

MASTER

A characterization of the special linear and unitary Lie algebra via its extremal geometry

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EINDHOVEN UNIVERSITY OF TECHNOLOGY

MASTER THESIS

**A characterization of the special
linear and unitary Lie algebra via
its extremal geometry**

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Chapter 1

Introduction

I would like to thank Prof. Dr. H. Cuypers for all his help and guidance during this master thesis.

1.1 Main results

Let \mathfrak{g} be a Lie algebra over a field \mathbb{F} . An element $x \in \mathfrak{g}$, with $x \neq 0$, is called *extremal* if for all $y \in \mathfrak{g}$ we have

$$[x, [x, y]] \subseteq \mathbb{F}x$$

(and, in case the characteristic of \mathbb{F} is 2, some additional conditions are satisfied).

An *extremal point* of the Lie algebra \mathfrak{g} is a 1-dimensional subspace spanned by an extremal element.

An *extremal line* of \mathfrak{g} is a 2-dimensional subspaces of \mathfrak{g} such that its nontrivial elements are extremal and pairwise commuting. The extremal geometry of \mathfrak{g} is then the geometry with as points the extremal points of \mathfrak{g} and as lines the extremal lines.

Examples of extremal elements are the long root elements in classical Lie algebras. In particular, if V is a (finite dimensional) vector space with dual space V^* , then the linear maps

$$\begin{aligned} t_{v,\phi} : V &\rightarrow V \\ w \in V &\mapsto \phi(w)v, \end{aligned}$$

where $0 \neq v \in V$ and $0 \neq \phi \in V^*$ with $\phi(v) = 0$, are extremal elements in $\mathfrak{gl}(V)$ generating the subalgebra $\mathfrak{sl}(V)$.

Lie algebras generated by extremal elements have been studied by Cohen et al., through their extremal geometry. See for example [9, 12, 5, 8, 6, 3].

The main result of [3] implies that the extremal geometry of a finite dimensional simple Lie algebra generated by extremal elements is a so-called root-shadow space of a spherical building, or this geometry contains no lines. If the extremal geometry is the root-shadow space of a spherical building of finite rank at least three, then the main result of Cuypers and Fleischmann [6] (see also Roberts [12] for buildings of type A_n and Cuypers, Roberts and Sphectorov [8] for buildings of type A_n , D_n or E_n) implies that the isomorphism type of the extremal geometry uniquely determines the isomorphism type of the Lie algebra.

In case of the Lie algebra $\mathfrak{psl}(V)$, the extremal geometry is isomorphic to the geometry $\Gamma(V, V^*)$ of incident point-hyperplane pairs of $\mathbb{P}(V)$, with as points the incident point-hyperplane pairs of $\mathbb{P}(V)$ and as lines the subsets of incident point-hyperplane pairs (p, H) , where either p is fixed and H runs over the hyperplanes containing a fixed codimension 2 space, or H is fixed and p runs over the 1-dimensional subspaces of a fixed 2-dimensional space.

The above example can be extended in the following way. Let V be a possibly infinite dimensional vector space and Π a subspace of V^* with $\text{Ann}_V(\Pi) = \{0\}$. Then the elements $t_{v,\phi} : V \rightarrow V$ with $0 \neq v \in V$ and $0 \neq \phi \in \Pi$ are extremal in the Lie subalgebra of $\mathfrak{gl}(V)$ which they generate and which will be denoted by $\mathfrak{fsl}(V, \Pi)$. This Lie algebra is, up to its center, simple. Its extremal geometry is isomorphic to the geometry $\Gamma(V, \Pi)$ of incident point-hyperplane pairs (p, H) of $\mathbb{P}(V)$ where p is a point and H a hyperplane which is the kernel of an element of Π .

In this thesis we want to extend the results of Cohen and Ivanyos and of Cuypers and Fleischmann. In particular we want to characterize the infinite dimensional Lie algebras $\mathfrak{psl}(V, \Pi)$ described above by its extremal geometry. We will prove the following theorem:

Theorem 1.1.1. *Let \mathfrak{h} be a simple Lie algebra of possibly infinite dimension, generated by extremal elements, such that its extremal geometry $\Gamma(\mathfrak{h})$ is isomorphic to the geometry $\Gamma(V, \Pi)$, for a vector space V of dimension at least 3, possibly infinite dimensional, and subspace Π of V^* such that $\text{Ann}_V(\Pi) = \{0\}$. Then $\mathfrak{h} \simeq \mathfrak{psl}(V, \Pi)$.*

This theorem will be proved in chapter 6.

The results of Cohen and Ivanyos [3] do not give information on Lie algebras generated by extremal elements, whose extremal geometry does not contain any line. This case has been considered in [7], where it is shown that a simple Lie algebra \mathfrak{g} over a field \mathbb{F} generated by extremal elements whose extremal geometry does not contain lines is either a symplectic Lie algebra or, there exists a quadratic extension \mathbb{K} of \mathbb{F} such that the Lie algebra $\mathfrak{g} \otimes_{\mathbb{F}} \mathbb{K}$

is generated by its extremal elements and the extremal geometry of $\mathfrak{g} \otimes_{\mathbb{F}} \mathbb{K}$ does contain lines.

An example of this last situation is provided by the special unitary algebras. Indeed, let V be a vector space over a field \mathbb{K} and h a nondegenerate Hermitian form (linear in the first coordinate) on V with respect to a field involution $\sigma : \mathbb{K} \rightarrow \mathbb{K}$ with fixed field \mathbb{F} . If $0 \neq v \in V$ is a singular vector, then the map

$$t_v : V \rightarrow V,$$

defined by $t_v(w) = h(w, v)v$ for all $w \in V$, is extremal in $\mathfrak{gl}(V)$. If V contains singular vectors, then the finitary special unitary Lie algebra $\mathfrak{fsu}(V, h)$ is the Lie algebra over \mathbb{F} generated by the extremal elements t_v , with $v \in V$ running over the singular vectors with respect to the form h .

The extremal geometry of this Lie algebra does not contain lines. However, $\mathfrak{fsu}(V, h) \otimes_{\mathbb{F}} \mathbb{K}$ is $\mathfrak{sl}(V, \Pi)$, where Π is the subspace of V^* generated by the maps $w \in V \mapsto h(v, w)$, with v running over the singular elements in V . So, the extremal geometry of $\mathfrak{fsu}(V, h) \otimes_{\mathbb{F}} \mathbb{K}$ is isomorphic to $\Gamma(V, \Pi)$.

Our second theorem provides a characterization of the finitary special unitary Lie algebras (up to centers):

Theorem 1.1.2. *Let \mathfrak{g} be a simple Lie algebra over \mathbb{F} generated by extremal elements, whose extremal geometry has no lines. Assume that \mathbb{F} can be quadratically extended to a field say \mathbb{K} , such that the extremal geometry $\Gamma(\hat{\mathfrak{g}})$ of the Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{F}} \mathbb{K}$ is isomorphic to the geometry of point-hyperplane pairs $\Gamma(V, \Pi)$ for some vector space V of dimension at least 4 over \mathbb{K} and subspace Π of V^* with $\text{Ann}_V(\Pi) = \{0\}$. Then \mathfrak{g} is isomorphic to $\mathfrak{pfsu}(V, h)$ for some nondegenerate Hermitian form h on the vector space V .*

This theorem will be proved in chapter 7.

1.2 Structure of the thesis

The thesis will start with some general theory on Lie algebras in the second chapter. In this chapter the definition of a Lie algebra will be given, together with the definitions of the classical linear Lie algebras $\mathfrak{gl}_n(\mathbb{F})$, $\mathfrak{sl}_n(\mathbb{F})$ and $(\mathfrak{s})\mathfrak{u}_n(\mathbb{F})$, where \mathbb{F} is some field.

In the third chapter we will start with the introduction of extremal elements of Lie algebras. After that we will introduce Lie algebras defined on a tensor product and link the above classical Lie algebras to these Lie algebras.

In the fourth chapter we will introduce point line spaces and root filtration spaces. The fifth chapter will be used to introduce the extremal geometry of a Lie algebra generated by its extremal elements. In this chapter the points and the lines of the extremal geometry will be introduced as defined

before. In chapter 6 we will prove theorem 1.1.1. This chapter will start with the finite dimensional case, which is proven by Roberts [12]. After that we introduce some general theory which will be needed for the infinite dimensional case. One of the main definitions is the one of a local system (definition 6.2.2). And that when two local systems are isomorphic then the direct limit is isomorphic. So the proof is based on constructing two isomorphic local systems, one for $\mathfrak{sl}(V, \Pi)$ and one for \mathfrak{h} which geometry is isomorphic to $\Gamma(V, \Pi)$.

In the last chapter, chapter 7, we give a proof for theorem 1.1.2.

Chapter 2

Lie Algebras

In this chapter we will introduce the notion of a Lie algebra and we will provide the theory, terminology and notation which will be used in this master thesis. All the theory, notation and terminology used in this chapter comes from chapter one of the PhD of Fleischmann [9].

2.1 General theory

Definition 2.1.1 (Lie Algebra). *A vector space \mathfrak{g} over a field \mathbb{F} together with a binary operation*

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g},$$

*is a **Lie Algebra** over \mathbb{F} if the following conditions are satisfied:*

(1) *Bilinearity: for all $\alpha, \beta \in \mathbb{F}$ and for all $x, y, z \in \mathfrak{g}$ we have*

$$\begin{aligned} [\alpha x + \beta y, z] &= \alpha[x, z] + \beta[y, z] \\ [z, \alpha x + \beta y] &= \alpha[z, x] + \beta[z, y]. \end{aligned}$$

(2) *Alternation: for all $x \in \mathfrak{g}$ the identity*

$$[x, x] = 0$$

holds.

(3) *Jacobi identity: for all $x, y, z \in \mathfrak{g}$ the following holds*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

The binary operation $[\cdot, \cdot]$ is called the **Lie bracket** of the Lie algebra \mathfrak{g} .

A fundamental example of a Lie algebra is the ring of endomorphisms of

a vector space V over a field \mathbb{F} , denoted by $\text{End}(V)$. The binary operation $[\cdot, \cdot]$ on $\text{End}(V)$ is defined as $[f, g] := fg - gf$ for $f, g \in \text{End}(V)$, where $fg := f \circ g$ is the composition of the maps f and g . The Lie algebra on the ring of endomorphisms is denoted by $\mathfrak{gl}(V)$ and is called the **general linear Lie algebra**.

Definition 2.1.2 (Lie subalgebra). *A linear subspace \mathfrak{h} of a Lie algebra \mathfrak{g} is called a Lie subalgebra if,*

$$\forall x, y \in \mathfrak{h}, \quad [x, y] \in \mathfrak{h}.$$

Any subalgebra of $\mathfrak{gl}(V)$ is called a **linear Lie algebra**. (Note that $\text{End}(V)$ is considered to be a vector space over a subfield $\mathbb{L} \subseteq \mathbb{F}$.)

An element of $f \in \mathfrak{gl}(V)$ is called **finitary**, if the kernel of f has finite codimension, that is $\dim V/\ker f < \infty$. The set $\mathfrak{fgl}(V) = \{f \in \mathfrak{gl}(V) : f \text{ is finitary}\}$ is a subalgebra of $\mathfrak{gl}(V)$. Any subalgebra of $\mathfrak{fgl}(V)$ is called **finitary linear Lie algebra**. Notice that when V is of finite dimension, every element of $\mathfrak{gl}(V)$ is finitary.

Definition 2.1.3 (Lie algebra homomorphism). *A linear map ϕ between two Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 is called a Lie algebra homomorphism if*

$$\phi([x, y]) = [\phi(x), \phi(y)], \quad \forall x, y \in \mathfrak{g}_1.$$

If a Lie algebra homomorphism ϕ is bijective, and its inverse also an Lie algebra homomorphism, then ϕ is called a **Lie algebra isomorphism**. If ϕ is an isomorphism between two Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 , we say that \mathfrak{g}_1 is isomorphic to \mathfrak{g}_2 , denoted by $\mathfrak{g}_1 \simeq \mathfrak{g}_2$.

Definition 2.1.4 (Adjoint map of an element). *Let \mathfrak{g} be a Lie algebra and $x \in \mathfrak{g}$. Then the linear map*

$$\begin{aligned} ad_x : \mathfrak{g} &\rightarrow \mathfrak{g} \\ y &\mapsto [x, y] \end{aligned}$$

is called the adjoint map of x .

With the adjoint map the **adjoint** representation of \mathfrak{g} is defined. The adjoint representation of \mathfrak{g} is defined by the Lie algebra homomorphism $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ defined by $x \mapsto ad_x$.

Definition 2.1.5 (Special linear Lie algebra). *Let V be a finite dimensional vectors space over a field \mathbb{F} . Then the special linear algebra $\mathfrak{sl}(V)$, is a subalgebra of $\mathfrak{gl}(V)$, where $\mathfrak{sl}(V)$ is the set of endomorphisms on V with trace zero. The trace of an element $x \in \mathfrak{gl}(V)$ is denoted by $\text{Tr}(x)$.*

Another notation for the special linear algebra is $\mathfrak{sl}_n(\mathbb{F})$, where $n = \dim(V)$. If V is of infinite dimension the trace map is defined on the finitary elements of $\mathfrak{gl}(V)$. In this case, $\mathfrak{fsl}(V)$ is defined as the subalgebra of $\mathfrak{gl}(V)$ which consists of finitary elements with trace zero.

The following theorem gives a classification of finite dimensional Lie algebras.

Theorem 2.1.6. *Every finite dimensional Lie algebra \mathfrak{g} is isomorphic to a subalgebra of $\mathfrak{gl}(V)$ for some vector space V over a field \mathbb{F} .*

Proof. The proof in case V is a vector space over a field \mathbb{F} of characteristic zero and p can be found in [11] chapter VI. \square

Like in ring theory, ideals exist also within the theory of Lie algebras.

Definition 2.1.7 (Ideal). *A subspace \mathcal{I} of a Lie algebra \mathfrak{g} is called an ideal if for all $x \in \mathfrak{g}$ and $y \in \mathcal{I}$ we have $[x, y] \in \mathcal{I}$.*

Let \mathfrak{g}_1 and \mathfrak{g}_2 be a Lie algebras. Then the kernel of a Lie algebra homomorphism $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is an ideal, since for all $x \in \mathfrak{g}_1$ and $y \in \ker \phi$ the following holds:

$$\phi([x, y]) = [\phi(x), \phi(y)] = [\phi(x), 0] = 0.$$

Proposition 2.1.8. *Let \mathfrak{g} be a Lie algebra and $\mathcal{I} \subset \mathfrak{g}$ an ideal. Then \mathfrak{g}/\mathcal{I} is a Lie algebra with the induced Lie bracket.*

Proof. Let $x, y \in \mathfrak{g}$ be arbitrary and $\mathcal{I} \subset \mathfrak{g}$ an ideal, then:

$$\begin{aligned} [x + \mathcal{I}, y + \mathcal{I}] &= \{[x + v, y + w] : v, w \in \mathcal{I}\} \\ &= \{[x, y] + [v, y] + [x, w] + [v, w] : v, w \in \mathcal{I}\} \end{aligned} \quad \square$$

We say that two elements x and y of a Lie algebra \mathfrak{g} **commute** if $[x, y] = 0$. The Lie algebra \mathfrak{g} is called **commutative** if every two elements of \mathfrak{g} commute. All the commutative Lie algebras form a class of Lie algebras, that is the class of **commutative Lie algebras**.

Definition 2.1.9 (Center of a Lie algebra). *Let \mathfrak{g} be a Lie algebra over a field \mathbb{F} . The center of a Lie algebra is defined as:*

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g} : [x, \mathfrak{g}] = 0\}.$$

Notice that the center of a Lie algebra is an ideal.

Definition 2.1.10 (Commutator subalgebra). *Let \mathfrak{g} be a Lie algebra, then the commutator subalgebra of \mathfrak{g} is defined as $[\mathfrak{g}, \mathfrak{g}]$ spanned by all elements $[x, y]$ where $x, y \in \mathfrak{g}$.*

Note that $[\mathfrak{g}, \mathfrak{g}]$ is a ideal of \mathfrak{g} . In general, if \mathcal{I} is an ideal of \mathfrak{g} , then $[\mathfrak{g}, \mathcal{I}]$ is an ideal of \mathfrak{g} spanned by $[x, i]$ where $x \in \mathfrak{g}$ and $i \in \mathcal{I}$.

Definition 2.1.11 (Simple Lie algebra). *Let \mathfrak{g} be a Lie algebra, then \mathfrak{g} is called a simple Lie algebra if $[\mathfrak{g}, \mathfrak{g}] \neq 0$ and there exists no ideal \mathcal{I} which is a non-trivial ideal of \mathfrak{g} .*

Note that if \mathfrak{g} is a simple Lie algebra then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, since there are no non-trivial ideals of \mathfrak{g} .

Further a Lie algebra which is the direct sum of simple Lie algebras is called a **semisimple Lie algebra**.

For a Lie algebra \mathfrak{g} the following sequence of ideals can be defined:

$$\mathfrak{g}^0 := \mathfrak{g}, \quad \mathfrak{g}^1 := [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^2 := [\mathfrak{g}, \mathfrak{g}^1] = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]], \quad \mathfrak{g}^3 := [\mathfrak{g}, \mathfrak{g}^2], \dots$$

A Lie algebra \mathfrak{g} is **nilpotent**, if there exist a $n \in \mathbb{N}$ such that $\mathfrak{g}^n = 0$. Another sequence of ideals for a Lie algebra \mathfrak{g} is:

$$\begin{aligned} \mathfrak{g}^{(0)} := \mathfrak{g}, \quad \mathfrak{g}^{(1)} := [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^{(2)} := [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]], \\ \mathfrak{g}^{(3)} := [\mathfrak{g}^{(2)}, \mathfrak{g}^{(2)}], \dots \end{aligned}$$

A Lie algebra \mathfrak{g} is **solvable**, if there exists a $n \in \mathbb{N}$ such that $\mathfrak{g}^{(n)} = 0$.

Proposition 2.1.12. *If \mathfrak{g} is a nilpotent Lie algebra, then \mathfrak{g} is solvable.*

Proof. Let \mathfrak{g} be an arbitrary Lie algebra. It is enough to show that $\mathfrak{g}^{(k)} \subseteq \mathfrak{g}^k$ for all $k \in \mathbb{N}$. This will be shown via induction.

If $k = 0$ the following hold by definition:

$$\mathfrak{g}^{(0)} = \mathfrak{g} = \mathfrak{g}^0.$$

So it holds for $k = 0$.

Assume that there is a k such that $\mathfrak{g}^{(k)} \subseteq \mathfrak{g}^k$, then

$$\mathfrak{g}^{(k+1)} = [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] \subseteq [\mathfrak{g}^k, \mathfrak{g}^k] = \mathfrak{g}^{k+1}.$$

Thus for all $k \in \mathbb{N}$, $\mathfrak{g}^{(k+1)} \subseteq \mathfrak{g}^{k+1}$ holds.

So if \mathfrak{g} is a nilpotent Lie algebra, then there exists a $n \in \mathbb{N}$ such that $\mathfrak{g}^n = 0$. Since $\mathfrak{g}^{(n)} \subseteq \mathfrak{g}^n = 0$, we have that $\mathfrak{g}^{(n)} = 0$. Thus \mathfrak{g} is solvable. \square

Proposition 2.1.13 (Proposition 2.3.1 of [10]). *Let \mathfrak{g} be a solvable Lie algebra with solvable ideals I and J . Then $I + J$ is a solvable ideal of \mathfrak{g} .*

Proof. See [10] proposition 2.3.1. \square

Definition 2.1.14 (Radical). *Let \mathfrak{g} be a finite dimensional Lie algebra, then a radical of \mathfrak{g} is a solvable ideal of maximal possible dimension denoted by $\text{Rad}(\mathfrak{g})$.*

Proposition 2.1.15. $\text{Rad}(\mathfrak{g})$ contains any solvable ideal of \mathfrak{g} and is unique.

Proof. Let \mathcal{I} be an arbitrary solvable ideal of \mathfrak{g} and $\text{Rad } \mathfrak{g}$ the radical of \mathfrak{g} . Then $\mathcal{I} + \text{Rad } \mathfrak{g}$ is solvable by proposition 2.1.13. By the definition of the $\text{Rad } \mathfrak{g}$ of \mathfrak{g} , we have that $\text{Rad } \mathfrak{g}$ is the biggest solvable ideal of \mathfrak{g} . This means that $\mathcal{I} + \text{Rad } \mathfrak{g} = \text{Rad } \mathfrak{g}$.

Now the uniqueness of the radical of \mathfrak{g} . Given two different maximal solvable ideals $\text{Rad } \mathfrak{g}$ and \mathcal{J} . Then $\text{Rad } \mathfrak{g} + \mathcal{J}$ is solvable by proposition 2.1.13. This means that $\text{Rad } \mathfrak{g} = \text{Rad } \mathfrak{g} + \mathcal{J} = \mathcal{J}$. This is in contradiction with the fact that $\text{Rad } \mathfrak{g}$ and \mathcal{J} are different.

So the radical of \mathfrak{g} is the unique maximal solvable ideal. \square

2.2 Linear Lie algebras

In this section we will introduce three families of Lie algebras, which are subalgebras of the general linear Lie algebra.

In the previous section, we already introduced the special linear Lie algebra $\mathfrak{sl}(V)$ in case V is a finite dimensional vector space over a field \mathbb{F} . This Lie algebra is isomorphic with the Lie algebra of $n \times n$ matrices with trace zero, where $n = \dim V$. This Lie algebra is spanned by the matrices

$$\begin{aligned} E_{i,j}, & \quad \text{with } i \neq j, \quad i, j \in \{1, \dots, n\}, \\ E_{i,i} - E_{i+1,i+1}, & \quad \text{with } i \in \{1, \dots, n-1\}, \end{aligned}$$

where

$$E_{i,j} = \begin{cases} 1, & \text{at position } (i, j), \\ 0 & \text{otherwise.} \end{cases}$$

The dimension of $\mathfrak{sl}(V)$ is $n^2 - 1$, since there are $n^2 - n$ matrices of the first type and $n - 1$ of the second type and they are independent.

Definition 2.2.1. Let V be a finite dimensional vector space over a field \mathbb{F} . The Lie algebra obtained as a quotient of $\mathfrak{sl}_n(\mathbb{F})$ by its center is denoted by $\mathfrak{psl}_n(\mathbb{F})$.

To introduce the other families, some additional theory will be needed.

