists a sequence of actions $w \in A^*$ such that $p \xrightarrow{w} p_n$. Moreover, $p \longrightarrow^n p_n$ is used to indicate that p_n is reachable in n steps from p.

2.2 Equivalence

In order to express that two closed terms are behaviourally equivalent, the notion of bisimilarity is introduced.

Definition 2.2.1. A binary relation R is a *bisimulation relation* iff R is symmetric and for all closed terms p and q such that $(p, q) \in R$:

1. If $p \xrightarrow{a} p'$, then there exists a term q', such that $q \xrightarrow{a} q'$, and $(p', q') \in R$.

2. If $p\downarrow$, then $q\downarrow$.

The terms p and q are bisimilar (notation: $p \leftrightarrow q$) iff there exists a bisimulation relation R such that $(p, q) \in R$.

The definition of bisimilarity is extended to open terms as follows.

Definition 2.2.2. Two open terms s and t are bisimilar (notation: $s \leftrightarrow t$) iff for every closed substitution σ it holds that $\sigma(t) \leftrightarrow \sigma(t)$.

Bisimilarity is compatible with the constructs of $\mathrm{TSP}^{\cdot}(A)$. This means that for all $\mathrm{TSP}^{\cdot}(A)$-terms p_1, p_2, q_1 and q_2:

1. if $p_1 \leftrightarrow q_1$, then $a.p_1 \leftrightarrow a.q_1$;

2. if $p_1 \leftrightarrow q_1$ and $p_2 \leftrightarrow q_2$, then $p_1 + p_2 \leftrightarrow q_1 + q_2$;

3. if $p_1 \leftrightarrow q_1$ and $p_2 \leftrightarrow q_2$, then $p_1 \cdot p_2 \leftrightarrow q_1 \cdot q_2$.

We summarize that bisimilarity is an equivalence relation on the set of closed $\mathrm{TSP}^{\cdot}(A)$-terms that is, moreover, compatible with the constructs of $\mathrm{TSP}^{\cdot}(A)$ by saying that bisimilarity is a *congruence* for $\mathrm{TSP}^{\cdot}(A)$.

$x + y$	$= y + x$	A1
$x + (y + z)$	$= (x + y) + z$	A2
$x + x$	$= x$	A3
$(x + y) \cdot z$	$= (x \cdot z) + (y \cdot z)$	A4
$(x \cdot y) \cdot z$	$= x \cdot (y \cdot z)$	A5
$x + \mathbf{0}$	$= x$	A6
$\mathbf{0} \cdot x$	$= \mathbf{0}$	A7
$x \cdot \mathbf{1}$	$= x$	A8
$\mathbf{1} \cdot x$	$= x$	A9
$a.x \cdot y$	$= a.(x \cdot y)$	A10

Table 2.1: Axiomatisation for TSP$^{\cdot}(A)$

2.3 Axiomatisation

The theory TSP$^{\cdot}(A)$ contains the set of equations as shown in Table 2.1. These equations are often referred to as *axioms*.

If s and t are TSP$^{\cdot}(A)$-terms, then $s = t$ is *derivable* from TSP$^{\cdot}(A)$, denoted TSP$^{\cdot}(A) \vdash s = t$, if and only if it follows from the rules as shown in Figure 2.2. Here, s, t and u range over the set of open TSP$^{\cdot}(A)$-terms. The rules on the first row represent represent reflexivity, symmetry and transitivity, respectively. The rules on the second row are the congruence rules for the constructs of TSP$^{\cdot}(A)$. On the last row we have the rules for substitution and application of an axiom, where σ is a substitution and E is the set of axioms of TSP$^{\cdot}(A)$.

$$\frac{}{t = t} \qquad \frac{s = t}{t = s} \qquad \frac{t = u \quad u = s}{t = s}$$

$$\frac{t = s}{a.t = a.s} \qquad \frac{t_1 = s_1 \quad t_2 = s_2}{t_1 + t_2 = s_1 + s_2} \qquad \frac{t_1 = s_1 \quad t_2 = s_2}{t_1 \cdot t_2 = s_1 \cdot s_2}$$

$$\frac{t = s}{\sigma(t) = \sigma(s)} \qquad \frac{t = s \in E}{t = s}$$

Figure 2.2: Derivability rules for TSP$^{\cdot}(A)$.

The equational theory TSP$^{\cdot}(A)$ is sound and ground-complete with respect to bisimilarity.

Definition 2.3.1. A set of axioms E is *sound* for TSP$^{\cdot}(A)$ with respect to bisimilarity, if for all closed TSP$^{\cdot}(A)$-terms p and q, $E \vdash p = q$ implies $p \leftrightarrows q$.

Definition 2.3.2. A set of axioms E is *ground-complete* for TSP$^{\cdot}(A)$ with respect to bisimilarity, if for all closed TSP$^{;}(A)$-terms p and q, $p \leftrightarrows q$ implies $E \vdash p = q$.

2.4 Recursion

Recursion is added to the theory of sequential processes, in order to be able to express infinite behaviour.

Definition 2.4.1. A *recursive equation* over $\text{TSP}^{\cdot}(A)$ and a set of recursion variables V is an equation of the form $X = p$, where $X \in V$ and p is a $\text{TSP}^{\cdot}(A)$-term in which all variables originate from V. The recursive equation $X = p$ *defines* X. A *recursive specification* Δ over $\text{TSP}^{\cdot}(A)$ and V, is a set of recursive equations that contains exactly one recursive equation $X = p$ for each $X \in V$.

Let Δ be a recursive specification. Note that for any equation $X = p \in \Delta$, while p may contain variables, these variables must all be defined by an equation in Δ and thus p is considered to be a closed term. We often write $V(\Delta)$ to denote the set of all recursion variables in Δ. The symbols α, β, γ and δ are used to range over possibly empty sequences of recursion variables, i.e. $\alpha, \beta, \gamma, \delta \in V^*$. The process represented by a sequence of variables is recursively defined. The empty sequence, denoted with ε, represents the process $\mathbf{1}$, and the sequence $\alpha\beta$ represents the sequential process $\alpha \cdot \beta$. Furthermore, $|\alpha|$ is used to denote the length of a sequence α.

The operational semantics for processes definable by a recursive specification Δ over $\text{TSP}^{\cdot}(A)$, are equal to the operational semantics for closed $\text{TSP}^{\cdot}(A)$-terms, extended with the rules for recursion as shown in Figure 2.3.

$$\frac{p \xrightarrow{a} p' \quad (X = p) \in \Delta}{X \xrightarrow{a} p'} \qquad \frac{p \downarrow \quad (X = p) \in \Delta}{X \downarrow}$$

Figure 2.3: Operational semantics for recursion

Often, a restricted set of recursive specifications is considered. To this end, the notion of guardedness is introduced.

Definition 2.4.2. A recursive equation $X = p$ over $\text{TSP}^{\cdot}(A)$ is *completely guarded* iff every occurrence of a variable in p occurs in the scope of an action prefix.

Definition 2.4.3. A recursive equation $X = p$ over $\text{TSP}^{\cdot}(A)$ is *guarded* iff there exists a completely guarded term q such that $(\text{TSP}^{\cdot}(A) + \Delta) \vdash p = q$.

Note that $(\text{TSP}^{\cdot}(A) + \Delta) \vdash p = q$ denotes that $p = q$ is derivable from the theory $\text{TSP}^{\cdot}(A)$ extended with all recursive equations in Δ.

Example 2.4.4. Consider the following recursive specification:

$$X = a.X + Y \qquad Y = b.Y$$

The term $a.X + Y$ is not completely guarded as Y is not in the scope of an action prefix. However, using the defining equation of Y, we derive $a.X + Y = a.X + b.Y$ which is a completely guarded term. Hence, we may conclude that $a.X + Y$ is guarded.

Definition 2.4.5. A recursive specification consisting of only guarded equations is called a *guarded recursive specification.*

Every process definable by a finite guarded recursive specification is image-finite.

Definition 2.4.6. A process t is *image-finite* iff for every process t' reachable from t there are only a finite number of actions that t' can execute.

2.5 Greibach Normal Form

A well known normal form for recursive process specifications is Greibach Normal Form (*GNF*). A recursive process specification is said to be in GNF if all recursive equations are of the form:

$$X = \sum_{i=1}^{n} a_i.\alpha_i \text{ for some } n \in \mathbb{N}^+$$

where $\mathbb{N}^+$ denotes the set of positive natural numbers, a_i denotes an action and α_i is a possibly empty sequence of variables. It has been shown [3], that every context-free process without deadlock and intermediate termination can be transformed to GNF. In order to support deadlock behaviour and intermediate termination, GNF is extended as follows.

Definition 2.5.1. A recursive process specification over $\text{TSP}^{\cdot}(A)$ is said to be in GNF if all recursive equations are of the form:

- $X = \mathbf{0}$; or
- $X = \mathbf{1}$; or
- $X = \sum_{i=1}^{n} a_i.\alpha_i(+\mathbf{1})$, for some $n \in \mathbb{N}^+$

where $\mathbb{N}^+$ denotes the set of positive natural numbers, a_i denotes an action and α_i is a possibly empty sequence of variables, and $(+\mathbf{1})$ denotes an optional $\mathbf{1}$-summand.

An interesting result arising from this normal form, is that, starting from a sequence of variables α, after executing an action the process always ends up in a new sequence of variables α'. Therefore, all processes defined by recursive specifications in GNF can be represented by a sequence of variables.

Proposition 2.5.2. *For every guarded recursive specification Δ over $\text{TSP}^{\cdot}(A)$, there exists a recursive specification Δ' in GNF such that for each equation $X = p \in \Delta$, there exists an equation $X = q \in \Delta'$ such that $p \leftrightarrow q$.*

The proof for this proposition can be found in Appendix D.

2.6 Norm

Definition 2.6.1. The *norm* of a $\text{TSP}^{\cdot}(A)$-term is the minimum number of actions needed to reach a state that can terminate. The norm of a closed $\text{TSP}^{\cdot}(A)$-term p is defined as:

$$n^{\cdot}(p) = \min(\{\mathit{length}(w) \mid p \xrightarrow{w} p' \text{ and } p' \downarrow\} \cup \{\infty\}) \ .$$

Note that bisimilar processes have the same norm. A process is called *normed* if it has a finite norm, i.e., its norm is not ∞, and *unnormed* otherwise. The norm of a process p without recursion variables can be computed by utilizing the following recursive definition:

- if $p \equiv \mathbf{0}$, then $n^{\cdot}(p) = \infty$;
- if $p \equiv \mathbf{1}$, then $n^{\cdot}(p) = 0$;

- if $p \equiv a.p'$, then $n^{\cdot}(p) = 1 + n^{\cdot}(p')$;
- if $p \equiv p_1 + p_2$, then $n^{\cdot}(p) = \min(n^{\cdot}(p_1), n^{\cdot}(p_2))$;
- if $p \equiv p_1 \cdot p_2$, then $n^{\cdot}(p) = n^{\cdot}(p_1) + n^{\cdot}(p_2)$.

By this straightforward recursive definition it immediately follows that $n^{\cdot}(p)$ is computable for all closed TSP$^{\cdot}(A)$-terms p without variables. In order to compute the norm of processes including recursion variables, first the norm of all recursion variables needs to be computed. As shown in [15], given a recursive specification in GNF, the set of normed variables V_n can be computed by utilizing the following recursive definition:

$$N_0 = \{X \in V \mid X \text{ can terminate}\}$$
$$N_{i+1} = N_i \cup \{X \in V \mid \exists_{\alpha \in N_i{}^*} X \xrightarrow{a} \alpha\} \ .$$

The set of normed variables V_n is defined as $V_n = N_i$ for the smallest i such that $N_i = N_{i+1}$. Given the set of normed variables V_n and the set of unnormed variables V_u, the norm of a variable X can be computed by utilizing the following definition:

- if $X \in V_u(\Delta)$, then $n^{\cdot}(X) = \infty$;
- if $X \in V_n(\Delta)$ and $X = p \in \Delta$, then $n^{\cdot}(X) = n^{\cdot}(p)$.

Hence, $n^{\cdot}(p)$ is computable for closed TSP$^{\cdot}(A)$-terms p. Finally, note that the norm of a sequence of variables α is defined as the norm of the sequential process represented by the sequence, e.g., $n^{\cdot}(XYZ) = n^{\cdot}(X) + n^{\cdot}(Y) + n^{\cdot}(Z)$.

Chapter 3

Revised semantics sequential composition

There are several open questions regarding context-free processes with intermediate termination. In [5] it is argued that the correspondence between context-free processes and pushdown processes fails modulo bisimulation, due to the transparency of process terms that is caused by the presence of intermediate termination in a sequential context. Also, the decidability problem of bisimilarity for context-free processes with intermediate termination remains unsolved. In [17] it is shown that transparency again poses a complicating factor. Baeten et al. [6] propose a solution to this problem of transparency, by introducing a revised operational semantics for the sequential composition operator.

In this section, the complication of transparency caused by intermediate termination in a sequential context is illustrated. Then, the process theory TSP$^{;}(A)$ as proposed in [6] is introduced; the theory of sequential processes, including intermediate termination and the revised operational semantics for sequential composition. Finally, a new definition of norm for TSP$^{;}(A)$-terms is introduced.

3.1 Transparency

Before introducing the theory of sequential processes using the revised operational semantics, the issue of transparency caused by intermediate termination in combination with the standard semantics for sequential composition will be briefly discussed.

Definition 3.1.1. A process term t contains *intermediate termination* if t can terminate and there exists an action a and term t' such that $t \xrightarrow{a} t'$.

This intermediate termination in a sequential context, results in transparency of TSP$^{\cdot}(A)$-terms.

Definition 3.1.2. For any TSP$^{\cdot}(A)$-term $t_1 \cdot t_2$, the term t_1 is *transparent* if t_1 can terminate and there exists an action a and term t_1' such that $t_1 \xrightarrow{a} t_1'$.

This notion of transparency can result in unbounded branching of transition systems corresponding to a TSP$^{\cdot}(A)$-processes defined by a recursive specification. This is illustrated in the following example, taken from [6].

Example 3.1.3. Consider the following recursive $\mathrm{TSP}^{\cdot}(A)$-specification:

$$X = a.XY + b.\mathbf{1} \qquad Y = c.\mathbf{1} + \mathbf{1}$$

The transition system that is associated with this recursive specification is shown in Figure 3.1. Every state labelled with Y^i has i outgoing transitions. As the transition system is infinite and i is not bounded, we say that the transition system contains unbounded branching.

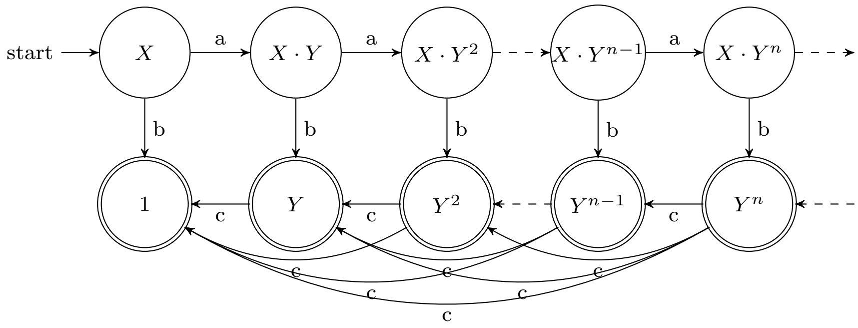

Figure 3.1: Unbounded branching

Since it is not possible to express unbounded branching behaviour by a pushdown process modulo bisimilarity, the correspondence between $\mathrm{TSP}^{\cdot}(A)$-processes defined by recursive specifications and pushdown processes with respect to bisimilarity immediately fails.

The main issue regarding decidability of bisimilarity caused by transparency, is the fact that the length of normed sequences cannot be bounded. More details on this issue can be found in Chapter 6.

3.2 Syntax and semantics $\mathrm{TSP}^{;}(A)$

Let A be a finite set of actions. Let the theory of sequential processes over A using the revised semantics for sequential composition be denoted with $\mathrm{TSP}^{;}(A)$. The set of $\mathrm{TSP}^{;}(A)$-terms is generated by the following syntax:

$$p ::= \mathbf{0} \mid \mathbf{1} \mid a.p \mid p + p \mid p\,;p \mid x$$

The only difference between this syntax and the syntax of $\mathrm{TSP}^{\cdot}(A)$, is the sequential composition operator that is replaced by ";". The operational semantics of $\mathrm{TSP}^{;}(A)$-terms are equal to those of $\mathrm{TSP}^{\cdot}(A)$-terms, except for the rules for sequential composition. The operational rules for sequential composition for $\mathrm{TSP}^{;}(A)$-terms can be found in Figure 3.2. Here, $p \not\rightarrow$ is used to indicate that there do not exist an action a and a term p' such that $p \xrightarrow{a} p'$.

In the operational semantics, a negative premisses is used. In order to verify that the semantics are well-defined, in the sense that with every guarded recursive specification a (unique) transition system is associated, we need to establish a *stratification* for the transitions [7]. Note that guarded recursive specifications over $\mathrm{TSP}^{;}(A)$ are defined similar to guarded recursive specifications over $\mathrm{TSP}^{\cdot}(A)$ (see Section 2.4).

$$\frac{p\downarrow \quad q\downarrow}{p\,;q\downarrow} \qquad \frac{p \xrightarrow{a} p'}{p\,;q \xrightarrow{a} p'\,;q} \qquad \frac{p\downarrow \quad q \xrightarrow{a} q' \quad p \not\rightarrow}{p\,;q \xrightarrow{a} q'}$$

Figure 3.2: Operational semantics of sequential composition for TSP$^{;}(A)$.

Definition 3.2.1. Let $\mathcal{P}$ be the set of process terms. A function $S : \mathcal{P} \to \lambda$, where λ is an ordinal, is called a *stratification* of $\mathcal{P}$, if for every rule r in the operational semantics and every substitution σ, it holds that:

$$\text{for all } p_1 \in pprem(\sigma(r)) : S(p_1) \leq s(conc(\sigma(r))); \text{ and}$$
$$\text{for all } p_1 \in nprem(\sigma(r)) : S(p_1) < s(conc(\sigma(r))).$$

Here *pprem* is used to denote the set of terms that appear on the left-hand side in a positive premises of a rule, *nprem* to denote the set of terms that appear on the left-hand side in a negative premises of a rule, and *conc* to denote the set of terms that appear on the left-hand side in a conclusion of a rule.

In [18], Yang has established such a stratification for guarded recursive specifications over TSP$^{;}(A)$. This stratification S is defined as follows:

$$\begin{aligned} S(\mathbf{0}) &= 0 \\ S(\mathbf{1}) &= 0 \\ S(a.p) &= 0 \\ S(p_1 + p_2) &= S(p_1) + S(p_2) + 1 \\ S(p_1\,; p_2) &= S(p_1) + S(p_2) + 1 \\ S(X) &= S(p), X = p \in \Delta \ . \end{aligned}$$

It is straightforward from Definition 3.2.1, that S is a valid stratification for guarded TSP$^{;}(A)$-specifications. Note that this stratification is well-defined for completely guarded recursive specifications, as for each $X = p \in \Delta$, p may only contain variables in the scope of an action prefix, which ensures $S(p)$ is defined. This is not the case for unguarded recursive specifications, since then, for example, given the equation $X = XY + a.\mathbf{1}$, the stratificiation $S(X)$ is not defined.

The derivability rules for TSP$^{;}(A)$ are equal to the rules for TSP$^{\cdot}(A)$. We show that the congruence result for sequential composition is still valid using the new semantics.

Lemma 3.2.2. *For all closed* TSP$^{;}(A)$*-terms* p_1, p_2, q_1 *and* q_2, *if* $p_1 \leftrightarrow q_1$ *and* $p_2 \leftrightarrow q_2$, *then* $p_1\,; p_2 \leftrightarrow q_1\,; q_2$.

Proof. Let p_1, p_2, q_1 and q_2 be closed TSP$^{;}(A)$-terms, and suppose that $p_1 \leftrightarrow q_1$ and $p_2 \leftrightarrow q_2$. Then, there exist bisimulation relations R_1 and R_2 such that $p_1 R_1 q_1$ and $p_2 R_2 q_2$. Using these bisimulation relations we define the following relation:

$$R = R_1 \cup R_2 \cup \{((p_1\,; p_2), (q_1\,; q_2)) \mid (p_1, q_1) \in R_1, (p_2, q_2) \in R_2\}$$

To this end, we verify that the symmetric closure of R is a bisimulation relation, by verifying that the properties of bisimilarity hold for all pairs $((p_1\,; p_2), (q_1\,; q_2))$, with $(p_1, q_1) \in R_1$ and

$(p_2, q_2) \in R_2$. The proof for all other pairs is symmetrical.
Suppose $p_1 \,;\, p_2 \xrightarrow{a} p'$. We distinguish two cases:

- Suppose $p_1 \xrightarrow{a} p_1'$ and $p' \equiv p_1' \,; p_2$. Then, since $p_1 R_1 q_1$ and R_1 is a bisimulation relation, $q_1 \xrightarrow{a} q_1'$ such that $p_1' R_1 q_1'$. Hence, $q_1 \,;\, q_2 \xrightarrow{a} q_1' \,;\, q_2$ and clearly $(p_1' \,;\, p_2) R (q_1' \,;\, q_2)$.

- Suppose $p_1 \downarrow$, $p_1 \not\rightarrow$, $p_2 \xrightarrow{a} p_2'$ and $p \equiv p_2'$. Then, since $p_1 R_1 q_1$ and R_1 is a bisimulation relation, we must have $q_1 \downarrow$. Moreover, since R_1 is symmetric, $q_1 R_1 p_1$, and thus $q_1 \not\rightarrow$ or else it cannot be the case that $p_1 \not\rightarrow$. Finally, since $p_2 R_2 q_2$ we have $q_2 \xrightarrow{a} q_2'$ such that $p_2' R_2 q_2'$ and thus $q_1 \,;\, q_2 \xrightarrow{q}{}'_2$ and $p_2' R q_2'$.

Suppose $p_1 \,;\, p_2 \downarrow$. Then $p_1 \downarrow$ and $p_2 \downarrow$ and thus since $p_1 R_1 q_1$ and $p_2 R_2 q_2$, we have $q_1 \downarrow$ and $q_2 \downarrow$ and hence also $q_1 \,;\, q_2 \downarrow$. □

3.3 Norm TSP$^{;}(A)$

The definition of norm as defined for TSP$^{\cdot}(A)$-terms in Section 2.6, is not suitable for TSP$^{;}(A)$-terms. This is illustrated by the following example.

Example 3.3.1. Consider the following recursive TSP$^{;}(A)$-specification:

$$X = a.X + \mathbf{1} \qquad Y = b.\mathbf{1} \ .$$

We have $n^{\cdot}(X) = 0$ and $n^{\cdot}(Y) = 1$, hence $n^{\cdot}(XY) = 1$. This suggests that an action exists that the process XY can execute after which it ends up in a state where it can terminate. However, according to the operational semantics of TSP$^{;}(A)$, the only action XY can execute is $XY \xrightarrow{a} XY$. As XY cannot terminate, clearly XY can never reach a state where it can terminate and thus XY should have an infinite norm.

To solve this issue, the definition of norm is redefined for TSP$^{;}(A)$-terms as follows.

Definition 3.3.2. The *norm* of a TSP$^{;}(A)$-term is the minimum number of actions needed to reach a state that has to terminate. The norm of a closed TSP$^{;}(A)$-term p is defined as:

$$n^{;}(p) = \min(\{\mathit{length}(w) \mid p \xrightarrow{w} p' \text{ and } p' \leftrightarrow \mathbf{1}\} \cup \{\infty\})$$

This new definition has an impact on the recursive definition that is used to compute the norm. The definition of norm for the alternative composition of two terms $p \equiv p_1 + p_2$, is adjusted as follows:

- if $n^{;}(p_1) = 0$, then $n^{;}(p) = n^{;}(p_2)$;
- if $n^{;}(p_2) = 0$, then $n^{;}(p) = n^{;}(p_1)$;
- else $n^{;}(p) = \min(n^{;}(p_1), n^{;}(p_2))$.

Note that in this definition we assume a term to be normalized and thus $p_1 \not\leftrightarrow \mathbf{0}$ and $p_2 \not\leftrightarrow \mathbf{0}$. Using this new definition of norm, the variable X from Example 3.3.1 has an infinite norm, as X is not able to reach a state where it has to terminate. As a result, XY has an infinite norm as well. The only downside to this definition is that now X has an infinite norm, while it is actually able to terminate. However, this definition is sufficient for our purpose, as will become clear in later sections.

Chapter 4

Axiomatisation $\mathrm{TSP}^{;}(A)$

It is well known that $\mathrm{TSP}^{\cdot}(A)$ affords a finite ground-complete axiomatisation modulo strong bisimilarity (Table 2.1). However, no results regarding the axiomatisation of $\mathrm{TSP}^{;}(A)$ have been established yet. In this chapter, a ground-complete axiomatisation is proposed for $\mathrm{TSP}^{;}_{+NT}(A)$, which is the theory $\mathrm{TSP}^{;}(A)$ expanded with an auxiliary operator. Moreover, it is shown that without an auxiliary operator, bisimilarity does not afford a finite and ground-complete axiomatisation for $\mathrm{TSP}^{;}(A)$. Finally, it is argued that the ground-complete axiomatisation for $\mathrm{TSP}^{;}_{+NT}(A)$ is not ω-complete and, moreover, it is not possible to define a finite ω-complete axiomatisation for $\mathrm{TSP}^{;}_{+NT}(A)$.

4.1 Ground-complete axiomatisation $\mathrm{TSP}^{;}_{+NT}(A)$

The axiomatisation for $\mathrm{TSP}^{\cdot}(A)$ as seen in Table 2.1 cannot easily be adopted to fit the syntax and semantics of $\mathrm{TSP}^{;}(A)$. This is caused by the fact that the revised semantics for the sequential composition do not allow regular distribution over $+$ as seen in A4. For example, consider the following equation:

$$(a.\mathbf{1} + \mathbf{1}) \,;\, b.\mathbf{1} = a.\mathbf{1} \,;\, b.\mathbf{1} + \mathbf{1} \,;\, b.\mathbf{1}$$

If A4 would be sound for $\mathrm{TSP}^{;}(A)$ with respect to bisimilarity, the left and right hand side of this equation would be bisimilar. However, clearly this is not the case as $a.\mathbf{1}\,;b.\mathbf{1}+\mathbf{1}\,;b.\mathbf{1} \xrightarrow{b} \mathbf{1}$, but $(a.\mathbf{1}+\mathbf{1})\,;b.\mathbf{1}\stackrel{b}{\nrightarrow}$. Hence, some other axioms are needed in order to be able to distribute the sequential composition over a choice. It turns out, it is not possible to express the distribution over $+$ for all $\mathrm{TSP}^{;}(A)$-terms in a finite number of axioms without using an auxiliary operator.

4.1.1 NT-operator

The operator $NT(p)$ is used to express the non-terminating part of a closed $\mathrm{TSP}^{;}(A)$-term p. The non-terminating operator $NT(p)$ removes all intermediate terminating behaviour from a term p. For example, $NT(a.p) = a.p$ and $NT(a.p + \mathbf{1}) = a.p$. The operational rule for this operator is defined in Figure 4.1. The set of closed $\mathrm{TSP}^{;}_{+NT}(A)$-terms is generated by the following syntax:

$$p ::= \mathbf{0} \mid \mathbf{1} \mid a.p \mid p + p \mid p \,;\, p \mid NT(p)$$

The operational semantics for $\mathrm{TSP}^{;}_{+NT}(A)$ are defined as the operational semantics for $\mathrm{TSP}^{;}(A)$ extended with the operational rule for the NT-operator as shown in Figure 4.1.

Note that for any term p, the term $NT(p)$ can never terminate as there is no rule for termination.

$$\frac{p \xrightarrow{a} p'}{NT(p) \xrightarrow{a} p'}$$

Figure 4.1: The operational rule for non-terminating

Finally, note that the congruence property for the NT-operator is valid.

Lemma 4.1.1. *For all closed* $\mathrm{TSP}^{;}(A)$*-terms* p *and* q*, if* $p \underline{\leftrightarrow} q$*, then* $NT(p) \underline{\leftrightarrow} NT(q)$*.*

Proof. Let p and q be closed $\mathrm{TSP}^{;}(A)$-terms, and suppose that $p \underline{\leftrightarrow} q$. Then, there exists a bisimulation relation R_1 such that pR_1q and. Using this bisimulation relation we define the following relation:

$$R = R_1 \cup \{(NT(p), NT(q)) \mid (p, q) \in R_1\}$$

To this end, we verify that R is a bisimulation relation, by verifying that the properties of bisimilarity hold for all pairs $(NT(p), NT(q))$, with $(p, q) \in R_1$. The proof for all other pairs is symmetrical.

Suppose $NT(p) \xrightarrow{a} p'$. Then, it must be the case that $p \xrightarrow{a} p'$. Since pR_1q, we have $q \xrightarrow{a} q'$ such that $p'R_1q'$. Hence, $NT(q) \xrightarrow{a} q'$ and $p'Rq'$. Furthermore, note that $NT(p) \not\downarrow$. □

4.1.2 Axiomatisation $\mathrm{TSP}^{;}_{+NT}(A)$

Using the NT-operator, it is possible to define a finite, sound and ground-complete axiomatisation. The axiomatisation for $\mathrm{TSP}^{;}_{+NT}(A)$ can be found in Table 4.1. All axioms are sound with respect to bisimilarity. The proofs for this result can be found in Appendix A.

In order to prove ground-completeness of the axiomiatisation, we utilize the well-known ground-completeness result of the theory $\mathrm{BSP}(A)$. The signature of $\mathrm{BSP}(A)$ consists of the constants $\mathbf{1}$ and $\mathbf{0}$, the unary prefix operator $a.$ for each $a \in A$, and the alternative composition operator $+$. The operational semantics of these operators confirm to semantics of these operators as defined for $\mathrm{TSP}^{;}_{+NT}(A)$. The axiomatisation of $\mathrm{BSP}(A)$ consists of the axioms $A1^{;}, A2^{;}, A3^{;}$ and $A6^{;}$ as shown in Table 4.1. In [2], it is shown that $\mathrm{BSP}(A)$ is sound and ground-complete with respect to bisimilarity. Now consider the following theorem.

