RESEARCH ARTICLE

Beyond Equidistant Sampling for Performance and Cost: A Loop-Shaping Approach Applied to a Motion System

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Summary
Non-equidistant sampling potentially enhances the performance/cost trade-off that is present in traditional equidistant sampling schemes. The aim of this paper is to develop a systematic feedback control design approach for systems that go beyond equidistant sampling. A loop-shaping design framework for such non-equidistantly sampled systems is developed that addresses both stability and performance. The framework only requires frequency response function measurements of the LTI system, while it appropriately addresses the LPTV behavior introduced by the non-equidistant sampling. Experimental validation on a motion system demonstrates the superiority of the design framework for non-equidistantly sampled systems compared to traditional designs that rely on equidistant sampling.

KEYWORDS:
linear periodically time-varying, loop-shaping, non-equidistant sampling, feedback control, frequency response functions, Nyquist test

1 | INTRODUCTION

Digital implementations of motion controllers provide a large design flexibility at a low cost1. Most of the digital implementations are based on fixed, equidistant sampling schemes. Such schemes are favorable from a control design perspective since time invariance of continuous-time systems is preserved. In particular, for linear time-invariant (LTI) systems, equidistant sampling allows the use of frequency-domain control design approaches, including the use of Bode plots and Nyquist diagrams2.

From the perspective of cost-effective and high performance control design, flexible sampling is preferred over fixed sampling. Nowadays, digital controllers are often embedded in software and task scheduling policies allocate resources to the different software applications. The scheduling is often periodic and generally leads to periodic, non-equidistant sampling of the individual applications. Due to the periodicity, equidistant sampling can always be obtained by simply discarding part of the sampling instances. However, such an approach goes at the expense of the achievable performance since not all data and decision variables are exploited. Flexible sampling, including non-equidistant sampling, is preferred since it allows to exploit all available data and decision variables with identical hardware cost and thereby improve the performance/cost trade-off compared to fixed sampling. Examples of flexible sampling include non-equidistant sampling3,4, multirate control5,6,7,8, and sparse control9.

Flexible sampling potentially improves the performance/cost trade-off, but is challenging from a control design perspective. In particular, flexible sampling of continuous-time LTI systems leads to linear periodically time-varying (LPTV) behavior3. Hence, typical frequency-domain control design techniques are not directly applicable. Most control designs for LPTV systems require a parametric model of the system, including pole placement10,11, linear quadratic regulator (LQR) control, linear quadratic Gaussian (LQG) control, $H_2/H_\infty$ approaches12,13,14, internal model principle15, and LTI approximations16. Also, designs based
on time-invariant reformulations are often based on parametric models, including Floquet-Lyapunov transformations\textsuperscript{17}, Section 1.2 and lifting approaches\textsuperscript{17}, Section 1.6, which enable the use of full state feedback\textsuperscript{18}, pole placement\textsuperscript{19}, model matching\textsuperscript{20}, LQR\textsuperscript{21}, LQG\textsuperscript{22}, and $H_\infty$ control\textsuperscript{23,24}. However, as is argued in\textsuperscript{25}, despite the availability of solid control theory, such model-based designs are demanding since (i) obtaining a parametric LPTV model is difficult; and (ii) typical LTI interpretations are not valid, leading to complications for the actual design\textsuperscript{26,27}.

Although non-equidistant sampling has a large potential and the underlying theory has been substantially developed, at present there is a lack of suitable control design techniques to address stability, performance, and robustness. The aim of this paper is to develop a non-parametric loop-shaping control design framework for non-equidistantly sampled systems based on frequency response function (FRF) measurements. Such a framework is well-developed for traditional equidistantly sampled, single-variable systems\textsuperscript{2}, Section 2.6;\textsuperscript{28,29} Chapter 6. The presented framework builds on the multirate approach in\textsuperscript{30,25} exploits \textit{w}-plane loop-shaping\textsuperscript{25}, Section 5.1, explicitly incorporates time-varying aspects, and addresses key objectives such as stability and performance.