Lemma 2.2.2. Let V be a vector space over a field \mathbb{F} . Let

$$h : V \times V \rightarrow \mathbb{F}$$

be a bi-additive form, such that

$$h(v, 0) = 0 = h(0, v), \quad \forall v \in V.$$

If for $f, g \in \mathfrak{gl}(V)$, such that for all $u, v \in V$

$$\begin{aligned} h(f(u), v) &= -h(u, f(v)), \\ h(g(u), v) &= -h(u, g(v)). \end{aligned}$$

Then

$$h([f, g](u), v) = -h(u, [f, g](v)).$$

Proof. Let $u, v \in V$ be arbitrary, then note that

$$\begin{aligned} 0 &= h(v, 0) = h(v, w - w) = h(v, w) + h(v, -w), \\ 0 &= h(0, w) = h(v - v, w) = h(v, w) + h(-v, w). \end{aligned}$$

This results into:

$$\begin{aligned} h(-v, w) &= -h(v, w), \\ h(v, -w) &= -h(v, w). \end{aligned}$$

Let now $f, g \in \mathfrak{gl}(V)$, then

$$\begin{aligned} h([f, g](u), v) &= h((fg)(u) - (gf)(u), v) \\ &= h(f(gu), v) - h(g(fu), v) \\ &= -h(g(u), f(v)) + h(f(u), g(v)) \\ &= h(u, (gf)(v)) - h(u, (fg)(v)) \\ &= h(u, (gf - fg)(v)) \\ &= h(u, -[f, g](v)) \\ &= -h(u, [f, g](v)), \end{aligned}$$

for all $u, v \in V$. □

This shows that a bi-additive form h on V , which satisfies $h(f(u), v) = -h(u, f(v))$ for an endomorphism f on V , is preserved by the commutator of $\mathfrak{gl}(V)$. In particular, the set

$$\{f \in \text{End}(V) : h(f(u), v) = -h(u, f(v))\}$$

forms a Lie subalgebra of $\mathfrak{gl}(V)$.

Definition 2.2.3 ((σ, ε) -sesquilinear form). Let V be a vector space of a field \mathbb{F} , σ an automorphism of \mathbb{F} and $0 \neq \varepsilon \in \mathbb{F}$.

A map

$$h : V \times V \rightarrow \mathbb{F}$$

is a (σ, ε) -sesquilinear form if for all $u, v, w \in V$ and $\lambda, \mu \in \mathbb{F}$

$$(a) \ h(v + w, u) = h(v, u) + h(w, u),$$

$$(b) \ h(\lambda v, \mu w) = \lambda h(v, w) \mu^\sigma,$$

$$(c) \ h(v, w) = \varepsilon h(w, v)^\sigma.$$

The (σ, ε) -sesquilinear form h is called **reflexive** if for all $v, w \in V$ $h(v, w) = 0$ implies $h(w, v) = 0$.

Lemma 2.2.4. *Let h be a nontrivial (σ, ε) -sesquilinear form, then the following holds*

1. $\varepsilon^\sigma = \varepsilon^{-1}$,
2. for all $\lambda \in \mathbb{F}$, $\lambda^{\sigma^2} = \varepsilon^{-1} \lambda \varepsilon = \lambda$.
3. if $\sigma = id$ then $\varepsilon = \pm 1$.

Proof. Let h be a nontrivial (σ, ε) -sesquilinear form, then there exists v, w such that $h(v, w) = 1$.

1.

$$1 = h(v, w) = \varepsilon h(w, v)^\sigma = \varepsilon (\varepsilon h(v, w)^\sigma)^\sigma = \varepsilon \varepsilon^\sigma,$$

$$\text{so } \varepsilon^\sigma = \varepsilon^{-1}.$$

2. Let $\lambda, \mu \in \mathbb{F}$ be arbitrary, then

$$\begin{aligned} \lambda \mu^\sigma &= \lambda h(v, w) \mu^\sigma \\ &= h(\lambda v, \mu w) \\ &= \varepsilon h(\mu w, \lambda v)^\sigma \\ &= \varepsilon (\mu h(w, v) \lambda^\sigma)^\sigma \\ &= \varepsilon \lambda^{\sigma^2} h(w, v)^\sigma \mu^\sigma \\ &= \varepsilon \lambda^{\sigma^2} \varepsilon^\sigma \mu^\sigma \\ &= \varepsilon \lambda^{\sigma^2} \varepsilon^{-1} \mu^\sigma. \end{aligned}$$

So σ^2 is conjugation by ε , that is $\lambda^{\sigma^2} = \sigma^2(\lambda) = \varepsilon^{-1} \lambda \varepsilon$.

3. If σ is the identity, then for $\lambda, \mu \in \mathbb{F}$

$$\lambda \mu = (\lambda \mu)^\sigma = \mu^\sigma \lambda^\sigma = \mu \lambda.$$

Thus $\varepsilon = \pm 1$.

□

Definition 2.2.5. Let V be a vector space over a field \mathbb{F} and h a (σ, ε) -sesquilinear form.

1. h is called symmetric if:

- (a) $\varepsilon = 1$,
- (b) $\sigma = id$,

2. h is called anti-symmetric if:

- (a) $\varepsilon = -1$,
- (b) $\sigma = id$,

3. h is called alternating or symplectic if:

- (a) $\varepsilon = -1$,
- (b) $\sigma = id$,
- (c) $h(v, v) = 0$ for all $v \in V$.

4. h is called Hermitian if:

- (a) $\varepsilon = 1$,
- (b) $\sigma^2 = id$ and $\sigma \neq id$.

5. h is called skew-Hermitian if:

- (a) $\varepsilon = -1$,
- (b) $\sigma^2 = id$ and $\sigma \neq id$.

6. h is called nondegenerate if:

- (a) for all $v \in V$, $h(v, w) = 0$ implies $w = 0$,
- (b) for all $w \in V$, $h(v, w) = 0$ implies $v = 0$.

Note that an anti-symmetric sesquilinear form is symplectic if $\text{char}(\mathbb{F}) \neq 2$.

Lemma 2.2.6. Let $h : V \times V \rightarrow \mathbb{F}$ be a (σ, ε) -sesquilinear form where σ is not the identity. If $\alpha \in \mathbb{F}$ with $\alpha \neq 0$, then αh is a (τ, η) -sesquilinear form with $\tau(\lambda) = (\alpha^{-1}\lambda\alpha)^\sigma$ for all $\lambda \in \mathbb{F}$ and $\eta = \alpha\varepsilon\alpha^{-\sigma}$.

Proof. Let $h : V \times V \rightarrow \mathbb{F}$ be a (σ, ε) -sesquilinear form where σ is not the identity. Let $\alpha \in \mathbb{F}$ with $\alpha \neq 0$ and $v, w \in V$, then

$$\begin{aligned}
(\alpha h)((v, w)) &= \alpha h(v, w) \\
&= \alpha \varepsilon h(w, v)^\sigma \\
&= \alpha \varepsilon h(w, v)^\sigma \alpha^\sigma \alpha^{-\sigma} \\
&= \alpha \varepsilon (\alpha h(w, v))^\sigma \alpha^{-\sigma} \\
&= \alpha \varepsilon \alpha^{-\sigma} \alpha^\sigma (\alpha h(w, v))^\sigma \alpha^{-\sigma} \\
&= (\alpha \varepsilon \alpha^{-\sigma}) (\alpha^{-1} (\alpha h(w, v)) \alpha)^\sigma \\
&= \eta (\alpha h(w, v))^\tau
\end{aligned}$$

□

Proposition 2.2.7. *Let $h : V \times V \rightarrow \mathbb{F}$ be a (σ, ε) -sesquilinear form where σ is not the identity and $h \neq 0$.*

If $\alpha \in \mathbb{F}$ such that $\alpha = \beta^\sigma - \varepsilon^\sigma \beta \neq 0$, then αh is a $(\tau, -1)$ -sesquilinear form where τ has order 2.

Proof. Let $h : V \times V \rightarrow \mathbb{F}$ be a (σ, ε) -sesquilinear form and $\alpha \in \mathbb{F}$ such that $\alpha = \beta^\sigma - \varepsilon^\sigma \beta \neq 0$ with $\beta \in \mathbb{F}$. Then we have that αh is a sesquilinear form by lemma 2.2.6. So we need to show that $\eta = -1$ and τ is of order 2.

We have that

$$\alpha^\sigma = \beta^{\sigma^2} - \beta^\sigma \varepsilon = \varepsilon^\sigma \beta \varepsilon - \beta^\sigma \varepsilon = -\alpha \varepsilon.$$

From the previous lemma we have that

$$\eta = \alpha \varepsilon \alpha^{-\sigma} = \alpha \varepsilon (-\alpha \varepsilon)^{-1} = -1.$$

To show that τ is of order two, we need to show that $\tau^2(\lambda) = \lambda$ for all $\lambda \in \mathbb{F}$. Let $\lambda \in \mathbb{F}$ be arbitrary then

$$\begin{aligned}
\tau^2(\lambda) &= \tau(\tau(\lambda)) = \tau(\alpha^\sigma \lambda^\sigma \alpha^{-\sigma}) = \tau(-\alpha \varepsilon \lambda^\sigma (-\alpha \varepsilon)^{-1}) \\
&= \alpha^\sigma (-\alpha \varepsilon \lambda^\sigma (-\alpha \varepsilon)^{-1})^\sigma \alpha^{-\sigma} \\
&= -\alpha \varepsilon (-\alpha \varepsilon)^{-\sigma} \lambda^{\sigma^2} (-\alpha \varepsilon)^\sigma (-\alpha \varepsilon)^{-1} \\
&= \alpha \varepsilon (\varepsilon^{-1} \alpha \varepsilon)^{-1} \varepsilon^{-1} \lambda \varepsilon (\varepsilon^{-1} \alpha \varepsilon) \varepsilon^{-1} \alpha^{-1} \\
&= \lambda.
\end{aligned}$$

□

Definition 2.2.8 (Singular/isotropic). *Let h be a (σ, ε) -sesquilinear form. A nonzero element $v \in V$ is called singular or isotropic if $h(v, v) = 0$. When $h(v, v) \neq 0$ we say that v is non-singular or anisotropic.*

Definition 2.2.9 (Hyperbolic pair). *Let h be a (σ, ε) -sesquilinear form. Let $v, w \in V$ such that its span is a two-dimensional subspace of V . If v and w are singular and $h(v, w) = 1$, then v, w is called a hyperbolic pair.*

The space spanned by a hyperbolic pair is called a hyperbolic 2-space. We call a pair (V, h) , where V is a vector space and h a (σ, ε) -sesquilinear form on V , a **sesquilinear space**.

Now that we have the sesquilinear forms, we can use them to construct Lie subalgebras of $\mathfrak{gl}(V)$, since the sesquilinear forms satisfy lemma 2.2.2. The first space we consider is a symplectic nondegenerate space (V, f) , this means that if f is a symplectic and nondegenerate sesquilinear form on the vector space V . Since the form is symplectic and nondegenerate, we have for all $0 \neq v \in V$ there exists a $0 \neq w \in V$ such that (v, w) is a hyperbolic pair. If we restrict the form f to the space

$$\langle v, w \rangle^\perp = \{u \in V : f(v, u) = 0, f(w, u) = 0\}.$$

Then f is again nondegenerate.

Definition 2.2.10 (Hyperbolic basis). *Let h be a (σ, ε) -sesquilinear form on a finite dimensional vector space V over a field \mathbb{F} . A basis $\{e_1, f_1, \dots, e_n, f_n\}$ is called a hyperbolic basis if*

$$h(e_i, e_j) = 0 = h(f_i, f_j), \quad h(e_i, f_j) = \delta_{ij}.$$

Note that if V is finite dimensional, a hyperbolic basis can be found only if $\dim V$ is even. In this case the symplectic form, h , on V can be defined by

$$h(v, w) = v^T F w, \tag{2.1}$$

where

$$F = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{2.2}$$

Now the theory about sesquilinear forms is given, we continue with the characterization of the Lie subalgebras of $\mathfrak{gl}(V)$.

Definition 2.2.11 (Symplectic Lie algebra). *Let (V, h) be a symplectic vector space over the field \mathbb{F} . Then the symplectic Lie algebra, denoted by $\mathfrak{sp}(V, h)$, is the Lie subalgebra of $\mathfrak{gl}(V)$ that consists of all endomorphisms $f \in \text{End}(V)$ that satisfy*

$$h(f(v), w) = -h(w, f(v)), \quad \text{for } v, w \in V.$$

When V is of finite dimension and h is as in (2.1) and nondegenerate, the symplectic Lie algebra is denoted with $\mathfrak{sp}_n(\mathbb{F})$. We can identify $\mathfrak{sp}_n(\mathbb{F})$ with the Lie algebra \mathfrak{g} of matrices A such that $A^T F = -FA$, where F is as in (2.2). A basis for $\mathfrak{sp}_n(\mathbb{F})$ is then given by

$$\begin{array}{ll} E_{i,i+m}, & 1 \leq i \leq m, \\ E_{i+m,i}, & 1 \leq i \leq m, \\ E_{i,j} - E_{m+j,m+i}, & 1 \leq i, j \leq m, \\ E_{i,m+j} + E_{j,m+1}, & 1 \leq i < j \leq m, \\ E_{m+i,j} + E_{m+j,i}, & 1 \leq i < j \leq m, \end{array}$$

where the dimension of $\mathfrak{sp}_n(\mathbb{F})$ equals $2m^2 + m$.

In the case that V has infinite dimension, we have the **finitary symplectic Lie algebra** $\mathfrak{fsp}(V)$. The finitary symplectic Lie algebra is defined as the intersection of $\mathfrak{sp}(V)$ with $\mathfrak{fgl}(V)$.

The next type of space we consider is the (skew)-Hermitian space (V, h) , where V is a vector space over a field \mathbb{F} and h a (skew)-Hermitian form on V .

We call the space **anisotropic** if V does not contain singular vectors.

If (V, h) is a nondegenerate (skew)-Hermitian space containing a singular vector v , then there is a w such that (v, w) form a hyperbolic pair. The subspace $\langle v, w \rangle^\perp = \{u \in V : h(v, u) = 0 \text{ and } h(w, u) = 0\}$ is again nondegenerate.

So we can decompose V into $V_1 \perp V_2$, where V_1 is a hyperbolic space and V_2 is anisotropic. This means that if V is finite dimensional, h is represented by

$$h(v, w) = v^T H w^\sigma, \quad \text{for } v, w \in V,$$

$$H = \begin{pmatrix} 0 & I_k & 0 \\ \pm I_k & 0 & 0 \\ 0 & 0 & \Delta_m \end{pmatrix}.$$

Here Δ_m is a diagonal $m \times m$ diagonal matrix is with on the diagonal $\lambda \in \mathbb{F}$, such that $\lambda^\sigma = \lambda$ or $\lambda^\sigma = -\lambda$ if h is Hermitian or skew-Hermitian respectively.

Definition 2.2.12 (Unitary Lie algebra). *Let (V, h) be a nontrivial Hermitian space over a field \mathbb{F} . Then the unitary Lie algebra $\mathfrak{u}(V, h)$ consists of all $f \in \text{End}(V)$, such that*

$$h(f(v), w) = -h(v, f(w)), \quad \forall v, w \in V.$$

By lemma 2.2.2 the unitary Lie algebra is a Lie algebra over a strict subfield \mathbb{L} of \mathbb{F} which is fixed element wise by σ .

In the case that V is of finite dimension and h is nondegenerate, we can identify $\mathfrak{u}(V, h)$ with the matrix Lie algebra \mathfrak{g} where $M^T H = -H M^\sigma$, where H as previous and $\Delta_m = \lambda I_m$.

One easily deduce that the dimension of $\mathfrak{u}(V, h)$ equals n^2 over $\mathbb{F}_\sigma = \{\mu \in \mathbb{F} : \mu^\sigma = \mu\}$, where $n = \dim(V) = 2k + m$.

For the Lie algebra \mathfrak{g} , we can denote a basis in case h is a skew-Hermitian form and μ is an element not fixed by σ . Then a basis is as follow:

$$\begin{aligned}
& E_{i,j} + E_{k+j,k+i}, & 1 \leq i, j \leq k \\
& \mu E_{i,j} + \mu^\sigma E_{k+j,k+i}, & 1 \leq i, j \leq k \\
& E_{k+i,j} + E_{k+j,i}, & 1 \leq i < j \leq k \\
& \mu E_{k+i,j} + \mu^\sigma E_{k+j,i}, & 1 \leq i < j \leq k \\
& E_{k+i,i}, & 1 \leq i \leq k, \\
& E_{i,k+j} + E_{j,k+i}, & 1 \leq i < j \leq k, \\
& \mu E_{i,k+j} + \mu^\sigma E_{j,k+i}, & 1 \leq i < j \leq k, \\
& E_{i,k+i}, & 1 \leq i \leq k, \\
& E_{2k+i,k+j} + \lambda E_{j,2k+i}, & 1 \leq i \leq m, \quad 1 \leq j \leq k \\
& \mu E_{2k+i,k+j} + \lambda \mu^\sigma E_{j,2k+i}, & 1 \leq i \leq m, \quad 1 \leq j \leq k \\
& E_{k+i,2k+j} + \lambda E_{2k+j,i}, & 1 \leq i \leq m, \quad 1 \leq j \leq k, \\
& \mu E_{k+i,2k+j} + \lambda \mu^\sigma E_{2k+j,i}, & 1 \leq i \leq m, \quad 1 \leq j \leq k, \\
& E_{2k+i,2k+j} - E_{2k+j,2k+i}, & 1 \leq i < j \leq m, \\
& \mu E_{2k+i,2k+j} - \mu^\sigma E_{2k+j,2k+i}, & 1 \leq i < j \leq m, \\
& \lambda E_{2k+i,2k+i}, & 1 \leq i \leq m.
\end{aligned}$$

Definition 2.2.13 (Special unitary Lie algebra). *Let $\mathfrak{u}(V, h)$ be a unitary Lie algebra, then the special unitary Lie algebra are those elements $f \in \mathfrak{u}(V, h)$ such that $f \in \mathfrak{sl}(V)$.*

The finitary unitary and finitary special unitary Lie algebras $\mathfrak{fu}(V, h)$ and $\mathfrak{fsu}(V, H)$ are the intersections of $\mathfrak{u}(V, h)$ with $\mathfrak{fgl}(V)$ and $\mathfrak{fsl}(V)$ respectively. As before, we denote $\mathfrak{psu}_n(\mathbb{F})$ as the Lie algebra obtained by the quotient of $\mathfrak{su}_n(\mathbb{F})$ by its center. The Lie algebras described in this section are referred to as **classical linear Lie algebras**.

Chapter 3

Extremal elements

In this chapter we give the notion of extremal elements of a Lie algebra. Furthermore we define the extremal form on a Lie algebras when they are generated by extremal elements. The theory, notation and terminology used in this chapter comes from chapter two of the PhD of Fleischmann [9].

3.1 General theory

Definition 3.1.1 (Extremal elements). *Let \mathfrak{g} be a Lie algebra. A nonzero element $x \in \mathfrak{g}$ is called an extremal element, if there exist a map $g_x : \mathfrak{g} \rightarrow \mathbb{F}$ such that*

$$[x, [x, y]] = 2g_x(y)x, \quad \forall y \in \mathfrak{g}.$$

Furthermore, if $\text{char}(\mathbb{F}) = 2$, then

- 1) $[[x, y], [x, z]] = g_x([y, z])x + g_x(z)[x, y] - g_x(y)[x, z],$
- 2) $[x, [y, [x, z]]] = g_x([y, z])x - g_x(z)[x, y] - g_x(y)[x, z],$

needs to hold for all $y, z \in \mathfrak{g}$.

If $\text{char}(\mathbb{F}) \neq 2$ then, this is equivalent with $[x, [x, \mathfrak{g}]] \subseteq \mathbb{F}x$, see [9, 5]. The map g_x is called the **extremal form** on x . The set of all extremal elements of \mathfrak{g} is denoted as $E(\mathfrak{g})$ or E if it is clear from the context what \mathfrak{g} is. The set $\mathcal{E}(\mathfrak{g}) = \{\mathbb{F}x : x \in E(\mathfrak{g})\}$ is the set of extremal points in the projective space on \mathfrak{g} . The set $\mathcal{E}(\mathfrak{g})$ is also denoted as \mathcal{E} if it is clear from the context what \mathfrak{g} is.

The latter two identities stated in definition 3.1.1, are called the Premet identities. If the characteristic of the underlying field is not two, then these identities are not part of the definition of the extremal elements, but consequences, see [9, 5].

Lemma 3.1.2 (Premet identities). *Let \mathfrak{g} be a Lie algebra over a field \mathbb{F} with $\text{char } \mathbb{F} \neq 2$. Let $x \in E(\mathfrak{g})$, then for all $y, z \in \mathfrak{g}$ the following holds*

- 1) $[[x, y], [x, z]] = g_x([y, z])x + g_x(z)[x, y] - g_x(y)[x, z],$
- 2) $[x, [y, [x, z]]] = g_x([y, z])x - g_x(z)[x, y] - g_x(y)[x, z].$

Proof. Let \mathfrak{g} be a Lie algebra, let $x \in E(\mathfrak{g})$ and $y, z \in \mathfrak{g}$ be arbitrary. Then

- 1) With the use of the Jacobi identity the following holds.

$$\begin{aligned}
[[x, y], [x, z]] &= -[[x, z], [x, y]] \\
&\stackrel{\text{Jacobi}}{=} [x, [y, [x, z]]] + [y, [[x, z], x]] \\
&= [x, [y, [x, z]]] - [[[x, z], x], y] \\
&= [x, [y, [x, z]]] + [[[x, [x, z]], y] \\
&= [x, [y, [x, z]]] + [2g_x(z)x, y] \\
&= [x, [y, [x, z]]] + 2g_x(z)[x, y]. \tag{3.1}
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
[[x, y], [x, z]] &\stackrel{\text{Jacobi}}{=} -[x, [z, [x, y]]] - [z, [[x, y], x]] \\
&= -[x, [z, [x, y]]] - [[[x, [x, y]], z] \\
&= -[x, [z, [x, y]]] - [2g_x(y)x, z] \\
&= -[x, [z, [x, y]]] - 2g_x(y)[x, z] \\
&\stackrel{\text{Jacobi}}{=} [x[x, [y, z]]] + [x, [y, [x, z]]] - 2g_x(y)[x, z] \\
&= 2g_x([y, z])x + [x, [y, [z, x]]] - 2g_x(y)[x, z]. \tag{3.2}
\end{aligned}$$

Adding equation (3.1) and (3.2) the first Premet identity is proven. So

$$[[x, y], [x, z]] = g_x([y, z])x + g_x(z)[x, y] - g_x(y)[x, z], \tag{3.3}$$

where g_x is the extremal form on x .