Theorem 4.1.2. *For every closed* $\mathrm{TSP}^{;}_{+NT}(A)$*-term* p*, there exists a closed BSP(A)-term* q *such that* $\mathrm{TSP}^{;}_{+NT}(A) \vdash p = q$*.*

The complete proof for this theorem can be found in Appendix B. Using this theorem, it is possible to prove that $\mathrm{TSP}^{;}_{+NT}(A)$is ground-complete for all closed $\mathrm{TSP}^{;}_{+NT}(A)$-terms with respect to bisimilarity.

$x + y$	$= y + x$	$A1^{;}$
$x + (y + z)$	$= (x + y) + z$	$A2^{;}$
$x + x$	$= x$	$A3^{;}$
$(x ; y) ; z$	$= x ; (y ; z)$	$A5^{;}$
$x + \mathbf{0}$	$= x$	$A6^{;}$
$\mathbf{0} ; x$	$= \mathbf{0}$	$A7^{;}$
$x ; \mathbf{1}$	$= x$	$A8^{;}$
$\mathbf{1} ; x$	$= x$	$A9^{;}$
$a.x ; y$	$= a.(x ; y)$	$A10^{;}$
$NT(x + y) ; z$	$= NT(x) ; z + NT(y) ; z$	$A11^{;}$
$(a.x + y + \mathbf{1}) ; NT(z)$	$= (a.x + y) ; NT(z)$	$A12^{;}$
$(a.x + y + \mathbf{1}) ; (z + \mathbf{1})$	$= (a.x + y) ; (z + \mathbf{1}) + \mathbf{1}$	$A13^{;}$
$NT(\mathbf{0})$	$= \mathbf{0}$	$NT1$
$NT(\mathbf{1})$	$= \mathbf{0}$	$NT2$
$NT(a.x)$	$= a.x$	$NT3$
$NT(x + y)$	$= NT(x) + NT(y)$	$NT4$

Table 4.1: Ground-complete axiomatisation for TSP$^{;}_{+NT}(A)$

Theorem 4.1.3. TSP$^{;}_{+NT}(A)$ *is ground-complete for all closed* TSP$^{;}_{+NT}(A)$*-terms with respect to bisimilarity.*

Proof. In order to prove ground-completeness of TSP$^{;}_{+NT}(A)$ with respect to bisimilarity, it suffices to show that for all closed TSP$^{;}_{+NT}(A)$-terms p and q, $p \leftrightarrow q$ implies TSP$^{;}_{+NT}(A) \vdash p = q$. Let p and q be closed $TSP^{;}_{+NT}(A)$-terms such that $p \leftrightarrow q$. By Theorem 4.1.2, there exist BSP(A)-terms p' and q', such that TSP$^{;}_{+NT}(A) \vdash p = p'$ and TSP$^{;}_{+NT}(A) \vdash q = q'$. Since TSP$^{;}_{+NT}(A)$ is sound with respect to bisimilarity, we derive $p' \leftrightarrow p \leftrightarrow q \leftrightarrow q'$, hence $p' \leftrightarrow q'$. Since the axiomatization for BSP(A) is ground-complete with respect to strong bisimilarity and $p' \leftrightarrow q'$, we derive BSP$(A) \vdash p' = q'$. By Theorem 3.2.17 and Theorem 3.2.19 from [2], we conclude that TSP$^{;}_{+NT}(A)$ is an operational conservative extension of BSP(A). Hence, TSP$^{;}_{+NT}(A) \vdash p' = q'$ and thus TSP$^{;}_{+NT}(A) \vdash p = p' = q' = q$. □

4.2 Bisimilarity is not finitely based over TSP$^{;}(A)$

In the previous sections it is shown that there exists a finite and ground-complete axiomatisation for TSP$^{;}_{+NT}(A)$ with respect to strong bisimilarity. The auxiliary operator NT is used to express the inability of a term to terminate. In this section it is shown that without this operator, given a finite set of axioms E, there always exists an equation that is sound with respect to bisimilarity but cannot be derived from E. First a sketch of the reasoning behind the proof is given.

Let $\tilde{a}, \tilde{b} \in A$ be two fixed actions, and consider the following equation:

$$(\tilde{a}.\mathbf{1} + \mathbf{1}) ; \tilde{b}.\mathbf{1} = \tilde{a}.\mathbf{1} ; \tilde{b}.\mathbf{1}$$

It is easy to verify that this equation is sound with respect to bisimilarity. When deriving this equation we use the fact that the right hand side of the sequential composition is not able to terminate and the left hand side is able to do a step, to remove the termination from the left hand side of the sequential composition. Hence, if we want to apply an axiom to derive this equation, it should contain the same properties. Intuitively, in order to be sure that for all closed substitutions the right hand side cannot terminate, it may not contain any variable summands. Now consider the following equation, with $n \in \mathbb{N}$.

$$(\tilde{a}.\mathbf{1} + \mathbf{1}) \,;\, \sum_{i=1}^{n} \tilde{b}.(\tilde{b}.\mathbf{1} + \mathbf{1})^i = \tilde{a}.\mathbf{1} \,;\, \sum_{i=1}^{n} \tilde{b}.(\tilde{b}.\mathbf{1} + \mathbf{1})^i$$

Again, it can easily be verified that this equation is sound, by using the same properties as before. However, since n is not bounded, the *width* of the right hand side of the sequential composition is not bounded either.

Definition 4.2.1. The *width* of a closed TSP$^{;}(A)$-term p, written as $width(p)$, is the cardinality of the set $\{p' \mid p \xrightarrow{a} p', \text{ for some } a \in A\}$.

Note that this definition can be extended to open TSP$^{;}(A)$-terms by defining $width(x) = 0$ for any variable x.

Now suppose that we have a finite and sound set of axioms E, such that for each axiom $t = u \in E$ it holds that $width(t') < n$ for each subterm t' of t, and $width(u') < n$ for each subterm u' of u. Intuitively, in order to derive $E \vdash (\tilde{a}.\mathbf{1}+\mathbf{1});\sum_{i=1}^{n} \tilde{b}.(\tilde{b}.\mathbf{1} + \mathbf{1})^i = \tilde{a}.\mathbf{1};\sum_{i=1}^{n} \tilde{b}.(\tilde{b}.\mathbf{1} + \mathbf{1})^i$, we need to apply an axiom of the following shape:

$$(t_1 + \mathbf{1}) \,;\, t_2 = t_1 \,;\, t_2$$

for which it must hold that for every closed substitution σ, we have that $\sigma(t_1)$ can do a step and $\sigma(t_2)$ cannot terminate. Suppose we want to apply such an axiom, then there must exist a substitution σ such that $\sigma(t_1) = \tilde{a}.\mathbf{1}$ and $\sigma(t_2) = \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} \,;\, (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$. Since $width(t_2) < n$ and $width(\sigma(t_2)) = n$, at least one of the actions that $\sigma(t_2)$ can execute must come from a substitution in some variable x, i.e. t_2 must contain some variable x such that $\sigma(x)\xrightarrow{\tilde{b}}$. Because of this variable x, we can reason that there exists a substitution σ' such that $\sigma'(t_2)$ can terminate and $\sigma'(t_1)$ cannot terminate. For this substitution we have $\sigma'(t_1 + \mathbf{1} \,;\, t_2) \downarrow$ and $\sigma'(t_1 \,;\, t_2) \not\downarrow$. Hence, the equation $(t_1 + \mathbf{1}) \,;\, t_2 = t_1 \,;\, t_2$ cannot be sound and thus E cannot contain such an axiom and $(\tilde{a}.\mathbf{1} + \mathbf{1}) \,;\, \sum_{i=1}^{n} \tilde{b}.(\tilde{b}.\mathbf{1} + \mathbf{1})^i = \tilde{a}.\mathbf{1} \,;\, \sum_{i=1}^{n} \tilde{b}.(\tilde{b}.\mathbf{1} + \mathbf{1})^i$ cannot be derived from E.

In order to formally prove this result, we show that, given a set of axioms E for which the width of all subterms of all equations is smaller than a natural number n, every term p that can be derived to be equal to $(\tilde{a}.\mathbf{1} + \mathbf{1}) \,;\, \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} \,;\, (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$, must be of a certain shape. This shape is defined by the following two properties.

Definition 4.2.2. Given a closed TSP$^{;}(A)$-term p, $\Phi_n(p)$ holds iff $p \equiv p_1 \,;\, p_2$ such that $p_1 \leftrightarroweq \tilde{a}.\mathbf{1} + \mathbf{1}$ and $p_2 \leftrightarroweq \sum_{i=1}^{n} \tilde{b}.(\tilde{b}.\mathbf{1} + \mathbf{1})^i$.

Definition 4.2.3. Given a closed TSP$^{;}(A)$-term p, $\Psi_n(p$ holds iff $p \leftrightarroweq \tilde{a}.\mathbf{1} \,;\, \sum_{i=1}^{n} \tilde{b}.(\tilde{b}.\mathbf{1} + \mathbf{1})^i$ and p contains a summand $p_1 \,;\, p_2$ such that one of the following cases holds:

1. $\Phi_n(p_1 \,;\, p_2)$.

2. $p_1 \leftrightarrow \mathbf{1}$ and $\Psi_n(p_2)$.
3. $\Psi_n(p_1)$ and $p_2 \leftrightarrow \mathbf{1}$.

It is not difficult to see that $\Psi_n((\tilde{a}.\mathbf{1}+\mathbf{1});\sum_{i=1}^{n}\tilde{b}.(\tilde{b}.\mathbf{1}+\mathbf{1})^i)$ holds, but $\Psi_n(\tilde{a}.\mathbf{1};\sum_{i=1}^{n}\tilde{b}.(\tilde{b}.\mathbf{1}+\mathbf{1}))$ does not. Hence, if we can show that for some n and any two closed TSP$^{;}(A)$-terms p and q, it holds that $\Psi_n(p)$ and $E \vdash p = q$ implies $\Psi_n(q)$, then clearly the equation $(\tilde{a}.\mathbf{1}+\mathbf{1})\,;\sum_{i=1}^{n}\tilde{b}.(\tilde{b}.\mathbf{1}+\mathbf{1})^i = \tilde{a}.\mathbf{1}\,;\sum_{i=1}^{n}\tilde{b}.(\tilde{b}.\mathbf{1}+\mathbf{1})^i$ cannot be derived from E. This is captured by the following theorem.

Theorem 4.2.4. *Let E be a finite set of equations, sound with respect to $\leftrightarrow$, and let n be a natural number such that for each axiom $t = u \in E$, for each subterm t' of t and each subterm u' of u, $width(t') < n$ and $width(u') < n$. Furthermore, let p and q be closed* TSP$^{;}(A)$*-terms such that $p \leftrightarrow q$ and suppose that $E \vdash p = q$. It then holds that if $\Psi_n(p)$, then $\Psi_n(q)$.*

The complete proof for this theorem can be found in Appendix C. Inspiration for the outline of this proof was drawn from [1]. Using this theorem it is now possible to prove that there does not exist a finite and ground-complete axiomatisation for TSP$^{;}(A)$ with respect to bisimilarity.

Theorem 4.2.5. *There does not exist a finite ground-complete axiomatisation for* TSP$^{;}(A)$ *with respect to bisimilarity.*

Proof. We give a proof by contradiction. Suppose E is a finite and ground-complete set of equations, sound with respect to bisimilarity. Let n be a natural number such that for each axiom $t = u \in E$, for each subterm t' of t and each subterm u' of u, $width(t') < n$ and $width(u') < n$. Note that since E is finite, such an n always exists. Now, consider the sound equation:

$$(\tilde{a}.\mathbf{1}+\mathbf{1})\,;\sum_{i=1}^{n}(\tilde{b}.\mathbf{1}\,;(\tilde{b}.\mathbf{1}+\mathbf{1})^i) = \tilde{a}.\mathbf{1}\,;\sum_{i=1}^{n}(\tilde{b}.\mathbf{1}\,;(\tilde{b}.\mathbf{1}+\mathbf{1})^i)$$

Since E is ground-complete, we should be able to derive this equation from E. By Theorem 4.2.4, it must be the case that if $\Psi_n((\tilde{a}.\mathbf{1}+\mathbf{1});\sum_{i=1}^{n}(\tilde{b}.\mathbf{1};(\tilde{b}.\mathbf{1}+\mathbf{1})^i))$ holds, $\Psi_n(\tilde{a}.\mathbf{1};\sum_{i=1}^{n}(\tilde{b}.\mathbf{1};(\tilde{b}.\mathbf{1}+\mathbf{1})^i))$ holds as well. However, this is clearly not the case, hence the equation could not have been derived from E, which contradicts E being ground-complete. We conclude by contradiction that there does not exist a finite ground-complete axiomatisation for TSP$^{;}(A)$ with respect to bisimilarity. □

4.3 ω-complete axiomatisation TSP$^{;}_{+NT}(A)$

In Section 4.1.2 it is shown that the axiomatisation as shown in Table 4.1 is ground-complete for all closed TSP$^{;}_{+NT}(A)$-terms with respect to bisimilarity. The definition of completeness is extended to open terms as follows.

Definition 4.3.1. A set of axioms E is *ω-complete* for TSP$^{;}_{+NT}(A)$ with respect to bisimilarity if, for all TSP$^{;}_{+NT}(A)$-terms s and t, $s \leftrightarrow t$ implies $E \vdash s = t$.

The theory TSP$^{;}_{+NT}(A)$is not ω-complete with respect to bisimilarity. For example, the sound equation $NT(NT(x)) = NT(x)$ cannot be derived from TSP$^{;}_{+NT}(A)$. In this section, it is argued that it is not possible to define a finite ω-complete axiomatisation for TSP$^{;}_{+NT}(A)$.

Additionally, Appendix E contains an argument that shows that a well known technique to prove ω-completeness cannot be applied for $\mathrm{TSP}^{;}_{+NT}(A)$. This result was established before finding that an ω-complete axiomatisation cannot exist.

The argument that an ω-complete axiomatisation does not exist for $\mathrm{TSP}^{;}_{+NT}(A)$ is very similar to the argument given in the previous section for ground-completeness of $\mathrm{TSP}^{;}(A)$. First, consider the following equation:

$$(x + \mathbf{1}) ; x = x ; x \ .$$

This equation sound with respect to bisimilarity, as is shown in the following lemma.

Lemma 4.3.2. *For all closed* $\mathrm{TSP}^{;}_{+NT}(A)$*-terms* t*,* $(t + \mathbf{1}) ; t \leftrightarrows t ; t$*.*

Proof. To this end, we verify that the symmetric closure of

$$R = \{(((t + \mathbf{1}) ; t), (t ; t)) \mid t \in \mathrm{TSP}^{;}_{+NT}(A)\} \cup \{(t, t) \mid t \in \mathrm{TSP}^{;}_{+NT}(A)\}$$

is a bisimulation relation, by verifying that the properties of bisimilarity hold for all pairs $(((t + \mathbf{1}) ; t), (t ; t))$. The proof for all other pairs is similar or trivial.

Suppose that $(t + \mathbf{1}) ; t \xrightarrow{a} s$. It cannot be the case that $(t + \mathbf{1})\not\rightarrow$, $(t + \mathbf{1}) \downarrow$ and $t \xrightarrow{a} s$, hence it must be the case that $t \xrightarrow{a} t'$ and $s \equiv t' ; t$. In this case, clearly also $t ; t \xrightarrow{a} t' ; t$ and $(t' ; t)R(t' ; t)$. If $(t + \mathbf{1}) ; t \downarrow$, then we must have $t \downarrow$ and thus also $t ; t \downarrow$. □

As can been seen, the **1**-summand can be removed from the left hand side of the sequential composition as it does not influence the behaviour of the term and is therefore redundant. Normally, in order to remove such redundant termination, it is required that the left hand side of the sequential composition can execute an action and the right hand side cannot terminate. Clearly, this cannot be guaranteed in this case, as there exist substitutions for which x cannot execute an action or can terminate, e.g. $\sigma(x) = \mathbf{0}$. However, for each substitution σ such that $\sigma(x)\not\rightarrow$, it holds that the entire term cannot execute an action and thus the **1**-summand is still redundant. Similarly, for each substitution σ such that $\sigma(x) \downarrow$, both sides of the equation can terminate and thus the **1**-summand is redundant as well. Now consider the following equation:

$$(x + \mathbf{1}) ; (x ; y) = x ; (x ; y) \ .$$

This equation is sound with respect to bisimilarity as well, which can be deduced by the same reasoning as the previous equation. This time, however, while the right hand side of the equation contains more behaviour than just x, it is "guarded" by x, which achieves the same result. Now finally, consider the following sound equation, with $n \in \mathbb{N}$:

$$(x + \mathbf{1}) ; \sum_{i=1}^{n} (x ; (a.\mathbf{1} + \mathbf{1})^i) = x ; \sum_{i=1}^{n} (x ; (a.\mathbf{1} + \mathbf{1})^i) \ .$$

Similarly to the equation used in Section 4.2, the size of the right hand side of this equation is not bounded. In order to properly define the size of the equation, we redefine the definition of width for (open) $\mathrm{TSP}^{;}_{+NT}(A)$-terms, such that the behaviour of variables is taken into account. We say that $x \xrightarrow{x} \bar{x}$ for every variable x, and redefine the definition of width as follows.

Definition 4.3.3. The *width* of a $\mathrm{TSP}^{;}_{+NT}(A)$-term t, written as *width*(t), is the cardinality of the set $\{t' \mid t \xrightarrow{a} t'$, for some $a \in A\} \cup \{t' \mid t \xrightarrow{x} t'$, for some $x \in V\}$

Since $\sum_{i=1}^{n}(x\,;(a.\mathbf{1}+\mathbf{1})^i) \xrightarrow{x} \bar{x}\,;(a.\mathbf{1}+\mathbf{1})^i$ for all $1 \le i \le n$, clearly $\mathit{width}(\sum_{i=1}^{n}(x\,;(a.\mathbf{1}+\mathbf{1})^i)) = n$. Now, we can apply the same reasoning as used in Section 4.2 and argue that if we want to remove the redundant termination on the left hand side of the sequential composition, we need to apply an axiom of the shape $t_1 + \mathbf{1}\,;t_2 = t_1\,;t_2$. Then, if $\mathit{width}(t_2) < n$, then t_2 must contain some variable summand, and there exists a substitution σ such that $\sigma(t_1) \not\downarrow$ and $\sigma(t_2)\downarrow$. Hence, such an axiom can never be sound and thus the equation cannot be derived. The proof for this result can be obtained by adapting the proof in Section 4.2.

Chapter 5

Decidability of bisimilarity for $\mathrm{TSP}^{;}(A)$

In this chapter a proof is presented for the decidability of bisimilarity for processes definable in $\mathrm{TSP}^{;}(A)$ with finite guarded recursive specifications, i.e., given a finite guarded recursive specification Δ, it is shown that for all sequences $\alpha, \beta \in V(\Delta)^*$, it can be decided in a finite amount of time if $\alpha \leftrightarrows \beta$ or not. The proof closely follows the structure of the decidability proof for BPA processes as presented in [9], which is based on the first decidability results for arbitrary BPA processes found by Christensen, Hüttel and Stirling [11].

A common approach to prove decidability of an equivalence is to prove the semi-decidability of the equivalence and of its complement. In case of bisimilarity, this means that a proof should be given for the semi-decidability of bisimilarity and for the semi-decidability of non-bisimilarity. It is a well known result that non-bisimilarity is semi-decidable, by utilizing the stratified bisimulation relation. The most obvious approach for the semi-decision procedure of bisimilarity would consist of producing a complete bisimulation relation. However, as processes generated from recursive specifications may contain an infinite number of states, this relation can be infinite and thus such a procedure may never terminate. For this reason, the proof focusses on showing that for any pair of bisimilar processes (α, β), there exists a finite relation containing the pair that is a *base* for a bisimulation relation, meaning that this finite relation, together with the congruence property will form a bisimulation.

This chapter is structured as follows. First, a semi-decision procedure for non-bisimilarity is briefly discussed. Second, a definition is given for a finite bisimulation base, and it is proven that if a finite bisimulation base exists for every pair of bisimilar $\mathrm{TSP}^{;}(A)$-terms, there exists a semi-decision procedure for bisimarity. Then, it is shown that every guarded recursive $\mathrm{TSP}^{;}(A)$-specification can be transformed into a convenient normal form. Finally, it is shown that for every pair of bisimilar $\mathrm{TSP}^{;}(A)$-terms, there exists a finite bisimulation base that contains the pair, concluding the proof for semi-decidability of bisimilarity.

5.1 Semi-decidability procedure non-bisimilarity

The semi-decision procedure for non-bisimilarity utilizes the so called stratified bisimulation relation that is defined on all image-finite processes.

Definition 5.1.1. A *stratified bisimulation relation* $\leftrightarrow_n$ for $n \in \mathbb{N}$ is defined as follows:

- $p \leftrightarrow_0 q$ for all terms p and q; and
- $p \leftrightarrow_{n+1} q$ iff
  - if $p \xrightarrow{a} p'$, then there exists a term q', such that $q \xrightarrow{a} q'$, and $p' \leftrightarrow_n q'$;
  - if $q \xrightarrow{a} q'$, then there exists a term p', such that $p \xrightarrow{a} p'$, and $p' \leftrightarrow_n q'$;
  - if $p\downarrow$, then $q\downarrow$;
  - if $q\downarrow$, then $p\downarrow$.

This stratified bisimulation relation can be used to decide if two processes are non-bisimilar in a stepwise manner. The following lemma substantiates this idea.

Lemma 5.1.2. *For all image-finite closed* $\mathrm{TSP}^{;}(A)$*-terms* p *and* q*,* $p \leftrightarrow q$ *iff* $p \leftrightarrow_n q$ *for all* $n \geq 0$*.*

Proof. Let p and q be closed $\mathrm{TSP}^{;}(A)$-terms and suppose $p \leftrightarrow q$. We show that $p \leftrightarrow_n q$ for all $n \geq 0$, by induction on n. If $n = 0$, then clearly $p \leftrightarrow_n q$. Suppose $n \geq 0$ and that for all $\mathrm{TSP}^{\cdot}(A)$-terms p' and q', $p' \leftrightarrow q'$ implies $p' \leftrightarrow_n q'$. We need to show that $p \leftrightarrow_{n+1} q$ holds, by showing that all four cases of the definition hold. The last two cases are immediately true since $p \leftrightarrow q$. For the first case, note that if $p \xrightarrow{a} q'$, then $q \xrightarrow{a} q'$ such that $p' \leftrightarrow q'$. Since $p' \leftrightarrow q'$, by induction $p' \leftrightarrow_n q'$ and thus the first case holds. For the second case a symmetric argument can be given, hence $p \leftrightarrow_{n+1} q$.

Conversely, it can be shown that $p \leftrightarrow_n q$ for all $n \geq 0$ implies $p \leftrightarrow q$, by verifying that $\{(p,q) \mid p$ and q are image-finite, and $p \leftrightarrow_n q$ for all $n \geq 0\}$ is a bisimulation. □

From the previous lemma it is clear that if two image-finite closed $\mathrm{TSP}^{;}(A)$-terms p and q are not bisimilar, then there must exist an n such that $p \leftrightarrow_n q$ does not hold. This gives rise to a technique for semi-decidability of non-bisimilarity. Given two closed $\mathrm{TSP}^{;}(A)$-terms p and q, the semi-decision procedure for non-bisimilarity consists of deciding if $p \leftrightarrow_i q$, starting from $i = 0$ and incrementing i as long as $p \leftrightarrow_i q$ holds. If $p \not\leftrightarrow q$, then there exists an n such that $p \not\leftrightarrow_n q$, and thus the procedure will eventually terminate.

Corollary 5.1.3. Non-bisimilarity is semi-decidable for all processes definable by finite guarded recursive specifications over $\mathrm{TSP}^{;}(A)$.

5.2 Semi-decidability procedure for bisimilarity

As stated before, in order to prove semi-decidability of bisimilarity, we need to somehow capture the (possibly infinite) bisimulation relation in a finite bisimulation base. In order to be able to do this we make use of the congruence property of bisimilarity.

Given a binary relation R on sequences of variables, let $\stackrel{R}{\equiv}$ denote the least congruence containing R, i.e. $\stackrel{R}{\equiv}$ is the least equivalence relation which contains R and all pairs $(\alpha\alpha', \beta\beta')$ whenever it contains the pairs (α, β) and (α', β').

Definition 5.2.1. A binary relation R on sequences of variables is a *bisimulation base* iff R is symmetric and for all pairs $(\alpha, \beta) \in R$, it holds that:

- if $\alpha \xrightarrow{a} \alpha'$, then $\beta \xrightarrow{a} \beta'$ for some β' such that $\alpha' \overset{R}{\equiv} \beta'$; and
- if $\alpha \downarrow$, then $\beta \downarrow$.

As can be seen from the definition, a bisimulation base R only differs from a bisimulation relation in the fact that the derivative states must be related by $\overset{R}{\equiv}$ instead of R. This gives rise to the idea that $\overset{R}{\equiv}$ is a bisimulation relation, as proven in the following lemma.

Lemma 5.2.2. *If R is a bisimulation base, then $\overset{R}{\equiv}$ is a bisimulation relation.*

Proof. Let R be a bisimulation base and let α and β be arbitrary sequences such that $\alpha \overset{R}{\equiv} \beta$. First note that $\overset{R}{\equiv}$ is symmetric. We prove by induction on the derivation of $\alpha \overset{R}{\equiv} \beta$ that the two clauses given by $\overset{R}{\equiv}$ is a bisimulation relation are true for the pairs (α, β) and (β, α).

- Suppose $(\alpha, \beta) \in R$, then, if $\alpha \xrightarrow{a} \alpha'$, by definition of R, $\beta \xrightarrow{a} \beta'$ such that $\alpha' \overset{R}{\equiv} \beta'$. Similarly, if $\alpha \downarrow$, then $\beta \downarrow$. Moreover, since R is symmetric, $(\beta, \alpha) \in R$ and thus the two clauses can be proven to be true for (β, α) in an analogous fashion.
- Suppose the last rule applied in the derivation is the reflexivity rule, then the two clauses are trivially satisfied for both pairs.
- Suppose the last rule applied in the derivation is the symmetry rule, then $\beta \overset{R}{\equiv} \alpha$ and thus the two clauses of $\overset{R}{\equiv}$ being a bisimulation relation are immediately satisfied for both pairs by an application of the induction hypothesis.
- Suppose the last rule applied in the derivation is the transitivity rule. Then there exists γ such that $\alpha \overset{R}{\equiv} \gamma \overset{R}{\equiv} \beta$. By induction, if $\alpha \xrightarrow{a} \alpha'$, then $\gamma \xrightarrow{a} \gamma'$ such that $\alpha' \overset{R}{\equiv} \gamma'$. Again, by induction, if $\gamma \xrightarrow{a} \gamma'$, then $\beta \xrightarrow{a} \beta'$ such that $\gamma' \overset{R}{\equiv} \beta'$, hence, by transitivity $\alpha' \overset{R}{\equiv} \beta'$. Similarly, if $\alpha \downarrow$, by induction $\gamma \downarrow$ and thus, again by induction, also $\beta \downarrow$. The two clauses can be proven to be true for (β, α) in an analogous fashion.
- Suppose the last rule applied is the congruence rule for sequential composition, then $\alpha = \alpha_1\alpha_2$ and $\beta = \beta_1\beta_2$ such that $\alpha_1 \overset{R}{\equiv} \beta_1$ and $\alpha_2 \overset{R}{\equiv} \beta_2$. Suppose $\alpha \xrightarrow{a} \alpha'$; we distinguish two cases. If $\alpha_1 \xrightarrow{a} \alpha_1'$ and $\alpha' = \alpha_1'\alpha_2$, then, by induction, $\beta_1 \xrightarrow{a} \beta_1'$ such that $\alpha_1' \overset{R}{\equiv} \beta_1'$ and thus $\beta \xrightarrow{a} \beta_1'\beta_2$. Moreover, since $\overset{R}{\equiv}$ is a congruence, $\alpha_1'\alpha_2 \overset{R}{\equiv} \beta_1'\beta_2$. If $\alpha_1 \not\rightarrow$, $\alpha_1 \downarrow$, $\alpha_2 \xrightarrow{a} \alpha_2'$ and $\alpha' = \alpha_2'$, then by induction, $\beta_2 \xrightarrow{a} \beta_2'$ such that $\alpha_2' \overset{R}{\equiv} \beta_2'$. Also by induction, since $\alpha_1 \downarrow$, $\beta_1 \downarrow$, and since $\alpha_1 \not\rightarrow$ also $\beta_1 \not\rightarrow$. Hence, $\beta \xrightarrow{a} \beta_2'$ and $\alpha_2' \overset{R}{\equiv} \beta_2'$. If $\alpha \downarrow$, then $\alpha_1 \downarrow$ and $\alpha_2 \downarrow$. Hence, by induction $\beta_1 \downarrow$ and $\beta_2 \downarrow$ and thus $\beta \downarrow$. The two clauses can be proven to be true for (β, α) in an analogous fashion.

□

Based on the previous lemma, the following corollary can be derived.

Corollary 5.2.3. $\alpha \leftrightarroweq \beta$ iff $\alpha \overset{R}{\equiv} \beta$ for some bisimulation base R.

Suppose that for every bisimilar pair (α, β), there exists a *finite* bisimulation base R such that $\alpha \overset{R}{\equiv} \beta$. Then, there is a semi-procedure for decidability, namely, enumerating all finite binary relations R and checking if R is a bisimulation base such that $\alpha \overset{R}{\equiv} \beta$. Determining if a finite binary relation is a bisimulation base is semi-decidable, as is proven in the following lemma.

Lemma 5.2.4. *It is semi-decidable whether a given finite binary relation R over V^* is a bisimulation base.*

Proof. In order to decide whether a finite binary relation R is a bisimulation base, we need to check that each pair $(\alpha, \beta) \in R$ satisfies the two clauses of the definition of a bisimulation base. Hence, we need to check if $\alpha \downarrow$ implies $\beta \downarrow$, and if each transition of $\alpha \xrightarrow{a} \alpha'$ has a matching transition $\beta \xrightarrow{a} \beta'$ such that $\alpha' \overset{R}{\equiv} \beta'$, and vice versa. Checking if $\alpha' \overset{R}{\equiv} \beta'$ is semi-decidable as the relation $\overset{R}{\equiv}$ itself is semi-decidable. □

Since determining if a relation is a bisimulation base is semi-decidable, the semi-decision procedure for checking if $\alpha \leftrightarroweq \beta$ would then consist of enumerating all finite binary relations R and checking, using a dovetailing procedure, if one of them is a bisimulation base and $\alpha \overset{R}{\equiv} \beta$. What remains to show is that for every bisimilar pair (α, β), there exists a *finite* bisimulation base R such that $\alpha \overset{R}{\equiv} \beta$.