The main contribution of this paper is a framework for LPTV loop-shaping feedback control design based on FRF measurements, which enables to exploit non-equidistant sampling for improved control performance. This paper has the following contributions: (I) development of a suitable stability test: an FRF measurement based Nyquist test for LPTV systems; (II) quantification of performance: LPTV generalizations of FRFs for non-equidistantly sampled systems; (III) design through loop-shaping: systematic framework based on FRF measurements and LTI insights; (IV) application of the design framework to a motion system demonstrating the potential of non-equidistant sampling; and (V) validation of the designed controllers through experiments. Preliminary results are reported in\textsuperscript{31}. The present paper contains substantial original contributions including proof of the Nyquist stability test (Contribution (I)), step-by-step LPTV loop-shaping design procedures (Contribution (III)), improved and more detailed control designs (Contribution (IV)), and experimental validation of the control designs (Contribution (IV)).

The outline of this paper is as follows. In Section 2, the potential of non-equidistant sampling is demonstrated via an illustrative example and the control objective is formulated. The Nyquist stability test for non-equidistantly sampled systems (Contribution (I)) is presented in Section 3. The performance quantification based on FRFs (Contribution (II)) is presented in Section 4. The loop-shaping design (Contribution (III)) is presented in Section 5. Application of the design framework to a motion system (Contribution (IV)) is presented in Section 6. In Section 7, the designed controllers are validated in experiments (Contribution (V)). Conclusions are presented in Section 8.

**Notation.** For notation convenience, single-input, single-output (SISO) systems are considered. The results can directly be generalized to multivariable systems. Let $\lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leq x\}$. Lifted variables are underlined, with $I_n$ the $n \times n$ identity matrix and 0 the zero matrix of suitable dimensions.

## 2 NON-EQUIDISTANT SAMPLING IN MOTION CONTROL

In this section, the potential of non-equidistant sampling in motion control applications is explored and the control objective is defined.

### 2.1 Non-equidistant sampling for cost-effective embedded implementations

Multiple software applications are often embedded on a single platform to reduce the overall implementation cost. An example of such a platform is CompSOC\textsuperscript{32}. A scheduling policy is used to allocate the platform resources to the different applications. The scheduling is often periodic and typically results in periodic, non-equidistant sampling of the individual applications as is illustrated in Figure 1.

Non-equidistant sampling introduces time variance, also for underlying time-invariant continuous-time dynamics, which poses challenges from a control design perspective. In particular, periodic, non-equidistant sampling of a linear time-invariant (LTI) system results in linear periodically time-varying (LPTV) behavior, see Section 2.3.

### 2.2 Exploiting non-equidistant sampling in control design

The potential of non-equidistant sampling in control is illustrated via the example in Figure 2. The figure shows a continuous-time sine wave with frequency $\frac{3}{8}$ Hz. The digital controller only has access to the non-equidistantly sampled signal. Control techniques
FIGURE 1 Example of resource allocation to a motion control application (●) and other applications (□). The communication (actuation and sensing) to the motion application at the end of each interval results in periodic (●) non-equidistant sampling of the motion system (●).

FIGURE 2 Example demonstrating the potential of non-equidistant sampling in control. The underlying continuous-time signal (—) is sampled non-equidistantly (○) with period 4s. Equidistant sampling (+) is obtained by discarding part of the samples, which obstructs reconstruction of the true continuous-time signal. Instead, the aliased signal (---) is observed, posing severe limitations for control. Control for the non-equidistant sampling sequence (○) has the potential to enhance the performance since the continuous-time signal can be reconstructed.

The typical way for traditional LTI control designs is to use the equidistant sampling sequence with the highest sampling frequency, i.e., $\frac{1}{2}$ Hz for the example in Figure 2. Clearly, such a design does not exploit all available data, which may yield suboptimal performance. In fact, for the example in Figure 2, aliasing occurs and a sine wave with frequency $\frac{1}{8}$ Hz instead of $\frac{3}{8}$ Hz can be observed. This poses substantial performance limitations for continuous-time performance. Indeed, typical LTI control designs may improve on-sample behavior, but often degrade intersample behavior. This observation is corroborated by experiments in Section 7.

In the proposed approach, the control design is explicitly based on the non-equidistant sampling sequence. Such an approach exploits all available data and design freedom and therefore has the potential to outperform traditional LTI control. The experiments in Section 7 confirm that control design on the non-equidistant rate is superior to LTI control on the equidistant rate in situations similar to that illustrated in Figure 2.