- 2) With the use of the Jacobi identity and the first Premet identity the following holds.

$$\begin{aligned}
[x, [y, [x, z]]] &\stackrel{\text{Jacobi}}{=} -[y, [[x, z], x]] - [[x, z], [x, y]] \\
&= [[x, y], [x, z]] - [y, [[x, z], x]] \\
&\stackrel{\text{1st Premet}}{=} g_x([y, z])x + g_x(z)[x, y] - g_x(y)[x, z] - [[[x, [x, z]], y] \\
&= g_x([y, z])x + g_x(z)[x, y] - g_x(y)[x, z] - 2g_x(z)[x, y] \\
&= g_x([y, z])x - g_x(z)[x, y] - g_x(y)[x, z], \tag{3.4}
\end{aligned}$$

where g_x is the extremal form on x . □

Proposition 3.1.3. *Let \mathfrak{g} be a Lie algebra and $x \in E(\mathfrak{g})$ an extremal element, then ad_x is nilpotent of at most order three.*

Proof. Let \mathfrak{g} be a Lie algebra, $x \in E(\mathfrak{g})$ and $y \in \mathfrak{g}$ arbitrary then

$$\text{ad}_x^3(y) = [x, [x, [x, y]]] = [x, \lambda x] = \lambda[x, x] = 0,$$

for some $\lambda \in \mathbb{F}$. □

Definition 3.1.4 (Sandwich). *Let \mathfrak{g} be a Lie algebra. An element $x \in \mathfrak{g}$ is called sandwich if for all $y, z \in \mathfrak{g}$ the following holds*

$$\begin{aligned} \text{ad}_x^2(y) &= 0, \\ (\text{ad}_x \text{ad}_y \text{ad}_x)(z) &= 0. \end{aligned}$$

If an element x from a Lie algebra is extremal and sandwich, we have that $\text{ad}_x^2(y) = [x, [x, y]] = 2g_x(y)x = 0$. Therefore we can choose the extremal g_x form to be zero. So we introduce the convention that if an element x is sandwich and extremal in \mathfrak{g} , then g_x is identically zero. This gives rise to the following lemma.

Lemma 3.1.5. *Let \mathfrak{g} be a Lie algebra and $x, y \in E(\mathfrak{g})$, then*

$$\begin{aligned} g_x(y) &= g_y(x), \\ g_x([y, z]) &= -g_y([x, z]). \end{aligned}$$

for all $z \in \mathfrak{g}$.

Proof. Let \mathfrak{g} be a Lie algebra, $x, y \in E(\mathfrak{g})$ and $z \in \mathfrak{g}$ be arbitrary.

The proof consists of two cases. The first case is where $[x, y] = 0$, and the second where $[x, y] \neq 0$.

Case 1. Let $[x, y] = 0$ and assume that $g_x(y) \neq 0$, a contradiction will be derived. Notice that in case the characteristic of the field \mathbb{F} is different from 2, there is a contradiction following from the definition of an extremal element. In the case that the characteristic of the field is equal to 2, we have the following.

The first Premet identity states

$$g_x(y)[x, z] = g_x([y, z])x. \tag{3.5}$$

Applying ad_x on both sides gives

$$\begin{aligned} g_x(y) \text{ad}_x^2(z) &= g_x(y)[x, [x, z]] = [x, g_y(x)[x, z]] \\ &= [x, g_x([y, z])x] = g_x([y, z])[x, x] = 0. \end{aligned}$$

$g_x(y) \neq 0$ implies that $\text{ad}_x^2(z) = 0$ for all $z \in \mathfrak{g}$.
From equation (3.5) follows that $[x, z] = \frac{g_x([y, z])}{g_x(y)}x$.
For all $u \in \mathfrak{g}$, we have that

$$\begin{aligned} (\text{ad}_x \text{ad}_u \text{ad}_x)(z) &= [x, [u, [x, z]]] = [x, [u, \frac{g_x([y, z])}{g_x(y)}x]] \\ &= -\frac{g_x([y, z])}{g_x(y)}[x, [x, u]] = \frac{g_x([y, z])}{g_x(y)} \text{ad}_x^2(u) = 0. \end{aligned}$$

This means that x is sandwich in \mathfrak{g} , thus $g_x(y) = 0$ by convention. This is in contradiction with the assumption $g_x(y) \neq 0$. So $g_x(y) = 0$. Since x is extremal, $x \neq 0$. Thus $g_x([y, z]) = 0$ for all $z \in \mathfrak{g}$.
If the role of x and y are swapped, we derive that $g_y(x) = 0$ and $g_y([x, z]) = 0$.
So the first case has been proved.

Case 2. Let $[x, y] \neq 0$. Then the second Premet identity states

$$[x, [y, [x, [y, z]]]] + g_x(y)[x, [y, z]] = g_x([y, [y, z]]x - g_x([y, z])[x, y]), \quad (3.6)$$

$$[y, [x, [y, [x, z]]]] + g_x(y)[y, [x, z]] = g_x([y, z])[x, z] - g_x(z)[y, [y, z]], \quad (3.7)$$

where (3.7) is the second Premet identity after applying ad_y at both sides. Exchanging x and y in (3.7) and then subtracting (3.6) from (3.7) the following holds

$$(g_x(y) - g_y(x))[x, [y, z]] = -(g_x([y, z]) + g_y([x, z]))[x, y] \subseteq \mathbb{F}[x, y]. \quad (3.8)$$

It is sufficient to show that $g_x(y) = g_y(x)$, since $g_x([y, z]) = -g_y([x, z])$ follows from the fact that $[x, y] \neq 0$.

In the case that $\text{char } \mathbb{F} \neq 2$ the Jacobi identity gives

$$\begin{aligned} 0 &= [y, [[x, y], x]] + [x[y, [x, y]]] + [[x, y], [x, y]] \\ &= -[y, 2g_x(y)x] - [x, 2g_y(x)y] \\ &= 2g_x(y)[x, y] - 2g_y(x)[x, y]. \end{aligned}$$

So $g_x(y) = g_y(x)$ since $[x, y] \neq 0$. From (3.8) follows that $g_x([y, z]) = -g_y([x, z])$.

If $\text{char } \mathbb{F} = 2$ equation 3.8 results into $[x, [y, \mathfrak{g}]] + [y, [x, \mathfrak{g}]] \subseteq \mathbb{F}[x, y]$. Applying ad_y on both sides, we have that

$$[y, [x, [y, \mathfrak{g}]]] + [y, [y, [x, \mathfrak{g}]]] \subseteq \mathbb{F}[y, [x, y]] = 2\mathbb{F}g_y(x)y = 0.$$

Since $[y, [y, [x, \mathfrak{g}]]] = 2g_y([x, \mathfrak{g}])y = 0$, also $[y, [x, [y, \mathfrak{g}]]] = 0$. Using the second Premet identity we get

$$0 = [y, [x, [y, \mathfrak{g}]]] = g_y([x, \mathfrak{g}])y - g_y(\mathfrak{g})[y, x] - g_y(x)[y, \mathfrak{g}].$$

This gives that $[y, \mathfrak{g}] \subseteq \mathbb{F}y + \mathbb{F}[x, y]$. By definition we have $[y, [y, x]] = 0$, so $\mathbb{F}y + \mathbb{F}[x, y]$ is a commutative Lie subalgebra of \mathfrak{g} and therefor $[y, \mathfrak{g}]$ as its subspace. The Premet identities states that

$$[y, [\mathfrak{g}, [y, \mathfrak{g}]]] = [[y, \mathfrak{g}], [y, \mathfrak{g}]] = 0.$$

So $(\text{ad}_y \text{ad}_u \text{ad}_y)(z) = 0$ for all $u, z \in \mathfrak{g}$. y is sandwich in \mathfrak{g} since $\text{ad}_y^2(z) = 0$ for all $z \in \mathfrak{g}$. Thus by convention $g_y(x) = 0$. The same arguments holds for $g_x(y)$. So $g_x(y) = g_y(x)$ and $g_x([y, z]) = -g_y([x, z])$. \square

The following three identities can be deduced as well.

Lemma 3.1.6. *Let \mathfrak{g} be a Lie algebra and $x, y \in E(\mathfrak{g})$ and $z \in \mathfrak{g}$. Then*

1)

$$[[x, y], [x, [y, z]]] = 2g_y(z)g_x(y)x + g_x([y, z])[x, y] - g_x(y)[x, [y, z]].$$

2)

$$\begin{aligned} [[x, y], [[x, y], z]] &= g_x(y)(2g_y(z)x - [x, [y, z]] + 2g_x(z)y - [y, [x, z]]) \\ &\quad + (g_x([y, z]) - g_y([x, z]))[x, y]. \end{aligned}$$

3) and if $z \in E(\mathfrak{g})$, then

$$\begin{aligned} [[x, [y, z]], [y, [x, z]]] &= -g_x([y, z])g_y(z)x - g_x([y, z])g_x(z)y \\ &\quad - g_x([y, z])g_x(y)z - 2g_x(z)g_y(z)[x, y] \\ &\quad + 2g_x(y)g_y(z)[x, z] - 2g_x(y)g_x(z)[y, z]. \end{aligned}$$

Proof. Let \mathfrak{g} be a Lie algebra, $x, y \in E(\mathfrak{g})$ and $z \in \mathfrak{g}$ be arbitrary then

1) Apply the first Premet identity, where z is replaced by $[y, z]$. Then

$$\begin{aligned} [[x, y], [x, [y, z]]] &= g_x([y, [y, z]])x + g_x([y, z])[x, y] - g_x(y)[x, [y, z]] \\ &= g_x(2g_y(z)y)x + g_x([y, z])[x, y] - g_x(y)[x, [y, z]] \\ &= 2g_x(y)g_y(z)x + g_x([y, z])[x, y] - g_x(y)[x, [y, z]]. \end{aligned}$$

2) Apply the Jacobi identity. Then

$$\begin{aligned} [[x, y], [[x, y], z]] &= -[[x, y], [z, [x, y]]] \\ &= [[x, y], [x, [y, z]]] + [[x, y], [y, [z, x]]] \\ &= [[x, y], [x, [y, z]]] + [[y, x], [y, [x, z]]]. \end{aligned}$$

Applying the first identity of this lemma, we get

$$\begin{aligned} [[x, y], [[x, y], z]] &= 2g_y(z)g_x(y)x + g_x([y, z])[x, y] - g_x(y)[x, [y, z]] \\ &\quad + 2g_x(z)g_y(x)y + g_y([x, z])[y, x] - g_y(x)[y, [x, z]] \\ &= g_x(y)(2g_y(z)x - [x, [y, z]] + 2g_x(z)y - [y, [x, z]]) \\ &\quad + (g_x([y, z]) - g_y([x, z]))[x, y]. \end{aligned}$$

3) Let now $z \in E(\mathfrak{g})$ as well. Then applying the Jacobi identity results into

$$\begin{aligned} [[x, [y, z]], [y, [x, z]]] &= -[[y, [x, z]], [x, [y, z]]] \\ &= [x, [[y, z], [y, [x, z]]]] - [[y, z], [x, [y, [x, z]]]]. \end{aligned}$$

Keeping $g_x([y, z]) = -g_y([x, z])$ and applying the first identity of this lemma, the following holds

$$\begin{aligned} [x, [[y, z], [y, [x, z]]]] &= [x, g_y([z, [x, z]])y + g_y([x, z])[y, z] - g_y(z)[y, [x, z]]] \\ &= -2g_y(z)g_x(z)[x, y] - g_x([y, z])[x, [y, z]] - g_y(z)[x, [y, [x, z]]] \\ &= -2g_y(z)g_x(z)[x, y] - g_x([y, z])[x, [y, z]] - g_y(z)g_x([y, z])x \\ &\quad + g_y(z)g_x(y)[x, z] + g_y(z)g_x(z)[x, y]. \end{aligned}$$

The following also holds

$$\begin{aligned} -[[y, z], [x, [y, [x, z]]]] &= -[[y, z], g_x([y, z])x - g_x(y)[x, z] - g_x(z)[x, y]] \\ &= g_x([y, z])[x, [y, z]] - g_x(y)g_x([y, z])z + g_x(y)g_y(z)[x, z] \\ &\quad - g_x(y)g_x(z)[y, z] - g_x(z)g_x([y, z])y \\ &\quad - g_x(y)g_x(z)[y, z] - g_x(z)g_y(z)[x, y]. \end{aligned}$$

So we have that

$$\begin{aligned} [[x, [y, z]], [y, [x, z]]] &= -g_x([y, z])g_y(z)x - g_x([y, z])g_x(z)y \\ &\quad - g_x([y, z])g_x(y)z - 2g_x(z)g_y(z)[x, y] \\ &\quad + 2g_x(y)g_y(z)[x, z] - 2g_x(y)g_x(z)[y, z]. \quad \square \end{aligned}$$

Definition 3.1.7. Let \mathfrak{g} be a Lie algebra and $x, y \in E(\mathfrak{g})$ extremal elements. Then for x, y we define

$$(x, y) \in \begin{cases} E_{-2}, & \iff \mathbb{F}x = \mathbb{F}y, \\ E_{-1}, & \iff [x, y] = 0, (x, y) \notin E_{-2} \text{ and } \mathbb{F}x + \mathbb{F}y \subseteq E(\mathfrak{g}) \cup \{0\}, \\ E_0, & \iff [x, y] = 0 \text{ and } (x, y) \notin E_{-2} \cup E_{-1}, \\ E_1, & \iff [x, y] \neq 0 \text{ and } g_x(y) = 0, \\ E_2 & \iff g_x(y) \neq 0. \end{cases}$$

For the corresponding extremal points $\langle x \rangle, \langle y \rangle$, we define

$$\langle \langle x \rangle, \langle y \rangle \rangle \in \mathcal{E}_i \iff (x, y) \in E_i.$$

For two distinct extremal points $\langle x \rangle, \langle y \rangle$ we say the pair $(\langle x \rangle, \langle y \rangle)$ is

hyperbolic,	if $(\langle x \rangle, \langle y \rangle) \in \mathcal{E}_2,$
special,	if $(\langle x \rangle, \langle y \rangle) \in \mathcal{E}_1,$
polar,	if $(\langle x \rangle, \langle y \rangle) \in \mathcal{E}_0,$
strongly commuting,	if $(\langle x \rangle, \langle y \rangle) \in \mathcal{E}_{-1},$
commuting,	if $(\langle x \rangle, \langle y \rangle) \in \mathcal{E}_i$ for $i \leq 0.$

Notice that for any two extremal elements x and y , we have that the pair $(\langle x \rangle, \langle y \rangle)$ is in one of the above situations. In the following we will often write x for $\langle x \rangle$ if it is clear that x is an extremal point.

3.2 The exponential map

Definition 3.2.1 (Exponential map). *Let \mathfrak{g} be a Lie algebra, $x \in E(\mathfrak{g})$ an extremal element. Then for all $y \in \mathfrak{g}$ and $\lambda \in \mathbb{F}$, the exponential map is defined as*

$$\exp(x, \lambda)(y) = \left(\sum_{n=0}^{\infty} \frac{(\lambda \operatorname{ad}_x)^n}{n!} \right) (y). \quad (3.9)$$

Since x is ad-nilpotent of at most order three (proposition 3.1.3), the exponential map reduces to

$$\exp(x, \lambda)(y) = y + \lambda[x, y] + \lambda^2 g_x(y)x. \quad (3.10)$$

Proposition 3.2.2. *Let \mathfrak{g} be a Lie algebra and $x \in E(\mathfrak{g})$ with extremal form $g_x : \mathfrak{g} \rightarrow \mathbb{F}$. Then the exponential map is an endomorphism on \mathfrak{g} for all $\lambda \in \mathbb{F}$. Furthermore, for all $\lambda, \mu \in \mathbb{F}$*

$$\exp(x, \lambda + \mu) = \exp(x, \lambda) \exp(x, \mu).$$

Proof. Let \mathfrak{g} be a Lie algebra, $x \in E(\mathfrak{g})$, $y, z \in \mathfrak{g}$ and $\lambda, \mu \in \mathbb{F}$ be arbitrary. We first show that $\exp(x, \lambda)$ is an endomorphism on \mathfrak{g} . Note that $\exp(x, \lambda)y \in \mathfrak{g}$ by definition. So it is sufficient to show that $\exp(x, \lambda)$ is a Lie algebra homomorphism, that is $\exp(x, \lambda)[y, z] = [\exp(x, \lambda)y, \exp(x, \lambda)z]$. So with the

use of the first Premet identity, the following holds:

$$\begin{aligned}
[\exp(x, \lambda)(y), \exp(x, \lambda)(z)] &= [y + \lambda[x, y] + \lambda^2 g_x(y)x, z + \lambda[x, z] + \lambda^2 g_x(z)x] \\
&= [y, z] + \lambda[y, [x, z]] + \lambda^2 [y, g_x(z)x] \\
&\quad + [\lambda[x, y], z] + [\lambda[x, y], \lambda[x, z]] + [\lambda[x, y], \lambda^2 g_x(z)x] \\
&\quad + [\lambda^2 g_x(y)x, z] + [\lambda^2 g_x(y)x, \lambda[x, z]] + [\lambda^2 g_x(y)x, \lambda^2 g_x(z)x] \\
&= [y, z] + \lambda[y, [x, z]] + \lambda^2 g_x(z)[y, x] \\
&\quad + \lambda[[x, y], z] + \lambda^2 [[x, y], [x, z]] + \lambda^3 g_x(z)[[x, y], x] \\
&\quad + \lambda^2 g_x(y)[x, z] + \lambda^3 g_x(y)[x, [x, z]] \\
&= [y, z] + \lambda([y, [x, z]] + [[x, y], z]) \\
&\quad + \lambda^2(-g_x(z)[x, y] + [[x, y], [x, z]] + g_x(y)[x, z]) \\
&= [y, z] + \lambda[x, [y, z]] + \lambda^2 g_x([y, z])x \\
&= \exp(x, \lambda)([y, z]).
\end{aligned}$$

Thus $\exp(x, \lambda)$ is an endomorphism on \mathfrak{g} .

Now we show $\exp(x, \lambda + \mu) = \exp(x, \lambda) \exp(x, \mu)$. So

$$\begin{aligned}
\exp(x, \lambda) \exp(x, \mu)(z) &= \exp(x, \lambda)(z + \mu[x, z] + \mu^2 g_x(z)x) \\
&= z + \mu[x, z] + \mu^2 g_x(z)x \\
&\quad + \lambda[x, z + \mu[x, z] + \mu^2 g_x(z)x] \\
&\quad + \lambda^2 g_x(z + \mu[x, z] + \mu^2 g_x(z)x)x \\
&= z + (\lambda + \mu)[x, z] + \lambda\mu[x, [x, z]] + \lambda\mu^2 g_x(z)[x, x] \\
&\quad + \lambda^2 g_x(z)x + \lambda^2 \mu g_x([x, z])x \\
&\quad + \lambda^2 \mu^2 g_x(z)g_x(x)x + \mu^2 g_x(z)x \\
&= z + (\lambda + \mu)[x, z] + 2\lambda\mu g_x(z)x + \lambda^2 g_x(z)x + \mu^2 g_x(z)x \\
&= z + (\lambda + \mu)[x, z] + (\lambda + \mu)^2 g_x(z)x \\
&= \exp(x, \lambda + \mu)z. \quad \square
\end{aligned}$$

Lemma 3.2.3. *Let \mathfrak{g} be a Lie algebra and $x \in E(\mathfrak{g})$. Then the set $U_x = \{\exp(x, \lambda) : \lambda \in \mathbb{F}\}$ is a subgroup of $\text{Aut}(\mathfrak{g})$ isomorphic to \mathbb{F} .*

Proof. Let \mathfrak{g} be a Lie algebra and $x \in E(\mathfrak{g})$ be arbitrary. Note that $\exp(x, \lambda)$ is an endomorphism on \mathfrak{g} . First we show that $\exp(x, \lambda)$ is injective for all $\lambda \in \mathbb{F}$.

Let $y, z \in \mathfrak{g}$ such that $\exp(x, \lambda)(y - z) = 0$ for $\lambda \in \mathbb{F}$. We show $y = z$, so

$$y - z = \exp(x, -\lambda)(\exp(x, \lambda)(y - z)) = 0$$

So $y = z$, thus $\exp(x, \lambda)$ is injective for all $\lambda \in \mathbb{F}$.

To show that $\exp(x, \lambda)$ is surjective, we show that its inverse is injective.

Claim: $\exp(x, -\lambda)$ is the inverse of $\exp(x, \lambda)$ for all $\lambda \in \mathbb{F}$. So let $z \in \mathfrak{g}$ be arbitrary then

$$\begin{aligned}\exp(x, \lambda) \exp(x, -\lambda)(z) &= \exp(x, \lambda - \lambda)(z) = \exp(x, 0)(z) = z, \quad \text{and} \\ \exp(x, -\lambda) \exp(x, \lambda)(z) &= \exp(x, -\lambda + \lambda)(z) = \exp(x, 0)(z) = z.\end{aligned}$$

So $\exp(x, -\lambda)$ is the inverse of $\exp(x, \lambda)$. Obviously $\exp(x, -\lambda)$ is injective, the proof of it is analogue to the proof of injectivity of $\exp(x, \lambda)$. So $\exp(x, \lambda)$ is an isomorphism of Lie algebras. This means that the set $U_x = \{\exp(x, \lambda) : \lambda \in \mathbb{F}\}$ is a subgroup of $\text{Aut}(\mathfrak{g})$.

Now we show that U_x is isomorphic to \mathbb{F} . So let the map

$$\begin{aligned}\phi : \mathbb{F} &\rightarrow U_x \\ \lambda &\mapsto \exp(x, \lambda)\end{aligned}$$

be a group homomorphism. Since for all $\lambda, \mu \in \mathbb{F}$ we have $\phi(\lambda + \mu) = \phi(\lambda)\phi(\mu)$. Clearly ϕ is injective, since for $\lambda \in \ker \phi$ the following holds for all $z \in \mathfrak{g}$

$$z = \phi(\lambda)(z) = z + \lambda[x, z] + \lambda^2 g_x(z)x \implies \lambda = 0.$$

Obviously ϕ is surjective. So U_x is isomorphic to \mathbb{F} . □

3.3 The extremal form

Lemma 3.3.1. *If a Lie algebra \mathfrak{g} is generated by extremal elements, then it is linearly spanned by the set $E(\mathfrak{g})$ of all its extremal elements.*

Proof. Let \mathfrak{g} be a Lie algebra spanned by extremal elements. This means that the Lie algebra is also spanned by the bracket of extremal elements. We show that every bracket is a linear combination of extremal elements. This by induction on the length of the bracket.

For $n = 1$, that is a bracket of length one, it is trivial. Now assume that for a $n \in \mathbb{N}_{\geq 1}$, the elements of bracket of length n is spanned by extremal elements. We show for $n + 1$.

Let $z = [\cdot, [\cdot, [\cdot, [\dots, [\cdot[x, y] \dots]]]]]$. Consider now $\exp(x, 1)y = y + [x, y] + g_x(y)x$. This is also extremal since the exponential map is an automorphism. Now we can express z as a sum of brackets of length n . By induction hypothesis we have that z is the linear combination of extremal elements. □

Theorem 3.3.2. *Let \mathfrak{g} be generated by $E(\mathfrak{g})$. Then there is a unique bilinear symmetric form $g : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{F}$ such that the linear form g_x coincides with $y \mapsto g(x, y)$ for each $x \in E(\mathfrak{g})$. Moreover, this form is associative in the sense that $g(x, [y, z]) = g([x, y], z)$ for all $x, y, z \in \mathfrak{g}$.*

Proof. Let \mathfrak{g} be a Lie algebra generated by $E(\mathfrak{g})$. So there is a basis $B = \{u_i : i \in I\}$, where each $u_i \in E(\mathfrak{g})$. Note that if x is extremal then λx is extremal for $\lambda \in \mathbb{F} \setminus \{0\}$. This, since $[\lambda x[\lambda x, y]] = 2\lambda g_x(y)\lambda x = 2\lambda g(x, y)\lambda x$ for all $y \in \mathfrak{g}$.