Before proving the existence of such a finite bisimulation base, it is shown that all recursive TSP$^{;}(A)$ specifications can be transformed to a convenient normal form.

5.3 Extended Greibach Normal Form

Similarly to TSP$^{\cdot}(A)$, every guarded recursive specification over TSP$^{;}(A)$ can be transformed to GNF as defined in Definition 2.5.1. The proof for recursive specifications over TSP$^{\cdot}(A)$ in Appendix D, can be easily adapted to fit TSP$^{;}(A)$-terms, by using the axioms of TSP$^{;}_{+NT}(A)$ as shown in Figure 4.1.

Proposition 5.3.1. *For every guarded recursive specification Δ over* TSP$^{;}(A)$*, there exists a recursive specification Δ' in GNF such that for each equation $X = p \in \Delta$, there exists an equation $X = q \in \Delta'$ such that $p \leftrightarroweq q$.*

In the proof of decidability of bisimilarity we would like to utilize two important properties of sequences of variables representing TSP$^{;}(A)$-terms:

1. **Maximum length:** Given a sequence α, $|\alpha| \leq n(\alpha)$.
2. **Cancellation:** Given normed sequences α, β and γ, if $\alpha\gamma \leftrightarroweq \beta\gamma$, then $\alpha \leftrightarroweq \beta$.

It turns out that in general these two properties do not hold for sequences of variables defined by a recursive specification over TSP$^{;}(A)$ in GNF. This is illustrated by the following example.

Example 5.3.2. Consider the recursive specification Δ in GNF:

$$X = a.YWZ + a.YW + a.ZZ \qquad Y = b.\mathbf{1} + \mathbf{1} \qquad Z = b.\mathbf{1} \qquad W = \mathbf{1}$$

Consider the sequence YWZ. Clearly, $|YWZ| = 3 > 2 = n(YWZ)$, invalidating the first property. Moreover, it is not difficult to see that $YWZ \leftrightarrow ZZ$, but $YW \not\leftrightarrow Z$, validating the second property.

The example shows that both properties do not hold on arbitrary sequences definable by a recursive $\text{TSP}^{;}(A)$ specification in GNF. In order to solve this issue, we propose a normal form for recursive $\text{TSP}^{;}(A)$ specifications, named Extended Greibach Normal Form (*EGNF*). For every recursive specification Δ in EGNF, it must holds that for every sequence α definable by Δ, there must exist a bisimilar sequence α' definable by Δ, for which both properties are valid.

First, as shown in Example 5.3.2, the maximum length of sequences representing normed $\text{TSP}^{;}(A)$-terms is not restricted, as equations of the form $X = \mathbf{1}$ are allowed in GNF. To prevent this, a recursive specification in EGNF may not contain any equations of the form $X = \mathbf{1}$. Note that it is still possible to express processes that are bisimilar to $\mathbf{1}$ by using the empty sequence.

Second, Example 5.3.2 shows that the cancellation property does not hold for normed $\text{TSP}^{;}(A)$-terms. This is caused by the presence of redundant intermediate termination.

Definition 5.3.3. A sequence α has *redundant intermediate termination* iff $\alpha \equiv \beta\gamma$ and β has intermediate termination, while γ cannot immediately terminate.

In Example 5.3.2, the intermediate termination of Y in YWZ is redundant, as WZ cannot terminate. In order to prevent this, for any sequence α definable by a recursive specification Δ in EGNF, there must exist a bisimilar sequence α' definable by Δ that does not contain any redundant intermediate termination.

5.3.1 Transformation to EGNF

We show that any recursive $\text{TSP}^{;}(A)$ specification in GNF can be transformed to EGNF. An informal application of the desired transformation is presented in the following example.

Example 5.3.4. Consider the recursive specification Δ in GNF with root X:

$$X = a.YWZ + a.YW + a.ZZ \qquad Y = b.\mathbf{1} + \mathbf{1} \qquad Z = b.\mathbf{1} \qquad W = \mathbf{1}$$

First, the equation $W = \mathbf{1}$ is removed and all occurrences of W in defining equations are removed as well, resulting in the following specification:

$$X = a.YZ + a.Y + a.ZZ \qquad Y = b.\mathbf{1} + \mathbf{1} \qquad Z = b.\mathbf{1}$$

Next, as Z cannot terminate, the intermediate termination of Y in $a.YZ$ is redundant, thus we want to remove this intermediate termination. However, in $a.Y$, the intermediate termination of Y is not redundant, hence, we cannot simply remove the intermediate termination from the defining equation of Y. Instead, we introduce a fresh variable $\bar{Y}$, that is equal to Y without the intermediate termination. Then, we replace all occurrences of Y of which the intermediate termination is redundant with $\bar{Y}$, as follows:

$$X = a.\bar{Y}Z + a.Y + a.ZZ \qquad Y = b.1 + \mathbf{1} \qquad \bar{Y} = b.\mathbf{1} \qquad Z = b.\mathbf{1}$$

In contrast to Example 5.3.2, the maximum length of each sequence that can be defined is now indeed equal to the norm of the sequence. For example, $|\bar{Y}Z| = 2 = n(\bar{Y}Z)$. Moreover, for every definable sequence, there exists a bisimilar definable sequence without redundant intermediate termination, e.g. YZ contains redundant intermediate termination, but is bisimilar to $\bar{Y}Z$ that does not contain redundant intermediate termination. As a result, we can now apply the cancellation property, e.g. $\bar{Y}Z \leftrightarrow ZZ$ and $\bar{Y} \leftrightarrow Z$.

As shown in the previous example, in order to remove the intermediate termination from a variable X, we need to introduce a fresh variable $\bar{X}$ and its defining equation. Given a recursive specification Δ in GNF, for any recursive equation $X = p \in \Delta$, we define the recursive equation $\bar{X} = \bar{p}$ without intermediate termination, where $\bar{X}$ is a fresh variable and $\bar{p}$ is defined as follows:

$$\bar{p} = \begin{cases} \sum_{i=1}^{n} a_i.\bar{\alpha_i} & \text{if } p = \sum_{i=1}^{n} a_i.\alpha_i(+\mathbf{1}) \text{ for some } n \in \mathbb{N}^+ \\ p & \text{otherwise.} \end{cases}$$

Here, $\bar{\alpha_i}$ represents the sequence of variables α_i where each variable $X \in \alpha_i$ is replaced with $\bar{X}$. For example, given $X = a.Y + \mathbf{1}$ and $Y = \mathbf{1}$, we get $\bar{X} = a.\bar{Y}$ and $\bar{Y} = \mathbf{1}$. It is not difficult to see that $\bar{X}$ has the same behaviour as X except that it does not have any reachable intermediate termination.

Given a recursive specification Δ in GNF, we define the specification $\bar{\Delta}$ without intermediate termination as follows:

$$\bar{\Delta} = \{\bar{X} = \bar{p} \mid X = p \in \Delta\}$$

The set of variables $V(\bar{\Delta})$ will often be abbreviated with $\bar{V}$, and thus all variables $X \in \bar{V}$ do not contain any intermediate termination and, moreover, are not able to reach a state with intermediate termination. Note that $\bar{\Delta}$ may still contain equations of the form $X = \mathbf{1}$.

Lemma 5.3.5. *Given a recursive specification $\bar{\Delta}$, for any sequence $\alpha \in \bar{V}^*$, if $\alpha \xrightarrow{a} \alpha'$, then $\alpha' \in \bar{V}^*$.*

Proof. Let $\alpha \in \bar{V}^*$ be an arbitrary sequence such that $\alpha \xrightarrow{a} \alpha'$ for some α'. Then $\alpha = \bar{X}_1 \dots \bar{X}_n$ for some $n \geq 1$, and for some $1 \leq i \leq n$, we have $\bar{X}_i \xrightarrow{a} \beta$ and thus $\alpha' = \beta\bar{X}_{i+1} \dots \bar{X}_n$. By definition of $\bar{X}_i$, we must have $\beta \in \bar{V}^*$ and thus $\alpha' \in \bar{V}^*$. □

We split the set of variables in two disjoint sets $V = V_{\downarrow} \cup V_{\not\downarrow}$, where $V_{\downarrow}$ denotes the set of variables that can terminate and $V_{\not\downarrow}$ the set of variables that cannot terminate. It is not difficult to see that any sequences of the form $\bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$does not contain any redundant intermediate termination. Hence, EGNF is formally defined as follows.

Definition 5.3.6. A recursive specification $\Delta \cup \bar{\Delta}$ is in *EGNF* iff all equations are of the form $X = \mathbf{0}$ or $X = \sum_{i=1}^{n} a_i.\alpha_i(+\mathbf{1})$ for some $n \in \mathbb{N}^+$, action a_i and sequence $\alpha_i \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$.

Let Δ be a recursive specification in GNF. We will show how to transform Δ to a recursive specification Δ' in EGNF, such that for each equation $X = p \in \Delta$, there exists an equation $X = q \in \Delta'$ such that $p \leftrightarrow q$.

First, all equations of the form $X = \mathbf{1}$ are removed from Δ and all occurrences of such variables X in other equations are replaced by the empty sequence. Next, we define $\Delta' = \Delta \cup \bar{\Delta}$ with the same root as Δ. For every equation in Δ', all redundant intermediate termination needs to be removed. The following lemma proves that any redundant intermediate termination can be removed from a sequence of variables. Note that in the lemma it is assumed that all sequences of variables do not contains variables that are bisimilar to $\mathbf{1}$.

Lemma 5.3.7. *Given a recursive specification $\Delta \cup \bar{\Delta}$, for any sequence α and variable $X \in V_{\not\downarrow}$, $\alpha X \leftrightarroweq \bar{\alpha} X$.*

Proof. We define $R = \{(\alpha X, \bar{\alpha} X) \mid X \in V_{\not\downarrow}\}$ and show R is a bisimulation relation.

Suppose $\alpha X \xrightarrow{a} \beta$. We distinguish two cases.

- Suppose $\alpha \xrightarrow{a} \alpha'$ and $\beta = \alpha' X$. Then, by definition, for any variable Y, if $Y \xrightarrow{a} \gamma$, then $\bar{Y} \xrightarrow{a} \bar{\gamma}$. Hence, by the operational semantics, since $\alpha \xrightarrow{a} \alpha'$, $\bar{\alpha} \xrightarrow{a} \bar{\alpha}'$ and thus $\bar{\alpha} X \xrightarrow{a} \bar{\alpha}' X$ and $(\alpha' X, \bar{\alpha}' X) \in R$.
- If $\alpha \not\rightarrow$, $\alpha \downarrow$ and $X \xrightarrow{a} \beta$, then $\alpha \leftrightarroweq \mathbf{1}$, hence α is the empty sequence and thus $\bar{\alpha}$ is empty as well. Hence, $\alpha X \leftrightarroweq X \leftrightarroweq \bar{\alpha} X$.

Also, since $X \not\downarrow$, we have $\alpha X \not\downarrow$ and hence the case where $\alpha X \downarrow$ vacuously holds. A symmetric argument applies for the case where $\bar{\alpha} X \xrightarrow{a} \beta$. □

From the previous lemma, and compatibility of bisimilarity with sequential composition, the following may be concluded.

Corollary 5.3.8. Given a recursive specification $\Delta \cup \bar{\Delta}$, for every sequence $\alpha \in V^*$ there exists a sequence $\beta \in \bar{V}^* V_{\not\downarrow} {V_{\downarrow}}^* \cup {V_{\downarrow}}^*$ such that $\alpha \leftrightarroweq \beta$.

Using this result it can be shown that all redundant intermediate termination can removed from the equations of Δ'. For each equation $X = p \in \Delta'$, replace each summand of p of the form $a.\alpha$ by $a.f(\alpha)$, where $f(\alpha)$ is defined as follows:

$$f(\alpha) = \begin{cases} \bar{\alpha}_1 \alpha_2 & \text{if } \alpha = \alpha_1 \alpha_2 \text{ for some } \alpha_1 \in V^* \text{ and } \alpha_2 \in V_{\not\downarrow} {V_{\downarrow}}^* \\ \alpha & \text{otherwise} \end{cases}$$

Clearly, the resulting recursive definition Δ' is in EGNF as it does not contain any equations of the form $X = \mathbf{1}$ and for every sequence α, $f(\alpha) \in \bar{V}^* V_{\not\downarrow} {V_{\downarrow}}^* \cup {V_{\downarrow}}^*$. Also, this procedure terminates as it consists of simply replacing a finite number of sequences with a different sequence. Finally, by Lemma 5.3.7, for every sequence α, $\alpha \leftrightarroweq f(\alpha)$, hence for every sequence $X = p \in \Delta$, there is a sequence $X = q \in \Delta'$ such that $p \leftrightarroweq q$.

Proposition 5.3.9. *For every recursive specification Δ in GNF there exists a recursive specification Δ' in EGNF such that for each equation $X = p \in \Delta$, there exists an equation $X = q \in \Delta'$ such that $p \leftrightarroweq q$.*

It remains to show that two important properties introduced at the beginning of Section 5.3 are valid for all sequences of the form $\bar{V}^* V_{\not\downarrow} {V_{\downarrow}}^* \cup {V_{\downarrow}}^*$defined by a recursive specification in EGNF. The first property is immediate from the fact that all recursion variables of a recursive specification in EGNF must have a norm of at least 1. Hence, what remains to show it that the cancellation property is valid for all sequences of the form $\bar{V}^* V_{\not\downarrow} {V_{\downarrow}}^* \cup {V_{\downarrow}}^*$.

5.3.2 Normed cancellation

In order to prove the cancellation property, we first show that for every sequence $\alpha \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ defined by a recursive specification in EGNF, it holds that if $\alpha \xrightarrow{a} \alpha'$, then $\alpha' \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$. We utilize the fact that every sequence $\alpha\beta \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ can be split in a specific way.

Lemma 5.3.10. *For any sequence $\alpha\beta \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, either $\alpha \in \bar{V}^*$ and $\beta \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^*$, or $\alpha \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ and $\beta \in V_{\downarrow}{}^*$.*

Proof. Suppose $\beta \in V_{\downarrow}{}^*$, then clearly it must be the case that $\alpha \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ and thus the claim holds. On the other hand, if $\beta \notin V_{\downarrow}{}^*$, then $\beta \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^*$ and we must have $\alpha \in \bar{V}^*$ and the claim holds. □

Using the previous lemma, we show that sequences of the form $\bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$maintain their shape when executing an action.

Lemma 5.3.11. *Let Δ be a recursive specification in EGNF. For any sequence $\alpha \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ definable by Δ, if $\alpha \xrightarrow{a} \alpha'$, then $\alpha' \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$.*

Proof. Let $\alpha \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ be a sequence definable by Δ and suppose $\alpha \xrightarrow{a} \alpha'$. Then we must have $\alpha = X\beta$ for some variable $X \in V(\Delta)$ and sequence $\beta \in V(\Delta)^*$, such that $X \xrightarrow{a} \gamma$ and $\alpha' = \gamma\beta$. Since Δ is in EGNF and $X \xrightarrow{a} \gamma$, $X = \sum_{i=1}^{n} a_i.\gamma_i(+\mathbf{1})$ for some $n \in \mathbb{N}^+$, with $\gamma_i \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ for all $1 \leq i \leq n$. Hence, we must have $\gamma = \gamma_i$ for some $1 \leq i \leq n$ and thus $\gamma \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$. Since $X\beta \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, by Lemma 5.3.10, we may distinguish two cases.

- Suppose $X \in \bar{V}$ and $\beta \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^*$. Then, by Lemma 5.3.5, $\gamma \in \bar{V}^*$ and thus $\gamma\beta \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$.
- Suppose $X \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ and $\beta \in V_{\downarrow}{}^*$. Then, since $\gamma \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ clearly also $\gamma\beta \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$.

□

Since for all sequences α definable by a recursive specification in EGNF, there exists a bisimilar sequence $\alpha' \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, it is sufficient to show that the cancellation property holds for normed sequences of this shape. This is proven in the following lemma.

Lemma 5.3.12. *Let Δ be a recursive specification in EGNF. For normed sequences $\alpha\gamma, \beta\gamma \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ definable by Δ, if $\alpha\gamma \leftrightarroweq \beta\gamma$, then $\alpha \leftrightarroweq \beta$.*

Proof. We define $R = \{(\alpha, \beta) \mid \alpha\gamma, \beta\gamma \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^* \text{ definable by } \Delta \text{ and } \alpha\gamma \leftrightarroweq \beta\gamma\}$ and show R is a bisimulation relation.

Suppose $\alpha \xrightarrow{a} \alpha'$, then $\beta\gamma \xrightarrow{a} \delta$ such that $\delta \leftrightarroweq \alpha'\gamma$. We distinguish two cases.

- If $\beta \xrightarrow{a} \beta'$ and $\delta = \beta'\gamma$, then since $\alpha\gamma, \beta\gamma \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, by Lemma 5.3.11, $\alpha'\gamma, \beta'\gamma \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, and thus $(\alpha', \beta') \in R$.
- If $\beta \downarrow$, $\beta \not\rightarrow$, $\gamma \xrightarrow{a} \gamma'$ and $\delta = \gamma'$, then $n(\beta\gamma) = n(\gamma) < n(\alpha\gamma)$, contradicting $\alpha\gamma \leftrightarroweq \beta\gamma$. Hence, this case vacuously holds.

Moreover, if $\alpha\downarrow$, then $\alpha \in V_{\downarrow}{}^*$ and since $\alpha\gamma \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ also $\gamma \in V_{\downarrow}{}^*$. Hence, we have $\alpha\gamma\downarrow$ and thus $\beta\gamma\downarrow$ and $\beta\downarrow$. A symmetric argument applies for the cases where $\beta \xrightarrow{a} \beta'$ and $\beta\downarrow$. □

Also, note that the lemma does not hold for unnormed sequences, as can be seen from the following example.

Example 5.3.13. Consider the following recursive specification in EGNF:

$$X = a.1 \qquad Y = a.Y$$

Clearly $XY \leftrightarrow Y$, but $X \not\leftrightarrow \mathbf{1}$.

5.3.3 Properties of sequences

It can now be shown that both important properties are valid for all sequences definable by a recursive specification in EGNF.

Proposition 5.3.14. *For any recursive specification Δ in EGNF, the following two properties must hold.*

1. ***Max length:*** *For any sequence $\alpha \in V(\Delta)$, $|\alpha| \leq n(\alpha)$.*
2. ***Cancellation:*** *For all normed sequences $\alpha\gamma, \beta\gamma \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, if $\alpha\gamma \leftrightarrow \beta\gamma$, then $\alpha \leftrightarrow \beta$.*

Proof. Let Δ be an arbitrary recursive specification in EGNF.

1. Let α be an arbitrary sequence reachable from the root of Δ. If α contains a variable X such that $X = \mathbf{0}$, then clearly $|\alpha| \leq n(\alpha) = \infty$. In case α does not contain such a variable, all variables must have a defining equation of the shape $\sum_{i=1}^{n} a_i.\alpha_i(+\mathbf{1})$ for some $n \geq 1$. Since every variable must be able to execute at least one action, the norm of each variable must be greater or equal than 1. Hence, in this case also $|\alpha| \leq n(\alpha)$.
2. Follows immediately from Lemma 5.3.12.

□

To conclude this section on EGNF, we introduce two more lemmas to show that sequences maintain there shape after certain operations. The first lemma states that given a sequence $\alpha \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, any subsequence of α must also be of the form $\bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$.

Lemma 5.3.15. *If $\alpha \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, then for any subsequence α' of α it holds that $\alpha' \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$.*

Proof. Let $\alpha \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ and let α' be an arbitrary subsequence of α. If α' is empty, then trivially $\alpha' \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, hence we consider the case where α' is not empty. Since $\alpha \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, we must have $\alpha' \in V_{\downarrow}{}^*$, $\alpha' \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^*$ or $\alpha' \in \bar{V}^*$. In the first two cases it is clear that $\alpha' \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$. In the last case, note that for every $X \in \bar{V}$ we have $X \in V_{\not\downarrow}$. Hence, since α' is not empty, we then have $\alpha' \in \bar{V}^* V_{\not\downarrow}$ and thus $\alpha' \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$. □

The second lemma states that given a sequence $X\alpha \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, if α is replaced with another bisimilar sequence $\beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, then $X\beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$.

Lemma 5.3.16. *Let* $X\alpha, \beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ *such that* $\alpha \mathrel{\underline{\leftrightarrow}} \beta$*, then* $X\beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$.

Proof. We distinguish two cases based on β. If $\beta \in V_{\downarrow}{}^*$, then since $X \in V_{\downarrow}$ or $X \in V_{\not\downarrow}$, clearly $X\beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$. If $\beta \notin V_{\downarrow}{}^*$, then $\beta \not\downarrow$ and since $\alpha \mathrel{\underline{\leftrightarrow}} \beta$ also $\alpha \not\downarrow$. Hence, $\alpha \notin V_{\downarrow}{}^*$ and thus since $X\alpha \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, by Lemma 5.3.10, it must be the case that $X \in \bar{V}$ and thus $X\beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$. □

From now on it is assumed that all recursive specifications are in EGNF unless stated otherwise. Moreover, all sequences of variables α are assumed to defined by a recursive specification in EGNF and to be of the form $\bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$.

5.4 Finite bisimulation base for normed $\mathrm{TSP}^{;}(A)$

Let Δ be a finite normed recursive specification in EGNF. In this section it is argued that for all sequences $\alpha, \beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ defined by Δ, whenever $\alpha \mathrel{\underline{\leftrightarrow}} \beta$ there exists a finite bisimulation base R such that $\alpha \overset{R}{\equiv} \beta$. This relation R is defined as the symmetric closure of the following set:

$$\{(X, \alpha) \mid X \text{ is normed } X \mathrel{\underline{\leftrightarrow}} \alpha\}$$

Lemma 5.4.1. *For each normed variable* $X \in V(\Delta)$ *the set* $\{(X, \alpha) \mid X \mathrel{\underline{\leftrightarrow}} \alpha\}$ *must be finite.*

Proof. Since Δ is in EGNF, we have $|\alpha| \leq n(\alpha)$. Hence, since X is normed and $n(X) = n(\alpha)$ we have $|\alpha| \leq n(X)$, and thus α has a finite maximum length. Hence, there can only be finitely many such α as $V(\Delta)$ is finite. □

By the previous lemma, clearly, R is a finite relation as $V(\Delta)$ is finite. What remains to show is that for all sequences $\alpha, \beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, $\alpha \overset{R}{\equiv} \beta$ if and only if $\alpha \mathrel{\underline{\leftrightarrow}} \beta$. The well-founded preorder $\preceq$ used in Lemma 5.4.2 is defined as follows:

$$(\alpha_1, \beta_1) \preceq (\alpha_2, \beta_2) \text{ iff } \max(n(\alpha_1), n(\beta_1)) \leq \max(n(\alpha_2), n(\beta_2))$$

Lemma 5.4.2. *Let* Δ *be a finite normed recursive specification in EGNF. Let* R *be the symmetric closure of* $\{(X, \alpha) \mid X \in V(\Delta)$ *and* $X \mathrel{\underline{\leftrightarrow}} \alpha\}$*. Then* R *is a finite relation satisfying* $\alpha \overset{R}{\equiv} \beta$ *if and only if* $\alpha \mathrel{\underline{\leftrightarrow}} \beta$ *for all sequences* $\alpha, \beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$..

Proof. First, note that by Lemma 5.4.1, R must be finite. Moreover, since for all pairs $(\alpha, \beta) \in R$, $\alpha \mathrel{\underline{\leftrightarrow}} \beta$ and $\mathrel{\underline{\leftrightarrow}}$ is a congruence for $\mathrm{TSP}^{;}(A)$, we have $\overset{R}{\equiv} \subseteq \mathrel{\underline{\leftrightarrow}}$. What remains to show is that $\overset{R}{\equiv} \supseteq \mathrel{\underline{\leftrightarrow}}$. We prove by induction on $\preceq$ that $X\alpha \mathrel{\underline{\leftrightarrow}} Y\beta$ implies $X\alpha \overset{R}{\equiv} Y\beta$, for all sequences $X\alpha, Y\beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$.

Suppose that α is empty and thus $X\alpha = X$. Then, clearly $(X, Y\beta) \in R$ and hence $X\alpha = X \overset{R}{\equiv} Y\beta$. A symmetric argument applies in case β is empty.

Now suppose both α and β are not empty, and without loss of generality assume $n(X) \leq n(Y)$. Then, $X\alpha \xrightarrow{\sigma} \alpha$ for some σ with length $n(X)$ and $Y\beta \xrightarrow{\sigma} \gamma\beta$ such that $\alpha \mathrel{\underline{\leftrightarrow}} \gamma\beta$, and thus $X\gamma\beta \mathrel{\underline{\leftrightarrow}} X\alpha \mathrel{\underline{\leftrightarrow}} Y\beta$. Sinxe $Y\beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, by Lemma 5.3.11, $\gamma\beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ and thus by Lemma 5.3.16, $X\gamma\beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$. Hence, by Lemma 5.3.12, we have $X\gamma \mathrel{\underline{\leftrightarrow}} Y$.

Then by Lemma 5.3.15, we have $X\gamma, Y \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ and thus $(X\gamma, Y) \in R$ and $X\gamma \overset{R}{\equiv} Y$. Furthermore, since $n(\gamma\beta) = n(\alpha) < n(X\alpha) = n(Y\beta)$, $(\alpha, \gamma\beta) \prec (X\alpha, Y\beta)$ and, moreover, by Lemma 5.3.15, $\alpha \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$. Hence, by the induction hypothesis, $\alpha \overset{R}{\equiv} \gamma\beta$ and thus $X\alpha \overset{R}{\equiv} X\gamma\beta \overset{R}{\equiv} Y\beta$. □

What remains to show before we can conclude that bisimilarity is semi-decidable for all normed $\mathrm{TSP}^{;}(A)$-terms of the form $\bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, is that R is a bisimulation base.

Lemma 5.4.3. *If R is a symmetric relation such that for each pair $(\alpha, \beta) \in R$ we have $\alpha, \beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ and, moreover, $\alpha \overset{R}{\equiv} \beta$ if and only if $\alpha \mathrel{\underline{\leftrightarrow}} \beta$ for all sequences $\alpha, \beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, then R is a bisimulation base.*

Proof. Consider an arbitrary pair $(\alpha, \beta) \in R$. We show that the two properties of R being a bisimulation base are satisfied. Suppose $\alpha \xrightarrow{a} \alpha'$. Then, since $\alpha \overset{R}{\equiv} \beta$ and $\alpha, \beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, we have $\alpha \mathrel{\underline{\leftrightarrow}} \beta$. Hence, $\beta \xrightarrow{a} \beta'$ such that $\alpha' \mathrel{\underline{\leftrightarrow}} \beta'$. Moreover, since $\alpha, \beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, by Lemma 5.3.11, $\alpha', \beta' \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ and thus $\alpha' \overset{R}{\equiv} \beta'$. In case $\alpha\downarrow$, clearly, since $\alpha \mathrel{\underline{\leftrightarrow}} \beta$, we have $\beta\downarrow$. Since R is symmetric and both cases are satisfied, we conclude that R is a bisimulation base. □

As we have shown that for every pair of bisimilar normed sequences $\alpha, \beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ there exists a finite bisimulation base R such that $\alpha \overset{R}{\equiv} \beta$, we conclude the following.

Corollary 5.4.4. Bisimilarity is semi-decidable for all processes of the form $\bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, definable by finite normed guarded recursive specifications over $\mathrm{TSP}^{;}(A)$-terms.

5.5 Finite bisimulation base for $\mathrm{TSP}^{;}(A)$

In this section, another bisimulation base is presented, this time to show that bisimilarity is semi-decidable for *all* (potentially unnormed) processes definable in $\mathrm{TSP}^{;}(A)$ with finite guarded recursive specifications. Let Δ be a finite recursive specification in EGNF. In this section it is argued that for all sequences $\alpha, \beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, whenever $\alpha \mathrel{\underline{\leftrightarrow}} \beta$, then there exists a finite bisimulation base R such that $\alpha \overset{R}{\equiv} \beta$. Before presenting this bisimulation base R, it is shown how to restrict the form of sequences generated from a specification in EGNF even further, based on their norm.

Lemma 5.5.1. *For any sequence $\alpha\beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, if α has an infinite norm, then $\alpha\beta \mathrel{\underline{\leftrightarrow}} \alpha$.*

Proof. We define $R = \{(\alpha, \alpha\beta) \mid \alpha\beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ and α has infinite norm$\}$ and show R is a bisimulation relation.

Suppose $\alpha \xrightarrow{a} \alpha'$, then $\alpha\beta \xrightarrow{a} \alpha'\beta$ and since α has an infinite norm, α' must also have an infinite norm. Moreover, since $\alpha\beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, by Lemma 5.3.11, $\alpha'\beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ and thus $(\alpha', \alpha'\beta) \in R$. Furthermore, if $\alpha\downarrow$, then $\alpha \in V_{\downarrow}{}^*$, hence $\beta \in V_{\downarrow}{}^*$ and thus $\alpha\beta\downarrow$. A symmetric argument applies for the cases where $\alpha\beta \xrightarrow{a} \alpha'\beta$ and $\alpha\beta\downarrow$. □

Given a sequence $\alpha\beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, from the previous lemma it can be deduced that if α has an infinite norm, β can never be reached and thus the sequence $\alpha\beta$ can be replaced by α. We split the set of variables in two disjoint sets $V = V_n \cup V_u$, where V_n denotes the set of variables with finite norm and V_u the set of variables with infinite norm. Then, by the lemma we only need to consider sequences of the form $V_n{}^*V_u \cup V_n{}^*$. Combined with the result arising from EGNF, this means that only sequences of the form $(\bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*) \cap (V_n{}^*V_u \cup V_n{}^*)$ need to be considered.