2.3 Non-equidistant control architecture

In this paper, the focus is on feedback control of non-equidistantly sampled LTI motion applications according to the control diagram in Figure 3. The following definitions are adopted.

Definition 1 (Linear system). Let $y_1 = Hu_1$ and $y_2 = Hu_2$, then $H$ is linear if $ay_1 + \beta y_2 = H(au_1 + \beta u_2)$, for all $a, \beta \in \mathbb{R}$.

Definition 2 (LPTV system). A system $H$ is LPTV with period $\tau \in \mathbb{N}$ if it is linear (Definition 1) and it commutes with the delay operator $D_\tau$ defined by $D_\tau u[k] = u[k - \tau]$, i.e., $D_\tau H = HD_\tau$.

Definition 3 (LTI system). A system $H$ is LTI if it is LPTV (Definition 2) with period $\tau = 1$.

The following two assumptions are made.

Assumption 1. The discrete-time system $G_{b,d}$ in Figure 3 is LTI (Definition 3).
Assumption 2. The base rate sampling sequence is given by

$$\Delta_b := (\delta_b, \delta_b, \ldots),$$  \hspace{1cm} (1)

with $\delta_b \in \mathbb{R}_{>0}$ and only available for dedicated identification experiments and performance evaluation, and not available for control. The non-equidistant sampling sequence with periodicity $r \in \mathbb{N}$ available for control is given by

$$\Delta_{ne} := (\delta_1, \delta_2, \ldots, \delta_r, \delta_1, \delta_2, \ldots, \delta_r, \ldots),$$  \hspace{1cm} (2)

with $\delta_i = \gamma_i \delta_b, \gamma_i \in \mathbb{N}, 1 \leq i \leq r$, and is defined by $\Gamma_{ne} := (\gamma_1, \gamma_2, \ldots, \gamma_r) \in \mathbb{N}^r$.

A key observation is that the non-equidistant sampling sequence $\Delta_{ne}$ in Figure 3 introduces periodic time-varying behavior. In particular, by Assumption 2, $\Delta_b$ has periodicity 1 and period time $\delta_b$, and $\Delta_{ne}$ has periodicity $\tau$ and period time $\sum_{i=1}^{\tau} \delta_i = T \delta_b$, with $T := \sum_{i=1}^{\tau} \gamma_i$. Hence, for linear controllers $C_d$, the system in Figure 3 is LPTV (Definition 2) with period time $T \delta_b$.

Traditional LTI approaches typically use the equidistant sampling sequence with the highest possible sampling frequency as given by Assumption 2. Note that by periodicity of $\Delta_{ne}$ in (2) such a sequence always exists since $\delta_{eq} \leq T \delta_b$. The sampling sequences are illustrated by Example 1.

**Definition 4.** Given Assumption 2, let $\Gamma_{ne} = \{ \sum_{i=1}^{\gamma_i} | 1 \leq j \leq r \}$, then the equidistant sampling sequence is defined as

$$\Delta_{eq} := (\delta_{eq}, \delta_{eq}, \ldots),$$  \hspace{1cm} (3)

where $\delta_{eq} = \gamma_{eq} \delta_b$, with

$$\gamma_{eq} := \min \{ \gamma \in \mathbb{N} | \forall 1 \leq n \leq \frac{T}{\gamma}, n \gamma \in \Gamma_{ne} \},$$  \hspace{1cm} (4)

and is defined by $\Gamma_{eq} = \gamma_{eq} \in \mathbb{N}$.

**Example 1.** The sampling sequences for the example in Figure 2 are illustrated in Figure 4. The non-equidistant sampling sequence has periodicity $r = 3$ with $\delta_1 = \delta_2 = 1 s$ and $\delta_3 = 2 s$. Let $\delta_b = 1 s$, then $\Gamma_{ne} = [1 \ 1 \ 2]$, i.e., $\gamma_1 = \gamma_2 = 1, \gamma_3 = 2$, and $T = 4$. By Definition 4, $\Gamma_{ne} = \{1, 2, 4\}$. For $\gamma = 1, \frac{T}{\gamma} = 4$, but $n \gamma \notin \Gamma_{ne}$ for $n = 3$. For $\gamma = 2, \frac{T}{\gamma} = 2$ and $n \gamma \in \Gamma_{ne}$ for $n = 1, 2$, hence $\Gamma_{eq} = 2$.