For $x \in \mathfrak{g}$ with $x = \sum_{i \in I} \lambda_i u_i$, we define the extremal form of x as $g_x = \sum_{i \in I} \lambda_i g_{u_i}$.

Let $y \in \mathfrak{g}$ such that $y = \sum_{i \in I} u_i$ and assume that there is a second way of writing, that is $y = \sum_{i \in I} v_i$ with $v_i \in E(\mathfrak{g})$. Then for all $z \in E(\mathfrak{g})$, we have

$$\begin{aligned} g_y(z) &= \sum_{i \in I} g_{u_i}(z) = \sum_{i \in I} g_z(u_i) = g_z\left(\sum_{i \in I} u_i\right) \\ &= g_z\left(\sum_{i \in I} v_i\right) = \sum_{i \in I} g_z(v_i) = \sum_{i \in I} g_{v_i}(z). \end{aligned}$$

Since \mathfrak{g} is spanned by $E(\mathfrak{g})$, we have that $\sum_{i \in I} g_{u_i} = \sum_{i \in I} g_{v_i}$. So g_x is well defined. We also have that $g(x, y) = g_x(y)$ defines a symmetric bilinear form.

To show the associative, let $x, y, z \in E(\mathfrak{g})$, then by lemma 3.1.5, we have that

$$g_x([y, z]) = g(x, [y, z]) = -g_y([x, z]) = -g(y, [x, z]).$$

By symmetry, we have

$$g(x, [z, y]) = -g(x, [y, z]) = g(y, [x, z]) = g([x, z], y).$$

So we have

$$g(x, [z, y]) = g([x, z], y).$$

Similarly we have

$$\begin{aligned} g(x, [y, z]) &= g([x, y], z), \\ g(y, [x, z]) &= g([y, x], z). \end{aligned} \quad \square$$

Definition 3.3.3 (Extremal form). *Let \mathfrak{g} be a Lie algebra generated by its extremal elements. The unique form of theorem 3.3.2, that is $g : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{F}$, is called the extremal form of \mathfrak{g} .*

Definition 3.3.4 (Radical of extremal form). *Let \mathfrak{g} be a Lie algebra generated by its extremal elements. Then the radical of the extremal form is defined as*

$$\text{Rad}(g) = \{u \in \mathfrak{g} : g(u, z) = 0, \quad \forall z \in \mathfrak{g}\}.$$

Lemma 3.3.5. *Let \mathfrak{g} be a Lie algebra and $B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{F}$ be a symmetric or antisymmetric bilinear form such that*

$$B(x, [y, z]) = B([x, y], z), \quad \forall x, y, z \in \mathfrak{g}.$$

Then $\text{Rad}(B)$ is an ideal of \mathfrak{g} .

Proof. Let \mathfrak{g} be a Lie algebra. To show that $\text{Rad}(B)$ is an ideal of \mathfrak{g} , we need to show that $\text{Rad}(B) \subset \mathfrak{g}$ is a linear subspace and for all $x \in \mathfrak{g}$ and $y \in \text{Rad}(B)$, $[x, y] \in \text{Rad}(B)$. It is obvious that $\text{Rad}(B)$ is a linear subspace of \mathfrak{g} . So let $x \in \mathfrak{g}$ and $y \in \text{Rad}(B)$ be arbitrary, then for all $z \in \mathfrak{g}$

$$B([x, y], z) = B(x, [y, z]) = 0.$$

Furthermore, we have

$$B([y, x], z) = \varepsilon B(z, [y, z]) = \varepsilon B([z, y], x) = 0, \quad \text{for } \varepsilon = \pm 1.$$

So $[x, y], [y, x] \in \text{Rad}(B)$. Thus $\text{Rad}(B)$ is an ideal of \mathfrak{g} . \square

Lemma 3.3.6 ([5] 9.8 and 9.12). *Let \mathfrak{g} be a Lie algebra and g the extremal form on \mathfrak{g} , then $\text{Rad}(\mathfrak{g}) \subseteq \text{Rad}(g)$. Moreover, if $\text{char}(\mathbb{F}) \neq 2, 3$, then $\text{Rad}(\mathfrak{g}) = \text{Rad}(g)$.*

Proposition 3.3.7 ([5] 9.14). *Let \mathfrak{g} be a Lie algebra and g the extremal form on \mathfrak{g} . Then $\text{Rad}(g) = 0$ if and only if \mathfrak{g} is the direct sum of simple ideals.*

Corollary 3.3.8 ([5] 9.15). *Let \mathfrak{g} be a Lie algebra and g the extremal form on \mathfrak{g} , then $\mathfrak{g}/\text{Rad}(\mathfrak{g})$ is a direct sum of simple ideals if and only if $\text{Rad}(\mathfrak{g}) = \text{Rad}(g)$.*

3.4 Tensors and extremal elements

In this section let V be a vector space over a field \mathbb{F} and Π be a subspace of V^* , the dual of V . Note that V^* is a vector space over \mathbb{F} .

Definition 3.4.1. *Let $\mathfrak{g} = V \otimes \Pi$. On \mathfrak{g} , we define an \mathbb{F} -bilinear product $[\cdot, \cdot]$ by*

$$[v \otimes \phi, w \otimes \psi] := \phi(w)(v \otimes w) - \psi(v)(w \otimes \phi)$$

with $v, w \in V$ and $\phi, \psi \in \Pi$.

Proposition 3.4.2. *The space \mathfrak{g} with the \mathbb{F} -bilinear product as in definition 3.4.1, defines a Lie algebra over \mathbb{F} .*

Proof. We have that $[\cdot, \cdot]$ is by definition a \mathbb{F} -bilinear product and alternating, since

$$[v \otimes \phi, v \otimes \phi] = \phi(v)(v \otimes \phi) - \phi(v)(v \otimes \phi) = 0.$$

So we only need to show the Jacobi identity. Let $u, v, w \in V$ and $\psi, \phi, \chi \in \Pi$, then

$$\begin{aligned} & [w \otimes \chi, [v \otimes \phi, u \otimes \psi]] \\ &= [w \otimes \chi, \phi(u)(v \otimes \psi) - \psi(u)(u \otimes \phi)] \\ &= \phi(u)\chi(v)(w \otimes \psi) - \psi(w)\phi(u)(v \otimes \chi) - \psi(v)\chi(u)(w \otimes \phi) + \phi(w)\psi(v)(u \otimes \chi). \end{aligned}$$

and

$$\begin{aligned} & [v \otimes \phi, [u \otimes \psi, w \otimes \chi]] \\ &= \psi(w)\phi(u)(v \otimes \chi) - \chi(v)\psi(w)(u \otimes \phi) - \chi(u)\chi(w)(v \otimes \psi) + \psi(v)\chi(u)(w \otimes \phi). \end{aligned}$$

and

$$\begin{aligned} & [u \otimes \psi, [w \otimes \chi, v \otimes \phi]] \\ &= \chi(v)\psi(w)(u \otimes \phi) - \phi(u)\chi(v)(w \otimes \psi) - \phi(w)\psi(v)(u \otimes \chi) + \chi(u)\phi(w)(v \otimes \psi). \end{aligned}$$

Adding the three identities together results into the Jacobi identity. \square

The Lie algebra induced by $[\cdot, \cdot]$ on the vector space $V \otimes \Pi$ is denoted by $\mathfrak{g}(V, \Pi)$.

Definition 3.4.3 (Singular). *A pure tensor $v \otimes \phi \in V \otimes V^*$ is called singular if $\phi(v) = 0$. When $\phi(v) \neq 0$, $v \otimes \phi$ is called nonsingular.*

Proposition 3.4.4. *Let g be an \mathbb{F} -bilinear form on $V \otimes \Pi$ given by*

$$g(v \otimes \phi, w \otimes \psi) = -\psi(v)\phi(w)$$

for $v \otimes \phi, w \otimes \psi \in V \otimes \Pi$.

Then for all singular pure tensors $v \otimes \phi$ and tensors $w \otimes \psi$ and $u \otimes \xi \in V \otimes \Pi$ we have

$$\begin{aligned} & [v \otimes \phi, [v \otimes \phi, w \otimes \psi]] = 2g(v \otimes \phi, w \otimes \psi)(v \otimes \phi), \\ & [[v \otimes \phi, w \otimes \psi], [v \otimes \phi, u \otimes \xi]] = g(v \otimes \phi, \psi(u)w \otimes \xi - \xi(w)u \otimes \psi)v \otimes \phi \\ & \quad + g(v \otimes \phi, u \otimes \xi)[v \otimes \phi, w \otimes \psi] \\ & \quad - g(v \otimes \phi, w \otimes \psi)[v \otimes \phi, u \otimes \xi], \end{aligned}$$

and

$$\begin{aligned} & [v \otimes \phi, [w \otimes \psi, [v \otimes \phi, u \otimes \xi]]] = g(v \otimes \phi, w \otimes \xi\psi(u) - u \otimes \psi\xi(w))v \otimes \phi \\ & \quad - g(v \otimes \phi, u \otimes \xi)[v \otimes \phi, w \otimes \psi] \\ & \quad - g(v \otimes \phi, w \otimes \psi)[v \otimes \phi, u \otimes \xi]. \end{aligned}$$

Proof. See proposition 2.4.3 of [9]. \square

Corollary 3.4.5. *Let \mathbb{F} be a field. Then the singular pure tensors $v \otimes \phi \in V \otimes \Pi$ are extremal. Moreover,*

$$\exp(v \otimes \phi, \lambda)(w \otimes \psi) = (w + \lambda\phi(w)v) \otimes (\psi - \lambda\psi(v)\phi)$$

for $\lambda \in \mathbb{F}$.

Proof. By proposition 3.4.4, we have that the singular pure tensors $v \otimes \phi \in V \otimes \Pi$ satisfy the definition, definition 3.1.1, of an extremal element. For the second part, let $v \otimes \phi \in E(V \otimes \Pi)$ and $w \otimes \psi \in V \otimes \Pi$ be arbitrary and $\lambda \in \mathbb{F}$. Then

$$\begin{aligned} \exp(v \otimes \phi, \lambda)(w \otimes \psi) &= w \otimes \psi + \lambda[v \otimes \phi, w \otimes \psi] + \lambda^2 g(v \otimes \phi, w \otimes \psi)(v \otimes \phi) \\ &= w \otimes \psi + \lambda(\phi(w)(v \otimes \psi) - \psi(v)(w \otimes \phi)) \\ &\quad - \lambda^2 \phi(w)\psi(v)(v \otimes \phi) \\ &= (w + \lambda\phi(w)v) \otimes (\psi - \lambda\psi(v)\phi). \end{aligned} \quad \square$$

3.4.1 General linear Lie algebras

Let V be a vector space over a field \mathbb{F} . Suppose that Π is a subspace of V^* . Let $\mathfrak{gl}(V)$ be the algebra of \mathbb{F} -linear maps from $V \rightarrow V$. Equipping the Lie bracket on the algebra, we have that $\mathfrak{gl}(V)$ is a Lie algebra over \mathbb{F} , since $\text{End}(V)$ is a vector space over \mathbb{F} .

Proposition 3.4.6. *The map*

$$\begin{aligned} \Phi : V \otimes \Pi &\rightarrow \mathfrak{gl}(V) \\ v \otimes \phi &\mapsto t_{v,\phi}, \end{aligned}$$

where

$$\begin{aligned} t_{v,\phi} : V &\rightarrow V \\ w &\mapsto \phi(w)v, \end{aligned}$$

is a Lie algebra homomorphism.

Proof. Denote for convenience $\mathfrak{g} = V \otimes \Pi$.

Let $v \otimes \phi, w \otimes \psi \in \mathfrak{g}$, $u \in V$ be arbitrary. We show that

$$\Phi([v \otimes \phi, w \otimes \psi]_{\mathfrak{g}})(u) = [\Phi(v \otimes \phi), \Phi(w \otimes \psi)]_{\mathfrak{gl}(V)}(u).$$

So

$$\begin{aligned} \Phi([v \otimes \phi, w \otimes \psi]_{\mathfrak{g}})(u) &= \Phi(\phi(w)(v \otimes \psi) - \phi(v)(w \otimes \phi))(u) \\ &= t_{\phi(w)v,\psi}(u) - t_{\psi(v)w,\phi}(u) \\ &= \psi(u)\phi(w)v - \phi(u)\psi(v)w \\ &= t_{v,\phi}(\psi(u)w) - t_{w,\psi}(\phi(u)v) \\ &= t_{v,\phi}t_{w,\psi}(u) - t_{w,\psi}t_{v,\phi}(u) \\ &= [t_{v,\phi}, t_{w,\psi}]_{\mathfrak{gl}(V)}(u). \end{aligned} \quad \square$$

In the case that V is finite dimensional and $\Pi = V^*$, we have that Φ is injective and thus an isomorphism. If V is infinite dimensional the following proposition shows to what Lie algebra $\mathfrak{g}(V \otimes V^*)$ is isomorphic to.

Proposition 3.4.7. *Suppose \mathbb{F} a field and V a vector space, then the map Φ induces a Lie algebra isomorphism of $\mathfrak{g}(V \otimes V^*)$ into $\mathfrak{fgl}(V)$.*

Proof. Let \mathbb{F} be a field and V a vector space. Then we have that Φ maps into $\mathfrak{fgl}(V)$ since every element from $x \in V \otimes V^*$ is a finite combination of pure tensors. For a pure tensor $v \otimes \phi \in V \otimes V^*$, we have $\text{codim}(\ker t_{v,\phi}) = \text{codim}(\ker \phi) \in \{0, 1\}$.

If V is of finite dimension, then Φ is a bijection. If V is of infinite dimension and $x \in \ker \Phi$, then it can be written as finite sum of pure tensors. This gives that there is a finite dimensional subspace V_0 of V such that these pure tensors are in $V_0 \otimes V_0^*$. So x is in the kernel of Φ restricted to this subspace, meaning that $x = 0$. Thus Φ is injective.

Now we show surjectivity. Let $f \in \mathfrak{fgl}(V)$, then we have that the codimension of the kernel of f is of finite dimension, that is $\dim(V/\ker f) < \infty$. This means that there is a subspace V_1 of V of finite dimension such that the $\text{Im}(f) \subseteq V_1$. This means that we can decompose V as $V_1 \oplus V_2$. Here V_1 is of finite dimensional and it contains $f(V)$ and V_2 is contained in the kernel of f . But then f is contained in $\Phi(V_1 \otimes V_1^*)$. \square

The subalgebra of $\mathfrak{g}(V, \Pi)$ generated by the elements $v \otimes \phi \in V \otimes \Pi$ with $\phi(v) = 0$ is denoted as $\mathfrak{g}_0(V, \Pi)$.

The elements $t_{v,\phi}$, where $v \otimes \phi$ is a singular pure tensor, are called **infinitesimal transvections**. We call $\langle v \rangle$ the **center** and $\langle \phi \rangle$ the **axis** of the infinitesimal transvection.

The images of nonsingular pure tensors, that is $t_{v,\phi}$ are called **infinitesimal reflections**.

For an element $t \in \mathfrak{gl}(V)$, an action can be defined on V^* . The action is defined as

$$t^*(\psi)(v) := -\psi(tv),$$

for $v \in V$ and $\psi \in V^*$.

Definition 3.4.8 (Dual action). *For $t_1, t_2 \in \mathfrak{gl}(V)$ and $w \in V$, $\phi \in V^*$ the dual action is the Lie algebra action defined by*

$$\begin{aligned} ([t_1, t_2])^*(\phi)(w) &= -\phi([t_1, t_2](w)) \\ &= -t_2^*\phi(t_1w) + t_1^*\phi(t_2w) \\ &= (t_1^*t_2^*\phi)(w) - (t_2^*t_1^*\phi)(w) \\ &= ([t_1^*, t_2^*](\phi))(w). \end{aligned}$$

This action extends to an action t^\otimes of t on $V \otimes V^*$ given by

$$\begin{aligned} t^\otimes : V \otimes V^* &\rightarrow V \otimes V^* \\ v \otimes \phi &\mapsto tv \otimes t^*\phi. \end{aligned}$$

As we have that the infinitesimal transvections are pure singular tensors, it follows from corollary 3.4.5 that the infinitesimal transvections are extremal. Furthermore, we have that the exponent of an infinitesimal transvection is a transvection, that is a linear transformation $T_{v,\phi} := 1 + t_{v,\phi}$ where $\phi(v) = 0$ and v, ϕ are nonzero.

Definition 3.4.9 (Transvection group). *A transvection group is a group of the form $\{1 + \lambda t_{v,\phi} : \lambda \in \mathbb{F}\}$, where the actions of a transvection on V^* and $V \otimes V^*$ are given by*

$$\begin{aligned} T_{v,\phi}^* : V^* &\rightarrow V^* \\ \psi &\mapsto \psi + t_{v,\phi}^*\psi = \psi - \phi\psi(v), \end{aligned}$$

$$\begin{aligned} T_{v,\phi}^\otimes : V \otimes V^* &\rightarrow V \otimes V^* \\ (w \otimes \psi) &\mapsto (w + t_{v,\phi}(w)) \otimes (\psi + t_{v,\phi}^*(\psi)) \\ &= (w + \phi(w)v) \otimes (\psi - \phi\psi(v)). \end{aligned}$$

This defines an action of the group generated by all transvection on $V \otimes V^*$ respecting the Lie product.

In the case $\text{char } \mathbb{F} \neq 2$ we can also associate an invertible linear map to an infinitesimal reflection $t_{v,\phi}$, namely the reflection $R_{v,\phi} := 1 - t_{v,\phi}$. This is studied in [2].

3.4.2 Special Linear Lie algebras

Let V be a vector space over a field \mathbb{F} . If V is of finite dimension n , then $\mathfrak{sl}(V)$ is the subalgebra of $\mathfrak{gl}(V)$ of elements with trace zero. The subalgebra $\mathfrak{sl}(V)$ is the Φ -image of the subspace $\mathfrak{g}_0(V, V^*)$ of $V \otimes V^*$ generated by singular pure tensors, that is

$$v \otimes \phi \in V \otimes V^* \quad \text{with} \quad \phi(v) = 0.$$

A basis for $\mathfrak{g}_0(V, V^*)$ is

$$\begin{aligned} v_i \otimes \phi_j, \quad &1 \leq i \neq j \leq n, \\ v_i + v_{i+1} \otimes \phi_i - \phi_{i+1}, \quad &1 \leq i \leq n-1, \\ \text{with } \phi_i(v_j) &= \delta_{i,j}. \end{aligned}$$

Note that we can identify $v_i \otimes \phi_j$ with the matrix $E_{i,j}$ for all $1 \leq i \neq j \leq n$ and $v_i + v_{i+1} \otimes \phi_i - \phi_{i+1}$, with $E_{i,i} - E_{i+1,i+1}$ for all $1 \leq i \leq n-1$ as

in section 2.2. The matrix algebra spanned by these matrix, is used as an identification for $\mathfrak{sl}(V)$. Note that $\mathfrak{g}_0(V, V^*)$ is of dimension $n^2 - 1$, so $\text{Im } \Phi$ is at least of dimension $n^2 - 1$. Thus $\text{Im } \Phi = \mathfrak{sl}(V)$.

As it is well known, the algebra $\mathfrak{sl}(V)$ and thus $\mathfrak{g}_0(V, V^*)$ is simple up to its center. The center of these algebras not trivial when the characteristic of \mathbb{F} divides the dimension of n .

In the case that V is infinite dimensional, more simple Lie algebras are encountered. This is in the following way.

Let Π be a subspace of V^* and consider the subalgebra $\mathfrak{fgl}(V, \Pi)$ of $\mathfrak{fgl}(V)$ generated by elements $t_{v, \phi}$ with $v \in V$ and $\phi \in \Pi$ with $\phi(v) = 0$. That is the Φ -image of $\mathfrak{g}_0(V, \Pi)$.

Definition 3.4.10 (Annihilator). *Let V be a vector space over a field \mathbb{F} and let Π be a subspace of V^* , then the annihilator of Π with respect to V is defined as*

$$\text{Ann}_V(\Pi) := \{u \in V : \psi(u) = 0, \forall \psi \in \Pi\}.$$

Let U be a subspace of $\text{Ann}_V(\Pi)$, then for all $u \otimes \psi$ with $u \in U$, we have that

$$[v \otimes \phi, u \otimes \psi] = -\psi(v)u \otimes \phi$$

by definition. So if $\{0\} \neq U \neq V$ we find $\langle u \otimes \psi : u \in U, \psi \in \Pi \rangle$ to be a proper ideal in $\mathfrak{g}(V, \Pi)$.

Lemma 3.4.11. *Let V be a vector space over a field \mathbb{F} , and $\Pi \subseteq V^*$ such that $\text{Ann}_V \Pi = \{0\}$. Then $\mathfrak{fsl}(V, \Pi)$ is simple.*

Proof. As we have that $\mathfrak{g}_0(V, \Pi)$ is isomorphic to $\mathfrak{fsl}(V, \Pi)$ it is sufficient to show that $\mathfrak{g}_0(V, \Pi)$ is simple.

Let $\mathcal{I} \subseteq \mathfrak{g}_0(V, \Pi)$ be a non zero ideal. Notice that $\mathfrak{g}_0(V, \Pi) = \langle v_i \otimes \phi_i : v_i \in V, \phi_i \in \Pi, \phi_i(v_i) = 0 \rangle$. Let $0 \neq x \in \mathcal{I}$ then $x = \sum_{i \in I} \alpha_i (v_i \otimes \phi_i)$ for some index set I and $\alpha_i \in \mathbb{F}$ for all $i \in I$. We show that every generator of \mathfrak{g}_0 is in \mathcal{I} and thus $\mathcal{I} = \mathfrak{g}_0$. So let $0 \neq v_l \phi_l \in \mathfrak{g}_0$, then

$$\begin{aligned} [v_l \otimes \phi_l : x] &= \sum_{i \in I} \alpha_i [v_l \otimes \phi_l : v_i \phi_i] \\ &= \sum_{i \in I} \alpha_i (\phi_l(v_i)(v_l \otimes \phi_i) - \phi_i(v_l)(v_i \otimes \phi_l)). \end{aligned}$$

As we have that \mathcal{I} is an ideal, we must have that $[v_l \otimes \phi_l : x] \in \mathcal{I}$, and also

$[v_l \otimes \phi_l[v_l \otimes \phi_l : x]] \in \mathcal{I}$. This means that

$$\begin{aligned} [v_l \otimes \phi_l[v_l \otimes \phi_l : x]] &= \sum_{i \in I} \alpha_i [v_l \otimes \phi_l; \phi_l(v_i)(v_l \otimes \phi_i) - \phi_i(v_l)(v_i \otimes \phi_l)] \\ &= \sum_{i \in I} \alpha_i (\phi_l(v_l) \phi_i(v_l)(v_l \otimes \phi_i) \\ &\quad - 2\phi_i(v_l) \phi_l(v_i)(v_l \otimes \phi_l) + \phi_i(v_l) \phi_l(v_l)(v_i \otimes \phi_l)) \\ &= \sum_{i \in I} -2\alpha_i \phi_i(v_l) \phi_l(v_i)(v_l \otimes \phi_l). \end{aligned}$$

If $[v_l \otimes \phi_l[v_l \otimes \phi_l : x]] = 0$, we have that $\phi_i(v_l) = 0$ for all $i \in I$. This means that $v_l = 0$, since $\text{Ann}_V \Pi = \{0\}$. This does not happen as $v_l \otimes \phi_l \neq 0$. So we have that $v_l \otimes \phi_l \in \mathcal{I}$.