Similar to the decidability proof for normed terms, a form of cancellation is needed. However, as shown in Example 5.3.13, the cancellation property does not hold for unnormed recursive $\mathrm{TSP}^{;}(A)$ terms. For this reason, a variant on the cancellation property is presented in the following two lemmas. The first lemma states that for any non-empty sequence γ, for any two sequences $\gamma\alpha, \gamma\beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, if $\alpha \leftrightarrow \gamma\alpha$ and $\beta \leftrightarrow \gamma\beta$, then $\alpha \leftrightarrow \beta$.

Lemma 5.5.2. *If $\alpha \leftrightarrow \gamma\alpha$ and $\beta \leftrightarrow \gamma\beta$ for some $\gamma \not\leftrightarrow \mathbf{1}$ and $\gamma\alpha, \gamma\beta \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, then $\alpha \leftrightarrow \beta$.*

Proof. We define $R = \{(\delta\alpha, \delta\beta) \mid \alpha \leftrightarrow \gamma\alpha$ and $\beta \leftrightarrow \gamma\beta$ for some $\gamma \not\leftrightarrow \mathbf{1}$ and $\alpha\gamma, \beta\gamma \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*\}$ and show $\leftrightarrow R \leftrightarrow$ is a bisimulation relation.

Let $(\alpha_1, \beta_1) \in (\leftrightarrow R \leftrightarrow)$. Then there exists a pair $(\delta\alpha_2, \delta\beta_2) \in R$ such that $\alpha_2 \leftrightarrow \gamma\alpha_2$ and $\beta_2 \leftrightarrow \gamma\beta_2$ for some non-empty sequence γ, and $\alpha_1 \leftrightarrow \delta\alpha_2 R \delta\beta_2 \leftrightarrow \beta_1$. Suppose $\alpha_1 \xrightarrow{a} \alpha_1'$, then $\delta\alpha_2 \xrightarrow{a} \zeta$ such that $\alpha_1' \leftrightarrow \zeta$. We distinguish two cases.

- If $\delta \xrightarrow{a} \delta'$ and $\zeta = \delta'\alpha_2$, then $\delta\beta_2 \xrightarrow{a} \delta'\beta_2$ and $(\delta'\alpha_2, \delta'\beta_2) \in R$. Moreover, $\beta_1 \xrightarrow{a} \beta_1'$ such that $\delta'\beta_2 \leftrightarrow \beta_1'$, hence $(\alpha_1', \beta_1') \in (\leftrightarrow R \leftrightarrow)$.
- If $\delta\downarrow$, $\delta \not\rightarrow$, $\alpha_2 \xrightarrow{a} \alpha_2'$ and $\zeta = \alpha_2'$, then since $\gamma \not\leftrightarrow \mathbf{1}$, we have $\gamma \xrightarrow{a} \gamma'$ and $\gamma\alpha_2 \xrightarrow{a} \gamma'\alpha_2$ such that $\alpha_2' \leftrightarrow \gamma'\alpha_2$. Moreover, $\gamma\beta_2 \xrightarrow{a} \gamma'\beta_2$ and thus $\beta_2 \xrightarrow{a} \beta_2'$ such that $\beta_2' \leftrightarrow \gamma'\beta_2$, and since $\delta \leftrightarrow \mathbf{1}$, $\beta_1 \leftrightarrow \beta_2$ and $\beta_1 \xrightarrow{a} \beta_1'$ such that $\beta_2' \leftrightarrow \beta_1'$. Now clearly $(\gamma'\alpha_2, \gamma'\beta_2) \in R$ and thus $\alpha_1' \leftrightarrow \alpha_2' \leftrightarrow \gamma'\alpha_2 R \gamma'\beta_2 \leftrightarrow \beta_2' \leftrightarrow \beta_1'$ and $(\alpha_1', \beta_1') \in (\leftrightarrow R \leftrightarrow)$.

Now suppose $\alpha_1\downarrow$. Then, $\delta\alpha_2\downarrow$, hence we must have $\delta\downarrow$ and $\alpha_2\downarrow$. Moreover, since $\alpha_2 \leftrightarrow \gamma\alpha_2$, also $\gamma\downarrow$ and thus we have $\gamma \in V_{\downarrow}{}^*$. Since $\gamma\beta_2 \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ also $\beta \in V_{\downarrow}{}^*$ and thus $\beta_2\downarrow$. Therefore, $\delta\beta_2\downarrow$ and thus also $\beta_1\downarrow$.

A symmetric argument applies for the cases where $\delta\beta \xrightarrow{a} \zeta$ and $\delta\beta\downarrow$.

□

Using this result, we will show that the cancellation property applies for (potentially unnormed) sequences of the form $\bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, if $\alpha\gamma \leftrightarrow \beta\gamma$ for infinitely many non-bisimilar γ.

Lemma 5.5.3. *Let α and β be arbitrary sequences. If for infinitely many non-bisimilar γ it holds that $\alpha\gamma, \beta\gamma \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ and $\alpha\gamma \leftrightarrow \beta\gamma$, then $\alpha \leftrightarrow \beta$.*

Proof. We define $R = \{(\alpha, \beta) \mid$ for infinitely many non-bisimilar γ, $\alpha\gamma, \beta\gamma \in \bar{V}^*V_{\not\downarrow}V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ and $\alpha\gamma \leftrightarrow \beta\gamma\}$ and show R is a bisimulation relation.

Suppose $\alpha \xrightarrow{a} \alpha'$. Then, for infinitely many non-bisimilar γ, $\alpha\gamma \xrightarrow{a} \alpha'\gamma$. For each of these γ, $\beta\gamma \xrightarrow{a} \delta$ such that $\delta \leftrightarrow \alpha'\gamma$. We distinguish two cases.

- If $\beta \xrightarrow{a} \beta'$ and $\delta = \beta'\gamma$, then by image-finiteness there can only be finitely many such β', hence there must be a β' such that $\alpha'\gamma \leftrightarrow \beta'\gamma$ for infinitely many non-bisimilar γ. Moreover, since for all these γ we have $\alpha\gamma, \beta\gamma \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, by Lemma 5.3.11, $\alpha'\gamma, \beta'\gamma \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ and thus $(\alpha', \beta') \in R$.
- If $\beta \downarrow$, $\beta \not\rightarrow$, $\gamma \xrightarrow{a} \gamma'$ and $\delta = \gamma'$, then $\beta \leftrightarrow \mathbf{1}$ and thus $\alpha\gamma \leftrightarrow \beta\gamma \leftrightarrow \gamma$ for infinitely many non-bisimilar γ. However, by Lemma 5.5.2, for every γ_1 and γ_2 such that $\alpha\gamma_1 \leftrightarrow \gamma_1$ and $\alpha\gamma_2 \leftrightarrow \gamma_2$ it holds that $\gamma_1 \leftrightarrow \gamma_2$. Hence, in this case there cannot exist infinitely many non-bisimilar γ such that $\alpha\gamma \leftrightarrow \beta\gamma$ and thus this case cannot occur.

Moreover, if $\alpha \downarrow$, then $\alpha \in V_{\downarrow}{}^*$ and since for infinitely many γ, $\alpha\gamma \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ also $\gamma \in V_{\downarrow}{}^*$. Hence, we have $\alpha\gamma \downarrow$ and thus $\beta\gamma \downarrow$ and $\beta \downarrow$. A symmetric argument applies for the cases where $\beta \xrightarrow{a} \beta'$ and $\beta \downarrow$. □

The definition of the finite bisimulation base relies on breaking down terms into simpler terms. In order to utilize this idea, a definition is given for when terms are decomposable.

Definition 5.5.4. A pair $(X\alpha, Y\beta)$ satisfying $X\alpha \leftrightarrow Y\beta$ is *decomposable* if X and Y have finite norm, and there exists some γ such that,

- $X \longrightarrow^* \gamma$, $X \leftrightarrow Y\gamma$ and $\gamma\alpha \leftrightarrow \beta$; or
- $Y \longrightarrow^* \gamma$, $Y \leftrightarrow X\gamma$ and $\gamma\beta \leftrightarrow \alpha$.

Now, using this definition and all previous technical results, we present a relation that will be the heart of the bisimulation base, and show that this relation must be finite. Note that we define two pairs $(X\alpha, Y\beta)$ and $(X\alpha', Y\beta')$ to be *distinct* if $\alpha \not\leftrightarrow \alpha'$ or $\beta \not\leftrightarrow \beta'$.

Lemma 5.5.5. *Let Δ be a finite recursive specification in EGNF. For any $X, Y \in V(\Delta)$, any set R of the form $\{(X\alpha, Y\beta) \mid X\alpha, Y\beta \in (\bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*) \cap (V_n{}^* V_u \cup V_n{}^*)$, $X\alpha \leftrightarrow Y\beta$, and $(X\alpha, Y\beta)$ is not decomposable$\}$ which contains only distinct pairs must be finite.*

Proof. If $X, Y \in V_u$, then R can contain at most the single pair (X, Y).

If $X \in V_u$ and $Y \in V_n$, then $R = \{(X, Y\beta_i) \mid i \in I\}$ for some set I. For each $i \in I$, we have $Y\beta_i \xrightarrow{\sigma} \beta_i$ such that the length of σ is $n(Y)$ and thus$X \xrightarrow{\sigma} \alpha_i$ such that $\alpha_i \leftrightarrow \beta_i$. By image-finiteness there can only be finitely many such α_i and thus also only a finite number of non-bisimilar such β_i can exist. As a result there can only be a finite number of distinct pairs $(X, Y\beta_i) \in R$ and thus R must be finite. The case where $X \in V_n$ and $Y \in V_u$ is symmetric to this case.

If $X, Y \in V_n$, then $R = \{(X\alpha_i, Y\beta_i) \mid i \in I\}$ for some set I. For the sake of contradiction, assume I is infinite, and, without loss of generality assume that $n(Y) \leq n(X)$ and that $Y\beta \xrightarrow{\sigma} \beta$ such that the length of σ is $n(Y)$. For each $i \in I$ we must have $X \xrightarrow{\sigma} \gamma_i$ such that $\gamma_i\alpha_i \leftrightarrow \beta_i$. By image-finiteness there can only be finitely many such γ_i, hence there must exist some γ such that $X \xrightarrow{\sigma} \gamma$ and $\gamma\alpha_i \leftrightarrow \beta_i$ for infinitely many $i \in I$. Now let $(X\alpha_i, Y\beta_i)$ and $(X\beta_j, Y\beta_j)$ be two distinct pairs in R such that $\gamma\alpha_i \leftrightarrow \beta_i$ and $\gamma\alpha_j \leftrightarrow \beta_j$. If $\alpha_i \leftrightarrow \alpha_j$, then $\beta_i \leftrightarrow \gamma\alpha_i \leftrightarrow \gamma\alpha_j \leftrightarrow \beta_j$, contradicting distinctness. Hence, for all $i \in I$ such that $\gamma\alpha_i \leftrightarrow \beta_i$, it must be the case that all these α_i are non-bisimilar. Hence we must have $X\alpha_i \leftrightarrow Y\gamma\alpha_i$ for infinitely many non-bisimilar α_i.

Since $X\alpha_i \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, by Lemma 5.3.11, $\gamma\alpha_i \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ and thus, by Lemma 5.3.16, $Y\gamma\alpha_i \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ for all $i \in I$. Hence, we can apply Lemma 5.5.3 to deduce $X \leftrightarrow Y\gamma$, contradicting non-decomposability. We conclude I cannot be infinite and thus R must be finite. □

Lastly, before presenting the final result, we define a pre-order that allows us to not only order terms based on their norm, but also allows us to distinguish between bisimilar terms with infinite norm.

Definition 5.5.6. The *finite prefix norm* $n_f(\alpha)$ of α is defined as follows:

$$n_f(\alpha) = \max(\{n(\beta) \mid n(\beta) < \infty \text{ and } \alpha = \beta\gamma \text{ for some } \gamma\}).$$

The pre-order $\preceq$ on pairs is defined as:

$$(\alpha_1, \alpha_2) \preceq (\beta_1, \beta_2) \text{ iff } \max(n_f(\alpha_1), n_f(\alpha_2)) \leq \max(n_f(\beta_1), n_f(\beta_2)).$$

The pre-order $\preceq$ is used to determine which of two non-distinct pairs should be put in the finite bisimulation base, as presented in the following lemma.

Lemma 5.5.7. *Let Δ be a finite recursive specification in EGNF. Let,*

- $R_0 = \{(X, \alpha) \mid X \in V_n,\ \alpha \in \bar{V}^* V_{\not\downarrow} {V_\downarrow}^* \cup {V_\downarrow}^*$ *and* $X \leftrightarrow \alpha\}$*; and*
- $R_1 = \{(X, XY) \mid X \in V_u$ *and* $XY \in \bar{V}^* V_{\not\downarrow} {V_\downarrow}^* \cup {V_\downarrow}^*\}$*; and*
- $R_2 = \{(X\alpha, Y\beta) \mid X\alpha, Y\beta \in (\bar{V}^* V_{\not\downarrow} {V_\downarrow}^* \cup {V_\downarrow}^*) \cap ({V_n}^* V_u \cup {V_n}^*),\ X\alpha \leftrightarrow Y\beta$ *and* $(X\alpha, Y\beta)$ *not decomposable*$\}$ *which contains only distinct pairs, and containing minimal elements with respect to* $\preceq$*.*

Let R be the symmetric closure of $R_0 \cup R_1 \cup R_2$, then R is a finite relation satisfying $\alpha \overset{R}{\equiv} \beta$ if and only if $\alpha \leftrightarrow \beta$ for all sequences $\alpha, \beta \in \bar{V}^* V_{\not\downarrow} {V_\downarrow}^* \cup {V_\downarrow}^*$.

Proof. First, by Lemma 5.4.1 and Lemma 5.5.5 and since V is finite, R must be finite. Moreover, by Lemma 5.5.1, all pairs in R_1 are bisimilar and thus clearly all pairs in R are bisimilar. Hence, since $\leftrightarrow$ is a congruence for TSP$^{;}(A)$, $\overset{R}{\equiv} \subseteq \leftrightarrow$. It remains to show is that $\overset{R}{\equiv} \supseteq \leftrightarrow$. We prove by induction on $\preceq$ that $X\alpha \leftrightarrow Y\beta$ implies $X\alpha \overset{R}{\equiv} Y\beta$, for all $X\alpha, Y\beta \in \bar{V}^* V_{\not\downarrow} {V_\downarrow}^* \cup {V_\downarrow}^*$

Suppose that $(X\alpha, Y\beta)$ is decomposable, then $X, Y \in V_n$ and, without loss of generality, assume that $X \longrightarrow^* \gamma$ such that $X \leftrightarrow Y\gamma$ and $\gamma\alpha \leftrightarrow \beta$. Then, $n_f(\gamma\alpha) < n_f(Y\gamma\alpha) = n_f(X\alpha)$ and $n_f(\beta) < n_f(Y\beta)$, so $(\gamma\alpha, \beta) \prec (X\alpha, Y\beta)$. Furthermore, since $X \longrightarrow^* \gamma$, $X\alpha \longrightarrow^* \gamma\alpha$ and thus, by Lemma 5.3.11, we have $\gamma\alpha \in \bar{V}^* V_{\not\downarrow} {V_\downarrow}^* \cup {V_\downarrow}^*$. Moreover, since $Y\beta \in \bar{V}^* V_{\not\downarrow} {V_\downarrow}^* \cup {V_\downarrow}^*$, by Lemma 5.3.15, $\beta \in \bar{V}^* V_{\not\downarrow} {V_\downarrow}^* \cup {V_\downarrow}^*$ and thus by induction $\gamma\alpha \overset{R}{\equiv} \beta$. Finally, since $\gamma\alpha \in \bar{V}^* V_{\not\downarrow} {V_\downarrow}^* \cup {V_\downarrow}^*$, by Lemma 5.3.15, $\gamma \in \bar{V}^* V_{\not\downarrow} {V_\downarrow}^* \cup {V_\downarrow}^*$ and thus by Lemma 5.3.16, $Y\gamma \in \bar{V}^* V_{\not\downarrow} {V_\downarrow}^* \cup {V_\downarrow}^*$. Hence, $(X, Y\gamma) \in R_0$ and thus $X\alpha \overset{R}{\equiv} Y\gamma\alpha \overset{R}{\equiv} Y\beta$.

Now, suppose that $(X\alpha, Y\beta)$ is not decomposable. Then $(X\alpha', Y\beta') \in R_2$ for some $\alpha' \leftrightarrow \alpha$ and $\beta' \leftrightarrow \beta$ with $(\alpha', \beta') \preceq (\alpha, \beta)$. We distinguish three cases.

- If $X, Y \in V_n$, then $(\alpha, \beta), (\alpha', \beta') \prec (X\alpha, Y\beta)$, so $(\alpha, \alpha'), (\beta, \beta') \prec (X\alpha, Y\beta)$. Hence, by induction $\alpha \overset{R}{\equiv} \alpha'$ and $\beta \overset{R}{\equiv} \beta'$, so $X\alpha \overset{R}{\equiv} X\alpha' R Y\beta' \overset{R}{\equiv} Y\beta$.
- If $X \in V_n$ and $Y \in V_u$, then since $\beta \equiv X_1 \ldots X_n$ for some $n \geq 0$ and $YX_i \overset{R}{\equiv} Y$ for each $0 \leq i \leq n$, we find $Y\beta \overset{R}{\equiv} Y$. Furthermore, $n_f(\alpha') \leq n_f(\alpha) < n_f(X\alpha)$, so

$(\alpha, \alpha') \prec (X\alpha, Y)$. Hence, by induction $\alpha \stackrel{R}{\equiv} \alpha'$, and since $(X\alpha', Y) \in R_2$ we find $X\alpha \stackrel{R}{\equiv} X\alpha' \stackrel{R}{\equiv} Y \stackrel{R}{\equiv} Y\beta$. A symmetric argument applies for the case when $X \in V_u$ and $Y \in V_n$.

- If $X, Y \in V_u$, then since $\alpha \equiv X_1 \ldots X_n$ for some $n \geq 0$ and $XX_i \stackrel{R}{\equiv} X$ for each $0 \leq i \leq n$, we find $X\alpha \stackrel{R}{\equiv} X$. Similarly, we find $Y\beta \stackrel{R}{\equiv} Y$ and thus since $(X, Y) \in R_2$, we derive $X\alpha \stackrel{R}{\equiv} X \stackrel{R}{\equiv} Y \stackrel{R}{\equiv} Y\beta$.

□

As we have shown that there exists a finite symmetric relation R satisfying $\alpha \stackrel{R}{\equiv} \beta$ if and only if $\alpha \leftrightarrow \beta$ for all sequences $\alpha, \beta \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ and, moreover, for every pair $(\alpha, \beta) \in R$ we have $\alpha, \beta \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, by Lemma 5.4.3, we conclude that R is a bisimulation base that satisfies $\alpha \stackrel{R}{\equiv} \beta$ for all bisimilar sequences $\alpha, \beta \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$.

Corollary 5.5.8. Bisimilarity is semi-decidable for all processes of the form $\bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$, definable by finite guarded recursive specifications over $\mathrm{TSP}^{;}(A)$-terms.

Using this result it can be shown that bisimilarity is decidable for processes definable in $\mathrm{TSP}^{;}(A)$ with finite guarded recursive specifications.

Theorem 5.5.9. *Bisimilarity is decidable for all processes definable in* $\mathrm{TSP}^{;}(A)$ *with finite guarded recursive specifications.*

Proof. Let Δ be a guarded recursive $\mathrm{TSP}^{;}(A)$-specification and let $\alpha, \beta \in V(\Delta)$ be two processes. As shown by Proposition 5.3.1 and 5.3.9, Δ can be transformed to a recursive specification Δ' in EGNF, such that $\alpha, \beta \in V(\Delta')$ and $\alpha \leftrightarrow \beta$ as defined by Δ if and only if $\alpha \leftrightarrow \beta$ as defined by Δ'. Moreover, by Corollary 5.3.8 there exist sequences $\alpha', \beta' \in \bar{V}^* V_{\not\downarrow} V_{\downarrow}{}^* \cup V_{\downarrow}{}^*$ definable by Δ' such that $\alpha \leftrightarrow \alpha'$ and $\beta \leftrightarrow \beta'$, hence it is sufficient to decide bisimilarity for the pair (α', β'). By Corollary 5.1.3 non-bisimilarity is semi-decidable for (α', β'), and, by Corollary 5.5.8, bisimilarity is semi-decidable for (α', β') as well. Hence a dovetailing procedure can be used to decide bisimilarity of the pair (α', β'). □

Chapter 6

Decidability of bisimilarity for $\mathrm{TSP}^{\cdot}(A)$

In the previous chapter it is shown that bisimilarity is decidable for all processes definable in $\mathrm{TSP}^{;}(A)$ with finite guarded recursive specifications. The proof is built on two important properties.

1. For all normed sequences α it holds that $|\alpha| \leq n^{;}(\alpha)$.
2. For normed sequences α, β and γ, if $\alpha\gamma \mathrel{\underline{\leftrightarrow}} \beta\gamma$, then $\alpha \mathrel{\underline{\leftrightarrow}} \beta$.

Although these properties do not hold in general for sequences definable in $\mathrm{TSP}^{;}(A)$ with finite guarded recursive specifications, all recursive guarded recursive $\mathrm{TSP}^{;}(A)$ specifications can be transformed to a normal form such that all sequences can be transformed to a shape for which these properties do always hold.

In this chapter it is shown that these properties do not hold for all sequences definable in $\mathrm{TSP}^{\cdot}(A)$ with finite guarded recursive specifications either. Moreover, in contrast to $\mathrm{TSP}^{;}(A)$, it is argued why in case of guarded recursive $\mathrm{TSP}^{\cdot}(A)$, it is not as straightforward to transform a guarded recursive specification to a normal form, such that these properties do hold for all reachable sequences.

6.1 Length of normed sequences

The first property states that for each normed sequence α it holds that $|\alpha| \leq n^{\cdot}(\alpha)$. This property is used in the decidability proof to show that the following relation is finite:

$$R = \{(X, \alpha) \mid X \mathrel{\underline{\leftrightarrow}} \alpha \text{ and } X \text{ has a finite norm}\} \ .$$

The following example shows that this property does not hold for normed sequences of variables definable in $\mathrm{TSP}^{\cdot}(A)$ with finite guarded recursive specifications.

Example 6.1.1. Consider the following recursive $\mathrm{TSP}^{\cdot}(A)$-specification:

$$X = a.X + a.\mathbf{1} + \mathbf{1}$$

Since X can terminate $n^{\cdot}(X) = 0$. From this it immediately follows that $|X| > n^{\cdot}(X)$, invalidating the first property. Moreover, it is not difficult to see that $X \mathrel{\underline{\leftrightarrow}} XX \mathrel{\underline{\leftrightarrow}} XXX \mathrel{\underline{\leftrightarrow}} \ldots$, which means that the set $\{(X, \alpha) \mid X \mathrel{\underline{\leftrightarrow}} \alpha\}$ is infinite.

From the previous example it can be seen that the length of sequences can not easily be bound by their norm. Redefining the norm for $\mathrm{TSP}^{\cdot}(A)$ as we did with $\mathrm{TSP}^{;}(A)$, prevents this problem as all variables (of a specification in EGNF) must have a norm of at least 1. However, this definition does not make sense for the semantics of $\mathrm{TSP}^{\cdot}(A)$. For example, considering the specification from Example 6.1.1; we would have $n^{;}(X) = 1$ and $n^{;}(XXX) = 3$, while in fact $XXX \xrightarrow{a} \mathbf{1}$ and thus should have a norm of 1 as well. This is caused by transparency of processes defined in $\mathrm{TSP}^{\cdot}(A)$.

Although it is not possible to use the technique that is used to bind normed $\mathrm{TSP}^{;}(A)$-sequences, to bind normed $\mathrm{TSP}^{\cdot}(A)$-sequences, we expect that every infinite set of bisimilar normed sequences can still be captured by a finite base. The reasoning and some results regarding this hypothesis is illustrated in Appendix F.

6.2 Cancellation

The second property used in the decidability proof for $\mathrm{TSP}^{;}(A)$-processes, is the cancellation property for normed terms. For normed $\mathrm{TSP}^{\cdot}(A)$-sequences, this property does not hold, which is shown in the following example.

Example 6.2.1. Consider the following normed recursive $\mathrm{TSP}^{\cdot}(A)$-specification:

$$X = a.\mathbf{1} + \mathbf{1} \qquad Y = b.\mathbf{1} + \mathbf{1} \qquad Z = a.Z + b.Z + \mathbf{1}$$

It is not difficult to see that $XZ \mathrel{\underline{\leftrightarrow}} YZ$. However, the cancellation property does not hold as clearly $X \not\mathrel{\underline{\leftrightarrow}} Y$.

Again, the intermediate termination in combination with sequential composition gives some undesired results. Even though $XZ \mathrel{\underline{\leftrightarrow}} YZ$, X and Y do not have the same behaviour. This is caused by the fact that if XZ executes some action from X that Y is not able to mimic, then YZ can simply skip Y and mimic the behaviour by executing the action from Z, and vice versa. This problem is solved in the semantics of $\mathrm{TSP}^{;}(A)$, as in this case skipping over a variable in a sequence is not allowed as long as it is possible to execute an action from that variable.

It is clear that the cancellation property does not hold for normed $\mathrm{TSP}^{\cdot}(A)$-sequences. Moreover, it can be shown that the unique solutions property used in the cancellation lemma for unnormed sequences, does not work for $\mathrm{TSP}^{\cdot}(A)$-sequences either.

Example 6.2.2. Consider the following recursive $\mathrm{TSP}^{\cdot}(A)$-specification:

$$X = a.X + b.X \qquad Y = a.Y + c.Y \qquad Z = a.\mathbf{1} + \mathbf{1}$$

Clearly $X \mathrel{\underline{\leftrightarrow}} ZX$ and $Y \mathrel{\underline{\leftrightarrow}} ZY$, but $X \not\mathrel{\underline{\leftrightarrow}} Y$.

The same issue that occurs for the normed cancellation happens when trying to apply this unique solutions lemma. Since Z is able to terminate, if X executes some actions that Z is not able to mimic, ZX can simply skip Z and perform the action from X. As a result, there is no guarantee ZY is able to mimic this action and thus Y does not have to be able to mimic it either.

As the cancellation property does not holds for normed $\mathrm{TSP}^{\cdot}(A)$-terms, this property cannot be used to show that the number of distinct sequences that cannot be decomposed must be finite, as is done in the proof of decidability for $\mathrm{TSP}^{;}(A)$-terms.

Chapter 7

Conclusions

In this master thesis we have done research on context-free processes with intermediate termination. In particular, we have answered several open research questions regarding the theory of sequential processes $\mathrm{TSP}^{;}(A)$. The starting point for this research was the paper published by Baeten et al. [6]. They introduced a revised semantics for the sequential composition and showed that these semantics eliminate the notion of transparency that exist with the standard operational semantics for sequential composition. Two important research questions were answered regarding $\mathrm{TSP}^{;}(A)$. The first question that we looked into is the following:

Does bisimilarity afford a ground-complete axiomatisation for $\mathrm{TSP}^{;}(A)$*?*

To answer this question, we attempted to adapt the ground-complete axiomatisation of $\mathrm{TSP}^{\cdot}(A)$ for $\mathrm{TSP}^{;}(A)$. It turned out that a straightforward adaptation is not possible, as the revised semantics do not allow distribution of sequential composition over alternative composition. To solve this issue, we introduced the auxiliary operator NT, that expresses the non-terminating part of a term, i.e., it removes intermediate termination. Using this auxiliary operator, we were able to distinguish three cases and express the distribution of the sequential composition over alternative composition for each of these cases with an axiom. As a result we were able to define a ground-complete axiomatisation for $\mathrm{TSP}^{;}_{+NT}(A)$. Additionally we showed that without an auxiliary operator, bisimilarity does not afford a finite ground-complete axiomatisation for $\mathrm{TSP}^{;}(A)$. In order to prove this result, we provided an infinite set of $\mathrm{TSP}^{;}(A)$-terms with redundant intermediate termination. It was shown that in order to derive that this redundant intermediate termination can be removed, an infinite set of axioms is needed. Finally, we attempted to expand the ground-complete axiomatisation for $\mathrm{TSP}^{;}_{+NT}(A)$ to an ω-complete axiomatisation. We found that it is not possible to define a finite ω-complete axiomatisation for $\mathrm{TSP}^{;}_{+NT}(A)$ with respect to bisimilarity. Similar to $\mathrm{TSP}^{;}(A)$, there exists an infinite set of sound $\mathrm{TSP}^{;}_{+NT}(A)$-terms with redundant intermediate termination, for which we need an infinite set of axioms to be able to derive that the intermediate termination can be removed.

The second question that is answered in this thesis is the following:

Is bisimilarity decidable for processes definable in $\mathrm{TSP}^{;}(A)$ *with finite guarded recursive specifications?*

As a starting point for this question we studied the proofs for decidability of bisimilarity for BPA-processes [11, 9]. These proofs all use two important properties of processes defined by a specification in GNF. The first property states that the length of all sequences representing normed processes is bounded. This is not the case for arbitrary $\mathrm{TSP}^{;}(A)$ processes. We solved this issue by redefining the norm of $\mathrm{TSP}^{;}(A)$-processes. Not only does this new definition ensure that all normed sequences are bounded, this definition is also more suitable for $\mathrm{TSP}^{;}(A)$, as with the other definition normed $\mathrm{TSP}^{;}(A)$ processes cannot always terminate. The second important property used in the decidability proofs, is the cancellation property. We showed that an arbitrary recursive $\mathrm{TSP}^{;}(A)$-specification can be transformed to an Extended Greibach Normal Form (EGNF). For all sequences defined by a specification in EGNF, there exists a bisimilar sequence defined by that same specification of a particular shape. All sequences of this shape do not contain any redundant termination. Moreover, after executing an action, the resulting sequence is still of this particular shape, and thus does not contain any redundant termination either. Finally, it is shown that for sequences of this shape, cancellation can be applied. As we have defined a normal form for which the two properties are valid, we were able to reuse the standard decidability proof to prove that bisimilarity is decidable for processes definable in $\mathrm{TSP}^{;}(A)$ with finite guarded recursive specifications.

Finally, we have shown that the techniques used in the proof of decidability of bisimilarity for $\mathrm{TSP}^{;}(A)$-terms, cannot be used to prove decidability of bisimilarity for $\mathrm{TSP}^{\cdot}(A)$-terms. This is caused by the fact that the set of terms that are bisimilar to a normed $\mathrm{TSP}^{\cdot}(A)$-term may be infinite and, moreover, the cancellation property does not holds for normed $\mathrm{TSP}^{\cdot}(A)$-terms.

7.1 Future work

In this thesis we have proposed a ground-complete axiomatisation for $\mathrm{TSP}^{;}_{+NT}(A)$. As it turned out that bisimilarity does not afford an ω-complete axiomatisation for $\mathrm{TSP}^{;}_{+NT}(A)$, one may question if there exists another auxiliary operator that does afford an ω-complete axiomatisation. A possible solution would be to add an operator that expresses the terminating part of a term, and an operator that expresses that a term cannot execute an action. Using these operators we should be able to express the distribution of sequential composition over alternative composition for all (open) terms.

Furthermore, we have shown that bisimilarity is decidable for processes definable in $\mathrm{TSP}^{;}(A)$ with finite guarded recursive specifications, yet the question of decidability for $\mathrm{TSP}^{\cdot}(A)$-processes remains unsolved. While we have argued that the properties used in the proof of decidability for $\mathrm{TSP}^{;}(A)$-processes cannot be used for $\mathrm{TSP}^{\cdot}(A)$-processes, we expect that every infinite set of bisimilar normed sequences can still be captured by a finite base. Appendix F illustrates the reasoning behind this hypothesis. Moreover, we have analysed several infinite sets of bisimilar normed terms and have shown that these sets can be captured by a finite base. The observations made in this appendix may be useful in finding a finite bisimulation base for normed $\mathrm{TSP}^{\cdot}(A)$-terms.

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Appendix A

Soundness of $\text{TSP}^{;}_{+NT}(A)$

This appendix contains the soundness proofs for the axioms of $\text{TSP}^{;}_{+NT}(A)$. The proofs for axioms $A\mathbf{1}^{;}$, $A2^{;}$, $A3^{;}$ and $A6^{;}$ are omitted as it is a well known result that these axioms are sound with respect to bisimilarity.

Lemma A.1 ($A5^{;}$)**.** For all closed $\text{TSP}^{;}_{+NT}(A)$-terms p, q and r, $(p\,;q)\,;r \leftrightarrows p\,;(q\,;r)$.

Proof. To this end, we verify that the symmetric closure of

$$R = \{(((p\,;q)\,;r),(p\,;(q\,;r))) \mid p,q,r \in \text{TSP}^{;}_{+NT}(A)\} \cup \{(p,p) \mid p \in \text{TSP}^{;}_{+NT}(A)\}.$$

is a bisimulation relation, by considering all pairs of the shape $((p;q);r),(p;(q;r))$. The proof for all other pairs in the symmetric closure of R is trivial or requires a similar argument.

Suppose $(p\,;q)\,;r \xrightarrow{a} s$. We distinguish two cases:

- If $(p\,;q) \xrightarrow{a} s'$ and $s = s'\,;r$, then, again, we distinguish two cases:
  - If $p \xrightarrow{a} p'$ and $s' = p'\,;q$ and thus $s = (p'\,;q)\,;r$. Then since $p \xrightarrow{a} p'$ we have $p\,;(q\,;r) \xrightarrow{a} p'\,;(q\,;r)$, and clearly $(((p'\,;q)\,;r),(p'\,;(q\,;r))) \in R$.
  - If $p\downarrow$, $q \xrightarrow{a} q'$ and $p\not\rightarrow$, then $s' = q'$ and thus $s = q'\,;r$. Then we also have $p\,;(q\,;r) \xrightarrow{a} q'\,;r$, and clearly $(q'\,;r,q'\,;r) \in R$.
- If $p\,;q\downarrow$, $r \xrightarrow{a} r'$ and $p\,;q\not\rightarrow$ and $s = r'$, then, since $p\,;q\downarrow$ we have $p\downarrow$ and $q\downarrow$. Furthermore, since $p\,;q\not\rightarrow$ and $p\downarrow$, we have $p\not\rightarrow$ and $q\not\rightarrow$. Therefore, $p\,;(q\,;r) \xrightarrow{a} r'$, and clearly $(r',r') \in R$.

Suppose $(p\,;q)\,;r\downarrow$. Then we must have $p\,;q\downarrow$ and $r\downarrow$. Since $p\,;q\downarrow$ we also have $p\downarrow$ and $q\downarrow$. Therefore, clearly $p\,;(q\,;r)\downarrow$. □

Lemma A.2 ($A7^{;}$)**.** For all closed $\text{TSP}^{;}_{+NT}(A)$-terms p, $\mathbf{0}\,;p \leftrightarrows \mathbf{0}$.

Proof. To this end, we verify that the symmetric closure of

$$R = \{((\mathbf{0}\,;p),\mathbf{0}) \mid p \in \text{TSP}^{;}_{+NT}(A)\}$$

is a bisimulation relation, by considering all pairs of the shape $((\mathbf{0}\,;p),\mathbf{0})$. The proof for all other pairs in the symmetric closure of R is similar.

It is sufficient to note that since $\mathbf{0}\not\rightarrow$ and $\mathbf{0}\not\downarrow$, we have $\mathbf{0}\,;p\not\rightarrow$ and $\mathbf{0}\,;p\not\downarrow$ and hence there is nothing to proof. □

Lemma A.3 ($A8^{;}$)**.** For all closed $\mathrm{TSP}^{;}_{+NT}(A)$-terms p, $p\,;\mathbf{1} \leftrightarrows p$.

Proof. To this end, we verify that the symmetric closure of

$$R = \{((p\,;\mathbf{1}), p) \mid p \in \mathrm{TSP}^{;}_{+NT}(A)\}$$

is a bisimulation relation, by considering all pairs of the shape $((p\,;\mathbf{1}), p)$. The proof for all other pairs in the symmetric closure of R is similar.

Suppose that $p\,;\mathbf{1} \xrightarrow{a} s$. Then we must have $p \xrightarrow{a} p'$ and $s = p'\,;\mathbf{1}$. Then $p \xrightarrow{a} p'$ and $((p'\,;\mathbf{1}), p') \in R$. If $p\,;\mathbf{1}\downarrow$, then we must have $p\downarrow$. □

Lemma A.4 ($A9^{;}$)**.** For all closed $\mathrm{TSP}^{;}_{+NT}(A)$-terms p, $\mathbf{1}\,;p \leftrightarrows p$.

Proof. To this end, we verify that the symmetric closure of

$$R = \{((\mathbf{1}\,;p), p) \mid p \in \mathrm{TSP}^{;}_{+NT}(A)\} \cup \{(p,p) \mid p \in \mathrm{TSP}^{;}_{+NT}(A)\}$$

is a bisimulation relation, by considering all pairs of the shape $((\mathbf{1}\,;p), p)$.The proof for all other pairs in the symmetric closure of R is trivial or similar.

Suppose that $\mathbf{1}\,;p \xrightarrow{a} s$. Then we must have $\mathbf{1}\downarrow$, $\mathbf{1}\not\rightarrow$, $p \xrightarrow{a} p'$ and $s = p'$. Then $p \xrightarrow{a} p'$ and $(p', p') \in R$. If $\mathbf{1}\,;p\downarrow$, then we must have $p\downarrow$. □

Lemma A.5 ($A\mathbf{10}^{;}$)**.** For all closed $\mathrm{TSP}^{;}_{+NT}(A)$-terms p and q, $a.p\,;q \leftrightarrows a.(p\,;q)$.

Proof. To this end, we verify that the symmetric closure of

$$R = \{((a.p\,;q), a.(p\,;q)) \mid p,q \in \mathrm{TSP}^{;}_{+NT}(A)\} \cup \{(p,p) \mid p \in \mathrm{TSP}^{;}_{+NT}(A)\}$$

is a bisimulation relation, by considering all pairs of the shape $((a.p\,;q), a.(p\,;q))$. The proof for all other pairs in the symmetric closure of R is trivial or similar.

Suppose $a.p\,;q \xrightarrow{b} s$. Then we must have $b = a$ and $s = p\,;q$. Now clearly, $a.(p\,;q) \xrightarrow{a} p\,;q$ and $(p\,;q, p\,;q) \in R$. Furthermore, note that since $a.p\not\downarrow$, we also have $a.p\,;q\not\downarrow$. □

Lemma A.6 ($A11^{;}$)**.** For all closed $\mathrm{TSP}^{;}_{+NT}(A)$-terms p, q and r, $NT(p+q)\,;r \leftrightarrows NT(p)\,;r + NT(q)\,;r$.

Proof. To this end, we verify that the symmetric closure of

$$R = \{((NT(p+q);r), (NT(p);r{+}NT(q);r)) \mid p,q,r \in \mathrm{TSP}^{;}_{+NT}(A)\} \cup \{(p,p) \mid p \in \mathrm{TSP}^{;}_{+NT}(A)\}$$

is a bisimulation relation, by considering all pairs of the shape $((NT(p+q)\,;r), (NT(p)\,;r + NT(q)\,;r))$. The proof for all other pairs in the symmetric closure of R is trivial or similar.

Suppose $NT(p+q)\,;r \xrightarrow{a} s$, then since $NT(p+q)\not\downarrow$ we must have $NT(p+q) \xrightarrow{a} s'$ and $s = s'\,;r$. Hence, we have $p + q \xrightarrow{a} s'$ and distinguish two cases:

- If $p \xrightarrow{a} p'$ and $s = p';r$, then $NT(p) \xrightarrow{a} p'$ and thus $NT(p)\,;r \xrightarrow{a} p'\,;r$ and also $NT(p)\,;r + NT(q)\,;r \xrightarrow{a} p'\,;r$ and clearly $(p'\,;r, p'\,;r) \in R$.
- If $q \xrightarrow{a} q'$ and $s = q';r$, then $NT(q) \xrightarrow{a} q'$ and thus $NT(q)\,;r \xrightarrow{a} q'\,;r$ and also $NT(p)\,;r + NT(q)\,;r \xrightarrow{a} q'\,;r$ and clearly $(q'\,;r, q'\,;r) \in R$.

Furthermore, note that since $NT(p+q)\not\downarrow$, we have $NT(p+q)\,;r\not\downarrow$. □

Lemma A.7 ($A12^{;}$)**.** For all closed $\mathrm{TSP}^{;}_{+NT}(A)$-terms p, q and r, $(a.p + q + \mathbf{1}) ; NT(r) \leftrightarrow (a.p + q) ; NT(r)$.

Proof. To this end, we verify that the symmetric closure of

$$R = \{((a.p{+}q{+}\mathbf{1});NT(r), (a.p{+}q);NT(r)) \mid p, q, r \in \mathrm{TSP}^{;}_{+NT}(A)\} \cup \{(p, p) \mid p \in \mathrm{TSP}^{;}_{+NT}(A)\}$$

is a bisimulation relation, by considering all pairs of the shape $((a.p + q + \mathbf{1}) ; NT(r), (a.p + q) ; NT(r))$. The proof for all other pairs in the symmetric closure of R is trivial or similar.

Suppose $(a.p + q + \mathbf{1}) ; NT(r) \xrightarrow{b} s$. Then we must have $a.p + q + 1 \xrightarrow{b} s'$, and thus $s = s' ; NT(r)$. Since $\mathbf{1}\not\rightarrow$, we must have $a.p + q \xrightarrow{b} s'$, and hence $(a.p + q) ; NT(r) \xrightarrow{b} s' ; NT(r)$, and clearly $(s' ; NT(r), s' ; NT(r)) \in R$. Furthermore, note that since $NT(r) \not\downarrow$, $(a.p + q + \mathbf{1}) ; NT(r) \not\downarrow$. ☐

Lemma A.8 ($A13^{;}$)**.** For all closed $\mathrm{TSP}^{;}_{+NT}(A)$-terms p, q and r, $(a.p + q + \mathbf{1}) ; (r + \mathbf{1}) \leftrightarrow (a.p + q) ; (r + \mathbf{1}) + \mathbf{1}$

Proof. To this end, we verify that the symmetric closure of

$$R = \{((a.p{+}q{+}\mathbf{1});(r{+}\mathbf{1}), (a.p{+}q);(r{+}\mathbf{1}){+}\mathbf{1}) \mid p, q, r \in \mathrm{TSP}^{;}_{+NT}(A)\} \cup \{(p, p) \mid p \in \mathrm{TSP}^{;}_{+NT}(A)\}$$

is a bisimulation relation, by considering all pairs of the shape $((a.p{+}q{+}\mathbf{1}) ; (r{+}\mathbf{1}), (a.p{+}q) ; (r + \mathbf{1}) + \mathbf{1})$. The proof for all other pairs in the symmetric closure of R is trivial or similar.

Suppose $(a.p + q + \mathbf{1}) ; (r + \mathbf{1}) \xrightarrow{b} s$. Then we must have $a.p + q + \mathbf{1} \xrightarrow{b} s'$, and thus $s = s' ; (r{+}\mathbf{1})$. Since $1\not\rightarrow$, we must have $a.p{+}q \xrightarrow{b} s'$, and hence $(a.p{+}q) ; (r{+}\mathbf{1}) \xrightarrow{b} s' ; (r{+}\mathbf{1})$, and clearly $(s' ; (r + \mathbf{1}), s' ; (r + \mathbf{1})) \in R$.

Suppose $(a.p{+}q{+}\mathbf{1}) ; (r{+}\mathbf{1}) \downarrow$, then clearly since $\mathbf{1} \downarrow$ we also have $(a.p{+}q) ; (r{+}\mathbf{1}){+}\mathbf{1} \downarrow$. ☐

Lemma A.9 ($NT1$)**.** It holds that $NT(\mathbf{0}) \leftrightarrow \mathbf{0}$.

Proof. To this end, we verify that the symmetric closure of

$$R = \{(NT(\mathbf{0}), \mathbf{0})\}$$

is a bisimulation relation, by considering the pair $(NT(\mathbf{0}), \mathbf{0})$. The proof for the other pair in the symmetric closure of R is similar.

It is sufficient to note that since $\mathbf{0}\not\rightarrow$ we have $NT(\mathbf{0})\not\rightarrow$. Also $NT(\mathbf{0}) \not\downarrow$ and hence there is nothing to proof. ☐

Lemma A.10 ($NT2$)**.** It holds that $NT(\mathbf{1}) \leftrightarrow \mathbf{0}$.

Proof. To this end, we verify that the symmetric closure of

$$R = \{(NT(\mathbf{1}), \mathbf{0})\}$$

is a bisimulation relation, by considering the pair $(NT(\mathbf{1}), \mathbf{0})$. The proof for the other pair in the symmetric closure of R is similar.

It is sufficient to note that since $\mathbf{1}\not\rightarrow$ we have $NT(\mathbf{1})\not\rightarrow$. Also $NT(\mathbf{1}) \not\downarrow$ and hence there is nothing to proof. ☐

Lemma A.11 ($NT3$)**.** For all closed $\mathrm{TSP}^{;}_{+NT}(A)$-terms p, $NT(a.p) \leftrightarrow a.p$.

Proof. To this end, we verify that the symmetric closure of

$$R = \{(NT(a.p), a.p) \mid p \in \mathrm{TSP}^{;}_{+NT}(A)\} \cup \{(p,p) \mid p \in \mathrm{TSP}^{;}_{+NT}(A)\}$$

is a bisimulation relation, by considering all pairs of the shape $(NT(a.p), a.p)$. The proof for all other pairs in the symmetric closure of R is trivial or similar.

Suppose $NT(a.p) \xrightarrow{b} s$. Then, we must have $b = a$ and $s = p$ and clearly $a.p \xrightarrow{a} p$ and $(p,p) \in R$. Furthermore, note that $NT(a.p) \not\downarrow$. □

Lemma A.12 ($NT4$)**.** For all closed $\mathrm{TSP}^{;}_{+NT}(A)$-terms p and q, $NT(p+q) \leftrightarrow NT(p) + NT(q)$.

Proof. To this end, we verify that the symmetric closure of

$$R = \{(NT(p+q), (NT(p) + NT(q))) \mid p, q \in \mathrm{TSP}^{;}_{+NT}(A)\} \cup \{(p,p) \mid p \in \mathrm{TSP}^{;}_{+NT}(A)\}$$

is a bisimulation relation, by considering all pairs of the shape $(NT(p+q), (NT(p)+NT(q)))$. The proof for all other pairs in the symmetric closure of R is trivial or similar.

Suppose $NT(p+q) \xrightarrow{a} s$. Then, $p+q \xrightarrow{a} s$ and thus either $p \xrightarrow{a} s$ and $NT(p) \xrightarrow{a} s$; or $q \xrightarrow{a} s$ and $NT(q) \xrightarrow{a} s$. In both cases $NT(p) + NT(q) \xrightarrow{a} s$ and clearly $(s,s) \in R$. Furthermore, note that $NT(p+q) \not\downarrow$. □

Appendix B

Ground-completeness of $\mathrm{TSP}^{;}_{+NT}(A)$

This appendix contains the full proof of Theorem B.8, that is used in the proof of ground-completeness of $\mathrm{TSP}^{;}_{+NT}(A)$. The theorem utilizes Lemma B.6 and Lemma B.7, that state that the sequential composition operator and NT-operator can be eliminated from $\mathrm{TSP}^{;}_{+NT}(A)$-terms. Lemma B.1 to Lemma B.5 are needed in the proof of Lemma B.6. Note that in this appendix, $\mathrm{TSP}^{;}_{+NT}(A) \vdash p = q$ is often abbreviated by writing $p = q$.

Lemma B.1. For every closed BSP(A)-term p, if $p \not\downarrow$ then $NT(p) = p$.

Proof. Let p be a closed BSP(A)-term such that $p \not\downarrow$. We prove by induction on the structure of p that $NT(p) = p$.

- If $p \equiv \mathbf{0}$, then by $NT1$, $NT(p) = \mathbf{0} \equiv p$.
- If $p \equiv \mathbf{1}$, then $p \downarrow$, so the implication vacuously holds.
- If $p \equiv a.p'$, then by $NT3$, $NT(p) = a.p' \equiv p$.
- Suppose $p \equiv p_1 + p_2$, and suppose that $p_1 \not\downarrow$ implies that $NT(p_1) = p_1$ and $p_2 \not\downarrow$ implies that $NT(p_2) = p_2$ (IH). Since $p \not\downarrow$ we must have $p_1 \not\downarrow$ and $p_2 \not\downarrow$. Hence, by $NT4$ and IH, $NT(p) = NT(p_1) + NT(p_2) = p_1 + p_2 \equiv p$.

□

Lemma B.2. For every closed BSP(A)-term p, if $p \downarrow$ then $\mathbf{1} + NT(p) = p$.

Proof. Let p be a closed BSP(A)-term such that $p \downarrow$. We prove by induction on the structure of p that $\mathbf{1} + NT(p) = p$.

- If $p \equiv \mathbf{0}$, then $p \not\downarrow$, so the implication vacuously holds.
- If $p \equiv \mathbf{1}$, then by $NT2$ and $A6^{;}$, $\mathbf{1} + NT(p) = \mathbf{1} + \mathbf{0} = \mathbf{1} \equiv p$.
- If $p \equiv a.p'$, then $p \not\downarrow$, so the implication vacuously holds.
- Suppose $p \equiv p_1 + p_2$, and suppose that $p_1 \downarrow$ implies that $\mathbf{1} + NT(p_1) = p_1$, and $p_2 \downarrow$ implies that $\mathbf{1} + NT(p_2) = p_2$ (IH). Since $p \downarrow$, we must have $p_1 \downarrow$ or $p_2 \downarrow$ or both. We distinguish three cases:

– If $p_1 \downarrow$ and $p_2 \not\downarrow$, then $p \downarrow$, and by $NT4$, IH and Lemma B.1, $\mathbf{1} + NT(p) = \mathbf{1} + NT(p_1) + NT(p_2) \equiv p_1 + NT(p_2) = p_1 + p_2 \equiv p$.

– If $p_1 \not\downarrow$ and $p_2 \downarrow$, then $p \downarrow$, and by $NT4$, $A1^{;}$, IH and Lemma B.1, $\mathbf{1} + NT(p) = \mathbf{1} + NT(p_1) + NT(p_2) = NT(p_1) + \mathbf{1} + NT(p_2) = NT(p_1) + p_2 = p_1 + p_2 \equiv p$.

– If $p_1 \downarrow$ and $p_2 \downarrow$, then $p \downarrow$, and by $NT4$, $A3^{;}$, $A1^{;}$ and IH, $1 + NT(p) = 1 + NT(p_1) + NT(p_2) = \mathbf{1} + \mathbf{1} + NT(p_1) + NT(p_2) = \mathbf{1} + NT(p_1) + \mathbf{1} + NT(p_2) = p_1 + p_2 \equiv p$.

□

Lemma B.3. For every closed BSP(A)-term p, if $p \not\rightarrow$ then $NT(p) = \mathbf{0}$.

Proof. Let p be a closed BSP(A)-term such that $p \not\rightarrow$. We prove by induction on the structure of p that $NT(p) = \mathbf{0}$.

- If $p \equiv \mathbf{0}$, then by $NT1$, $NT(p) = \mathbf{0}$.
- If $p \equiv \mathbf{1}$, then by $NT2$, $NT(p) = \mathbf{0}$.
- If $p \equiv a.p'$, then $p \xrightarrow{a} p'$, so the implication vacuously holds.
- Suppose $p \equiv p_1 + p_2$, and suppose that $p_1 \not\rightarrow$ implies that $NT(p_1) = \mathbf{0}$, and $p_2 \not\rightarrow$ implies that $NT(p_2) = \mathbf{0}$ (IH). Since $p \not\rightarrow$, we must have $p_1 \not\rightarrow$ and $p_2 \not\rightarrow$, hence by $NT4$, IH and $A3^{;}$, $NT(p) = NT(p_1) + NT(p_2) = \mathbf{0} + \mathbf{0} = \mathbf{0}$.

□

Lemma B.4. For every closed BSP(A)-term p, if $p \xrightarrow{a} p'$ then $p = a.p' + p$.

Proof. Let p be a closed BSP(A)-term such that $p \xrightarrow{a} p'$. We prove by induction on the structure of p that $p = a.p' + p$.

- If $p \equiv \mathbf{0}$, then $p \not\rightarrow$, so the implication vacuously holds.
- If $p \equiv \mathbf{1}$, then $p \not\rightarrow$, so the implication vacuously holds.
- If $p \equiv b.p'$, then $p \xrightarrow{b} p'$ and by $A3^{;}$, $p = p + p \equiv b.p' + p$
- Suppose $p \equiv p_1 + p_2$, and suppose that $p_1 \xrightarrow{a} p_1'$ implies that $p_1 = a.p_1' + p_1$, and $p_2 \xrightarrow{a} p_2'$ implies that $p_2 = a.p_2' + p_2$ (IH). Suppose $p \xrightarrow{a} p'$, then either
  – $p_1 \xrightarrow{a} p'$, and by IH, $p = a.p' + p_1 + p_2 = a.p' + p$; or
  – $p_2 \xrightarrow{a} p'$, and by IH and $A1^{;}$, $p = p_1 + a.p' + p_2 = a.p' + p$.

□

Lemma B.5. For all closed BSP(A)-terms p and q, there exists a closed BSP(A)-term r such that $NT(p)\,;q = r$.

Proof. Let p and q be closed BSP(A)-terms. We will prove by induction on the structure of p that there exists a closed BSP(A)-term r such that $NT(p)\,;q = r$.

- If $p \equiv \mathbf{0}$, then by $NT1$ and $A7^{;}$, $NT(p)\,;q = \mathbf{0}\,;q = \mathbf{0}$, which is a BSP($A$)-term.

- If $p \equiv \mathbf{1}$, then by $NT2$ and $A7^{;}$, $NT(p)\,;q = \mathbf{0}\,;q = \mathbf{0}$, which is a BSP($A$)-term.
- Suppose $p \equiv a.p'$, and suppose that for every closed BSP(A)-term q there exists a BSP(A)-term r' such that $NT(p')\,;q = r'$ (IH). Then by $NT3$ and $A10^{;}$, $NT(p)\,;q = a.p'\,;q = a.(p'\,;q)$. We distinguish two cases:
  - If $p' \not\downarrow$, then by Lemma B.1 and IH, $NT(p)\,;q = a.(p'\,;q) = a.(NT(p')\,;q) = a.r'$, which is a BSP($A$)-term.
  - If $p' \downarrow$, then by Lemma B.2, $NT(p)\,;q = a.((\mathbf{1} + NT(p'))\,;q)$. We distinguish two cases:
    * If $p' \not\rightarrow$, then by Lemma B.3, $A6^{;}$ and $A9^{;}$, $NT(p);q = a.((\mathbf{1}+\mathbf{0});q) = a.(\mathbf{1};q) = a.q$, which is a BSP($A$)-term.
    * If $p' \xrightarrow{b} p''$ for some action $b \in A$ and some closed BSP(A)-term p'', we derive:

$$\begin{aligned} NT(p)\,;q &= a.((\mathbf{1} + NT(b.p'' + p'))\,;q) && \{\text{Lemma B.4}\}\\ &= a.((\mathbf{1} + NT(b.p'') + NT(p'))\,;q) && \{NT4\}\\ &= a.((\mathbf{1} + b.p'' + NT(p'))\,;q) && \{NT3\}\\ &= a.((b.p'' + NT(p') + \mathbf{1})\,;q) && \{A1^{;}\} \end{aligned}$$

      We distinguish two cases:

      · If $q \not\downarrow$, we derive:

$$\begin{aligned} NT(p)\,;q &= a.((b.p'' + NT(p') + \mathbf{1})\,;NT(q)) && \{\text{Lemma B.1}\}\\ &= a.((b.p'' + NT(p'))\,;NT(q)) && \{A12^{;}\}\\ &= a.((NT(b.p'') + NT(p'))\,;q) && \{NT3\}\\ &= a.(NT(b.p'' + p')\,;q) && \{NT4\}\\ &= a.(NT(p')\,;q) && \{\text{Lemma B.4}\}\\ &= a.r' && \{\text{IH}\} \end{aligned}$$

      Hence, $NT(p)\,;q = a.r'$, which is a BSP(A)-term.

      · If $q \downarrow$, we derive:

$$\begin{aligned} NT(p)\,;q &= a.((b.p'' + NT(p') + \mathbf{1})\,;(NT(q) + \mathbf{1})) && \{\text{Lemma B.2}\}\\ &= a.((b.p'' + NT(p'))\,;(NT(q) + \mathbf{1}) + \mathbf{1}) && \{A13^{;}\}\\ &= a.(((b.p'' + NT(p'))\,;q) + \mathbf{1}) && \{\text{Lemma B.2}\}\\ &= a.(((NT(b.p'') + NT(p'))\,;q) + \mathbf{1}) && \{NT3\}\\ &= a.((NT(b.p'' + p')\,;q) + \mathbf{1}) && \{NT4\}\\ &= a.((NT(p')\,;q) + \mathbf{1}) && \{\text{Lemma B.4}\}\\ &= a.(r' + \mathbf{1}) && \{\text{IH}\} \end{aligned}$$

      Hence, $NT(p)\,;q = a.(r' + \mathbf{1})$, which is a BSP($A$)-term.

- Suppose $p \equiv p_1 + p_2$, and suppose that for every closed BSP(A)-term q there exist BSP(A)-terms r_1 and r_2 such that $NT(p_1)\,;q = r_1$ and $NT(p_2)\,;q = r_2$ (IH). Then by $A11^{;}$ and IH, $NT(p)\,;q = NT(p_1)\,;q + NT(p_2)\,;q = r_1 + r_2$, which is a BSP($A$)-term.

□

Lemma B.6. For all closed BSP(A)-terms p and q, there exists a closed BSP(A)-term r such that $p\,;q = r$.

Proof. Let p and q be closed BSP(A)-terms. We will prove by case distinction that there exists a closed BSP(A)-term r such that $p\,;q = r$.

- If $p \not\downarrow$, then by Lemma B.1, $p\,;q = NT(p)\,;q$. By Lemma B.5 there exists a BSP(A)-term r such that $NT(p)\,;q = r$, and thus $p\,;q = r$, which is a BSP(A)-term.
- If $p \downarrow$, then by Lemma B.2, $p\,;q = (\mathbf{1} + NT(p))\,;q$. We distinguish 2 cases:
  - If $p \not\rightarrow$, then by Lemma B.3, $A6^{;}$ and $NT2$, $p\,;q = (\mathbf{1} + \mathbf{0})\,;q = \mathbf{1}\,;q = q$, which is a BSP($A$)-term.
  - If $p \xrightarrow{a} p'$, we derive:

$$\begin{aligned} p\,;q &= (\mathbf{1} + NT(a.p' + p))\,;q && \{\text{Lemma B.4}\} \\ &= (\mathbf{1} + NT(a.p') + NT(p))\,;q && \{NT4\} \\ &= (\mathbf{1} + a.p' + NT(p))\,;q && \{NT3\} \\ &= (a.p' + NT(p) + \mathbf{1})\,;q && \{A1\} \end{aligned}$$

    We distinguish two cases:

    * If $q \not\downarrow$, we derive:

$$\begin{aligned} p\,;q &= (a.p' + NT(p) + \mathbf{1})\,;NT(q) && \{\text{Lemma B.1}\} \\ &= (a.p' + NT(p))\,;NT(q) && \{A12^{;}\} \\ &= (NT(a.p') + NT(p))\,;q && \{\text{Lemma B.1}\} \\ &= NT(a.p' + p)\,;q && \{NT4\} \\ &= NT(p)\,;q && \{\text{Lemma B.4}\} \end{aligned}$$

      By lemma B.5 there exists a BSP(A)-term r such that $NT(p)\,;q = r$, and thus $p\,;q = r$, which is a BSP(A)-term.