This for all generators of $\mathfrak{g}_0(V, \Pi)$. So $\mathcal{I} = \mathfrak{g}_0(V, \Pi)$. Thus $\mathfrak{g}_0(V, \Pi)$ is simple. \square

3.4.3 Symplectic Lie algebras

Suppose that the pair (V, f) is a symplectic space, then the symplectic Lie algebra $\mathfrak{fsp}(V, f)$ is the image under Φ spanned by the elements $v \otimes f(v, \cdot) \in \mathfrak{g}$. This follows from the next two statements:

Lemma 3.4.12. *Let $f : V \times V \rightarrow \mathbb{F}$ be a symplectic form. Then with $t_v := \Phi(v \otimes f(v, \cdot))$ as in proposition 3.4.6, we have*

$$f(t_v(w), u) = -f(w, t_v(u)), \quad \forall u, v, w \in V.$$

Proof. Let $u, v, w \in V$ be arbitrary, then

$$\begin{aligned} f(t_v(w), u) &= f(vf(v, w), u) = f(v, w)f(v, u) = -f(w, v)f(v, u) \\ &= f(w, f(v, u)v) = -f(w, t_v(u)). \end{aligned}$$

\square

This lemma shows that $t_v \in \mathfrak{sp}(V)$, with $0 \neq v \in V$. To show that these elements also generate the finitary part is shown in the following proposition.

Proposition 3.4.13. *Let (V, f) be a nondegenerate symplectic space over a field \mathbb{F} . Then the finitary symplectic Lie algebra $\mathfrak{fsp}(V)$ is generated by its extremal elements t_v with $0 \neq v \in V$.*

Proof. Let the Lie algebra \mathfrak{g} defined by $\mathfrak{g} = \langle t_v : v \in V \setminus \{0\} \rangle$. By the previous lemma we have that $\mathfrak{g} \subseteq \mathfrak{fsp}(V)$. We will show that $\mathfrak{fsp}(V) \subseteq \mathfrak{g}$.

First let V be of finite dimension $n = 2m$. So there is a hyperbolic basis $v_1, \dots, v_m, v_{m+1}, \dots, v_{2m}$ for V with

$$f(v_i, v_j) = \begin{cases} 1, & \text{if } 1 \leq i \leq m \text{ and } j = m + 1 \\ 0, & \text{otherwise.} \end{cases}$$

Consider now the elements t_{v_i} and $t_{v_i+v_j}$ where $1 \leq i < j \leq 2m$. These elements are linearly independent. Indeed, suppose

$$\sum_{i=1}^{2m} \lambda_i t_{v_i} + \sum_{1 \leq i < j \leq 2m} \lambda_{i,j} t_{v_i+v_j} = 0.$$

Evaluating this in v_l with $1 \leq l \leq m$ gives

$$\lambda_{l+j} v_{l+m} \sum_{k < l+m} \lambda_{k(l+m)} (v_k + v_{l+m}) + \sum_{k > l+m} \lambda_{k(l+m)} (v_k + v_{l+m}) = 0.$$

This results into $\lambda_{k(l+m)} = 0$, $\lambda_{k'(l+m)} = 0$ for $k < l+m$ and $k' > l+m$ and $\lambda_{l+m} = 0$, since v_i is a basis.

Evaluating in v_l with $m+1 \leq l \leq 2m$ gives $\lambda_{k(l-m)} = 0$ and $\lambda_{k'(l-m)} = 0$ for $k < l-m$ and $k' > l-m$ and $\lambda_{l-m} = 0$.

So the elements t_{v_i} and $t_{v_i+v_j}$ where $1 \leq i < j \leq 2m$ form an independent set. Thus the subspace \mathfrak{g} has dimension $m(2m+1)$, hence equal to $\mathfrak{sp}(V)$. Now assume that V is of infinite dimension. Let $x \in \mathfrak{fsp}(V)$, then there exists a nondegenerate finite dimensional subspace V_0 of V such that $x \in \mathfrak{sp}(V_0)$. So by the previous we have $x \in \mathfrak{g}$. \square

The elements t_v are called **symplectic infinitesimal transvections**. Exponentiation of symplectic infinitesimal transvections leads to symplectic transvections on V .

3.4.4 Unitary Lie algebras

In the way we defined the symplectic infinitesimal transvections, we can also define unitary infinitesimal transvections. So let V be a vector space over a field \mathbb{K} . Let h be a skew-Hermitian form on V with respect to some automorphism σ of \mathbb{K} . Consider the elements $v \otimes h(\cdot, v) \in \mathfrak{g}(V, V^*)$.

Lemma 3.4.14. *Let t_v be defined as $t_v := \Phi(v \otimes h(\cdot, v))$. Then we have that $h(t_v(w), u) = -h(w, t_v(u))$ for all $u, v, w \in V$.*

Proof. Let $u, v, w \in V$ be arbitrary then

$$\begin{aligned} h(t_v(w), u) &= h(h(w, v)v, u) = h(w, v)h(v, u) \\ &= h(w, (h(v, u)v)^\sigma) = h(w, -h(u, v)v) \\ &= -h(w, t_v(u)). \end{aligned} \quad \square$$

The elements t_v , with $0 \neq v \in V$ a singular vector, are called **unitary infinitesimal transvection**. If v is nonsingular, then t_v is called **unitary infinitesimal reflection**.

For two unitary infinitesimal transvection t_v and t_w , we have that

$$[t_v, [t_v, t_w]] = -2h(v, w)h(w, v)t_v.$$

As $\mathfrak{u}(V, h)$ is considered as an algebra over a field \mathbb{F} inside $\mathbb{K}_\sigma = \{\alpha \in \mathbb{K} : \alpha^\sigma = \alpha\}$, then for t_v to be extremal we should have that $h(v, w)h(w, v) \in \mathbb{F}$ for all t_w . In this case, an extremal form g can be defined on $\mathfrak{u}(V, h)$ as

$$g(t_v, t_w) = -h(v, w)h(w, v).$$

Proposition 3.4.15. *Let V be a vector space over a field \mathbb{K} . Let h be a nondegenerate skew-Hermitian form on V with respect to the field automorphism σ of order two. Then the Lie algebra $\mathfrak{fu}(V, h)$ over \mathbb{K}_σ is generated by t_v for $v \in V$.*

Proof. First consider the case where $\dim V = n < \infty$. Then we have that $\dim \mathfrak{u}(V, h) = n^2$.

Let v_1, \dots, v_n be a basis of V such that the form is represented as

$$h(v, w) = v^T H w^\sigma, \quad \text{for } v, w \in V,$$

$$H = \begin{pmatrix} 0 & I_k & 0 \\ \pm I_k & 0 & 0 \\ 0 & 0 & \Delta_m \end{pmatrix},$$

where $n = 2m + k$ and $\lambda \in \mathbb{F}$ such that $\lambda^\sigma = -\lambda$.

Consider the elements $t_{v_i}, t_{v_i+v_j}$ and $t_{v+\mu v_j}$ for $1 \leq i < j \leq n$ and $\mu \in \mathbb{F}$ fixed such that $\mu^\sigma \neq \mu$.

As in the symplectic case, it is easily verified that these n^2 elements form an independent set in $\mathfrak{u}(V, h)$. This means that $\mathfrak{u}(V, h)$ is generated by its infinitesimal transvections and reflections.

If V is of infinite dimension, the same strategy is applied as in the symplectic case. \square

Proposition 3.4.16. *Let (V, h) be a nondegenerate unitary space over a field \mathbb{K} containing isotropic points. Then $\mathfrak{fsu}(V, h)$ is generated by its infinitesimal transvections.*

3.5 The \mathfrak{sl}_2 -relation

Let \mathfrak{g} be a Lie algebra over a field \mathbb{F} generated by $E(\mathfrak{g})$ and g the extremal form on \mathfrak{g} .

The first point of interest is what subalgebras of \mathfrak{g} can be generated by two extremal elements of \mathfrak{g} .

Proposition 3.5.1. *Let \mathfrak{g} be a Lie algebra over a field \mathbb{F} and $x, y \in E(\mathfrak{g})$. Denote the subalgebra generated by x and y as $S = \langle x, y \rangle$. Then exactly one of the following holds:*

(1) $g(x, y) = 0$ and $S = \mathbb{F}x + \mathbb{F}y$ is abelian.

- (2) $g(x, y) = 0$ and $S = \mathbb{F}x + \mathbb{F}y + \mathbb{F}z$, where $z = [x, y] \neq 0$ and $E(\mathfrak{g})$ contains all elements $S \setminus \langle z \rangle$.
- (3) $g(x, y) \neq 0$, $S = \mathbb{F}x + \mathbb{F}y + \mathbb{F}z$ and $S \simeq \mathfrak{sl}_2$. The set $E(\mathfrak{g})$ contains all elements that are mapped by this isomorphism onto infinitesimal transvections of \mathfrak{sl}_2 .

Proof. Let $x, y \in E(\mathfrak{g})$ and define $z = [x, y]$.

- (1) Let $z = 0$ and $g(x, y) = 0$. From lemma 3.3.1 we have that S is linearly spanned by x and y , thus $S = \mathbb{F}x + \mathbb{F}y$ and clearly abelian.
- (2) Let $z \neq 0$ and $g(x, y) = 0$. Then since S is a subalgebra, we have that $[x, y] = z \in S$.
So $\mathbb{F}x + \mathbb{F}y + \mathbb{F}z \subseteq S$. Note that $\mathbb{F}x + \mathbb{F}y + \mathbb{F}z$ is closed under multiplication (use of Lie bracket), so $S = \mathbb{F}x + \mathbb{F}y + \mathbb{F}z$. Furthermore, we have that for all $\lambda \in \mathbb{F}$ that $\exp(x, \lambda)(y) = t + \lambda[x, y] + \lambda^2 g_x(y)x = y + \lambda z$ and likewise $\exp(y, \lambda)(x) = x + \lambda z$ are extremal.
- (3) Let $g(x, y) \neq 0$. Like in the previous part we have that $S = \mathbb{F}x + \mathbb{F}y + \mathbb{F}z$. Since $g(x, y) \neq 0$ we can assume with out loss of generality that $g(x, y) = 1$.

Condisder the two-dimensional vector space V over \mathbb{F} with basis v_1, v_2 and dual basis ϕ_1 and ϕ_2 . Let $\hat{x} = v_1 \otimes \phi_2$ and $\hat{y} = v_2 \otimes \phi_1$ and $\hat{z} = [\hat{x}, \hat{y}]$. Then, we find that the structure of x, y, z and $\hat{x}, \hat{y}, \hat{z}$ are the same. So we have that $S \simeq \mathfrak{g}_0(V) \simeq \mathfrak{sl}_2$. Under this isomorphism we find that $\exp(x, s(y))$ is mapped to $t_{v_2 + sv_1, \phi_1 - s\phi_2}$. This implies that all elements are mapped to infinitesimal transvections are in E . \square

Note that $E(\mathfrak{g}) \cap \langle x, y \rangle$ may contains more elements than indicated in the above proposition. Since all non-zero elements from $\langle x, y \rangle$ may be extremal in the first and second case. However in the case that $\text{char}(\mathbb{F}) \neq 2$ and $\mathfrak{sl}_2 \simeq \langle x, y \rangle$, there are no other extremal elements in $\langle x, y \rangle$.

Definition 3.5.2 (\mathfrak{sl}_2 -relation). Let \mathcal{E} be the set of extremal points of the Lie algebra \mathfrak{g} . We define the relation

$$x \underset{\mathfrak{sl}_2}{\sim} y \iff g_{x_1}(y_1) \neq 0 \iff g(x_1, y_1) \neq 0$$

for some extremal $x_1 \in \langle x \rangle$ and $y_1 \in \langle y \rangle$ with $x, y \in \mathcal{E}$.

In the case that $\text{char}(\mathbb{F}) \neq 2$ this is equivalent with $\langle x_1, y_1 \rangle \simeq \mathfrak{sl}_2$. With the $\underset{\mathfrak{sl}_2}{\sim}$ relation a graph structure can be defined on \mathcal{E} , where \mathcal{E} are the vertices, and two x, y are connected if and only if $x \underset{\mathfrak{sl}_2}{\sim} y$. The graph $(\mathcal{E}, \underset{\mathfrak{sl}_2}{\sim})$ will be denoted as $\Gamma_{\mathfrak{sl}_2}(\mathfrak{g})$ or, if \mathfrak{g} is clear from the context by $\Gamma_{\mathfrak{sl}_2}$.

Lemma 3.5.3. *Let $\Gamma_{\mathfrak{sl}_2}(\mathfrak{g})$ have at least two connected components Γ_1 and Γ_2 with the corresponding sets \mathcal{E}_1 and \mathcal{E}_2 . Then*

$$[x, y] \in \text{Rad}(\mathfrak{g})$$

for all $x \in \mathcal{E}_1$ and $y \in \mathcal{E}_2$.

Proof. Let $x \in \mathcal{E}_1$ and $y \in \mathcal{E}_2$ be arbitrary, and let $z = [x, y] \neq 0$. Assume that $z \notin \text{Rad}(g)$, then there is an $u \in \mathcal{E}$ such that $g(u, z) = g_u(z) \neq 0$. Consider the two dimensional space $\langle x, z \rangle$. By proposition 3.5.1 all one spaces in $\langle x, z \rangle$ except for possibly $\langle z \rangle$ are extremal points. If there are two extremal elements $x_1, x_2 \in \langle x, z \rangle$ with $g_u(x_1) = g_u(x_2) = 0$, then by linearity of g also $g_u(z) = 0$, so all extremal points except for maybe one in $\langle x, z \rangle$ must be in \mathfrak{sl}_2 connection with u . So choose such an element $x_3 \notin \langle z \rangle$ with $g(x_3, u) \neq 0$. Now fix an element $a \in \exp(y)$ with $x_3^a = x$ and an $b \in \exp(x)$ with $y_3^b = y$. Then u^{ab} is connected to $x_3^{ab} = x^a = x$ and to $y_3^{ab} = y_3^b = y$ proving x and y must be in the same connected component of $\Gamma_{\mathfrak{sl}_2}(\mathfrak{g})$. This is a contradiction. So $z \in \text{Rad}(\mathfrak{g})$. \square

Corollary 3.5.4. *Let E_0 be a subset of extremal elements in \mathfrak{g} and \mathcal{E}_0 the corresponding set of extremal elements. Assume that $\langle E_0 \rangle = \mathfrak{g}$ and \mathcal{E}_0 is a connected component of the \mathfrak{sl}_2 -graph of \mathfrak{g} . If $E_1 = E \setminus E_0$ is non-empty, it consists of sandwich elements of \mathfrak{g} and $\langle E_1 \rangle$ is an ideal of \mathfrak{g} contained in $\text{Rad}(\mathfrak{g})$.*

Proof. For all $x \in E_0$ and $y \in E_1$ we have that $x \not\sim_{\mathfrak{sl}_2} y$, since $E_0 \cap E_1 = \emptyset$. This means that $g(x, y) = g(y, x) = 0$. Because $\langle E_0 \rangle = \mathfrak{g}$, $g(y, z) = 0$ for all $z \in \mathfrak{g}$. Thus $y \in \text{Rad}(g)$. Since for all $y \in E_1$ we have $y \in \text{Rad}(g)$, we have that

$$[y, [y, z]] = 2g(y, z)y = 0.$$

for all $z \in \mathfrak{g}$. By the second Premet identity we have that

$$[y, [x, [y, z]]] = gy([x, z])y - g_y(z)[y, x] - g_y(x)[y, z] = 0$$

for all $z \in \mathfrak{g}$. Meaning that y is sandwich.

Furthermore, we have for all $x \in E_0$ and $y \in E_1$ that $[x, y] = \exp(x, 1)y - y \in \langle E_1 \rangle$. So $\langle E_1 \rangle$ is an ideal contained in $\text{Rad}(g)$. \square

Corollary 3.5.5. *If \mathfrak{g} is simple and the bilinear form g is not trivial, then \mathcal{E} is connected with respect to the relation $\sim_{\mathfrak{sl}_2}$. In particular, the group $G = \langle \exp(x, \lambda) : x \in E(\mathfrak{g}), \lambda \in \mathbb{F} \rangle$ is transitive on the points in \mathcal{E} .*

Proof. Let \mathfrak{g} be a simple Lie algebra and g not trivial, then by proposition 3.3.7, we have that $\text{Rad}(g) = \{0\}$. Then lemma 3.5.3 implies that \mathcal{E} is

connected with respect to then $\sim_{\mathfrak{sl}_2}$ relation.

Let $x, y \in E$ with $(x, y) \in E_2$, then we can assume that $g_x(y) = g_y(x) = 1$. Then we have that $\exp(x, 1)y = y + [x, y] + x = y - [y, x] + x = \exp(y, -1)x$. This results into $\exp(y, 1)\exp(x, 1)y = x$. Meaning that $x, y \in \mathcal{E}$ are in the same orbit. Since we have that $\Gamma_{\mathfrak{sl}_2}(\mathfrak{g})$ is connected we have that all elements have the same orbit. Thus G is transitive on \mathcal{E} . \square

Theorem 3.5.6. *Suppose $\text{Rad}(g) = 0$ and $\text{char}(\mathbb{F}) \neq 2$. Then \mathfrak{g} is semisimple.*

Proof. By lemma 3.5.3 we have that \mathfrak{g} can be written as the direct sum of Lie subalgebras, each generated by the corresponding extremal sets of the connected parts. Let \mathfrak{g}_1 be a subalgebra of the directed sum generated by its set of extremal elements \mathcal{E}_1 . Suppose that \mathcal{I} is a nonzero ideal of \mathfrak{g}_1 . Let $0 \neq i \in \mathcal{I}$, then since $\text{Rad}(g) = 0$, there is an element $x \in \mathfrak{g}_1$ such that $g_{x_1}(i) \neq 0$. Moreover, as \mathfrak{g}_1 is generated by \mathcal{E}_1 , we can assume that $x \in \mathcal{E}_1$. Since $\text{char}(\mathbb{F}) \neq 2$, $x \in [x, [x, \mathcal{I}]] \subseteq \mathcal{I}$. But then for all $y \in \mathcal{E}_1$ with $g_x(y) \neq 0$ $y \in \mathcal{I}$. Then by the connectedness of \mathcal{I} with respect to $\sim_{\mathfrak{sl}_2}$, we have that $\mathcal{I} = \mathfrak{g}_1$. \square

Chapter 4

Geometry

In this chapter a general notion will be introduced for point line spaces and root filtration spaces. This chapter is based on chapter four of [9], and we will follow her theory, terminology and notations.

4.1 Point-line spaces

The theory will start with some general definitions used in geometry.

Definition 4.1.1 (Point-line space). *A point-line space $(\mathcal{P}, \mathcal{L})$ is a pair of a set \mathcal{P} of points and a set \mathcal{L} of lines, where each element of \mathcal{L} is a subset of \mathcal{P} of at least size two.*

A point-line space will be called **partial linear space** if any two points are on at most one line, and **linear space** if any two points are on exactly one line.

Definition 4.1.2 (Collinearity graph). *The collinearity graph of a point-line space $(\mathcal{P}, \mathcal{L})$ is the graph with vertex set \mathcal{P} and where two points are connected if and only if there is a line in \mathcal{L} containing both points.*

Let $(\mathcal{P}, \mathcal{L})$ be a partial linear space. Two points $p, q \in \mathcal{P}$ are called **collinear** if they are adjacent in the collinearity graph, the line through p and q is denoted by pq . The set of points that are collinear with a point $p \in \mathcal{P}$ is denoted as p^\perp .

The point-line space $(\mathcal{P}, \mathcal{L})$ is called **connected** if and only if the collinearity graph is connected.

A **subspace** of \mathcal{P} is a subset \mathcal{P}' of \mathcal{P} such that collinearity is preserved, that is for all collinear $p, q \in \mathcal{P}'$ we have that $x \in \mathcal{P}'$ for all $x \in pq$. This implies that the point-line space $(\mathcal{P}', \mathcal{L}')$, where \mathcal{P}' is a subspace and \mathcal{L}' is the set of lines that meet \mathcal{P}' in at least two points, is a partial linear space. It may be clear that the intersection of any collection of subspaces is again a subspace.

Let \mathcal{X} be any subset of \mathcal{P} , then the subspace **generated by** \mathcal{X} is the intersection of all subspaces containing \mathcal{X} and denoted by $\langle \mathcal{X} \rangle$.

If any two points of a subspace of a space are collinear, then it is called a **singular subspace**, and the **singular rank**, defined if all singular spaces are projective spaces, of the space is the supremum of all ranks of maximal singular subspaces.

If we have a point-line space $(\mathcal{P}, \mathcal{L})$ and if $n \in \mathbb{N}$ is the minimal number of generating elements of $(\mathcal{P}, \mathcal{L})$, then n is the generating rank of $(\mathcal{P}, \mathcal{L})$.

Definition 4.1.3 (Projective geometry). *A linear space $(\mathcal{P}, \mathcal{L})$ is called a projective geometry if*

- (1) *every line contains at least three points, and*
- (2) *if, whenever x, y , and z are noncollinear points and a line l meets the line xy and the line xz in distinct points, we have that l meets the line yz .*

Definition 4.1.4. *Let V be a vector space over the field \mathbb{F} . The projective space $\mathbb{P}(V)$ of V is the point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$, where the projective points \mathcal{P} are the 1-dimensional subspaces and the projective lines \mathcal{L} are the 2-dimensional subspaces of V , with the natural incidence.*

Definition 4.1.5 (Polar space). *A polar space is a partial linear space $(\mathcal{P}, \mathcal{L})$ satisfying the so called 'one-or-all' or Buekenhout-Shult axiom:*

A point p is collinear with one or all points of a line l

For points p, q of a polar space, we define $p \perp q$ as $p = q$ or p and q are collinear. We denote the set of all collinear points of a point p by p^\perp . A polar space is called **nondegenerate** if for all $p \in \mathcal{P}$, we have $p^\perp \neq \mathcal{P}$.

A way to construct a polar space, is by the use of a vector space V with a sesquilinear form f . Then the points are the singular 1-spaces and the lines are defined to be the singular 2-spaces of V .