    * If $q \downarrow$, we derive:

$$\begin{aligned} p\,;q &= (a.p' + NT(p) + \mathbf{1})\,;(\mathbf{1} + NT(q)) && \{\text{Lemma B.2}\} \\ &= (a.p' + NT(p))\,;(\mathbf{1} + NT(q)) + \mathbf{1} && \{A13^{;}\} \\ &= (a.p' + NT(p))\,;q + \mathbf{1} && \{\text{Lemma B.2}\} \\ &= (NT(a.p') + NT(p))\,;q + \mathbf{1} && \{NT3\} \\ &= NT(a.p' + p)\,;q + \mathbf{1} && \{A4^{;}\} \\ &= NT(p)\,;q + \mathbf{1} && \{\text{Lemma B.4}\} \end{aligned}$$

      By lemma B.5 there exists a BSP(A)-term r such that $NT(p)\,;q = r$, and thus $p\,;q = r + \mathbf{1}$, which is a BSP(A)-term.

□

Lemma B.7. For every closed BSP(A)-term p, there exists a closed BSP(A)-term q such that $NT(p) = q$.

Proof. Let p be a closed BSP(A)-term. We will prove induction on the structure of p that there exists a closed BSP(A)-term q such that $NT(p) = q$.

- If $p \equiv \mathbf{0}$, then by $NT1$, $NT(p) = \mathbf{0}$, which is a BSP(A)-term.
- If $p \equiv \mathbf{1}$, then by $NT2$, $NT(p) = \mathbf{0}$, which is a BSP(A)-term.
- If $p \equiv a.p'$, then by $NT3$, $NT(p) = a.p' \equiv p$, which is a BSP(A)-term.
- Suppose $p \equiv p_1 + p_2$ and suppose there exist closed BSP(A)-terms q_1 and q_2 such that $NT(p_1) = q_1$ and $NT(p_2) = q_2$ (IH). Then by $NT4$ and IH, $NT(p) = NT(p_1) + NT(p_2) = q_1 + q_2$, which is a BSP($A$)-term.

□

As we have shown how to eliminate the sequential composition and NT-operator from $\text{TSP}^{;}_{+NT}(A)$-terms, we can now prove that every $\text{TSP}^{;}_{+NT}(A)$-term can be derived to be equal to a BSP(A)-term.

Theorem B.8. For every closed $\text{TSP}^{;}_{+NT}(A)$-term p, there exists a closed BSP(A)-term q such that $p = q$.

Proof. Let p be an arbitrary closed $\text{TSP}^{;}_{+NT}(A)$-term. We will prove by induction on the structure of p that there exists a closed BSP(A)-term q such that $p = q$.

- If $p \equiv \mathbf{0}$, then p itself is a BSP(A)-term.
- If $p \equiv \mathbf{1}$, then p itself is a BSP(A)-term.
- Suppose that $p \equiv a.p'$, and suppose that there exists a BSP(A)-term q' such that $p' = q'$ (IH). Then, by IH, $p = a.q'$, which is a BSP(A)-term.
- Suppose that $p \equiv p_1 + p_2$, and suppose that there exist BSP(A)-terms q_1 and q_2 such that $p_1 = q_1$ and $p_2 = q_2$ (IH). Then, by IH, $p = q_1 + q_2$, which is a BSP(A)-term.
- Suppose that $p \equiv p_1 \,;\, p_2$, and suppose that there exist BSP(A)-terms q_1 and q_2 such that $p_1 = q_1$ and $p_2 = q_2$ (IH). Then, by IH, $p = q_1 \,;\, q_2$, and by Lemma B.6, there exists a BSP(A)-term r such that $p = r$.
- Suppose that $p \equiv NT(p')$, and suppose that there exists a BSP(A)-term q' such that $p' = q'$ (IH). Then, by IH, $p = NT(q')$, and by Lemma B.7, there exists a BSP(A)-term q such that $p = q$.

□

Appendix C

Bisimilarity is not finitely based over $\mathrm{TSP}^{;}(A)$

This appendix contains the complete proof for Theorem C.13, which is the heart of the proof of Theorem 4.2.5 that states that there does not exist a finite and ground-complete axiomatisation for $\mathrm{TSP}^{;}(A)$ with respect to bisimilarity.

In order to proof Theorem C.13, we need to be able to associate behaviour of $\mathrm{TSP}^{;}(A)$-terms with variables. For this reason, we assume an extended syntax in which a constant $\bar{x}$ is added for every variable x. For every variable x, we inductively define the set of $\mathrm{TSP}^{;}(A,\bar{x})$-terms as follows:

1. the constant $\bar{x}$ is a $\mathrm{TSP}^{;}(A,\bar{x})$-term; and
2. if t_1 is a $\mathrm{TSP}^{;}(A,\bar{x})$-term and t_2 is a $\mathrm{TSP}^{;}(A)$-term, then $t_1 \,;\, t_2$ is a $\mathrm{TSP}^{;}(A,\bar{x})$-term.

If t is a $\mathrm{TSP}^{;}(A,\bar{x})$-term and p is a closed $\mathrm{TSP}^{;}(A)$-term, then by $t[\bar{x} := p]$ we denote the $\mathrm{TSP}^{;}(A)$-term obtained by replacing $\bar{x}$ by p. The following rule is added to the operational semantics of (open) $\mathrm{TSP}^{;}(A)$-terms:

$$\frac{}{x \xrightarrow{x} \bar{x}} \ .$$

Note that while every variable can now "take a step", we do not let this contribute to the width of a term, so $\mathit{width}(x) = 0$ for every variable x.

Lemma C.1. Let t be a $\mathrm{TSP}^{;}(A)$-term.

1. If $t \xrightarrow{a} t'$ for some action a, then t' is a $\mathrm{TSP}^{;}(A)$-term.
2. If $t \xrightarrow{x} t'$ for some variable x, then t' is a $\mathrm{TSP}^{;}(A,\bar{x})$-term.

Proof.

1. The proof is straightforward by induction on the structure of t.
2. We give a proof by induction on the structure of t.

   If $t \equiv \mathbf{0}$, or $t \equiv \mathbf{1}$ or $t \equiv a.t'$, then there does not exist t' such that $t \xrightarrow{x} t'$, so the lemma vacuously holds.

Suppose that $t \equiv t_1 + t_2$, and suppose $t \xrightarrow{x} t'$. Then, according to the operational semantics, either $t_1 \xrightarrow{x} t'$ or $t_2 \xrightarrow{x} t'$. In both cases, by the induction hypothesis, t' is a $\mathrm{TSP}^{;}(A, \bar{x})$-term.

Suppose that $t \equiv t_1 \,;\, t_2$, and suppose that $t \xrightarrow{x} t'$. Then, according to the operational semantics, there are two cases: either $t_1 \xrightarrow{x} t_1'$ for some $\mathrm{TSP}^{;}(A)$-term and $t' \equiv t_1' \,;\, t_2$, or $t_1 \downarrow$, $t_1 \not\rightarrow$ and $t_2 \xrightarrow{x} t_2'$ for some $\mathrm{TSP}^{;}(A)$-term t_2'. In the first case, by the induction hypothesis t_1', and hence also $t_1' \,;\, t_2$, is a $\mathrm{TSP}^{;}(A, \bar{x})$-term. In the second case, by the induction hypothesis t_2' is a $\mathrm{TSP}^{;}(A, \bar{x})$-term.

□

We can now establish a relationship between transitions from t and transitions from $\sigma(t)$, where σ is a closed substitution. However, we cannot yet fully express that a transition originates from a substitution in a variable. For example, consider the $\mathrm{TSP}^{;}(A)$-term $t \equiv x \,;\, y$ and the closed substitution σ, where $\sigma(x) = \mathbf{1}$ and $\sigma(y) = a.\mathbf{1}$. Clearly, $\sigma(t) = \mathbf{1};a.\mathbf{1}$ and hence $\sigma(t) \xrightarrow{a} \mathbf{1}$. However, we cannot express that this a-transition originates from the substitution in y, as $t \stackrel{y}{\not\rightarrow}$. In order to be able to express this, we define the following substitution.

Definition C.2. Given a substitution σ and variable x, the substitution $\mu_{\sigma x}$ is defined as:

$$\mu_{\sigma x}(y) = \begin{cases} y & \text{if } y = x \\ \sigma(y) & \text{otherwise.} \end{cases}$$

Considering the example from the previous paragraph, we can now express that in $\sigma(t)$ a transition is possible from $\sigma(y)$, as $\mu_{\sigma y}(t) \xrightarrow{y} \bar{y}$ and $\sigma(y) \xrightarrow{a} \mathbf{1}$. We can establish several useful relationships between $\sigma(t)$ and $\mu_{\sigma x}(t)$.

Lemma C.3. Let t be a $\mathrm{TSP}^{;}(A)$-term, σ a closed substitution, x a variable, p a closed $\mathrm{TSP}^{;}(A)$-term and a an action such that $\sigma(x) \xrightarrow{a} p$. Then:

1. if $\sigma(t) \not\downarrow$, then $\mu_{\sigma x}(t) \not\downarrow$;
2. if $\sigma(t) \not\rightarrow$, then $\mu_{\sigma x}(t) \not\rightarrow$;
3. if $\sigma(t) \not\rightarrow$ and $\sigma(t) \downarrow$, then $\mu_{\sigma x}(t) \downarrow$.

Proof. We prove the lemma with induction on the structure of t.

- If $t \equiv \mathbf{0}$, then $\sigma(t) \equiv \mu_{\sigma x}(t) \equiv \mathbf{0}$. Clearly, the first two statements of the lemma hold, and since $\mathbf{0} \not\downarrow$, the third statement vacuously holds.
- If $t \equiv \mathbf{1}$, then $\sigma(t) \equiv \mu_{\sigma x}(t) \equiv \mathbf{1}$. Clearly, the last two statements of the lemma hold, and since $\mathbf{1} \downarrow$, the first statement vacuously holds.
- Suppose $t \equiv b.t'$ for some action b and $\mathrm{TSP}^{;}(A)$-term t'. Then, $\mu_{\sigma x}(t) \equiv b.\mu_{\sigma x}(t')$ and thus since $\mu_{\sigma x}(t) \not\downarrow$ the first statement of the lemma holds. Moreover, since $\sigma(t) \equiv b.\sigma(t')$, we have $\sigma(t) \xrightarrow{b} \sigma(t')$ and thus the last two statements of the lemma vacuously hold.

- Suppose $t \equiv y$ for some variable y. If $y \not\equiv x$, then $\mu_{\sigma x}(t) \equiv \sigma(t)$ and thus clearly all three statements of the lemma hold. On the other hand, if $y \equiv x$, then $\sigma(t) \xrightarrow{a} p$ and thus the last two statements of the lemma vacuously hold. Moreover, since then $\mu_{\sigma x}(t) \equiv x$ and since $x \not\downarrow$, the first statement of the lemma holds as well.

- Suppose $t \equiv t_1 + t_2$ for some TSP$^{;}(A)$-terms t_1 and t_2. If $\sigma(t) \not\downarrow$, then $\sigma(t_1) \not\downarrow$ and $\sigma(t_2) \not\downarrow$. Hence, by application of the induction hypothesis, $\mu_{\sigma x}(t_1) \not\downarrow$ and $\mu_{\sigma x}(t_2) \not\downarrow$, and thus $\mu_{\sigma x}(t) \not\downarrow$ and statement 1 of the lemma holds. We can conclude statement 2 and 3 of the lemma hold in an analogous fashion.

- Suppose $t \equiv t_1 \,;\, t_2$ for some TSP$^{;}(A)$-terms t_1 and t_2. If $\sigma(t) \not\downarrow$, then $\sigma(t_1) \not\downarrow$ or $\sigma(t_2) \not\downarrow$. Hence, by application of the induction hypothesis, $\mu_{\sigma x}(t_1) \not\downarrow$ or $\mu_{\sigma x}(t_2) \not\downarrow$, and thus $\mu_{\sigma x}(t) \not\downarrow$ and statement 1 of the lemma holds. We can conclude that statement 3 of the lemma holds in an analogous fashion. For the second statement, assume that $\sigma(t)\not\rightarrow$. Then $\sigma(t_1)\not\rightarrow$ and hence by the induction hypothesis $\mu_{\sigma x}(t_1)\not\rightarrow$. Now, in case $\sigma(t_1) \downarrow$ we must also have $\sigma(t_2)\not\rightarrow$, and hence, by a second application of the induction hypothesis, $\mu_{\sigma x}(t_2)\not\rightarrow$ and thus $\mu_{\sigma x}(t)\not\rightarrow$. On the other hand, if $\sigma(t_1) \not\downarrow$, by statement 1 of the lemma, $\mu_{\sigma x}(t_1) \not\downarrow$. Hence, in that case it immediately follows that $\mu_{\sigma x}(t)\not\rightarrow$.

□

Given a TSP$^{;}(A)$-term t such that $width(t') < n$ for each subterm t' of t, and a substitution σ such that $\Psi_n(\sigma(t))$, we want to establish certain properties that t must comply to. Since $\Psi_n(\sigma(t))$, t must contain a subterm x such that $\Psi_n(\sigma(x))$, or t contains a subterm $t_1 \,;\, t_2$ such that $\sigma(t_1) \leftrightarrow \tilde{a}.\mathbf{1} + \mathbf{1}$ and $\sigma(t_2) \leftrightarrow \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} \,;\, (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$. In the latter case, since $width(\sigma(t_2)) > width(t_2)$, t_2 must contain some variable x such that $\sigma(x)\xrightarrow{\tilde{b}}$. In both cases we want to show that there exists a substitution σ' such that $\sigma'(t) \downarrow$.

In the following lemma it is proven that if t contains a subterm t_2 such that $width(\sigma(t_2)) > width(t_2)$, then one of the actions that can be executed by $\sigma(t_2)$ must come from a substitution in some variable x.

Lemma C.4. Let t be a TSP$^{;}(A)$-term and σ a closed substitution. If $width(\sigma(t)) > width(t)$, then there must exist an action a, closed TSP$^{;}(A)$-terms p and p', a TSP$^{;}(A, \bar{x})$-term t', and a variable x such that $\sigma(t) \xrightarrow{a} p$, $\mu_{\sigma x}(t) \xrightarrow{x} t'$, $\sigma(x) \xrightarrow{a} p'$ and $p \equiv \sigma(t'[\bar{x} := p'])$.

Proof. We proceed with induction on the structure of t.

- If $t \equiv \mathbf{0}$, $t \equiv \mathbf{1}$, or $t \equiv b.t'$ for some action b and TSP$^{;}(A)$-term t', then $width(\sigma(t)) = width(t)$, so the implication vacuously holds.

- Let $t \equiv y$ for some variable y. Suppose $width(\sigma(t)) > width(t) \geq 1$, then $\sigma(t) \equiv \sigma(y)\xrightarrow{b}q$ for some action b and closed TSP$^{;}(A)$-term q. Then, $x = y$, $a = b$ and $p = p' = q$ and clearly $t \xrightarrow{x} \bar{x}$ and thus $\mu_{\sigma x}(t) \xrightarrow{x} \bar{x}$ and $p \equiv \sigma(\bar{x}[\bar{x} := p'])$.

- Let $t \equiv t_1 + t_2$ for some TSP$^{;}(A)$-terms t_1 and t_2. Suppose $width(\sigma(t)) > width(t)$, then either $width(\sigma(t_1)) > width(t_1)$ or $width(\sigma(t_2)) > width(t_2)$. In the first case, by the induction hypothesis, there exist action a, closed TSP$^{;}(A)$-terms p and p', a TSP$^{;}(A, \bar{x})$-term t'_1, and a variable x such that $\sigma(t_1) \xrightarrow{a} p$, $\mu_{\sigma x}(t_1) \xrightarrow{x} t'_1$, $\sigma(x) \xrightarrow{a} p'$ and $p \equiv \sigma(t'_1[\bar{x} := p'])$. What remains to notice is that $\sigma(t_1) \xrightarrow{a} p$ implies $\sigma(t) \xrightarrow{a} p$ and $\mu_{\sigma x}(t_1) \xrightarrow{x} t'_1$ implies $\mu_{\sigma x}(t) \xrightarrow{x} t'_1$. In the second case, we can arrive at the same conclusion in an analogous fashion.

- Let $t \equiv t_1 \,;\, t_2$ for some $\mathrm{TSP}^{;}(A)$-terms t_1 and t_2. Suppose $\mathit{width}(\sigma(t)) > \mathit{width}(t)$, then either $\mathit{width}(\sigma(t_1)) > \mathit{width}(t_1)$ or $\sigma(t_1)\not\rightarrow$, $\sigma(t_1)\downarrow$ and $\mathit{width}(\sigma(t_2)) > \mathit{width}(t_2)$. In the first case, by the induction hypothesis, there exist action a, closed $\mathrm{TSP}^{;}(A)$-terms p and p', a $\mathrm{TSP}^{;}(A,\bar{x})$-term t_1', and a variable x such that $\sigma(t_1) \xrightarrow{a} p$, $\mu_{\sigma x}(t_1) \xrightarrow{x} t_1'$, $\sigma(x) \xrightarrow{a} p'$ and $p \equiv \sigma(t_1'[\bar{x} := p'])$. What remains to notice is that $\sigma(t_1) \xrightarrow{a} p$ implies $\sigma(t) \xrightarrow{a} p \,;\, \sigma(t_2)$, $\mu_{\sigma x}(t_1) \xrightarrow{x} t_1'$ implies $\mu_{\sigma x}(t) \xrightarrow{x} t_1' \,;\, \mu_{\sigma x}(t_2)$ and $p \,;\, \sigma(t_2) \equiv \sigma(t_1'[\bar{x} := p']) \,;\, \sigma(t_2) \equiv \sigma(t_1'[\bar{x} := p']) \,;\, \sigma(\mu_{\sigma x}(t_2)[\bar{x} := p']) \equiv \sigma(t_1' \,;\, \mu_{\sigma x}(t_2)[\bar{x} := p'])$. In the second case, by the induction hypothesis, there exist action a, closed $\mathrm{TSP}^{;}(A)$-terms p and p', a $\mathrm{TSP}^{;}(A,\bar{x})$-term t_2', and a variable x such that $\sigma(t_2) \xrightarrow{a} p$, $\mu_{\sigma x}(t_2) \xrightarrow{x} t_2'$, $\sigma(x) \xrightarrow{a} p'$ and $p \equiv \sigma(t_2'[\bar{x} := p'])$. Since $\sigma(t_1)\not\rightarrow$ and $\sigma(t_1)\downarrow$, by Lemma C.3(2, 3), $\mu_{\sigma x}(t_1)\not\rightarrow$ and $\mu_{\sigma x}(t_1)\downarrow$, and thus $\sigma(t) \xrightarrow{a} p$ and $\mu_{\sigma x}(t) \xrightarrow{x} t_2'$.

□

Given a $\mathrm{TSP}^{;}(A)$-term t, a variable x and a substitution σ such that $t \xrightarrow{x} t'$, $\sigma(x) \xrightarrow{a} p$ and $\sigma(t'[\bar{x} := p])\downarrow$, we aim to prove that there must exist a substitution σ' such that $\sigma'(t)\downarrow$. The substitution we use for this is defined as follows.

Given a substitution σ and variable x, the substitution $\rho_{\sigma x}$ is defined as follows:

$$\rho_{\sigma x}(y) = \begin{cases} \sigma(y) + \mathbf{1} & \text{if } y = x \\ \sigma(y) & \text{otherwise.} \end{cases}$$

We define several apparent properties with respect to these substitutions and termination.

Lemma C.5. Let t be a $\mathrm{TSP}^{;}(A)$-term, x a variable and σ a closed substitution. Then:

1. if $t\downarrow$, then $\sigma(t)\downarrow$;
2. if $\sigma(t)\downarrow$, then $\rho_{\sigma x}(t)\downarrow$;
3. if $\rho_{\sigma x}(\mu_{\sigma x}(t))\downarrow$, then $\rho_{\sigma x}(t)\downarrow$.

Proof. By a straightforward induction on the structure of t. □

Using these properties we show that given a term t, variable x and substitution σ as described above, $\rho_{\sigma x}(t)\downarrow$ indeed holds.

Lemma C.6. Let t be a $\mathrm{TSP}^{;}(A)$-term, let x be a variable, and let σ be a substitution. If there exist a $\mathrm{TSP}^{;}(A,\bar{x})$-term t' and a closed $\mathrm{TSP}^{;}(A)$-term p such that $t \xrightarrow{x} t'$, $\sigma(x) \xrightarrow{a} p$ and $\sigma(t'[\bar{x} := p])\downarrow$, then $\rho_{\sigma x}(t)\downarrow$.

Proof. We proceed with induction on the structure of t.

- If $t \equiv \mathbf{0}$, $t \equiv \mathbf{1}$, or $t \equiv a.t'$ for some action a and $\mathrm{TSP}^{;}(A)$-term t'', then there does not exist a $\mathrm{TSP}^{;}(A,\bar{x})$-term t' such that $t \xrightarrow{x} t'$, so the implication vacuously holds.
- Let $t \equiv y$ for some variable y, and suppose that there exists a $\mathrm{TSP}^{;}(A,\bar{x})$-term t' such that $t \xrightarrow{x} t'$. Then $y \equiv x$, so $\rho_{\sigma x}(t) \equiv \sigma(x) + \mathbf{1}$, and hence $\rho_{\sigma x}(t)\downarrow$.

- Let $t \equiv t_1 + t_2$ for some $\mathrm{TSP}^{;}(A)$-terms t_1 and t_2. Suppose that there exist a $\mathrm{TSP}^{;}(A, \bar{x})$-term t' and closed $\mathrm{TSP}^{;}(A)$-term p sch that $t \xrightarrow{x} t'$ and $\sigma(t'[\bar{x} := p]) \downarrow$. Then, from $t \xrightarrow{x} t'$ it follows that either $t_1 \xrightarrow{x} t'$ or $t_2 \xrightarrow{x} t'$. In both cases, by the induction hypothesis, $\rho_{\sigma x}(t) \downarrow$.

- Let $t \equiv t_1 ; t_2$ for some $\mathrm{TSP}^{;}(A)$-terms t_1 and t_2. Suppose that there exist a $\mathrm{TSP}^{;}(A, \bar{x})$-term t' and a closed $\mathrm{TSP}^{;}(A)$-term p such that $t \xrightarrow{x} t'$ and $\sigma(t'[\bar{x} := p]) \downarrow$. We distinguish two cases according to which operational rule is applied last in the derivation of $t \xrightarrow{x} t'$. First, if $t_1 \xrightarrow{x} t'_1$ for some $\mathrm{TSP}^{;}(A)$-term t'_1 such that $t' \equiv t'_1 ; t_2$, then, since t_2 is a $\mathrm{TSP}^{;}(A)$-term and hence $\bar{x}$ does not occur in t_2, we have $\sigma(t'[\bar{x} := p]) = \sigma(t'_1[\bar{x} := p]) ; \sigma(t_2)$. Moreover, since $\sigma(t'[\bar{x} := p]) \downarrow$, it follows that both $\sigma(t'_1[\bar{x} := p]) \downarrow$ and $\sigma(t_2) \downarrow$. By the induction hypothesis we then get that $\rho_{\sigma x}(t_1) \downarrow$, and from $\sigma(t_2) \downarrow$ it immediately follows, in accordance with the definition of $\rho_{\sigma x}$, that $\rho_{\sigma x}(t_2) \downarrow$; hence, $\rho_{\sigma x}(t) \downarrow$. Second, if $t_1 \downarrow$, $t_1 \not\rightarrow$ and $t_2 \xrightarrow{x} t'$, then by the induction hypothesis we have that $\rho_{\sigma x}(t_2) \downarrow$. Furthermore, from $t_1 \downarrow$ we conclude by Lemma C.5(1,2) that $\rho_{\sigma x}(t_1) \downarrow$. It follows that $\rho_{\sigma x}(t) \downarrow$.

□

By utilizing the results from Lemma C.4 and Lemma C.6, we show that given a $\mathrm{TSP}^{;}(A)$-term t and substitution σ, if $\Psi_n(\sigma(t))$ holds, then t must contain some variable x such that $\rho_{\sigma x}(t) \downarrow$ and either $\Psi_n(\sigma(x))$ or $\sigma(x) \xrightarrow{\tilde{b}}$.

Lemma C.7. Let t be a $\mathrm{TSP}^{;}(A)$-term and let n be a natural number such that $\mathit{width}(t') < n$ for every subterm t' of t. If $\Psi_n(\sigma(t))$ for some closed substitution σ, then there is a variable x such that $\rho_{\sigma x}(t) \downarrow$ and either $\Psi_n(\sigma(x))$ or $\sigma(x) \xrightarrow{\tilde{b}}$.

Proof. We proceed with induction on the structure of t.

- If $t \equiv \mathbf{0}$, $t \equiv \mathbf{1}$ or $t \equiv a.t'$ for some action a and $\mathrm{TSP}^{;}(A)$-term t', then $\sigma(t)$ cannot have a summand of the form $t_1 ; t_2$, so $\Psi_n(\sigma(t))$ does not hold for any substitution σ. Hence the implication vacuously holds.

- Let $t \equiv y$ for some variable y, and suppose that $\Psi_n(\sigma(t))$ holds for some closed substitution σ. Then clearly from $\Psi_n(\sigma(t))$ it follows that $\Psi_n(\sigma(y))$. Furthermore, since $t \xrightarrow{y} \bar{y}$ and since $\sigma(\bar{y}[\bar{y} := \mathbf{1}]) \equiv \mathbf{1}$, we have that $\sigma(\bar{y}[\bar{y} := \mathbf{1}]) \downarrow$. Hence, by Lemma C.6, we have that $\rho_{\sigma y}(t) \downarrow$ and thus $x = y$.

- Let $t \equiv t_1 + t_2$ for some $\mathrm{TSP}^{;}(A)$-terms t_1 and t_2. If $\Psi_n(\sigma(t))$, then either $\sigma(t_1)$ or $\sigma(t_2)$ must contain a summand p such that one of the three cases of the definition of Ψ_n applies. We proceed to consider the case that p is a summand of $\sigma(t_1)$; the proof in the case that p is a summand of $\sigma(t_2)$ proceeds analogously. Note that, since $p \leftrightarrow (\tilde{a}.\mathbf{1}+\mathbf{1}) ; \sum_{i=1}^{n}(\tilde{b}.\mathbf{1};(\tilde{b}.\mathbf{1}+\mathbf{1})^i)$, we find that $\sigma(t_1) \xrightarrow{\tilde{a}} p'$ with $p' \leftrightarrow \sum_{i=1}^{n}(\tilde{b}.\mathbf{1};(\tilde{b}.\mathbf{1}+\mathbf{1})^i)$. Moreover, since $\sigma(t_1)$ is a summand of $\sigma(t)$ and also $\sigma(t) \leftrightarrow (\tilde{a}.\mathbf{1} + \mathbf{1}) ; \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} ; (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$, we find that $\sigma(t_1) \leftrightarrow (\tilde{a}.\mathbf{1} + \mathbf{1}) ; \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} ; (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$, and hence $\Psi_n(\sigma(t_1))$. Since every subterm of t_1 is a subterm of t, we also have that $\mathit{width}(t') < n$ for every subterm t' of t_1. Therefore, we may now apply the induction hypothesis to conclude that either $\Psi_n(\sigma(x))$ or $\sigma(x) \xrightarrow{b}$, and $\rho_{\sigma x}(t_1) \downarrow$; clearly, from the latter it follows that $\rho_{\sigma x}(t) \downarrow$.

- Let $t \equiv t_1 \,;\, t_2$ for some TSP$^{;}(A)$-terms t_1 and t_2, and suppose that $\Psi_n(\sigma(t))$. Then, considering the definition of Ψ_n, one of the following three cases must apply:

  1. If $\Phi_n(\sigma(t_1) \,;\, \sigma(t_2))$, then $\sigma(t_1) \leftrightarrow \tilde{a}.+\mathbf{1}$ and $\sigma(t_2) \leftrightarrow \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} \,;\, (\tilde{b}.\mathbf{1}+\mathbf{1})^i)$. Then $\sigma(t_2) \xrightarrow{\tilde{b}} (\tilde{b}.\mathbf{1}+\mathbf{1})^i$ for all $1 \leq i \leq n$. Clearly, if $i \neq j$, then $(\tilde{b}.\mathbf{1}+\mathbf{1})^i \not\leftrightarrow (\tilde{b}.\mathbf{1}+\mathbf{1})^j$, so $\mathit{width}(\sigma(t_2)) \geq n > \mathit{width}(t_2)$. It follows by Lemma C.4 that there exist an action a, closed TSP$^{;}(A)$-terms p and p', a TSP$^{;}(A, \bar{x})$-term t' and a variable x such that $\sigma(t_2) \xrightarrow{a} p$, $\mu_{\sigma x}(t_2) \xrightarrow{x} t'$, $\sigma(x) \xrightarrow{a} p'$ and $p \equiv \sigma(t'[\bar{x} := p'])$. Clearly, we must have $a = \tilde{b}$ and $p \leftrightarrow (\tilde{b}.\mathbf{1}+\mathbf{1})^i$ for some $1 \leq i \leq n$. To see that $\rho_{\sigma x}(t)\downarrow$, note that, since $\sigma(t_1) \leftrightarrow \tilde{a}.\mathbf{1}+\mathbf{1}$, we have that $\sigma(t_1)\downarrow$ and hence $\rho_{\sigma x}(t_1)\downarrow$. Moreover, since $\sigma(t'[\bar{x} := p']) \leftrightarrow (\tilde{b}.\mathbf{1}+\mathbf{1})^i$, we find that $\sigma(t'[\bar{x} := p'])\downarrow$, and hence, by Lemma C.6, we get that $\rho_{\sigma x}(\mu_{\sigma x}(t_2))\downarrow$. Finally, by Lemma C.5(3), we conclude that $\rho_{\sigma x}(t_2)\downarrow$ and thus $\rho_{\sigma x}(t)\downarrow$
  2. If $\sigma(t_1) \leftrightarrow \mathbf{1}$ and $\Psi_n(\sigma(t_2))$, then since every subterm of t_2 is a subterm of t we find that $\mathit{width}(t_2') < n$ for all subterms t_2' of t_2. Hence, by the induction hypothesis, for some variable x we have that either $\Psi_n(\sigma(x))$ or $\sigma(x)\xrightarrow{\tilde{b}}$ and, moreover, $\rho_{\sigma x}(t_2)\downarrow$. From $\sigma(t_1) \leftrightarrow \mathbf{1}$ it follows that $\sigma(t_1)\downarrow$, so, by Lemma C.5(2), $\rho_{\sigma x}(t_1)\downarrow$, and hence $\rho_{\sigma x}(t)\downarrow$.
  3. If $\Psi_n(\sigma(t_1))$ and $\sigma(t_2) \leftrightarrow \mathbf{1}$, then the proof that $\rho_{\sigma x}(t)\downarrow$ is analogous to the previous case.

□

We have established that if $\Psi_n(\sigma(t))$ holds for some substitution σ and TSP$^{;}(A)$-term t such that $\mathit{width}(t') < n$ for each subterm t' of t, then t must confirm to certain properties. Now, for any term u such that $t \leftrightarrow u$, these properties must be valid as well. Hence, what remains to show is that if u contains these properties, then $\Psi_n(\sigma(u))$ must hold as well. This is shown in Lemma C.12. In order to prove this result, some useful properties are established in Lemma C.8 to Lemma C.11.

Lemma C.8. Let p and q be closed TSP$^{;}(A)$-terms and suppose that $p \leftrightarrow q$. Then $\mathit{depth}(p) = \mathit{depth}(q)$.

Proof. Assume that $p \leftrightarrow q$ and, for the sake of contradiction, suppose that $\mathit{depth}(p) = n$ and $\mathit{depth}(q) = m$, for some $n > m$. Then, by definition, $p \longrightarrow^n p'$ and since $p \leftrightarrow q$, $q \longrightarrow^n q'$, such that $p' \leftrightarrow q'$. Clearly, since $n > m$, this contradicts $\mathit{depth}(q) = m$. Hence, we conclude $\mathit{depth}(p) = \mathit{depth}(q)$. □

Lemma C.9. For all closed TSP$^{;}(A)$-terms p_1 and p_2, if $p_1 \,;\, p_2 \leftrightarrow \tilde{a}.\mathbf{1} \,;\, \sum_{i=1}^{n}\tilde{b}.(\tilde{b}.\mathbf{1}+\mathbf{1})^i$, then one of the following cases must hold:

1. $p_1 \leftrightarrow \mathbf{1}$ and $p_2 \leftrightarrow \tilde{a}.\mathbf{1} \,;\, \sum_{i=1}^{n}\tilde{b}.(\tilde{b}.\mathbf{1}+\mathbf{1})^i$; or
2. $p_1 \leftrightarrow \tilde{a}.\mathbf{1}$ and $p_2 \leftrightarrow \sum_{i=1}^{n}\tilde{b}.(\tilde{b}.\mathbf{1}+\mathbf{1})^i$; or
3. $p_1 \leftrightarrow \tilde{a}.\mathbf{1}+\mathbf{1}$ and $p_2 \leftrightarrow \tilde{a}.\mathbf{1} \,;\, \sum_{i=1}^{n}\tilde{b}.(\tilde{b}.\mathbf{1}+\mathbf{1})^i$; or
4. $p_1 \leftrightarrow \tilde{a}.\mathbf{1} \,;\, \sum_{i=1}^{n}\tilde{b}.(\tilde{b}.\mathbf{1}+\mathbf{1})^i$ and $p_2 \leftrightarrow \mathbf{1}$.

Proof. Let p_1 and p_2 be closed $\mathrm{TSP}^{;}(A)$-terms such that $p_1 \,;\, p_2 \leftrightarrow \tilde{a}.\mathbf{1} \,;\, \sum_{i=1}^{n} \tilde{b}.(\tilde{b}.\mathbf{1} + \mathbf{1})^i$. Consider $\tilde{a}.\mathbf{1} \,;\, \sum_{i=1}^{n} \tilde{b}.(\tilde{b}.\mathbf{1} + \mathbf{1})^i \xrightarrow{\tilde{a}} q'$ and $q' = \sum_{i=1}^{n} \tilde{b}.(\tilde{b}.\mathbf{1} + \mathbf{1})^i$. Then, since $p_1 \,;\, p_2 \leftrightarrow \tilde{a}.\mathbf{1} \,;\, \sum_{i=1}^{n} \tilde{b}.(\tilde{b}.\mathbf{1} + \mathbf{1})^i$, $p_1 \,;\, p_2 \xrightarrow{\tilde{a}} p'$ such that $p' \leftrightarrow q'$. We distinguish two cases.

- If $p_1 \downarrow$, $p_1 \not\rightarrow$ and $p_2 \xrightarrow{\tilde{a}} p'$, then $p_1 \leftrightarrow \mathbf{1}$ and thus $p_2 \leftrightarrow \mathbf{1};p_2 \leftrightarrow p_1;p_2 \leftrightarrow \tilde{a}.\mathbf{1};\sum_{i=1}^{n} \tilde{b}.(\tilde{b}.\mathbf{1}+\mathbf{1})^i$. Hence, case 1 of the lemma holds.

- If $p_1 \xrightarrow{\tilde{a}} p_1'$ and $p' = p_1' \,;p_2$. Then, since $q' \xrightarrow{\tilde{b}} \mathbf{1}$ and $p' \leftrightarrow q'$, $p' \xrightarrow{\tilde{b}} p''$ such that $p'' \leftrightarrow \mathbf{1}$. Again, we consider two cases.

  – If $p_1' \downarrow$, $p_1' \not\rightarrow$ and $p_2 \xrightarrow{\tilde{b}} p''$, then $p_1' \leftrightarrow \mathbf{1}$ and thus $p_2 \leftrightarrow \mathbf{1} \,;\, p_2 \leftrightarrow p_1' \,;\, p_2 \leftrightarrow \sum_{i=1}^{n} \tilde{b}.(\tilde{b}.\mathbf{1} + \mathbf{1})^i$.
  We will show that in this case $p_1 \leftrightarrow \tilde{a}.\mathbf{1}$ or $p_1 \leftrightarrow \tilde{a}.\mathbf{1} + \mathbf{1}$. We already know that in this case $p_1 \xrightarrow{\tilde{a}} p_1'$ and $p_1' \leftrightarrow \mathbf{1}$. Now suppose $p_1 \xrightarrow{a} r$ for some action $a \in A$ and closed $\mathrm{TSP}^{;}(A)$-term r. Then we must have $a = \tilde{a}$. Moreover, if $r \not\leftrightarrow \mathbf{1}$, then either $r \longrightarrow$ or $r \not\downarrow$. If $r \longrightarrow^m r'$ for some $m > 0$ such that $r' \not\rightarrow$ and $r' \downarrow$, then $depth(p_1) \geq m+1$ and $depth(p_2) = n+1$. This means $depth(p_1 \,; p_2) \geq m+n+2 > n+2 \geq depth(\tilde{a}.\mathbf{1} \,;\, \sum_{i=1}^{n} \tilde{b}.(\tilde{b}.\mathbf{1} + \mathbf{1})^i$, which, by Lemma C.8, contradicts $p_1 \,;\, p_2 \leftrightarrow \tilde{a}.\mathbf{1} \,; \sum_{i=1}^{n} \tilde{b}.(\tilde{b}.\mathbf{1}+\mathbf{1})^i$. If $r' \not\downarrow$, then $r' \leftrightarrow \mathbf{0}$, and since $\tilde{a}.\mathbf{1} \,; \sum_{i=1}^{n} \tilde{b}.(\tilde{b}.\mathbf{1}+\mathbf{1})^i \not\rightarrow^{m+1} s$ such that $s \leftrightarrow \mathbf{0}$, this again contradicts $p_1 \,;\, p_2 \leftrightarrow \tilde{a}.\mathbf{1} \,;\, \sum_{i=1}^{n} \tilde{b}.(\tilde{b}.\mathbf{1} + \mathbf{1})^i$. Similarly, if $r \not\rightarrow$ and $r \not\downarrow$, $r \leftrightarrow \mathbf{0}$ and thus again $p_1 \,;\, p_2$ can reach a deadlock state while $\tilde{a}.\mathbf{1} \,;\, \sum_{i=1}^{n} \tilde{b}.(\tilde{b}.\mathbf{1} + \mathbf{1})^i$ cannot. Furthermore, it does not matter if $p_1 \downarrow$ or $p_1 \not\downarrow$. In both cases $p_1 \,; p_2 \leftrightarrow \tilde{a}.\mathbf{1};\sum_{i=1}^{n} \tilde{b}.(\tilde{b}.\mathbf{1}+\mathbf{1})^i$. Hence, we conclude that either $p_1 \leftrightarrow \tilde{a}.\mathbf{1}$ if $p_1 \not\downarrow$, or $p_1 \leftrightarrow \tilde{a}.\mathbf{1} + \mathbf{1}$ if $p_1 \downarrow$, which means that either case 2 or 3 of the lemma holds.

  – If $p_1' \xrightarrow{\tilde{b}} p_1''$ and $p'' = p_1'' \,; p_2$, then, since $p'' \leftrightarrow \mathbf{1}$, $p_1'' \leftrightarrow \mathbf{1}$ and $p_2 \leftrightarrow \mathbf{1}$. Furthermore, $p_1 \leftrightarrow p_1 \,;\, \mathbf{1} \leftrightarrow p_1 \,;\, p_2 \leftrightarrow \tilde{a}.\mathbf{1} \,;\, \sum_{i=1}^{n} \tilde{b}.(\tilde{b}.\mathbf{1} + \mathbf{1})^i$. Hence, we conclude that case 4 of the lemma holds.

□

Lemma C.10. For any $\mathrm{TSP}^{;}(A)$-term t, variable x and closed substitution σ, if $\sigma(t) \not\downarrow$ and $\rho_{\sigma x}(t) \downarrow$, then t contains x.

Proof. Let t be a $\mathrm{TSP}^{;}(A)$-term, x a variable and σ a closed substitution such that $\sigma(t) \not\downarrow$ and $\rho_{\sigma x}(t) \downarrow$. Now suppose t does not contain x. Then, by the definition of $\rho_{\sigma x}$, $\rho_{\sigma x}(t) \equiv \sigma(t)$, which means that if $\sigma(t) \not\downarrow$ we should also have $\rho_{\sigma x}(t) \not\downarrow$. Since this contradicts $\rho_{\sigma x}(t) \downarrow$, we conclude t must contain x. □

Lemma C.11. For any $\mathrm{TSP}^{;}(A)$-term t, variable x and closed substitution σ, if t contains x and $\sigma(x) \xrightarrow{a_1 ... a_n} p$ for some sequence of actions $a_1 ... a_n$ and closed $\mathrm{TSP}^{;}(A)$-term p, then

1. either $\sigma(t) \longrightarrow^* p' \xrightarrow{a_1 ... a_n} p''$, for some p' and p'',
2. or $\sigma(t) \longrightarrow^* p'$, for some p' such that $p' \leftrightarrow \mathbf{0}$.

Proof. Let t be a $\mathrm{TSP}^{;}(A)$-term, x a variable and σ a closed substitution such that t contains x and $\sigma(x) \xrightarrow{a_1 ... a_n} p$ for some sequence of actions $a_1 ... a_n$ and closed $\mathrm{TSP}^{;}(A)$-term p. We prove by induction on the structure of t that one of the two cases of the lemma must hold.

- If $t \equiv \mathbf{0}$ or $t \equiv \mathbf{1}$ then t cannot contain x, hence the implication vacuously holds.
- Suppose $t \equiv a.t'$ and suppose the lemma holds for t' (IH). If t contains x, then t' must contain x and thus, by the induction hypothesis, one of the two cases of the lemma must hold for t'. Since $\sigma(t) \xrightarrow{a} \sigma(t')$, if a case of the lemma holds for t', that same case must also hold for t and thus the lemma must hold for t.
- If $t \equiv y$ for some variable y, then in order for t to contain x, we must have $y \equiv x$. Hence, clearly case 1 of the lemma holds.
- Suppose $t \equiv t_1 + t_2$ and suppose that the lemma holds for t_1 and t_2 (IH). If t contains x, then either t_1 or t_2 must contain x. Without loss of generality suppose t_1 contains x, then, by the induction hypothesis, one of the two cases of the lemma must hold for t_1. Since t_1 is a summand of t, if a case of the lemma holds for t_1, that same case must also hold for t and thus the lemma must hold for t.
- Suppose $t \equiv t_1 \,;t_2$ and suppose that the lemma holds for t_1 and t_2 (IH). If t contains x, then either t_1 or t_2 must contain x.

  Suppose t_1 contains x, then, by the induction hypothesis, one of the two cases of the lemma must hold for t_1. If case 1 holds, then $\sigma(t_1) \longrightarrow^* p' \xrightarrow{a_1 \ldots a_n} p''$, for some p' and p''. Then, $\sigma(t) \longrightarrow^* p' ; \sigma(t_2) \xrightarrow{a_1 \ldots a_n} p'' ; \sigma(t_2)$, which means the lemma also holds for t. If case 2 holds for t_1, then $\sigma(t_1) \longrightarrow^* p'$, for some p' such that $p' \leftrightarroweq \mathbf{0}$. Then, $\sigma(t) \longrightarrow^* p' \,; \sigma(t_2)$ and since by $A7^{;}$, $p' \,; \sigma(t_2) \leftrightarroweq \mathbf{0}$, case 2 also holds for t.

  Suppose t_2 contains x, then, by the induction hypothesis, one of the two cases of the lemma must hold for t_2. Since $\sigma(t_1)$ is closed, $\sigma(t_1) \longrightarrow^* p$ such that $p \not\rightarrow$. Hence, then either $p \leftrightarroweq \mathbf{0}$ or $p \leftrightarroweq \mathbf{1}$. In the first case it is easy to see that case 2 of the lemma holds for t. In the second case, we must have $\sigma(t) \longrightarrow^* \sigma(t_2)$. As a consequence, since one of the two cases of the lemma must hold for t_2, that same case must then also hold t. Hence, we conclude that in all cases the lemma holds for t.

□

Lemma C.12. For any $\mathrm{TSP}^{;}(A)$-term t, variable x and closed substitution σ, if $\sigma(t) \leftrightarroweq (\tilde{a}.\mathbf{1} + \mathbf{1}) \,; \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} \,; (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$, $\rho_{\sigma x}(t) \downarrow$ and either $\Psi_n(\sigma(x))$ or $\sigma(x) \xrightarrow{\tilde{b}}$, then $\Psi_n(\sigma(t))$.

Proof. Let t be a $\mathrm{TSP}^{;}(A)$-term, x a variable and let σ be closed substitution such that $\sigma(t) \leftrightarroweq (\tilde{a}.\mathbf{1} + \mathbf{1}) \,; \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} \,; (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$, $\rho_{\sigma x}(t) \downarrow$ and either $\Psi_n(\sigma(x))$ or $\sigma(x) \xrightarrow{b}$. We prove by induction on the structure of t that $\Psi_n(\sigma(t))$ must hold.

- If $t \equiv \mathbf{0}$ or $t \equiv \mathbf{1}$, then $\sigma(t) \not\leftrightarroweq (\tilde{a}.\mathbf{1} + \mathbf{1}) \,; \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} \,; (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$, hence, the implication vacuously holds.
- If $t \equiv a.t'$, then $\rho_{\sigma x}(t) \not\downarrow$ which contradicts $\rho_{\sigma x}(t) \downarrow$, hence, the implication vacuously holds.
- If $t \equiv y$ for some variable y, then since $\sigma(t) \leftrightarroweq (\tilde{a}.\mathbf{1} + \mathbf{1}) \,; \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} \,; (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$, we must have $\sigma(t) \not\downarrow$. Moreover, since $\rho_{\sigma x}(t) \downarrow$, by Lemma C.10, $\sigma(t)$ contains x, thus we must have $y \equiv x$. Now suppose $\sigma(x) \xrightarrow{\tilde{b}}$, then $\sigma(t) \xrightarrow{\tilde{b}}$, contradicting $\sigma(t) \leftrightarroweq (\tilde{a}.\mathbf{1} + \mathbf{1}) \,; \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} \,; (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$. Hence, it must be the case that $\Psi_n(\sigma(x))$ holds and thus also $\Psi_n(\sigma(t))$ holds.

- Suppose $t \equiv t_1 + t_2$ and suppose that the lemma holds for t_1 and t_2 (IH). Since $\sigma(t) \leftrightarrow (\tilde{a}.\mathbf{1} + \mathbf{1}) \,;\, \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} \,;\, (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$, we have $\sigma(t) \not\downarrow$ and thus both $\sigma(t_1) \not\downarrow$ and $\sigma(t_2) \not\downarrow$. Moreover, since $\rho_{\sigma x}(t) \downarrow$ either $\rho_{\sigma x}(t_1) \downarrow$ or $\rho_{\sigma x}(t_2) \downarrow$. Without loss of generality assume $\rho_{\sigma x}(t_1) \downarrow$. Then, by Lemma C.10, t_1 must contain x. Since either $\sigma(x) \xrightarrow{\tilde{b}}$ or $\Psi_n(\sigma(x))$ and thus $\sigma(x) \xrightarrow{\tilde{a}}$, by Lemma C.11, either $\sigma(t_1) \longrightarrow^* q_1 \xrightarrow{a}$ for some action a and closed TSP$^{;}(A)$-term q_1, or $\sigma(t_1) \longrightarrow^* q_2$ for some closed TSP$^{;}(A)$-term q_2 such that $q_2 \leftrightarrow \mathbf{0}$. The second case clearly contradicts $\sigma(t) \leftrightarrow (\tilde{a}.\mathbf{1} + \mathbf{1}) \,;\, \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} \,;\, (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$. Hence, it must be the case that $\sigma(t_1) \longrightarrow^* q_1 \xrightarrow{\tilde{a}}$. Since $\sigma(t_1)$ is able to execute an action and $\sigma(t_1)$ is a summand of $\sigma(t)$, we must have $\sigma(t_1) \xrightarrow{a} p$ such that $p \leftrightarrow \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} \,;\, (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$. Hence, we must have $\sigma(t_1) \leftrightarrow (\tilde{a}.\mathbf{1}+\mathbf{1}) ; \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} ; (\tilde{b}.\mathbf{1}+\mathbf{1})^i)$, and thus by the induction hypothesis we conclude $\Psi_n(\sigma(t_1))$, and since $\sigma(t_1)$ is a summand of $\sigma(t)$ also $\Psi_n(\sigma(t))$.

- Suppose $t \equiv t_1 \,;\, t_2$ and suppose that the lemma holds for t_1 and t_2 (IH). Since $\sigma(t) = \sigma(t_1 \,;\, t_2) = \sigma(t_1) \,;\, \sigma(t_2)$, we have $\sigma(t_1) \,;\, \sigma(t_2) \leftrightarrow (\tilde{a}.\mathbf{1} + \mathbf{1}) \,;\, \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} \,;\, (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$ and thus by Lemma C.9, one of the following cases must hold:

  1. If $\sigma(t_1) \leftrightarrow \mathbf{1}$ and $\sigma(t_2) \leftrightarrow (\tilde{a}.\mathbf{1} + \mathbf{1}) \,;\, \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} \,;\, (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$, then, by the induction hypothesis, $\Psi_n(\sigma(t_2))$. Moreover, since $\sigma(t_1) \leftrightarrow \mathbf{1}$, by case 2 of Ψ_n we conclude $\Psi_n(\sigma(t))$.
  2. If $\sigma(t_1) \leftrightarrow \tilde{a}.\mathbf{1}$ and $\sigma(t_2) \leftrightarrow \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} \,;\, (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$, then $\sigma(t_1) \not\downarrow$. Moreover, since $\rho_{\sigma x}(t) \downarrow$ we must have $\rho_{\sigma x}(t_1) \downarrow$, and thus by by Lemma C.10, t_1 must contain x. We distinguish two cases.

     If $\Psi_n(\sigma(x))$, then $\sigma(x) \leftrightarrow (\tilde{a}.\mathbf{1} + \mathbf{1}) \,;\, \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} \,;\, (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$ and, by Lemma C.11, either $\sigma(t_1) \longrightarrow^* q_1 \xrightarrow{\tilde{a}} q_1' \xrightarrow{\tilde{b}} q_1''$ for some closed TSP$^{;}(A)$-terms q_1, q_1' and q_1'', or $\sigma(t_1) \longrightarrow^* q_2$ for some closed TSP$^{;}(A)$-term q_2 such that $q_2 \leftrightarrow \mathbf{0}$. Both cases clearly contradict $\sigma(t_1) \leftrightarrow \tilde{a}.\mathbf{1}$.

     If $\sigma(x) \xrightarrow{b}$, then, by Lemma C.11, either $\sigma(t_1) \longrightarrow^* q_1 \xrightarrow{\tilde{b}}$ for some closed TSP$^{;}(A)$-term q_1, or $\sigma(t_1) \longrightarrow^* q_2$ for some closed TSP$^{;}(A)$-term q_2 such that $q_2 \leftrightarrow \mathbf{0}$. Again, both cases contradict $\sigma(t_1) \leftrightarrow \tilde{a}.\mathbf{1}$, hence the case where $\sigma(t_1) \leftrightarrow \tilde{a}.\mathbf{1}$ and $\sigma(t_2) \leftrightarrow (\tilde{a}.\mathbf{1} + \mathbf{1}) \,;\, \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} \,;\, (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$ can never occur.
  3. If $\sigma(t_1) \leftrightarrow \tilde{a}.\mathbf{1} + \mathbf{1}$ and $\sigma(t_2) \leftrightarrow \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} ; (\tilde{b}.\mathbf{1}+\mathbf{1})^i)$, then clearly $\Phi_n(\sigma(t_1) ; \sigma(t_2))$ and thus, by case 1 of Ψ_n we conclude $\Psi_n(\sigma(t))$.
  4. If $\sigma(t_1) \leftrightarrow (\tilde{a}.\mathbf{1} + \mathbf{1}) \,;\, \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} \,;\, (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$ and $\sigma(t_2) \leftrightarrow \mathbf{1}$, then, by the induction hypothesis, $\Psi_n(\sigma(t_1))$. Moreover, since $\sigma(t_2) \leftrightarrow \mathbf{1}$, by case 3 of Ψ_n we conclude $\Psi_n(\sigma(t))$.

□

Finally, using Lemma C.7 and Lemma C.12, we prove the following theorem.

Theorem C.13. Let E be a finite set of equations, sound with respect to $\leftrightarrow$, and let n be a natural number such that for each axiom $t = u \in E$, for each subterm t' of t and each subterm u' of u, $\mathit{width}(t') < n$ and $\mathit{width}(u') < n$. Furthermore, let p and q be closed TSP$^{;}(A)$-terms such that $p \leftrightarrow q$ and suppose that $E \vdash p = q$. It then holds that if $\Psi_n(p)$, then $\Psi_n(q)$.

Proof. We give a proof by induction on a derivation of the equation $p = q$ from E. We distinguish cases, depending on the last rule used in this derivation, and assume that for each derivation of $p' = q'$ that is a sub-derivation of the derivation of $p = q$, if $\Psi_n(p')$ then $\Psi_n(q')$ (IH).

Case 1: The derivation consists of a combined application of the substitution rule and an axiom in E. Then, there exist $\mathrm{TSP}^{;}(A)$-terms t and u and a closed substitution σ such that $\sigma(t) = p$, $\sigma(u) = q$ and $t = u \in E$. If $\Psi_n(p)$, then $\Psi_n(\sigma(t))$ and thus $\sigma(t) \leftrightarroweq (\tilde{a}.\mathbf{1} + \mathbf{1}) ; \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} ; (\tilde{b}.\mathbf{1}+\mathbf{1})^i)$. Since $t = u$ is sound with respect to bisimilarity, $\sigma(u) \leftrightarroweq \sigma(t) \leftrightarroweq (\tilde{a}.\mathbf{1}+\mathbf{1});\sum_{i=1}^{n}(\tilde{b}.\mathbf{1};(\tilde{b}.\mathbf{1}+\mathbf{1})^i)$. Furthermore, by Lemma C.7, there must be some variable x such that $\rho_{\sigma x}(t)\downarrow$ (see Definition **??**) and either $\Psi_n(\sigma(x))$ or $\sigma(x) \xrightarrow{\tilde{b}} p$ for some closed $\mathrm{TSP}^{;}(A)$-term p. Hence, since $\rho_{\sigma x}(t) \leftrightarroweq \rho_{\sigma x}(u)$, also $\rho_{\sigma x}(u)\downarrow$. Then, since $\sigma(u) \leftrightarroweq (\tilde{a}.\mathbf{1}+\mathbf{1});\sum_{i=1}^{n}(\tilde{b}.\mathbf{1};(\tilde{b}.\mathbf{1}+\mathbf{1})^i)$, and $\rho_{\sigma x}(u)\downarrow$, by Lemma C.12, we conclude that $\Psi_n(\sigma(u))$ holds, and thus $\Psi_n(q)$ holds.

Case 2: The last rule applied is the reflexivity rule. Then, if $\Psi_n(p)$, it immediately follows that $\Psi_n(q)$.

Case 3: The the last rule applied is the transitivity rule. Then, there exist a term r and derivations of $p = r$ and $r = q$ that are proper sub-derivations of the derivation of $p = q$. If $\Psi_n(p)$, then by the induction hypothesis $\Psi_n(r)$. Hence, again by the induction hypothesis $\Psi_n(q)$.

Case 4: The last rule applied is the congruence rule for a. Then, there exist terms p' and q' such that $p \equiv a.p'$ and $qu \equiv a.q'$. This contradicts $\Psi_n(p)$, hence there is nothing to prove.

Case 5: The last rule applied is the congruence rule for $+$. Then, $p \equiv p_1 + p_2$ and $uq \equiv q_1 + q_2$, and there exist derivations of the equations $p_1 = q_1$ and $p_2 = q_2$ that are proper sub-derivations of $p = q$. If $\Psi_n(p)$ holds, either p_1 or p_2 must contain a summand p' such that one of the three cases from the definition of Ψ_n holds. Suppose p_1 contains p' as a summand. Then, since $p' \leftrightarroweq (\tilde{a}.\mathbf{1} + \mathbf{1}) ; \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} ; (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$, it must be the case that $p_1 \xrightarrow{\tilde{a}} p_1'$ such that $p_1' \leftrightarroweq \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} ; (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$. Moreover, since p_1 is a summand of p and $p \leftrightarroweq (\tilde{a}.\mathbf{1} + \mathbf{1}) ; \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} ; (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$, we must have $p_1 \leftrightarroweq (\tilde{a}.\mathbf{1} + \mathbf{1}) ; \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} ; (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$. Hence, $\Psi_n(p_1)$ holds, and, by the induction hypothesis, $\Psi_n(q_1)$ holds. Moreover, since $q \leftrightarroweq p \leftrightarroweq (\tilde{a}.\mathbf{1} + \mathbf{1}) ; \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} ; (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$, we then derive $\Psi_n(q)$. In case p_2 contains p' we can derive $\Psi_n(p_2)$, $\Psi_n(q_2)$ and $\Psi_n(q)$ in a similar way.

Case 6: The last rule applied is the congruence rule for ;. Then, $p \equiv p_1;p_2$ and $q \equiv q_1;q_2$, and there exist derivations of the equations $p_1 = q_1$ and $p_2 = q_2$ that are proper sub-derivations of $p = q$. If $\Psi_n(p)$, then one of the three cases from the definition of Ψ_n must hold for $p_1 ; p_2$.

1. If $\Phi_n(p_1 ; p_2)$, then by definition of Φ_n, $p_1 \leftrightarroweq \tilde{a}.\mathbf{1} + \mathbf{1}$ and $p_2 \leftrightarroweq \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} ; (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$. Since $p_1 = q_1$ and $p_2 = q_2$ are sound with respect to bisimilarity, $p_1 \leftrightarroweq \tilde{a}.\mathbf{1} + \mathbf{1}$ and $q_2 \leftrightarroweq \sum_{i=1}^{n}(\tilde{b}.\mathbf{1} ; (\tilde{b}.\mathbf{1} + \mathbf{1})^i)$. Hence, $\Phi_n(q_1 ; q_2)$ and since $q \equiv q_1 ; q_2$ we conclude $\Psi_n(q)$.
2. If $p_1 \leftrightarroweq \mathbf{1}$ and $\Psi_n(p_2)$, then since $p_1 = q_1$ is sound with respect to bisimilarity $q_1 \leftrightarroweq \mathbf{1}$. Moreover, by the induction hypothesis, since $\Psi_n(p_2)$ we have $\Psi_n(q_2)$. Hence, since $q \equiv q_1 ; q_2$ we conclude $\Psi_n(q)$.

3. If $\Psi_n(p_1)$ and $p_2 \leftrightarrow \mathbf{1}$, then since $p_2 = q_2$ is sound with respect to bisimilarity $q_2 \leftrightarrow \mathbf{1}$. Moreover, by the induction hypothesis, since $\Psi_n(p_1)$ we have $\Psi_n(q_1)$. Hence, since $q \equiv q_1 \,;\, q_2$ we conclude $\Psi_n(q)$.

□

Appendix D

Greibach normal form for $\mathrm{TSP}^{\cdot}(A)$

This appendix contains the full proof for Proposition 2.5.2. It is shown how any completely guarded recursive over $\mathrm{TSP}^{\cdot}(A)$ can be transformed to GNF.

To this end, we define define a second normal form, named head normal form.

Definition D.1. A $\mathrm{TSP}^{;}(A)$-term p is in *Head Normal Form* (HNF), if it is of the form $p \equiv \mathbf{0}$, $p \equiv \mathbf{1}$ or $p \equiv \sum_{i=1}^{n} a_i.p_i(+\mathbf{1})$ for some $n \in \mathbb{N}^+$, actions a_i and $\mathrm{TSP}^{;}(A)$-terms p_i.

Proposition D.2. For every completely guarded $\mathrm{TSP}^{\cdot}(A)$-term p there exists a $\mathrm{TSP}^{\cdot}(A)$-term q in HNF such that $p \leftrightarrow q$.

Proof. Let p be a completely guarded $\mathrm{TSP}^{\cdot}(A)$-term. We prove by induction on the structure of p that there exists a $\mathrm{TSP}^{\cdot}(A)$-term q in HNF such that $p \leftrightarrow q$.