Theorem 4.1.6 (Theorem 4.2.5 of [9]). *Let $(\mathcal{P}, \mathcal{L})$ be a nondegenerate polar space such that*

- (1) *all lines have at least three points.*
- (2) *there exist two nonintersecting lines l, m such that $p \perp q$ for all $p \in l, q \in m$.*

Then $(\mathcal{P}, \mathcal{L})$ is isomorphic to the polar space of 1- and 2- dimensional singular subspaces of vector space V with respect to a sesquilinear or pseudo-quadratic form on V .

4.1.1 Root filtration spaces

Definition 4.1.7 (Root filtration space). *Let $(\mathcal{E}, \mathcal{F})$ be a partial linear space. For $\{\mathcal{E}_i\}_{-2 \leq i \leq 2}$ a quintuple of symmetric relations partitioning $\mathcal{E} \times \mathcal{E}$, we call $(\mathcal{E}, \mathcal{F})$ a root filtration space with filtration $\{\mathcal{E}_i\}_{-2 \leq i \leq 2}$ if the following properties are satisfied*

- (A) *The relation \mathcal{E}_{-2} is equality on \mathcal{E} .*
- (B) *The relation is \mathcal{E}_{-1} is collinearity of distinct points of \mathcal{E} .*
- (C) *There is a map $\mathcal{E}_1 \rightarrow \mathcal{E}$, denoted by $(u, v) \mapsto [u, v]$, such that, if $(u, v) \in \mathcal{E}_1$ and $x \in \mathcal{E}_i(u) \cap \mathcal{E}_j(v)$, then $[u, v] \in \mathcal{E}_{\leq i+j}(x)$.*
- (D) *For each $(x, y) \in \mathcal{E}_2$, we have $\mathcal{E}_{\leq 0}(x) \cap \mathcal{E}_{\leq -1}(y) = \emptyset$.*
- (E) *For each $x \in \mathcal{E}$ the subsets $\mathcal{E}_{\leq -1}(x)$ and $\mathcal{E}_{\leq 0}(x)$ are subspaces of $(\mathcal{E}, \mathcal{F})$.*
- (F) *For each $x \in \mathcal{E}$, the subset $\mathcal{E}_{\leq 1}(x)$ is a geometric hyperplane of $(\mathcal{E}, \mathcal{F})$.*

Where $\mathcal{E}_{\leq i} = \cup_{j \leq i} \mathcal{E}_j$ and $\mathcal{E}_i(x) = \{y \in \mathcal{E} : (x, y) \in \mathcal{E}_i\}$.

A pair $(x, y) \in \mathcal{E}_i$ is called **hyperbolic** if $i = 2$, **special** if $i = 1$, **polar** if $i = 0$, **collinear** if $i = -1$ and **commuting** if $i \leq 0$.

According to the previous definition of the collinearity graph of $(\mathcal{E}, \mathcal{F})$ we have that $\mathcal{F} = \mathcal{E}_{-1}$, since a pair $(x, y) \in \mathcal{E}$ are connected if they are collinear that is $(x, y) \in \mathcal{E}_{-1}$. We call two points (x, y) inside $(\mathcal{E}, \mathcal{E}_{-1})$ which are joined by an edge **neighbours**.

In addition we call a root filtration space **nondegenerate** if the following two condition holds as well

- (G) For each $x \in \mathcal{E}$ the set $\mathcal{E}_2(x)$ is not empty.
- (H) The collinearity graph $(\mathcal{E}, \mathcal{E}_{-1})$ is connected.

Lemma 4.1.8 (Lemma 4.2.8 of [12]). *Let $(\mathcal{E}, \mathcal{F})$ be a nondegenerate root filtration space. Then its defining relations can be characterized by the collinearity graph $(\mathcal{E}, \mathcal{E}_{-1})$ in the following way.*

- (-2) $(x, y) \in \mathcal{E}_{-2}$ if and only if $x = y$
- (-1) $(x, y) \in \mathcal{E}_{-1}$ if and only if x and y are distinct collinear points.
- (0) $(x, y) \in \mathcal{E}_0$ if and only if x and y have at least two common neighbours.
- (1) $(x, y) \in \mathcal{E}_1$ if and only if x and y have a unique common neighbour.
- (2) $(x, y) \in \mathcal{E}_2$ if and only if x and y have no common neighbours.

Proof. A proof can be found in [12], lemma 4.2.8. □

Example 4.1.9. Let V be a vector space and Π a subspace of V^* . Consider the point-line geometry $\Gamma(V, \Pi)$ with as point set $\mathcal{E} = \{(p, H) : p \in H\}$ of point-hyperplane pairs of the projective space $\mathbb{P}(V)$, where the hyperplane H the kernel of an element form Π . Collinearity of $(p, H), (q, K) \in \mathcal{E}$ is given if $p = q$ or $H = K$. The lines in this space are given as follows: Let (p, H) and (q, K) be collinear, then the line through them consists of all pairs (r, H) with $r \in \langle p, q \rangle$ if $H = K$, and of all point-hyperplane pairs (p, M) with $H \cap K \subseteq M$ if $p = q$. This is a root filtration space, where the relations on pairs of points $x := (p, H), y := (q, K)$ are defined as follows:

- (-2) $x \sim_{-2} y \iff p = q, H = K$
- (-1) $x \sim_{-1} y \iff p = q$ or $H = K$ but not $(p, H) = (q, K)$
- (0) $x \sim_0 y \iff p \in K, q \in H$ but $H \neq K$ and $p \neq q$
- (1) $x \sim_1 y \iff q \in H$ but $p \notin K$ or $p \in K$ by $q \notin H$
- (2) $x \sim_2 y \iff q \notin K, q \notin H$.

The geometry of all subspaces of $\mathbb{P}(V)$, where V has finite dimension $n+1$, is a so-called spherical building of type A_n . The root filtration space of point-hyperplane pairs is a particular example of a root shadow space of this building. Any nondegenerate root filtration space with finite singular rank can be identified with a so-called root shadow space related to a spherical building. This result is stated and proven by Cohen and Ivanyos in [3].

Theorem 4.1.10 (Cohen and Ivanyos [3]). *A nondegenerate root filtration space with finite singular rank is isomorphic to the root shadow space of type $A_{1, \{1, n\}}, BC_{n, 2}, D_{n, 2}, E_{6, 2}, E_{7, 1}, E_{8, 8}, F_{4, 1}$, or $G_{2, 2}$.*

Spherical buildings of finite rank at least 3 are classified by Tits [13]. This implies that root filtration spaces that are nondegenerate and have finite rank at least 3 are known. For further theory on buildings and their root shadow spaces, we refer to [13], or chapter one and two of the Roberts [12].

4.1.2 Polarized embeddings

Definition 4.1.11 (Projective embedding). *Let Γ be the point-line geometry. A projective embedding of Γ over \mathbb{F} is an injective map e from \mathcal{P} to a set of points that span $\mathbb{P}(V)$, such that the image of point-shadow of each line comprises all projective points of a projective line.*

Note that this induces an injection from \mathcal{L} into the line set of $\mathbb{P}(V)$.

Let e be a projective embedding and $t : V \rightarrow W$ a surjective semilinear transformation, with the property that $K = \ker(t)$ intersects any span $\langle p, q \rangle$ for any pair $(p, q) \in \mathcal{P} \times \mathcal{P}$ trivially. Then e can be continued on the coset

of K , and the embedding $e' : \Gamma \rightarrow \mathbb{P}(W)$ is obtained. The embedding e' is called the **morphic image** of e , or we say that e' is **derived** from e or e **covers** e' . In particular, $e'(p) := t(e(p)) \in W$ is a 1-space for all $p \in \mathcal{P}$. In the case that all embeddings e' can be obtained in such a way from e , then e is called **absolute** or **absolute universal**.

For the rest of this section, we take $\Gamma = (\mathcal{E}, \mathcal{F})$ to be a nondegenerate root filtration space.

Definition 4.1.12 (Polarized embedding). *Let $\psi : \Gamma \rightarrow \mathbb{P}$ be an arbitrary projective embedding of Γ .*

The projective embedding ψ is called polarized if and only if $\psi(\mathcal{E}_{\leq 1}(x))$ is contained in a hyperplane of \mathbb{P} for all $x \in \mathcal{E}$.

The radical R_ψ of a polarized embedding ψ is defined as

$$R_\psi := \bigcap_{x \in \mathcal{E}} \langle \psi(\mathcal{E}_{\leq 1}(x)) \rangle,$$

where $\langle \psi(\mathcal{E}_{\leq 1}(x)) \rangle$ denotes the subspace of \mathbb{P} generated by $\psi(\mathcal{E}_{\leq 1}(x))$.

Lemma 4.1.13. *Let $\psi : \Gamma \rightarrow \mathbb{P}$ be a projective embedding that covers a polarized embedding ϕ . Then ψ is polarized.*

Moreover the kernel of the projection of ψ to ϕ is contained in the radical of ψ .

Proof. See lemma 4.4.2 of [9]. □

Proposition 4.1.14. *Let ψ be a cover of a polarized embedding ϕ of Γ . If the radical of ϕ is trivial, then ϕ is isomorphic to ψ modulo its radical R_ψ .*

Proof. See proposition 4.4.3 of [9] □

Theorem 4.1.15. *Suppose Γ admits an absolute universal embedding and a polarized embedding ϕ with trivial radical. Then any polarized embedding ψ of Γ covers ϕ .*

Proof. See theorem 4.4.4 of [9]. □

Theorem 4.1.16 (Theorem 4.4.5 [9]). *Suppose $(\mathcal{P}, \mathcal{L})$ is a point-line geometry admitting an absolute universal embedding ψ . If ϕ is a polarized embedding of $(\mathcal{P}, \mathcal{L})$, then*

$$\phi/R_\phi \cong \psi/R_\psi.$$

Proof. See theorem 4.4.5 in [9]. □

Theorem 4.1.17 (Theorem 2.8 of [6]). *Let $\Gamma = (\mathcal{E}, \mathcal{F})$ be an embeddable root shadow space of type $BC_{n,2}$ ($n \geq 3$), $D_{n,2}$ ($n \geq 4$), $E_{6,2}$, $E_{7,1}$, $E_{8,8}$ or $F_{4,1}$. Then Γ admits an absolutely universal embedding.*

Proof. See theorem 2.8 of [6]. □

Corollary 4.1.18. *Let Γ be a root shadow space of type $BC_{n,2}$ ($n \geq 3$), $D_{n,2}$ ($n \geq 4$), $E_{6,2}$, $E_{7,1}$, $E_{8,8}$ or $F_{4,1}$ admitting a polarized embedding. Then Γ admits, up to isomorphism, a unique polarized embedding with trivial radical.*

Chapter 5

Geometry of the Lie algebra

This chapter is based on chapter five of the PhD of Fleischmann [9]. In this section the extremal geometry will be introduced for Lie algebras which are generated by its extremal elements.

5.1 The extremal geometry

In order to define a geometry for a Lie algebra generated by extremal elements, we need to be able to talk about points and lines. This will be done with the extremal points of the Lie algebra \mathfrak{g} . In definition 3.1.7 names are introduced for pairs of extremal elements. The relations were defined as follow.

- (-2) $(x, y) \in E_{-2}$ if and only if x and y are linearly dependent.
- (-1) $(x, y) \in E_{-1}$ if and only if x and y are linearly independent and commuting. Furthermore every linear combination $\lambda x + \mu y$ is an extremal element. (By definition $(\lambda, \mu) \neq (0, 0)$.)
- (0) $(x, y) \in E_0$ if and only if x and y commute and $(x, y) \notin E_{-2} \cup E_{-1}$.
- (1) $(x, y) \in E_1$ if and only if x and y do not commute and are zero on the extremal form, that is $g_x(y) = g(x, y) = 0$.
- (2) $(x, y) \in E_2$ if and only if they are mapped to a nonzero by the extremal form, that is $g_x(y) = g(x, y) \neq 0$.

Note that $(x, y) \in E_{\leq 1}$ implies that $g(x, y) = 0$ and $(x, y) \in E_{\geq 1}$ implies that they do not commute, where $(x, y) \in E_{\leq i}$ is defined as $(x, y) \in \cup_{j \leq i} E_j$. As we want to construct a point-line space $(\mathcal{P}, \mathcal{L})$, we take as points the projective extremal points of the Lie algebra \mathfrak{g} . The projective extremal points correspond in a natural way with the extremal points, as defined directly after definition 3.1.7.

Definition 5.1.1 (Extremal geometry). *Let \mathcal{E} be the projective extremal points of the Lie algebra \mathfrak{g} and let \mathcal{L} be the set of projective lines $\mathbb{F}x + \mathbb{F}y$ for $(x, y) \in \mathcal{E}_{-1}$. Hereby, we identify a 2-space with the set of 1-spaces it contains. Then the point-line space $(\mathcal{E}, \mathcal{L})$ together with the previously defined relation \mathcal{E}_i , $i \in \{-2, \dots, 2\}$ on \mathcal{E} define the extremal geometry of \mathfrak{g} . Denoted by $\Gamma(\mathfrak{g})$.*

Remark, an extremal geometry is a partial linear space. This, because the unique line in \mathcal{L} containing two incident points $\mathbb{F}x$ and $\mathbb{F}y$ is $\mathbb{F}x + \mathbb{F}y$.

Example 5.1.2. *Assume that V is a vector space over a field \mathbb{F} and let Π be a subspace of V^* such that $\text{Ann}_V(\Pi) = \{0\}$. Then we have that the Lie algebra $\mathfrak{g}_0(V, \Pi)$ is generated by $\{v \otimes \phi, : v \in V, \phi \in \Pi, \phi(v) = 0\}$. Note that the elements $v \otimes \phi$ are the extremal elements of $\mathfrak{g}_0(V, \Pi)$. But then it is straightforward to check that the extremal geometry $\Gamma(\mathfrak{g}_0(V, \Pi))$ is isomorphic to the root filtration space $\Gamma(V, \Pi)$ defined in 4.1.9. Moreover, we notice that $\Gamma(\mathfrak{g}_0(V, \Pi))$ is embedded in the projective space of $\mathfrak{g}_0(V, \Pi)$ as its points and lines are 1- and 2-dimensional subspaces.*

The following important results connects Lie algebras generated by its extremal elements and root filtration spaces.

Theorem 5.1.3 (Theorem 28 of [4]). *Suppose that \mathfrak{g} is a Lie algebra, generated by its extremal elements $\mathcal{E}(\mathfrak{g})$ and with extremal form g with trivial radical. Then the extremal geometry $(\mathcal{E}, \mathcal{L})$ of \mathfrak{g} is a root filtration space with filtration $\{\mathcal{E}_i\}_{i=-2}^2$ as defined before. Let \mathcal{B}_i be the connected components of $(\mathcal{E}, \mathcal{E}_2)$ and \mathfrak{g}_i the Lie subalgebra generated by \mathcal{B}_i of \mathfrak{g} . Then each \mathcal{B}_i is a nondegenerate root filtration space or a root filtration space without lines, \mathfrak{g} is the direct sum of Lie subalgebras \mathfrak{g}_i and $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ for $i \neq j$. In particular \mathfrak{g}_i is an ideal of \mathfrak{g} .*

Corollary 5.1.4. *Let \mathfrak{g} be a simple Lie algebra generated by its extremal elements E . Then $(\mathcal{E}, \mathcal{L})$ is a nondegenerate root filtration space with thick lines or a root filtration space without lines.*

As we already saw before, nondegenerate root filtration spaces of finite rank at least 3 are classified. So, in this case the extremal geometry of a Lie algebra is known.

5.2 The embedding

Setting 5.2.1. *Let \mathfrak{g} be a Lie algebra generated by its extremal elements E , with a nondegenerate extremal form g . By $\Gamma = (\mathcal{E}, \mathcal{F})$, we denote the extremal geometry of \mathfrak{g} . Assume that Γ is nondegenerate and connected, so $\mathcal{E}_{-1} \neq \emptyset$.*

In the previous section we already saw that Γ is a root filtration space whose points and lines are 1- and 2-dimensional subspaces of \mathfrak{g} .

The natural projective embedding from the extremal geometry into the projective space on \mathfrak{g} is defined to be the injection

$$\begin{aligned}\phi : \mathcal{E} &\hookrightarrow \text{projective points of } \mathbb{P}(\mathfrak{g}) \\ x &\mapsto x\end{aligned}$$

For any line $l \in \mathcal{F}$, the restriction of ϕ to all the points of l is the full set $\phi(l)$ of points of some projective line. So we find

$$\begin{aligned}\phi : \Gamma &\hookrightarrow \mathbb{P}(\mathfrak{g}) \\ p \in \mathcal{E} &\mapsto \text{1-spaces} = \text{points} \\ l \in \mathcal{F} &\mapsto \text{2-spaces} = \text{lines}.\end{aligned}$$

The embedding ϕ is polarized, since for every extremal point $x \in \mathcal{E}$ we find $\phi(\mathcal{E}_{\leq 1}(x))$ to be contained in the hyperplane $\{y \in \mathfrak{g} : g(x, y) = 0\}$.

As the radical of the \mathfrak{g} is trivial, the embedding into $\mathbb{P}(\mathfrak{g})$ is the unique polarized embedding with trivial radical, provided that the extremal geometry Γ admits an absolute universal embedding. In particular, using 4.1.18, we find the following.

Theorem 5.2.2. *Let \mathfrak{g} be a simple Lie algebra generated by its extremal elements and with extremal geometry isomorphic to a root shadow space of type $BC_{n,2}$, ($n \geq 3$) $D_{n,2}$, ($n \geq 4$), $E_{6,2}$, $E_{7,1}$, $E_{8,8}$ or $F_{4,1}$. Then the embedding of $\Gamma(\mathfrak{g})$ into $\mathbb{P}(\mathfrak{g})$ is uniquely determined.*

5.3 Uniqueness of the Lie Product

In the previous section we have seen that, in many cases, the extremal geometry of a simple Lie algebra generated by its extremal elements is a known geometry whose embedding into the vector space underlying \mathfrak{g} is unique determined, up to isomorphism. It remains to prove in these cases that the Lie product is determined (up to a scalar) by the extremal geometry and its embedding. For, in that case the Lie algebra \mathfrak{g} is determined up to isomorphism.

Theorem 5.3.1 (Theorem 5.3.9 of [9]). *Let \mathfrak{g} be a Lie algebra generated by its set of extremal elements E , equipped with the Lie product denoted by $[\cdot, \cdot]$ and a nondegenerate extremal form g . Assume that there is a second Lie product $[\cdot, \cdot]_1$ defined on the underlying vector space, with corresponding nondegenerate extremal form g^1 , giving rise to the same extremal geometry Γ . Then, there is a $\lambda \in \mathbb{F}^*$ such that $[x, y]_1 = \lambda[x, y]$ for all $x, y \in \mathfrak{g}$.*

Proof. Theorem 5.3.9 of [9]. □

The main result of this section and the previous one is stated in the following theorem

Theorem 5.3.2 (Theorem 5.4.1 of [9]). *Let \mathfrak{g} be a Lie algebra generated by its set E of extremal elements with respect to the extremal form g with trivial radical. If $\Gamma(\mathfrak{g})$ is nondegenerate and the natural embedding of the extremal geometry $\Gamma(\mathfrak{g})$ into $\mathbb{P}(\mathfrak{g})$ admits an absolute universal cover, then \mathfrak{g} is determined, up to isomorphism, by $\Gamma(\mathfrak{g})$.*

Proof. This is a combination of previous results. So let \mathfrak{g}_1 be a second Lie algebra with isomorphic extremal geometry $\Gamma(\mathfrak{g}) \simeq \Gamma(\mathfrak{g}_1)$. By theorem 5.2.3 of Fleischmann [9] the projective embeddings of \mathfrak{g} and \mathfrak{g}_1 are equivalent and therefore they have the same Lie structure by theorem 5.3.1. \square

Using 5.2.2 we obtain:

Theorem 5.3.3. *Let \mathfrak{g} be a simple Lie algebra generated by its extremal elements and with extremal geometry isomorphic to a root shadow space of type $BC_{n,2}$, ($n \geq 3$) $D_{n,2}$, ($n \geq 4$), $E_{6,2}$, $E_{7,1}$, $E_{8,8}$ or $F_{4,1}$. Then \mathfrak{g} is determined, up to isomorphism, by $\Gamma(\mathfrak{g})$.*

Chapter 6

Characterization of $\mathfrak{psl}(V)$

In this chapter we want to prove Theorem 1.1.1. This means we want to prove that a simple Lie algebra \mathfrak{h} generated by its extremal elements and its extremal geometry isomorphic to $\Gamma(V, \Pi)$ for some vector space V and subspace Π of V^* with $\text{Ann}_V(\Pi) = \{0\}$ is isomorphic to the projective special linear Lie algebra $\mathfrak{psl}(V, \Pi)$.

Our proof is divided into two parts. The first part is concerned with the case that the Lie algebra \mathfrak{h} is of finite dimension. In this case we will show \mathfrak{h} to be isomorphic to $\mathfrak{psl}(V)$ for a finite dimensional vector space V . The second part is concerned with case the where the Lie algebra is not of finite dimension. We will prove that in this case \mathfrak{h} is (up to the center) isomorphic to $\mathfrak{psl}(V, \Pi)$ for some $\Pi \subseteq V^*$ such that $\text{Ann}_V(\Pi) = \{0\}$.

6.1 Finite dimensional case

We first treat the finite dimensional case.

So assume that \mathfrak{h} is a simple Lie algebra generated by its extremal elements and assume that its extremal geometry $\Gamma(\mathfrak{h})$ is isomorphic to $\Gamma(V, V^*)$, where V is a finite dimensional vector space of dimension at least 3.

This implies that the extremal geometry of \mathfrak{h} is isomorphic to the extremal geometry of $\mathfrak{psl}(V)$, which is a root shadow space of type $A_{n, \{1, n\}}$, where V is of dimension $n + 1$.

This situation has been considered by Roberts in his PhD-thesis [12], and also in Cuypers, Roberts, and Shpectorov [8], where the case of a root shadow space of a spherical building of simply laced type (and hence also A_n) is considered. The results of [12] or [8] imply:

Theorem 6.1.1. *The Lie algebra \mathfrak{h} is isomorphic to $\mathfrak{psl}(V)$.*

6.2 Infinite dimensional case

We now focus on the infinite dimensional case. So, assume that \mathfrak{h} is a simple Lie algebra generated by its extremal elements, whose extremal geometry $\Gamma = \Gamma(\mathfrak{h})$ is isomorphic to $\Gamma(V, \Pi)$, where V is an infinite dimensional vector space and Π is a subspace of V^* with $\text{Ann}_V(\Pi) = \{0\}$.

It is our goal to prove that \mathfrak{h} is isomorphic to $\mathfrak{fs}\mathfrak{l}(V, \Pi)$.

To obtain this result we approximate both \mathfrak{h} and the simple Lie algebra $\mathfrak{fs}\mathfrak{l}(V, \Pi)$ with a so called local system (Definition 6.2.2), and prove that both local systems are isomorphic, resulting in the isomorphism of the Lie algebras \mathfrak{h} and $\mathfrak{fs}\mathfrak{l}(V, \Pi)$.