- If $p \equiv \mathbf{0}$, $p \equiv \mathbf{1}$, or $p \equiv a.p'$, then p is in HNF.
- If $p \equiv X$ for some variable X, then p is not completely guarded, hence the case vacuously holds.
- If $p \equiv p_1 + p_2$, then p_1 and p_2 must both be completely guarded in order for p to be completely guarded. Then, by the induction hypothesis, there exist terms q_1 and q_2 in HNF such that $p_1 \leftrightarrow q_1$ and $p_2 \leftrightarrow q_2$. Hence, clearly, after application of $A3$ and $A6$, $q = q_1 + q_2$ is in HNF and $p \leftrightarrow q$.
- If $p \equiv p_1 \cdot p_2$, then p_1 and p_2 must be completely guarded in order for p to be completely guarded. Then, by the induction hypothesis, there exist terms q_1 and q_2 in HNF such that $p_1 \leftrightarrow q_1$, $p_2 \leftrightarrow q_2$ and thus $p \leftrightarrow q_1 \,;\, q_2$. We distinguish three cases:
  - If $q_1 \equiv \mathbf{0}$, then by $A7$, $p \leftrightarrow \mathbf{0}$, and clearly $\mathbf{0}$ is in HNF.
  - If $q_1 \equiv \mathbf{1}$, then by $A9$, $p \leftrightarrow q_2$, and q_2 is in HNF.
  - If $q_1 \equiv \sum_{i=1}^{n} a_i.q_i$ for some $n \geq 1$, then by $A4$ and $A10$, $p \leftrightarrow \sum_{i=1}^{n} a_i.(q_i \cdot q_2)$, which is in HNF.
  - If $q_1 \equiv \sum_{i=1}^{n} a_i.q_i + \mathbf{1}$ for some $n \geq 1$, then by $A4$ and $A10$, $p \leftrightarrow \sum_{i=1}^{n} a_i.(q_i \cdot q_2) + q_2$. Now, if $q_2 \equiv \mathbf{0}$, then we apply $A6$ to derive that $p \leftrightarrow \sum_{i=1}^{n} a_i.(q_i \cdot q_2)$, which is in HNF. If $q_2 \equiv \mathbf{1}$, then $p \leftrightarrow \sum_{i=1}^{n} a_i.(q_i \cdot q_2) + \mathbf{1}$, which is in HNF. Finally, if $q_2 \equiv \sum_{i=1}^{m} a_i.q_i'(+\mathbf{1})$ for some $m \geq 1$, then, $p \leftrightarrow \sum_{i=1}^{n} a_i.(q_i \cdot q_2) + \sum_{i=1}^{m} a_i.q_i'(+\mathbf{1})$, which is in HNF

□

Let Δ be a completely guarded recursive process specification over TSP$^{;}(A)$. We show how to create a recursive specification Δ' in GNF such that for any equation $X = p \in \Delta$, there exists an equation $X = q \in \Delta'$ such that $p \leftrightarrow q$. As a result, any sequence α as defined by Δ is bisimilar to the sequence α as defined by Δ'.

We start by defining $\Delta' = \Delta'_1 \cup \Delta'_2$ where $\Delta'_1 = \Delta$ and $\Delta'_2 = \emptyset$ and the root of Δ' is the root of Δ'_1. Next, we execute the following two steps once.

1. For each equation $X = p \in \Delta'_1$, transform p to HNF by applying the axioms as shown in the proof of Proposition D.2.

2. For each equation in Δ'_1, replace all summands $a.p$ where $p \notin V^*$ by $a.X'$ and add the equation $X' = p$ to Δ'_2, where X' is a fresh variable.

Note that all equations in Δ'_1 are now of the shape $X = \mathbf{0}$, $X = \mathbf{1}$ or $X = \sum_{i=1}^{n} a_i.\alpha_i(+\mathbf{1})$ for some $n \in \mathbb{N}^+$, and that for all equations $X = p \in \Delta'_2$, p contains only variables from $V(\Delta'_1)$. Since the equations in Δ'_2 are not necessarily guarded, we cannot simply transform the equations to HNF again. Therefore, for each equation in Δ'_2, we replace all variables X with p where $X = p \in \Delta'_1$. Since all equations in Δ'_1 are of the shape $X = \mathbf{0}$, $X = \mathbf{1}$ or $X = \sum_{i=1}^{n} a_i.\alpha_i(+\mathbf{1})$ for some $n \in \mathbb{N}^+$, all sequences of variables in equations in Δ'_2 are now directly preceded by an action. It is not difficult to verify that this property is maintained when transforming a process to HNF by applying the axioms as shown in the proof of Proposition D.2.

Now, in order to transform all equations in Δ'_2 to the correct form, we repeatedly execute the following steps

1. Transform all equations in Δ'_2 to HNF.

2. For each equation in Δ'_2, replace all summands $a.p$ where p is guarded, by $a.X'$ and add the equation $X' = p$ to Δ'_2, where X' is a fresh variable.

3. For each equation in Δ'_2, for all summands $a.p$ where p is unguarded, since all sequences of variables in $a.p$ are directly preceded by an action, we must have $p \in V^*$ or $p = p_1 ; p_2$ for some $p_1 \in V^*$ and guarded TSP$^{;}(A)$-term p_2. In the latter case, replace p_2 by X' and add the equation $X' = p_2$ to Δ'_2, where X' is a fresh variable.

4. Repeat the procedure from step 1 until all equations in Δ'_2 are of the form $X = \mathbf{0}$, $X = \mathbf{1}$ or $X = \sum_{i=1}^{n} a_i.\alpha_i(+\mathbf{1})$ for some $n \in \mathbb{N}^+$.

At the start of the procedure, $\Delta'_1 = \Delta$ and thus for each equation $X = p \in \Delta$ there is an equation $X = p \in \Delta'$. During the procedure, the right hand side of these equations are transformed by applying axioms. Since all applied axioms are valid with respect to bisimilarity, at the end of the procedure, for each equation $X = p \in \Delta$ there is an equation $X = q \in \Delta'$, such that $p \leftrightarrow q$. The procedure terminates as in each newly created equation $X = p$, p is strictly smaller than the summand it stems from. Hence, after a finite number of steps we end up with a recursive specification Δ' in GNF.

Appendix E

A technique for proving ω-completeness

A technique that can be used to proof ω-completeness is presented by J.F. Groote in [12]. The technique is based on transformation strategies and proceeds as follows. Suppose that we have a ground-complete set of axioms E and an arbitrary equation $t = u$ that is sound with respect to bisimilarity, i.e. for all closed substitutions σ it holds that $\sigma(t) \mathrel{\underline{\leftrightarrow}} \sigma(u)$. We define a closed substitution ρ that maps each variable in $t = u$ to a unique closed term representing this variable. Since E is assumed to be ground-complete, there exists a derivation for $\rho(t) = \rho(u)$. The goal is to show that this derivation can be transformed to a derivation of $t = u$, by a translation $\tilde{\rho}$, which replaces each subterm representing a variable by its original variable. A proof for ω-completeness can be given, if the following requirements are satisfied.

1. The equations $\tilde{\rho}(\rho(t)) = t$ and $\tilde{\rho}(\rho(u)) = u$ can be derived from E.
2. For every operational symbol f and closed terms $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$, where n is the arity of f, the equation $\tilde{\rho}(f(p_1, \ldots, p_n)) = \tilde{\rho}(f(q_1, \ldots, q_n))$ is provable from E and the equations $p_i = q_i$ and $\tilde{\rho}(p_i) = \tilde{\rho}(q_i)$ for $1 \leq i \leq n$.
3. For every equation $t_1 = u_1 \in E$ and closed substitution σ, the equation $\tilde{\rho}(\sigma(t_1)) = \tilde{\rho}(\sigma(u_1))$ can be derived from E.

If these three requirements are satisfied, the following lemma can be proven:

Lemma E.1. If p and q are two closed $\mathrm{TSP}^{;}_{+NT}(A)$-terms such that $E \vdash p = q$, then $E \vdash \tilde{\rho}(p) = \tilde{\rho}(q)$.

It follows directly from this lemma that E is ω-complete.

Theorem E.2. Let E be a set of equations that is ground-complete for $\mathrm{TSP}^{;}_{+NT}(A)$ with respect to bisimilarity. If for each pair of bisimilar terms t and u, there exist a substitution ρ and a translation $\tilde{\rho}$, satisfying requirement (1), (2) and (3), then E is ω-complete.

Proof. Let t and u be two arbitrary term such that $t \mathrel{\underline{\leftrightarrow}} u$. By definition, $\rho(t) \mathrel{\underline{\leftrightarrow}} \rho(u)$ and thus, as E is ground-complete, we must have $E \vdash \rho(t) = \rho(u)$. Hence, by Lemma E.1, we have $E \vdash \tilde{\rho}(\rho(t)) = \tilde{\rho}(\rho(u))$, and thus, by requirement (1) it follows that $E \vdash t = \tilde{\rho}(\rho(t)) = r\tilde{h}o(\rho(u)) = u$. □

We will show that this technique is not convenient to prove ω-completeness of $\mathrm{TSP}^{;}_{+NT}(A)$. In order for this technique to be convenient, there should exist a substitution ρ and translation $\tilde{\rho}$ such that $\tilde{\rho}(\rho(t)) = t$ for all $\mathrm{TSP}^{;}_{+NT}(A)$-terms t. Consider the following equation:

$$NT(x) = x \ .$$

Clearly, this equation is not sound with respect to bisimilarity, as, for example, defining $\sigma(x) = \mathbf{1}$ gives $NT(\mathbf{1}) = \mathbf{1}$ which is not true as $\mathbf{1}$ can terminate and $NT(\mathbf{1})$ cannot. Now suppose we would like to apply the technique to proof ω-completeness. If we define ρ such that $\rho(x) = p$ for some closed $\mathrm{TSP}^{;}_{+NT}(A)$-term p such that $p \not\downarrow$ and $\tilde{\rho}$ could be defined such that $\tilde{\rho}(\rho(NT(x))) = NT(x)$ and $\tilde{\rho}(\rho(x)) = x$. Then, $NT(p) \leftrightarrows p$ and since $\mathrm{TSP}^{;}_{+NT}(A)$ is ground-complete, there exists a derivation for $NT(p) = p$. Hence, it cannot be the case that both requirement (2) and (3) are satisfied, otherwise, by Lemma E.1, it must be the case that $\mathrm{TSP}^{;}_{+NT}(A) \vdash \tilde{\rho}(NT(p)) = \tilde{\rho}(p)$, and thus $\mathrm{TSP}^{;}_{+NT}(A) \vdash NT(x) = \tilde{\rho}(\rho(NT(x))) = \tilde{\rho}(NT(p)) = p = \tilde{\rho}(\rho(x)) = x$.

Therefore, we must define ρ such that $\rho(x) = p$ for some closed $\mathrm{TSP}^{;}_{+NT}(A)$-term p such that $p \downarrow$. Now consider the following equation:

$$x + \mathbf{1} = x.$$

Again, it is easy to see that this equation is not sound with respect to bisimilarity and therefore should not be derivable from $\mathrm{TSP}^{;}_{+NT}(A)$. However, with this definition for ρ we have $\rho(x + \mathbf{1}) \leftrightarrows \rho(x)$ and thus $\rho(x + \mathbf{1}) = \rho(x)$ is derivable from $\mathrm{TSP}^{;}_{+NT}(A)$. If we define $\tilde{\rho}$ such that $\tilde{\rho}(\rho(x + \mathbf{1})) = x + \mathbf{1}$ and $\tilde{\rho}(\rho(x)) = x$, then since $\rho(x + \mathbf{1}) = \rho(x)$ is derivable from $\mathrm{TSP}^{;}_{+NT}(A)$, again one of the requirements cannot be satisfied, otherwise, by Lemma E.1 we would have $\mathrm{TSP}^{;}_{+NT}(A) \vdash x + \mathbf{1} = \tilde{\rho}(\rho(x + \mathbf{1})) = \tilde{\rho}(\rho(x))) = x$.

Appendix F

Capturing normed $\mathrm{TSP}^{\cdot}(A)$-sequences with a finite base

One of the issues regarding decidability of bisimilarity for $\mathrm{TSP}^{\cdot}(A)$, is the fact that the number of different sequences with a certain norm is no longer finite. Given a normed variable X, there can be infinitely many sequences α such that $X \leftrightarrow \alpha$. If the norm of such a variable is n, then there can still be at most n variables in α with a norm greater or equal than 1, and all other variables in α must have a norm of 0. For example, given the defining equation $X = a.X + \mathbf{1}$, we see that $X \leftrightarrow XX \leftrightarrow XXX \leftrightarrow \ldots$, which means that the set $B = \{\alpha \mid X \leftrightarrow \alpha\}$ is infinite. Let $C(\alpha)$ denote the collection of variables of a sequence α (duplicate variables are allowed). If we have infinitely many bisimilar sequences, there must exist sequences α and β such that $\alpha \leftrightarrow \beta$ and $C(\alpha) \subset C(\beta)$. In other words, there exists sequences α and β such that β is build from the same variables of α and some extra variables. Intuitively, these extra variables, $C(\beta) \setminus C(\alpha)$, do not add any valuable behaviour as the same behaviour could already be achieved with the set of variables in α. Hence, the behaviour of these variables should somehow be covered by a subsequence of β. This property can help us construct a finite bisimulation base. We illustrate this principle by means of an example.

Example F.1. Consider the following recursive $\mathrm{TSP}^{\cdot}(A)$-specification:

$$X = a.\mathbf{1} + \mathbf{1} \qquad Y = b.\mathbf{1} + \mathbf{1} \qquad Z = b.Z + \mathbf{1}$$

It is not difficult to see that $XZ \leftrightarrow XYZ \leftrightarrow XYYZ \leftrightarrow \ldots$. As $C(XYZ) \setminus C(XZ) = \langle Y \rangle$, intuitively the sequence XYZ contains an Y that does not add any valuable behaviour. In this case, indeed Y can be removed, as $XYZ \leftrightarrow XZ$. More specifically, we see that $YZ \leftrightarrow Z$. As a result, the congruence of the finite relation $R = \{(YZ, Z)\}$ can be used to derive $XZ \overset{R}{\equiv} XY^iZ$ for all $i \in \mathbb{N}$.

In this appendix we analyse several infinite sets of bisimilar sequences B, and argue why they can be captured by a finite bisimulation base. For these sets, we show that there exists a finite relation R, such that all pairs of bisimilar sequences in B are related by the congruence $\overset{R}{\equiv}$, and moreover that R can be part of a full bisimulation base by showing that if $\alpha \overset{R}{\equiv} \beta$, then $\alpha \leftrightarrow \beta$.

F.1 Sequences with a single variable

Suppose we have a sequence X and an infinite set of natural numbers I, such that for each pair $(i,j) \in I \times I$, we have $X^i \leftrightarrow X^j$. Then $B = \{(X^i, X^j) \mid i,j \in I\}$ is an infinite set of bisimilar pairs. Now, let m be the minimum natural number in I and consider the finite relation $R = \{(X^m, X^{m+1})\}$. It is not difficult to see that the congruence containing R is $\stackrel{R}{\equiv} = \{(X^i, X^j) \mid i,j \geq m\}$. Now clearly, since m is the minimum of I, for every pair $(X^i, X^j) \in B$ we have $X^i \stackrel{R}{\equiv} X^j$. What remains to show, is that for every pair (X^i, X^j) such that $X^i \stackrel{R}{\equiv} X^j$, we have $X^i \leftrightarrow X^j$. Since bisimilarity is a congruence for TSP$^{\cdot}(A)$, it is sufficient to show that every pair in R is bisimilar, i.e., $X^m \leftrightarrow X^{m+1}$.

Lemma F.2. For all normed TSP$^{\cdot}(A)$-sequences α, β and γ, if $\alpha \leftrightarrow \gamma\beta\alpha$, then $\alpha \leftrightarrow \beta\alpha$.

Proof. We define $R = \{(\alpha, \beta\alpha) \mid \alpha \leftrightarrow \gamma\beta\alpha \text{ for some } \gamma\}$ and show R is a bisimulation relation. First note that since $\alpha \leftrightarrow \gamma\beta\alpha$, it must be the case that $n(\gamma) = n(\beta) = 0$ and thus $\gamma \downarrow$ and $\beta \downarrow$.

Suppose $\alpha \xrightarrow{a} \alpha'$, then since $\beta \downarrow$, $\beta\alpha \xrightarrow{a} \alpha'$ and clearly by reflexivity $\alpha' \leftrightarrow \alpha'$. If $\beta\alpha \xrightarrow{a} \delta$, then since $\gamma \downarrow$, $\gamma\beta\alpha \xrightarrow{a} \delta$ and thus $\alpha \xrightarrow{a} \alpha'$ such that $\alpha' \leftrightarrow \delta$. Also, since $\beta \downarrow$, if $\alpha \downarrow$ then $\alpha\beta \downarrow$, and vice versa.

□

Since m is the minimum number of I, there must be a pair $(X^m, X^n) \in S$ for some $m < n$, and thus $X^m \leftrightarrow X^n$. Hence, we have $X^m = X^{n-m}X^m$ and thus, by Lemma F.2, it follows that $X^m \leftrightarrow XX^m$.

F.2 Second variable at the end

We extend the previous result by adding a second variable Y to the end of a sequence of X's. Suppose we have an infinite set of natural numbers I, such that for each pair $(i,j) \in I \times I$, we have $X^i \cdot Y \leftrightarrow X^j \cdot Y$. Then $B = \{(X^i \cdot Y, X^j \cdot Y) \mid i,j \in I\}$ is an infinite set of bisimilar pairs. Again, let m be the minimum natural number in I and consider the finite relation $R = \{(X^m \cdot Y, X^{m+1} \cdot Y)\}$. It is easy to verify that for each pair $(X^i \cdot Y, X^j \cdot Y) \in B$ we have $X^i \cdot Y \stackrel{R}{\equiv} X^j \cdot Y$. Moreover, we can again apply Lemma F.2 to show that $X^m \cdot Y \leftrightarrow X^{m+1} \cdot Y$.

F.3 Second variable at the beginning

We attempt to do the same by adding a variable Y at the beginning of the sequence of X's. Again, let I be an infinite set of natural numbers, such that for each pair $(i,j) \in I \times I$, we have $Y \cdot X^i \leftrightarrow Y \cdot X^j$. Then $B = \{(Y \cdot X^i, Y \cdot X^j) \mid i,j \in I\}$ is an infinite set of bisimilar pairs. Let k be some natural number and consider the finite relation $R = \{(Y \cdot X^k, Y \cdot X^{k+1})\}$. Clearly, for each pair $(Y \cdot X^i, Y \cdot X^j) \in B$ with $i,j > k$ we have $Y \cdot X^i \stackrel{R}{\equiv} Y \cdot X^j$. There can only be finitely many pairs in B that are not related by $\stackrel{R}{\equiv}$, hence, by simply adding all these pairs to R we have a finite relation that satisfies $Y \cdot X^i \stackrel{R}{\equiv} Y \cdot X^j$ for all pairs $(Y \cdot X^i, Y \cdot X^j) \in B$. It remains to show is that there exists a $k \in \mathbb{N}$ such that $Y \cdot X^k \leftrightarrow Y \cdot X^{k+1}$.

In order to prove this result, we will first show that, given $Y \cdot X^n \leftrightarrow Y \cdot X^m$ for some $m < n$, for any sequence α reachable in one step from X, there exist $i < j$ such that $\alpha \cdot X^i \leftrightarrow \alpha \cdot X^j$.

For any sequence α, we define S^i_α as the set of sequences reachable in i steps from α, i.e., $S^i_\alpha = \{\alpha' \mid \alpha \xrightarrow{w} \alpha' \text{ for some } w \text{ of length } i\}$. Furthermore we use $\bigcup_{i=1}^n S^i_\alpha$ to denote the union $S^1_\alpha \cup S^2_\alpha \cup \cdots \cup S^n_\alpha$. Note that by image-finiteness, for each i and α the set S^i_α is finite.

Lemma F.3. Let α and β be TSP$^\cdot(A)$-sequences such that $\alpha\beta^n \leftrightarrow \alpha\beta^m$ for some $m < n$. Then for each $\gamma \in (\bigcup_{i=1}^{n(\alpha)+1} S^i_\alpha) \cup S^1_\beta$, there exists a sequence $\gamma' \in (\bigcup_{i=1}^{n(\alpha)+1} S^i_\alpha) \cup S^1_\beta$ such that $\gamma\beta^n \leftrightarrow \gamma'\beta^i$ for some $i < n$.

Proof. Let $\gamma \in (\bigcup_{i=1}^{n(\alpha)+1} S^i_\alpha) \cup S^1_\beta$, then either $\gamma \in \bigcup_{i=1}^{n(\alpha)+1} S^i_\alpha$ or $\gamma \in S^1_\beta$.

- If $\gamma \in \bigcup_{i=1}^{n(\alpha)+1} S^i_\alpha$, then $\alpha \xrightarrow{w} \gamma$ for some sequence of actions $w = a_1 \ldots a_j$ with $1 \le j \le n(\alpha)+1$. Hence, $\alpha\beta^n \xrightarrow{w} \gamma\beta^n$ and thus $\alpha\beta^m \xrightarrow{w} \delta$ such that $\gamma\beta^n \leftrightarrow \delta$. We must have either $\alpha \xrightarrow{w} \alpha'$ and $\delta = \alpha'\beta^m$, or $j = n(\alpha)+1$, $\alpha \xrightarrow{a_1 \ldots a_{j-1}} \alpha'$ such that $\alpha' \downarrow$, $\beta \xrightarrow{a_j} \beta'$ and $\delta = \beta'\beta^i$ for some $i < m$. In the first case $\alpha' \in S^j_\alpha$ and $m < n$, and in the second case $\beta' \in S^1_\beta$ and $i < n$. Hence, in both cases the claim clearly holds.

- If $\gamma \in S^1_\beta$, then $\beta \xrightarrow{a} \gamma$ for some action a. Since α is normed $\alpha \xrightarrow{w} \alpha'$ for some w with length $n(\alpha)$ such that $\alpha' \downarrow$. Hence, $\alpha\beta^n \xrightarrow{w} \alpha'\beta^n \xrightarrow{a} \gamma\beta^{n-1}$ and thus $\alpha\beta^m \xrightarrow{wa} \delta$ such that $\gamma\beta^{n-1} \leftrightarrow \delta$. We must have either $\alpha \xrightarrow{wa} \alpha''$ and $\delta = \alpha''\beta^m$, or $\beta \xrightarrow{a} \beta'$ and $\delta = \beta'\beta^i$ for some $i < m$. In the latter case, the claim clearly holds as then $\gamma\beta^n \leftrightarrow \beta'\beta^{i+1}$, $\beta' \in S^1_\beta$ and $i+1 \le m < n$. In the first case, we have $\gamma\beta^n \leftrightarrow \alpha''\beta^{m+1}$. If $m+1 < n$, the claim holds as $\alpha'' \in S^{n(\alpha)+1}_\alpha$. If $m+1 \ge n$, then since $m < n$ we must have $m+1 = n$. Then, as proven in case (1), there exists a $\gamma' \in (\bigcup_{i=1}^{n(\alpha)+1} S^i_\alpha) \cup S^1_\beta$ such that $\alpha''\beta^n \leftrightarrow \gamma'\beta^j$ for some $j < n$. Hence, in this case we conclude that $\gamma\beta^n \leftrightarrow \alpha'\beta^{m+1} \leftrightarrow \gamma'\beta^j$ and thus the claim holds as well.

□

Lemma F.4. Let B be a finite set of TSP$^\cdot(A)$-sequences and γ a TSP$^\cdot(A)$-sequence. If there exists an $n \in \mathbb{N}$ such that for each $\alpha_i \in B$ there exists a sequence $\alpha_j \in B$ such that $\alpha_i\gamma^n \leftrightarrow \alpha_j\gamma^m$ for some $m < n$, then there exist some $k, l \in \mathbb{N}$ with $k < l$ such that for each $\alpha \in S$, $\alpha\gamma^k \leftrightarrow \alpha\gamma^l$.

Proof. Let $n \in \mathbb{N}$ such that for each $\alpha_i \in B$ there exists a sequence $\alpha_j \in B$ such that $\alpha_i\gamma^n \leftrightarrow \alpha_j\gamma^m$ for some $m < n$.

Let $\alpha_1 \in B$, then there exists a sequence $\alpha_2 \in B$ such that $\alpha_1\beta^n \leftrightarrow \alpha_2\beta^m$ for some $m < n$. Since bisimilarity is a congruence for TSP$^\cdot(A)$, we have $\alpha_1\beta^n\beta^{n-m} \leftrightarrow \alpha_2\beta^m\beta^{n-m} \equiv \alpha_2\beta^n$, and thus since $\alpha_2 \in B$, there exists a sequence $\alpha_3 \in B$ such that $\alpha_1\beta^n\beta^{n-m} \leftrightarrow \alpha_2\beta^n \leftrightarrow \alpha_3\beta^k$ for some $k < n$. We can continue this process to end up with an infinite sequence of bisimilar terms of the shape $\alpha_1\beta^{n_1} \leftrightarrow \alpha_2\beta^{n_2} \leftrightarrow \ldots$, with $n_1 > n_2 > \ldots$. Since B is finite, there must be some $i < j$ such that $\alpha_i \equiv \alpha_j$. Since $n_1 \ge n_i > n_j$, we have $\alpha_1\beta^{n_1} \leftrightarrow \alpha_i\beta^{n_i} \equiv \alpha_j\beta^{n_j}\beta^{n_i-n_j} \leftrightarrow \alpha_1\beta^{n_1}\beta^{n_i-n_j}$, with $n_1 < n_1 + (n_i - n_j)$.

We conclude that for every $\alpha \in B$ there exist $k_\alpha, l_\alpha \in \mathbb{N}$ such that $\alpha\beta^{k_\alpha} \leftrightarrow \alpha\beta^{l_\alpha}$. Hence, picking $k = \max(\{k_\alpha \mid \alpha \in B\})$, gives $\alpha\beta^k \leftrightarrow \alpha\beta^{l_\alpha+(k-k_\alpha)}$ for all $\alpha \in B$. □

Using the result from the previous two lemmas, we can now show that given $Y \cdot X^n \leftrightarrow Y \cdot X^m$ for some $m < n$, there exists a $k \in \mathbb{N}$, such that $Y \cdot X^k \leftrightarrow Y \cdot X^{k+1}$.

Lemma F.5. If $\alpha\beta^n \leftrightarrows \alpha\beta^m$ for some $m < n$, then there exist a $k \in \mathbb{N}$ such that $\alpha\beta^k \leftrightarrows \alpha\beta^{k+1}$.

Proof. Let α and β be TSP$^{\cdot}(A)$-sequences such that $\alpha\beta^n \leftrightarrows \alpha\beta^m$ for some $m < n$. By Lemma F.3, it must be the case that for each $\gamma \in (\bigcup_{i=1}^{n(\alpha)+1} S_\alpha^i) \cup S_\beta^1$, there exists a sequence $\gamma' \in (\bigcup_{i=1}^{n(\alpha)+1} S_\alpha^i) \cup S_\beta^1$ such that $\gamma\beta^n \leftrightarrows \gamma'\beta^i$ for some $i < n$. Moreover, by image-finiteness, $(\bigcup_{i=1}^{n(\alpha)+1} S_\alpha^i) \cup S_\beta^1$ must be finite and thus we can apply Lemma F.4 to derive that there exists a $i \in \mathbb{N}$ such that for every $\gamma \in (\bigcup_{i=1}^{n(\alpha)+1} S_\alpha^i) \cup S_\beta^1$, there exists a $j > i$ such that $\gamma\beta^i \leftrightarrows \gamma\beta^j$. We will prove that for $k = i + 1$ we must have $\alpha\beta^k \leftrightarrows \alpha\beta^{i+1}$, by showing that the following relation R is a bismulation relation.

$$\begin{aligned} R = \{(\alpha\beta^n, \alpha\beta^m) \mid\ & \text{there exists an } i < n, m \text{ such that for every } \beta' \in S_\beta^1 \\ & \text{there exists a } j > i \text{ such that } \beta'\beta^i \leftrightarrows \beta'\beta^j\} \end{aligned}$$

If $n = m$, then clearly $\alpha\beta^n \leftrightarrows \alpha\beta^m$. Hence, we consider the case where $n \neq m$ and without loss of generality assume $n < m$. First, note that since $\alpha\beta^n \leftrightarrows \alpha\beta^m$ for some $m < n$, we must have $n(\beta) = 0$ and thus $\beta\downarrow$. Suppose $\alpha\beta^n \xrightarrow{a} \gamma$, then we distinguish two cases.

- If $\alpha \xrightarrow{a} \alpha'$ and $\gamma = \alpha'\beta^n$, then $\alpha\beta^m \xrightarrow{a} \alpha'\beta^m$ and clearly $(\alpha'\beta^n, \alpha'\beta^m) \in R$.
- If $\alpha\downarrow$, $\beta \xrightarrow{a} \beta'$ and $\gamma = \beta'\beta^k$ for some $k < n$, then since $k < n < m$ and $\beta\downarrow$ we simply have that $\alpha\beta^m \xrightarrow{a} \beta'\beta^k$ and clearly $\beta'\beta^k \leftrightarrows \beta'\beta^k$.

Now suppose $\alpha\beta^m \xrightarrow{a} \gamma$. We again distinguish two cases.

- If $\alpha \xrightarrow{a} \alpha'$ and $\gamma = \alpha'\beta^m$, then $\alpha\beta^n \xrightarrow{a} \alpha'\beta^n$ and clearly $(\alpha'\beta^n, \alpha'\beta^m) \in R$.
- Suppose $\alpha\downarrow$, $\beta \xrightarrow{a} \beta'$ and $\gamma = \beta'\beta^k$ for some $k < m$. Then, if $k < n$, we have $\alpha\beta^n \xrightarrow{a} \beta'\beta^k$ and clearly $\beta'\beta^k \leftrightarrows \beta'\beta^k$. If $k \geq n > i$, then $\alpha\beta^n \xrightarrow{a} \beta'\beta^i$ and there exists a $j > i$ such that $\beta'\beta^i \leftrightarrows \beta'\beta^j$. Hence, then $\beta'\beta^i \leftrightarrows \beta'\beta^j$ and since $k, j > i$ we have $(\alpha'\beta^k, \alpha'\beta^j) \in R$.

Finally note that if $\alpha\beta^n\downarrow$, $\alpha\downarrow$ and $\beta\downarrow$ and thus also $\alpha\beta^m\downarrow$, and vice versa.

As we have shown that R is a bisimulation relation and $(\alpha\beta^k, \alpha\beta^{k+1}) \in R$, we conclude that $\alpha\beta^k \leftrightarrows \alpha\beta^{k+1}$. □