6.2.1 Local systems

In order to define local systems for the simple Lie algebras \mathfrak{h} and $\mathfrak{fs}\mathfrak{l}(V, \Pi)$, we make use of directed sets.

Definition 6.2.1 (Directed set). *A directed set is a pair (I, \sqsubseteq) such that \sqsubseteq defines a partial order on I with the following properties:*

1. *For all $i, j \in I$ there is a $k \in I$ with $i, j \sqsubseteq k$.*
2. *For every $k \in I$ any ascending chain $i_0 \sqsubseteq i_1 \sqsubseteq \dots$ of elements $i_j \sqsubseteq k$ is of finite length.*

Now we are in a position to define a local system.

Definition 6.2.2 (Local system). *Let I be a directed set. Then a local system of a Lie algebra \mathfrak{g} over the directed set (I, \sqsubseteq) is a collection $(\mathfrak{g}_i)_{i \in I}$ of Lie subalgebras of \mathfrak{g} such that*

1. *if $i \sqsubseteq j$ then $\mathfrak{g}_i \subseteq \mathfrak{g}_j$;*
2. $\mathfrak{g} = \bigcup_{i \in I} \mathfrak{g}_i$.

If \mathfrak{g} contains a local system $(\mathfrak{g}_i)_{i \in I}$, then it is determined up to isomorphism by this local system, as it is (isomorphic to) the direct limit

$$\lim_{i \in I} \mathfrak{g}_i.$$

This implies that we have the following result.

Proposition 6.2.3. *Let I be a directed set. Suppose \mathfrak{g} and \mathfrak{h} are two Lie algebras with local systems $\mathfrak{g}_{i \in I}$ and $\mathfrak{h}_{i \in I}$ over I such that for each $i \in I$ there exist isomorphisms*

$$\phi_i : \mathfrak{g}_i \rightarrow \mathfrak{h}_i$$

with the property that for $j, k \in I$ with $j \sqsubseteq k$ we have

$$\phi_j = \phi_k|_{\mathfrak{g}_j}.$$

Then \mathfrak{g} and \mathfrak{h} are isomorphic.

We will now construct local systems for $\mathfrak{sl}(V, \Pi)$ and \mathfrak{h} . We start with the construction of a directed set based on subspaces of V and Π .

Lemma 6.2.4. *Let V_0 be a finite dimensional vector space over a field \mathbb{F} with basis $\{v_1, \dots, v_n\}$ and Π_0 be a vector space with dual basis $\{\phi_1, \dots, \phi_n\}$, that is $\phi_j(v_i) = \delta_{ij}$. Then we have $\text{Ann}_{V_0}(\Pi_0) = \{0\}$.*

Proof. If $v \in \text{Ann}_{V_0}(\Pi_0)$ then for all j we have

$$\begin{aligned} 0 &= \phi_j(v) \\ &= \phi_j\left(\sum_{i=1}^n \lambda_i v_i\right) \\ &= \sum_{i=1}^n \lambda_i \phi_j(v_i) \\ &= \lambda_j \phi_j(v_j) \\ &= \lambda_j. \end{aligned}$$

This implies that $\lambda_j = 0$ for all $1 \leq j \leq n$, so $v = 0$.

So we have that $\text{Ann}_{V_0}(\Pi_0) = \{0\}$. \square

Lemma 6.2.5. *Let V_0 be an infinite dimensional vector space over a field \mathbb{F} and $\Pi_0 \subset V_0^*$ a subspace such that Π_0 is annihilated by V_0 . Let $V_1 \subseteq V_0$ be of finite dimension and $\Pi_1 \subseteq \Pi_0$ be of finite dimension. Then there exist subspaces V_2 and Π_2 such that*

1. $V_1 \subseteq V_2 \subseteq V_0$
2. $\Pi_1 \subseteq \Pi_2 \subseteq \Pi_0$
3. $\text{Ann}_{V_2}(\Pi_2) = \{0\}$
4. $\dim(V_2) = \dim(\Pi_2) < \infty$.

Proof. Let V_0 be an infinite dimensional vector space over a field \mathbb{F} and $\Pi_0 \subset V_0^*$ a subspace such that Π_0 is annihilated by V_0 . Let $V_1 \subseteq V_0$ and $\Pi_1 \subseteq \Pi_0$ be subspaces such that $\dim V_1 = n < \infty$ and $\dim \Pi_1 = m < \infty$.

Assume that V_1 is spanned by the linear independent vectors $\{v_1, \dots, v_n\}$ and Π_1 is spanned by the linear independent vectors $\{\phi_1, \dots, \phi_m\}$. Now choose vectors w_1, \dots, w_m in $V_0 \setminus V_1$ such that $\phi_i(w_j) = \delta_{ij}$ and define $V_2 = V_1 + \langle w_1, \dots, w_m \rangle$. Now extend $\Pi_1 = \langle \phi_1, \dots, \phi_m \rangle$ with vectors ψ_1, \dots, ψ_n such that $\psi_i(v_j) = \delta_{ij}$ to Π_2 . Define $\Pi_2 = \Pi_1 + \langle \psi_1, \dots, \psi_n \rangle$.

By construction we have that $\Pi_1 \subseteq \Pi_2$ and $V_1 \subseteq V_2$ and $\dim V_2 = \dim \Pi_2$ and $\text{Ann}_{V_2}(\Pi_2) = \{0\}$. \square

Let \mathcal{I} be the set of pairs (U, Φ) of subspaces U of V and Φ of Π of the same finite dimension but not divisible by the characteristic of the underlying field and with $\text{Ann}_U(\Phi) = \{0\}$.

Proposition 6.2.6. \mathcal{I} is a directed set, with \sqsubseteq defined as $(U, \Phi) \sqsubseteq (P, \Psi) \Leftrightarrow U \subseteq P$ and $\Phi \subseteq \Psi$.

Proof. Given a pair $(U, \Phi), (P, \Psi) \in \mathcal{I}$. Then we construct a new pair $(U + P, \Phi + \Psi)$. Now we can apply proposition 6.2.5 and find a new pair (S, Λ) such that $\dim S = \dim \Lambda$, $\text{Ann}_S(\Lambda) = \{0\}$, $U, P \subseteq S$ and $\Phi, \Psi \subseteq \Lambda$. In the case that $\dim \Lambda$ is a multiple of the characteristic we can increase the dimensions of both spaces S and Λ by one by extending both bases.

Since all spaces in \mathcal{I} are of finite dimension, all ascending chains in \mathcal{I} withing an element $(U, \Phi) \in \mathcal{I}$ are of finite length.

This proves \mathcal{I} to be a directed set. \square

Now that we have a directed set \mathcal{I} , we can construct a local system for the Lie algebra $\mathfrak{g} = \mathfrak{sl}(V, \Pi)$ indexed by \mathcal{I} . For $i = (U, \Phi) \in \mathcal{I}$ let \mathfrak{g}_i be the Lie subalgebra $\langle t_{w, \psi} : w \in U, \psi \in \Pi, \psi(v) = 0 \rangle \simeq \mathfrak{sl}(U, \Phi)$ of \mathfrak{g} . This defines a local system $(\mathfrak{g}_{i \in \mathcal{I}})$ for \mathfrak{g} . Note that since the dimension of spaces $(U, \Phi) \in \mathcal{I}$ are not divisible by the characteristic of the underlying field, we have that $\mathfrak{g}_i \not\subseteq \mathfrak{psl}(U, \Phi)$. So for all $i \sqsubseteq j$ we have that $\mathfrak{g}_i \subseteq \mathfrak{g}_j$, and as every element in \mathfrak{g} is a finite linear combination of extremal elements each element of \mathfrak{g} is contained in $\bigcup_{i \in \mathcal{I}} \mathfrak{g}_i$. Thus $\mathfrak{g} = \bigcup_{i \in \mathcal{I}} \mathfrak{g}_i$.

Now we have a local system $(\mathfrak{g}_{i \in \mathcal{I}})$ for \mathfrak{g} , it is time to construct an isomorphic local system for \mathfrak{h} .

To do so, we identify the extremal geometries of \mathfrak{h} and of \mathfrak{g} with $\Gamma(V, \Pi)$ and denote them by Γ .

To construct the local system for \mathfrak{h} , we consider for each element $i = (U, \Phi)$ of \mathcal{I} the subgeometry $\Gamma_i \simeq \Gamma(U, \Phi)$ of Γ , where Γ_i consists of pairs $(\langle v \rangle, \langle \phi \rangle)$ where $\langle v \rangle \in \mathbb{P}(U)$ and $\langle \phi \rangle \in \mathbb{P}(\Phi)$.

Let \mathcal{E}_{Γ_i} be the extremal points in the extremal geometry Γ_i of \mathfrak{g}_i . Let the Lie subalgebra \mathfrak{h}_i of \mathfrak{h} be generated by \mathcal{E}_{Γ_i} , that is $\mathfrak{h}_i = \langle \mathcal{E}_{\Gamma_i} \rangle$.

Proposition 6.2.7. The collection of Lie algebras $(\mathfrak{h}_{i \in \mathcal{I}})$ obtained in the above way, is a local system for \mathfrak{h} .

Proof. Let $i, j \in \mathcal{I}$ such that $i \sqsubseteq j$. By construction we have that $\mathfrak{h}_i \subseteq \mathfrak{h}_j$.

We still have to show that \mathfrak{h} is $\bigcup_{i \in \mathcal{I}} \mathfrak{h}_i = \mathfrak{h}$.

But as each extremal point of \mathfrak{h} is in some Γ_i , we find that any element of \mathfrak{h} , as it is a finite sum of extremal elements, is inside some Γ_k . This gives that $(\mathfrak{h}_{i \in \mathcal{I}})$ is a local system for \mathfrak{h} . \square

6.2.2 The characterization

In the previous section, we have constructed two local systems $(\mathfrak{g}_{i \in \mathcal{I}})$ and $(\mathfrak{h}_{i \in \mathcal{I}})$ for \mathfrak{g} and \mathfrak{h} respectively. To conclude that \mathfrak{h} and \mathfrak{g} are isomorphic, we need to show that both local systems are isomorphic. We will do this in two steps. The first one is to show that \mathfrak{g}_i is isomorphic to \mathfrak{h}_i for each $i \in \mathcal{I}$. This is proven in the next proposition.

Proposition 6.2.8. *For each $i \in \mathcal{I}$ we have \mathfrak{h}_i is isomorphic to \mathfrak{g}_i .*

Proof. Let $i = (U, \Phi) \in \mathcal{I}$ be arbitrary. In order to show that \mathfrak{h}_i is isomorphic to \mathfrak{g}_i we first construct a Lie subalgebra \mathfrak{l} of \mathfrak{h}_i , which is isomorphic to $\mathfrak{sl}(U, \Phi) \simeq \mathfrak{g}_i$. After that we show that \mathfrak{l} equals \mathfrak{h}_i .

Since the extremal geometry of \mathfrak{h}_i contains Γ_i (but might be bigger), we can find $\mathcal{B}_U = \{u_1, \dots, u_n\}$, a basis for U and $\mathcal{B}_\Phi = \{\phi_1, \dots, \phi_n\}$, a basis for Φ , such that $\phi_i(u_j) = \delta_{ij}$.

As is shown in Roberts [12, Section 5.1], or Section 5 of [8], we can pick extremal elements corresponding to the points $(\langle u_i \rangle, \langle \phi_j \rangle)$, with $i \neq j$ forming a Chevalley type basis of type A_{n-1} for a subalgebra \mathfrak{l} . As Roberts shows in [12, 5.2.2] or Cuypers, Roberts and Shpectorov in Section 5 of [8], this subalgebra \mathfrak{l} is isomorphic to $\mathfrak{sl}(U, \Phi)$. (Notice, that here we use that the dimension of U is not divisible by the characteristic of the field, so that the center of $\mathfrak{sl}(U, \Phi)$ is trivial.) Thus we have that \mathfrak{h}_i contains a Lie subalgebra \mathfrak{l} that is isomorphic to $\mathfrak{sl}(U, \Phi)$.

Roberts now shows in the proof of Theorem 5.2.15 of [12], see also Section 6 of [8], that \mathfrak{l} contains \mathcal{E}_{Γ_i} and hence $\mathfrak{h}_i = \mathfrak{l}$. This shows that $\mathfrak{h}_i \simeq \mathfrak{sl}(U, \Phi) \simeq \mathfrak{g}_i$. \square

The last step will be shown in the main theorem of this chapter.

Theorem 6.2.9. *The Lie algebra \mathfrak{h} is isomorphic to \mathfrak{g} .*

Proof. In order to prove that \mathfrak{h} is isomorphic to \mathfrak{g} , we need to show that both local systems are isomorphic. Since we have that the local system $(\mathfrak{h}_{i \in \mathcal{I}})$ is element wise isomorphic to $(\mathfrak{g}_{i \in \mathcal{I}})$, we need to show that if $\mathfrak{g}_i \subseteq \mathfrak{g}_k$ and $\mathfrak{h}_i \subseteq \mathfrak{h}_k$, where $\phi_i : \mathfrak{g}_i \xrightarrow{\sim} \mathfrak{h}_i$ and $\phi_k : \mathfrak{g}_k \xrightarrow{\sim} \mathfrak{h}_k$, that $\phi_k|_{\mathfrak{g}_i} = \phi_i$.

Let \mathcal{E}_i and \mathcal{E}_k be the extremal points of \mathfrak{g}_i and \mathfrak{g}_k respectively. Since we have that $\mathfrak{g}_i \subseteq \mathfrak{g}_k$, we have that $\mathcal{E}_i \subseteq \mathcal{E}_k$. This implies that $\Gamma_i \subseteq \Gamma_k$. As we have that \mathfrak{h}_i and \mathfrak{h}_k are generated by the extremal points of Γ_i and Γ_k respectively, we have that $\mathfrak{h}_i \subseteq \mathfrak{h}_k$. Notice that for a $\mathfrak{h}_i, \mathfrak{h}_k, \mathfrak{g}_i$ and \mathfrak{g}_k are generated by extremal elements we have that ϕ_k maps extremal elements of \mathfrak{g}_k to extremal elements \mathfrak{h}_k . If we restrict the Lie algebra isomorphism ϕ_k to \mathfrak{g}_i , it maps the extremal elements of \mathfrak{g}_i to extremal elements of \mathfrak{h}_i one-to-one. As we have that ϕ_i maps the extremal elements of \mathfrak{g}_i isomorphic to the extremal elements of \mathfrak{h}_i , we have that $\phi_k|_{\mathfrak{g}_i} = \phi_i$.

Thus we have that both local systems are isomorphic. So we have that \mathfrak{h} and \mathfrak{g} are isomorphic. \square

Chapter 7

Characterization of $\mathfrak{pfsu}(V)$

Let \mathfrak{g} be a simple Lie algebra over the field \mathbb{F} generated by its extremal elements, whose extremal geometry does not contain lines. Then, as is shown in [7], either \mathfrak{g} is a finitary symplectic Lie algebra, or there exists a quadratic extension \mathbb{K} of the field \mathbb{F} such that the Lie algebra $\mathfrak{g}_{\mathbb{F}} \otimes \mathbb{K}$ is generated by its extremal elements and its extremal geometry does contain lines.

Let V be a vector space over the field \mathbb{K} and h be a skew-Hermitian form on V with respect to some automorphism σ of order 2 of \mathbb{K} , furthermore let $\mathbb{F} = \mathbb{K}_{\sigma} = \{\mu \in \mathbb{K} : \mu^{\sigma} = \mu\}$. Assume V contains singular vectors. In this chapter we provide a characterization of the Lie algebra $\mathfrak{g} = \mathfrak{fsu}(V, h)$ over \mathbb{F} , where h is a nondegenerate skew-Hermitian form on the \mathbb{K} vector space V by its extremal geometry, which has no lines. In this case the Lie algebra $\mathfrak{g} \otimes_{\mathbb{F}} \mathbb{K}$ is isomorphic to $\mathfrak{sl}(V, \Pi)$, where Π is the subspace of V^* generated by the forms $h(-, v)$, where v is singular.

7.1 The finitary special unitary Lie algebra and its extremal geometry

As in the introduction, let V be a vector space over the field \mathbb{K} and suppose h is a skew-Hermitian form on V with respect to some automorphism σ of order 2 of \mathbb{K} . Denote by $\mathbb{F} = \mathbb{K}_{\sigma} = \{\mu \in \mathbb{K} : \mu^{\sigma} = \mu\}$ the subfield of \mathbb{K} fixed by σ . Assume V contains singular vectors.

Let \mathfrak{g} be the finitary special unitary Lie algebra $\mathfrak{su}(V, h)$. As we have seen in section 3.4.15, the Lie algebra \mathfrak{g} is generated by its infinitesimal transvections

$$t_v : V \rightarrow V,$$

where v is singular, defined by

$$t_v(w) = h(w, v)v$$

for all $w \in V$.

Notice that the unitary infinitesimal transvections are extremal. In fact they are the only extremal elements in $\mathfrak{su}(V, h)$. Denote by $\Gamma = \Gamma(\mathfrak{g})$ the extremal geometry of \mathfrak{g} , see proposition 3.4.16.

Lemma 7.1.1. *The extremal geometry Γ of $\mathfrak{f}\mathfrak{su}(V, h)$ does not contains lines, that is $\mathcal{E}_{-1} = \emptyset$.*

Proof. Let $t_v, t_w \in \mathcal{E}$ be arbitrary. If $g(t_v, t_w) = h(v, w) = h(w, v) = 0$ we have that $h(v, w) = 0$. So we have that $[t_v, t_w] = 0$. From proposition 3.5.1, we have that $(t_v, t_w) \in \mathcal{E}_0$.

In the case that $h(v, w) \neq 0$, we have that $(t_v, t_w) \in \mathcal{E}_2$.

This proves that $\mathcal{E}_{-1} = \emptyset$. □

A result from the $\mathcal{E}_{-1} = \emptyset$ is that $\mathcal{E}_1 = \emptyset$.

Define now the Lie algebra

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{K}_\sigma} \mathbb{K},$$

The Lie algebra $\hat{\mathfrak{g}}$ is spanned by $\{x \otimes \lambda : x \in \mathfrak{g}, \lambda \in \mathbb{K}\}$, and its Lie bracket is induced by $[x \otimes \lambda, y \otimes \mu] = [x, y] \otimes (\lambda\mu)$.

We identify an element x from \mathfrak{g} with the element $x \otimes 1$ from $\hat{\mathfrak{g}}$.

Lemma 7.1.2. *Let x be an extremal element of \mathfrak{g} and $\alpha \in \mathbb{K}$, then $x \otimes \alpha$ is extremal in $\hat{\mathfrak{g}}$.*

Proof. For $x \otimes \alpha, y \otimes \beta$, we define $g(x \otimes \alpha, y \otimes \beta) = g(x, y)\alpha\beta$, where g is the extremal form of \mathfrak{g} .

Let $x \in \mathfrak{g}$ be extremal and define $x \otimes \alpha$ with $\alpha \in \mathbb{K}$, then for all pure tensors $y \otimes \beta, z \otimes \gamma \in \hat{\mathfrak{g}}$ we have that for $\text{char } \mathbb{K} \neq 2$ that

$$\begin{aligned} [x \otimes \alpha, [x \otimes \alpha, y \otimes \beta]] &= [x \otimes \alpha, [x, y] \otimes (\alpha\beta)] \\ &= [x, [x, y]] \otimes (\alpha\alpha\beta) \\ &\subseteq \mathbb{K}_\sigma(x \otimes (\alpha\alpha\beta)) \\ &\subseteq \mathbb{K}(x \otimes \alpha) \end{aligned}$$

In case $\text{char } \mathbb{K} = 2$ we have

$$\begin{aligned} [x \otimes \alpha, [x \otimes \alpha, y \otimes \beta]] &= [x, [x, y]] \otimes (\alpha\alpha\beta) \\ &= 2g(x, y)(x \otimes (\alpha\alpha\beta)) \\ &= 2g(x \otimes \alpha, y \otimes \beta)(x \otimes \alpha). \end{aligned}$$

To be extremal in characteristic two, the Premet identities needs to hold as well. So

$$\begin{aligned}
[[x \otimes \alpha, y \otimes \beta], [x \otimes \alpha, z \otimes \gamma]] &= [[x, y], [x, z]] \otimes (\alpha\alpha\beta\gamma) \\
&= (g(x, [y, z])x + g(x, z)[x, y] - g(x, y)[x, z]) \otimes (\alpha\alpha\beta\gamma) \\
&= g(x, [y, z])\alpha\beta\gamma(x \otimes \alpha) + g(x, z)\alpha\gamma[x \otimes \alpha, y \otimes \beta] \\
&\quad - g(x, y)\alpha\beta[x \otimes \alpha, z \otimes \gamma] \\
&= g(x \otimes \alpha, [y \otimes \beta, z \otimes \gamma])(x \otimes \alpha) \\
&\quad + g(x \otimes \alpha, z \otimes \gamma)[x \otimes \alpha, y \otimes \beta] \\
&\quad - g(x \otimes \alpha, y \otimes \beta)[x \otimes \alpha, z \otimes \gamma].
\end{aligned}$$

Furthermore we have

$$\begin{aligned}
[x \otimes \alpha, [y \otimes \beta[x \otimes \alpha, z \otimes \gamma]]] &= [x, [y, [x, z]]] \otimes (\alpha\beta\alpha\gamma) \\
&= (g(x, [y, z])x - g(x, z)[x, y] - g(x, y)[x, z]) \otimes (\alpha\beta\alpha\gamma) \\
&= g(x \otimes \alpha, [y \otimes \beta, z \otimes \gamma])(x \otimes \alpha) \\
&\quad - g(x \otimes \alpha, z \otimes \gamma)[x \otimes \alpha, y \otimes \beta] \\
&\quad - g(x \otimes \alpha, y \otimes \beta)[x \otimes \alpha, z \otimes \gamma].
\end{aligned}$$

This concludes the proof in case of characteristic two. \square

Proposition 7.1.3. $\hat{\mathfrak{g}} \simeq \mathfrak{fs}\mathfrak{l}(V, \Pi)$, where $\Pi = \langle h(-, v) : v \in V \setminus \{0\} \rangle_{\mathbb{K}}$.

Proof. We first show this in case V is of finite dimension. Notice that every element of $\hat{\mathfrak{g}}$ is a finite linear combination of pure tensors $x \otimes \alpha$. As we have that all element of $\hat{\mathfrak{g}}$ are maps on the underlying vector space V , we can define an action on V by $(x \otimes \alpha)(v) = \alpha x(v)$ for all $x \otimes \alpha \in \hat{\mathfrak{g}}$ and $v \in V$. This defines a homomorphism from $\hat{\mathfrak{g}}$ into $\mathfrak{sl}(V)$. Notice that it is sufficient to define the homomorphism on the pure tensors.

So let $\Xi : \hat{\mathfrak{g}} \rightarrow \mathfrak{sl}(V)$ given by $\Xi(x \otimes \alpha) = \alpha x$. Notice that x is a finite linear combination of infinitesimal transvections $t_v \in \mathfrak{g}$. We first show that Ξ is a Lie algebra homomorphism. Let $x \otimes \alpha, y \otimes \beta \in \hat{\mathfrak{g}}$ then

$$\begin{aligned}
[\Xi(x \otimes \alpha), \Xi(y \otimes \beta)] &= [\alpha x, \beta y] \\
&= [x, y](\alpha\beta) \\
&= \Xi([x, y] \otimes (\alpha\beta)).
\end{aligned}$$

Now we show that Ξ is injective. Let $x, y \in \hat{\mathfrak{g}}$ such that $\Xi(x) = \Xi(y)$. As $\hat{\mathfrak{g}}$ is of finite dimension we can write $x = \sum_{i \in I} x_i \otimes \gamma_i$ and $y = \sum_{i \in I} x_i \otimes \beta_i$, with all x_i independent. So we have that

$$\begin{aligned}
\sum_{i \in I} \gamma_i x_i &= \sum_{i \in I} \beta_i x_i \\
\sum_{i \in I} (\gamma_i - \beta_i) x_i &= 0
\end{aligned}$$

This implies that $\gamma_i - \beta_i = 0$, so $\gamma_i = \beta_i$ for all $i \in I$. This shows that $x = y$. So we have that Ξ is injective.

Notice that $\hat{\mathfrak{g}}$ over \mathbb{K} is of dimension $n^2 - 1$ as well $\mathfrak{sl}(V)$. This implies that Ξ is an isomorphism of Lie algebras. Thus $\hat{\mathfrak{g}} \simeq \mathfrak{sl}(V)$.

In the case that V is not of finite dimension, we can create a local system of finite dimensional Lie subalgebras of $\hat{\mathfrak{g}}$, which is element wise equal that $\mathfrak{sl}(W, \Psi)$ for some $W \subseteq V$ and $\Psi \subseteq \Phi$. This can be done along the lines of chapter 6. Details are left to the reader. This gives that in the non finite dimensional case that $\hat{\mathfrak{g}} \simeq \mathfrak{sl}(V, \Pi)$. \square

As we have that $\hat{\mathfrak{g}} \simeq \mathfrak{sl}(V, \Pi)$, it follows directly that $\Gamma(\hat{\mathfrak{g}})$ is isomorphic with the geometry of point-hyperplanes pairs $\Gamma(V, \Pi)$.

Since we have that $\hat{\Gamma} = \Gamma(\hat{\mathfrak{g}})$ is isomorphic to the geometry of point hyperplane pairs $\Gamma(V, \Pi)$, we can embed the extremal points of \mathfrak{g} into $\Gamma(V, \Pi)$. The extremal points $\langle t_w \rangle$ from \mathfrak{g} in $\Gamma(V, \Pi)$ correspond to the points $(\langle w \rangle, \langle \ker h(-, w) \rangle)$ of $\Gamma(V, \Pi)$, which are invariant under the automorphism $\bar{\sigma} : \hat{\Gamma} \rightarrow \hat{\Gamma}$ induced by the map $v \in V \mapsto h(-, v) \in V^*$.

7.2 Proof of the characterization

In this section we provide a proof of Theorem 1.1.2, the characterization of $\mathfrak{pfsu}(V, h)$ by its extremal geometry. Before we start with this proof, we first provide some preliminary definitions and result.

7.2.1 Preliminaries

Definition 7.2.1 (Field extension). *Let \mathbb{F} be a field, and denote the ring of polynomials in the unknown t over this field by $\mathbb{F}[t]$. Let $p(t)$ be an irreducible polynomial of degree n in $\mathbb{F}[t]$. Then $\mathbb{F}[t]/(p(t))$ is called a field extensions of degree n .*

In the case that $p(t)$ is an irreducible quadratic polynomial, we say that $\mathbb{F}[t]/(p(t))$ is a quadratic extension.

Let α be a zero of $p(t)$, then $\mathbb{F}[t]/(p(t))$ is often denoted as $\mathbb{F}[\alpha]$.

When \mathbb{F} is extended quadratically to a field $\mathbb{K} = \mathbb{F}[\alpha]$, we can define a field automorphism σ in a natural way. This is done by defining σ on \mathbb{F} as the identity and $\sigma(\alpha) = \bar{\alpha}$, where $\bar{\alpha}$ is the other zero of $p(t)$. Note that $\alpha \neq \bar{\alpha}$, otherwise $-2\alpha \in \mathbb{F}$ and thus $\alpha \in \mathbb{F}$. This contradicts that $p(t)$ is irreducible over \mathbb{F} .

Definition 7.2.2. *Let $\pi : \mathbb{P}(V) \rightarrow \{\text{Hyperplanes of } V\}$. We call π a polarity if for all points $p, q \in \mathbb{P}(V)$, we have*

$$q \in \pi(p) \Leftrightarrow p \in \pi(q)$$

Definition 7.2.3 (Definition 7.1.9 of [1]). *A map π on a projective space \mathbb{P} sending each point x of \mathbb{P} to either a hyperplane of \mathbb{P} is called a quasi-polarity if it satisfies the following property for all points x, y of \mathbb{P}*

$$x \in \pi(y) \Rightarrow y \in \pi(x).$$

The absolute with respect to a quasi polarity π , is the line space whose points are projective points x with $x \in \pi(x)$ and whose lines are the projective lines l such that $l \subseteq \pi(x)$. for each $x \in l$.

Corollary 7.2.4 (Corollary 10.1.9 of [1]). *Let $\dim V \geq 1$. Every quasi-polarity on the projective space $\mathbb{P}(V)$ whose absolute spans \mathbb{P} falls into one of the three following mutually exclusive cases.*

- (1) *A quasi-polarity determined by a nonzero alternating form over a field.*
- (2) *A quasi-polarity determined by a symmetric bilinear form over a field of characteristic distinct from two.*
- (3) *A quasi-polarity determined by a nonzero σ -Hermitian form such that $\sigma \neq id$.*

7.2.2 Proof of the characterization

We start with our assumptions for the remainder of this section:

Assumption 7.2.5. *Let \mathfrak{g} be a simple Lie algebra over a field \mathbb{F} generated by extremal elements, such that for $x, y \in \mathcal{E}$ the following holds*

$$\begin{aligned} x &= y; \\ (x, y) &\in \mathcal{E}_0; \\ (x, y) &\in \mathcal{E}_2. \end{aligned}$$

Assume that \mathbb{F} can be quadratically extended to a field say \mathbb{K} , such that the Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{F}} \mathbb{K}$, and the geometry $\Gamma(\hat{\mathfrak{g}})$ is isomorphic to the geometry of point-hyperplane pairs $\Gamma(V, \Pi)$, for some (V, Π) , where V has dimension at least 3 and Π is a subspace of V^ with $\text{Ann}_V(\Pi) = \{0\}$.*

It is our goal to show that there is a Hermitian form h on V such that the Lie algebra \mathfrak{g} is isomorphic to $\mathfrak{su}(V, h)$.

As we have that \mathbb{K} is a quadratic extension of \mathbb{F} , there exists a quadratic monic irreducible polynomial $p(t) \in \mathbb{F}[t]$ such that $\mathbb{K} = \mathbb{F}[t]/(p(t))$.

Let α_1 and α_2 be the zeros of $p(t)$, then we have that $\alpha_1, \alpha_2 \notin \mathbb{F}$.

Define now the field automorphism $\sigma : \mathbb{K} \rightarrow \mathbb{K}$ such that $\sigma(\alpha_1) = \alpha_2$ and $\sigma_{\mathbb{F}} = id$.

Lemma 7.2.6. *The map*

$$\begin{aligned}\hat{\sigma} : \hat{\mathfrak{g}} &\rightarrow \hat{\mathfrak{g}} \\ x \otimes \alpha &\mapsto x \otimes \sigma(\alpha)\end{aligned}$$

is a Lie algebra automorphism. Furthermore $\text{Fix}(\hat{\sigma}) = \mathfrak{g} \otimes 1$.

Proof. To prove that $\hat{\sigma}$ is a homomorphism, it is sufficient to show that $\hat{\sigma}$ is a Lie algebra homomorphism for the pure tensors.

Let $x \otimes \alpha, y \otimes \beta \in \hat{\mathfrak{g}}$ then

$$\begin{aligned}[\hat{\sigma}(x \otimes \alpha), \hat{\sigma}(y \otimes \beta)] &= [x \otimes \sigma(\alpha), y \otimes \sigma(\beta)] \\ &= [x, y] \otimes (\sigma(\alpha)\sigma(\beta)) \\ &= \hat{\sigma}([x, y] \otimes (\alpha\beta)).\end{aligned}$$

This shows that $\hat{\sigma}$ is a Lie algebra homomorphism.

To show that $\text{Fix}(\hat{\sigma}) = \mathfrak{g} \otimes 1$, it is sufficient to show it for $x \otimes \alpha$ and $x \otimes 1 + y \otimes \beta$ of $\hat{\mathfrak{g}}$, with x and y linear independent over \mathbb{K}_σ . Notice that $\mathfrak{g} \otimes 1 \subseteq \text{Fix}(\hat{\sigma})$. Let $x \otimes \alpha \in \text{Fix}(\hat{\mathfrak{g}})$ then $x \otimes \alpha = \hat{\sigma}(x \otimes \alpha) = x \otimes \sigma(\alpha)$. This implies that $\sigma(\alpha) = \alpha$, and this $\alpha \in \mathbb{F}$. So $x \otimes \alpha \in \mathfrak{g} \otimes 1$.

Now let $x \otimes 1 + y \otimes \beta \in \text{Fix}(\hat{\sigma})$, with x and y linear independent over \mathbb{K}_σ . Then

$$\begin{aligned}x \otimes 1 + y \otimes \beta &= \hat{\sigma}(x \otimes 1 + y \otimes \beta) \\ &= \hat{\sigma}(x \otimes 1) + \hat{\sigma}(y \otimes \beta) \\ &= x \otimes 1 + y \otimes \sigma(\beta).\end{aligned}$$

From this we conclude that $\beta = \sigma(\beta)$, so $\beta \in \mathbb{K}_\sigma$. This means that $x \otimes 1 + y \otimes \beta \in \mathfrak{g} \otimes 1$. Thus we have that $\text{Fix}(\hat{\sigma}) = \mathfrak{g} \otimes 1$. \square

Lemma 7.2.7. *For all $x \in \hat{\mathfrak{g}}$ we have $\hat{\sigma}^2(x) = x$.*

Proof. Notice that since ever $x \in \hat{\mathfrak{g}}$ is a finite linear combination of pure tensors, it is enough to show that $\hat{\sigma}^2(x) = x$ for all pure tensors x .

So let $x \otimes \alpha \in \hat{\mathfrak{g}}$ then we have that

$$\hat{\sigma}^2(x \otimes \alpha) = \hat{\sigma}(x \otimes \sigma(\alpha)) = x \otimes \sigma^2(\alpha) = x \otimes \alpha.$$

\square

We denote the set of extremal elements of $\hat{\mathfrak{g}}$ by \hat{E} .

Lemma 7.2.8. *Let $x \in \hat{E}$, then $\hat{\sigma}(x) \in \hat{E}$.*

Proof. This is a direct consequence of the fact that $\hat{\sigma}$ is an automorphism. \square

Since $\hat{\sigma}$ is an automorphism we have that

$$(x, y) \in \hat{\mathcal{E}}_i \iff (\hat{\sigma}(x), \hat{\sigma}(y)) \in \hat{\mathcal{E}}_i.$$

This implies that $\hat{\sigma}$ induces an automorphism $\bar{\sigma}$ of the extremal geometry $\Gamma(\hat{\mathfrak{g}})$ of $\hat{\mathfrak{g}}$, leaving the extremal points of $\hat{\mathfrak{g}}$ invariant.

As we assumed that the extremal geometry of $\hat{\mathfrak{g}}$ is isomorphic with the extremal geometry of point-hyperplanes $\Gamma(V, \Pi)$, we have the following commuting diagram

$$\begin{array}{ccc} \Gamma(\hat{\mathfrak{g}}) & \xrightarrow{\hat{\sigma}} & \Gamma(\hat{\mathfrak{g}}) \\ \downarrow \simeq & & \downarrow \simeq \\ \Gamma(V, \Pi) & \xrightarrow{\bar{\sigma}} & \Gamma(V, \Pi) \end{array}$$

where $\bar{\sigma}$ is uniquely determined by $\hat{\sigma}$.

Now in $\Gamma(V, \Pi)$, we can recover points of $\mathbb{P}(V)$ and hyperplanes in $\mathbb{P}(\Pi)$ in the following way

$$\begin{aligned} \text{a point } p \in \mathbb{P}(V) &\longleftrightarrow \Gamma_p = \{(p, \pi) : \forall \pi, \text{ with } p \in \pi\} \\ \text{a hyperplane } \pi \in \mathbb{P}(\Pi) &\longleftrightarrow \Gamma_\pi = \{(p, \pi) : \forall p, \text{ with } p \in \pi\} \end{aligned}$$

Lemma 7.2.9. *Let $p, q \in \mathbb{P}(V)$ and $\pi, \omega \in \mathbb{P}(\Pi)$. Then we have*

- (1) $p = q \iff \Gamma_p = \Gamma_q$
- (2) $\pi = \omega \iff \Gamma_\pi = \Gamma_\omega$

Proof. Let $p, q \in \mathbb{P}(V)$ and $\pi, \omega \in \mathbb{P}(\Pi)$ be arbitrary then.

- (1) If $p = q$ then it is clear that $\Gamma_p = \Gamma_q$. So assume that $\Gamma_p = \Gamma_q$. Then we have that all the hyperplanes that contains p contains also q , and all the hyperplanes that contains q contains p . This only occurs when $p = q$.
- (2) If $\pi = \omega$ then it is clear that $\Gamma_\pi = \Gamma_\omega$. So assume that $\Gamma_\pi = \Gamma_\omega$. Then we have that all points $r \in \pi$ are contained also in ω , and all points in ω are contained in π . This only occurs when $\pi = \omega$. \square

In the next lemma we list some properties:

Properties 7.2.10. (1) *Let p be a point and π a hyperplane of $\mathbb{P}(V)$ then*

$$p \in \pi \iff \Gamma_p \cap \Gamma_\pi \neq \emptyset$$

(2) *If X is a maximal clique in the collinearity graph of $\Gamma(V, \Pi)$, then $X = \Gamma_p$ or $X = \Gamma_\pi$ for some p or π .*

(3) *The subsets Γ_p for $p \in \mathbb{P}(V)$ partition Γ .*

(4) The subsets Γ_π for $\pi \in \mathbb{P}(\Pi)$ partition Γ .

On the maximal cliques in Γ we define an equivalence relation.

Definition 7.2.11. Given two maximal cliques M_1 and M_2 of Γ , then we define the relation \sim as follows

$$M_1 \sim M_2 \iff \begin{cases} M_1 \cap M_2 = \emptyset \text{ or } M_1 = M_2 \text{ and} \\ \exists \text{ Maximal clique } M_3 \text{ of } \Gamma \text{ such that } M_1 \cap M_3 \neq \emptyset \text{ and } M_2 \cap M_3 \neq \emptyset \end{cases}$$

Lemma 7.2.12. The relation \sim from definition 7.2.11 is an equivalence relation.

Proof. Let M_1 and M_2 be two maximal cliques. Then by the previous lemma, we have one of the following situations (up to permutation of 1 and 2 and of points and hyperplanes): $M_1 = \Gamma_p$ and $M_2 = \Gamma_q$ for some points p and q in $\mathbb{P}(V)$,

or $M_1 = \Gamma_p$ for some point $p \in \mathbb{P}(V)$ and $M_2 = \Gamma_\pi$ for some $\pi \in \mathbb{P}(\Pi)$.

In the first case take a hyperplane $\rho \in \mathbb{P}(\Pi)$ on p, q . Let M_3 be the maximal clique Γ_ρ . Then $M_1 \cap M_2 = \emptyset$ or $M_1 = M_2$ and $M_3 \cap M_1 \neq \emptyset \neq M_2 \cap M_3$.

In the second case, we can have that $p \in \pi$ and then $M_1 \cap M_2 \neq \emptyset$ or $p \notin \pi$ and then any maximal clique M_3 meeting M_1 nontrivially is of the form $M_3 = \Gamma_\rho$ and hence does not meet M_2 .

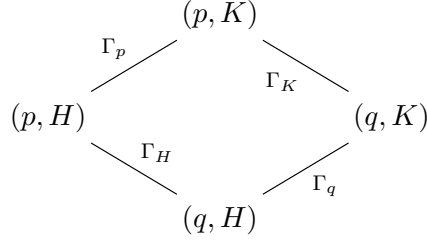
This shows that $M_1 \sim M_2$ if and only if M_1 and M_2 are maximal cliques both related to a point or both related to a hyperplane of $\mathbb{P}(V)$. In particular, \sim is an equivalence relation. \square

The two classes correspond to maximal cliques of "points" or of "hyperplane" type.

Lemma 7.2.13. If p is a point in $\mathbb{P}(V)$, then the automorphism $\bar{\sigma}$ maps Γ_p to Γ_π for some $\pi \in \mathbb{P}(\Pi)$.

If π is a hyperplane in $\mathbb{P}(V)$, then the automorphism $\bar{\sigma}$ maps Γ_π to Γ_p for some $p \in \mathbb{P}(V)$.

Proof. Let $(x, y) \in \mathcal{E}_0(\mathfrak{g})$. Then we have that x and y corresponds with (p, H) and (q, K) with $p \in H$ and $q \in K$ in $\Gamma(V, \Pi)$ respectively. This means that x and y have two unique neighbours v and w in $\Gamma(\hat{\mathfrak{g}})$. So (p, H) and (q, K) have two unique neighbours, (p, K) corresponding with v and (q, H) corresponding with w in $\Gamma(V, \Pi)$. This leads to the following diagram



As we have that $x, y \in \mathfrak{g}$, we must have that $\hat{\sigma}(v) \neq v$, because $\mathcal{E}_{-1} = \emptyset$. As we have that $(x, y) \in \hat{\mathcal{E}}_i \iff (\hat{\sigma}(x), \hat{\sigma}(y)) \in \hat{\mathcal{E}}_i$ we must have that $\hat{\sigma}(v) = w$. This implies that $\bar{\sigma}(\Gamma_p) = \Gamma_H$, and $\bar{\sigma}(\Gamma_H) = \Gamma_p$.

Since $\bar{\sigma}$ is an automorphism, it means that equivalence relations are preserved. This implies that for every point-hyperplane pair (p, π) , we have that $\bar{\sigma}(\Gamma_p) = \Gamma_\pi$ and $\bar{\sigma}(\Gamma_\pi) = \Gamma_q$, for some $\pi \in \mathbb{P}(\Pi)$ and some $p, q \in \mathbb{P}(V)$. This implies that points are mapped to hyperplanes and hyperplanes to points. \square

As we have that $\hat{\sigma}$ is of order two, we have that $\bar{\sigma}$ is of order two.

Proposition 7.2.14. *The map $\bar{\sigma}$ induces a polarity on $\mathbb{P}(V)$ to $\mathbb{P}(\Pi)$.*

Proof. We need to show that for all $p, q \in \mathbb{P}(V)$ that

$$\bar{\sigma}(\Gamma_p) = \Gamma_\pi \ni q \iff \bar{\sigma}(\Gamma_q) = \Gamma_\omega \ni p.$$

As we have that $\bar{\sigma}$ is an automorphism and thus it preserves the equivalence relations described above, it is sufficient to show that for all $x, y \in \mathcal{E}_0$ with x corresponding with (p, H) and y with (q, K) such that $q \in \Gamma_H$.

Notice that we have that $\bar{\sigma}(\Gamma_p) = \Gamma_H$. Since $q \in \Gamma_H$, we have that $q \in H$. This implies that $\bar{\sigma}(\Gamma_q) = \Gamma_H$. As we have that $p \in H$ we have that $p \in \Gamma_H$. This implies that $\bar{\sigma}$ induces a polarity $\psi : \mathbb{P}(V) \rightarrow \mathbb{P}(\Pi)$. \square

As we have that ψ is a polarity on $\mathbb{P}(V)$ to $\mathbb{P}(\Pi)$, it is also a quasi-polarity. As we have that $\bar{\sigma}$ maps maximal cliques to maximal cliques corresponding to points and hyperplanes, we also have that the absolute points of the quasi-polarity spans $\mathbb{P}(V)$.

Proposition 7.2.15. *Let $\bar{\sigma}$ as before, then ψ is a unitary quasi-polarity.*

Proof. It is sufficient to show that in case $\hat{\mathfrak{g}} = \langle x, y, z \rangle$, we have that ψ can not be an alternating or symmetric polarity.

Assume that ψ is an alternating polarity. Then it is determined by a nonzero alternating form f on V .

As we have that the geometry of $\hat{\mathfrak{g}}$ is isomorphic to $\Gamma(V, \Pi)$ for some vector space V and $\Pi \subseteq V^*$ such that $\text{Ann}_V(\Pi) = \{0\}$, we can identify x with $v \otimes f(-, v)$, y with $w \otimes f(-, w)$ and z with $u \otimes f(-, u)$. But then we have that $\hat{\mathfrak{g}} \simeq \mathfrak{sp}(V, f) \not\simeq \mathfrak{sl}(V, \Pi)$. We have a similar result in case that ψ is a symmetric polarity. This means that ψ must be a unitary polarity. \square

As we have that $\bar{\sigma}$ induces a unitary polarity, we have by corollary 7.2.4, that the polarity is induced by a Hermitian sesquilinear form h .

Theorem 7.2.16. *Let \mathfrak{g} be a Lie algebra satisfying assumptions 7.2.5. Then \mathfrak{g} is isomorphic to the $\mathfrak{pfsu}(V, h)$, for some vector space V with a Hermitian form h .*

Proof. We have that the quasi-polarity ψ induced by $\bar{\sigma}$ is induced by a Hermitian sesquilinear form h due to corollary 7.2.4.

As we have that $\Gamma(\hat{\mathfrak{g}}) \simeq \Gamma(V, \Pi)$ for some vector space V and $\Pi \subseteq V^*$ such that $\text{Ann}_V(\Pi) = \{0\}$, we have that $\hat{\mathfrak{g}}$ is, up to its center, isomorphic to $\mathfrak{pfs}\mathfrak{l}(V, \Pi)$. Then the extremal elements fixed by $\hat{\sigma}$ are those elements $\alpha(v \otimes h(-, v))$ with $\alpha \in \mathbb{F}$. These extremal elements are exactly the elements that generate the Lie subalgebra of $\hat{\mathfrak{g}} \simeq \mathfrak{sl}(V, \Pi)$ isomorphic $\mathfrak{pfsu}(V, h)$. This concludes that the Lie algebra \mathfrak{g} is isomorphic to $\mathfrak{pfsu}(V, h)$. \square

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