In the next section, the control objective is presented.

### 2.4 Control objective

The control objective considered in this paper is given as follows.
**Main problem.** Let the control diagram in Figure 3 and a frequency response function measurement $G_{b,d}(e^{j\omega_b})$ be given, and let Assumption 1 and Assumption 2 be satisfied. Design a feedback controller $C_d$ that provides

(A) robust stability, and

(B) robust performance in terms of $\epsilon_b$,

with robust stability and performance according to Section 6.

In this paper, the feedback control design is based on loop-shaping techniques since these are directly applicable to FRF measurements, which are fast, accurate, and inexpensive to obtain for motion systems, in contrast to parametric identification methods. The key challenge in this paper is that conventional loop-shaping for LTI systems is performed in the frequency domain, whereas the non-equidistantly sampled systems considered in this paper are time-varying. In this paper, the frequency-domain insights for LTI systems are generalized to non-equidistantly sampled systems.

The stability and performance aspects are addressed in Section 3 and Section 4, respectively. The loop-shaping design framework is presented in Section 5. Application and experimental validation of the framework is presented in Section 6 and Section 7, respectively.

**Remark 1.** Although $C_d$ in Figure 3 uses the signal $\epsilon$ operating on the non-equidistant sampling sequence $\Delta_{ne}$, the loop-shaping nominal performance goal of this paper addresses the fictitious signal $\epsilon_b$ operating on sampling sequence $\Delta_b$ to also take intersample behavior into account. This is also illustrated in Section 7.

**Remark 2.** The FRF measurement $G_{b,d}(e^{j\omega_b})$ is assumed to be sufficiently dense in view of integral behavior.

## 3 STABILITY: NYQUIST TEST FOR LPTV SYSTEMS

In this section, a stability test for the closed-loop system in Figure 3 is presented, which addresses subproblem (A). The proposed stability test is a Nyquist stability test for LPTV systems based on FRF measurements and constitutes Contribution (1).

### 3.1 LPTV stability

Consider the LPTV open-loop transfer function $L_{b,d} = G_{b,d}H_{c,d}D$ in Figure 3 and assume that there are no pole/zero cancellations. Then, internal closed-loop stability in Figure 3 is equivalent to stability of $S_{b,d} = (1 + L_{b,d})^{-1}$, see also Section 5.3. Let $S_{b,d}(z) = (A[i], B[i], C[i], D[i]), i = 1, 2, \ldots, T$, then closed-loop stability can be directly analyzed based on the monodromy matrix of $S_{b,d}$ given by

$$\Psi = A[0][A[1] \ldots A[T]].$$

(5)

More specific, the closed-loop system is stable if and only if

$$|\lambda_i(\Psi)| < 1, \text{ for all } i,$$

(6)

where the eigenvalues $\lambda_i(\Psi)$ are the roots of the characteristic polynomial

$$\phi(z) = \det(zI - \Psi).$$

(7)

Condition (6) provides a stability test for parametric models based on (7). However, there is no parametric model of $G_{b,d}$ available, see Section 2.4. Therefore, a Nyquist stability test based on FRF measurement $G_{b,d}(e^{j\omega_b})$ is proposed instead.

### 3.2 Towards a Nyquist stability test for $\phi(z)$

Nyquist stability tests for LTI systems are not directly applicable to the LPTV system in Figure 3 due to the time-varying behavior. The main idea is to connect the characteristic polynomial $\phi(z)$ in (7) to a Nyquist stability test. This is achieved through lifting of which preliminary results are presented in this section.
Let $u[k] \in \mathbb{R}$ and

$$u[k] = \begin{bmatrix} u[kT] \\ u[kT + 1] \\ \vdots \\ u[kT + T - 1] \end{bmatrix} \in \mathbb{R}^T,$$

with $T \in \mathbb{N}$. The lifting operator $\mathcal{L}_T$ is defined to be the map $u \mapsto u$, with inverse given by $u = \mathcal{L}_T^{-1}u$. Let $y = Hu$ with a linear system $\mathcal{H}$, then $y = \mathcal{L}_T y = (\mathcal{L}_T H \mathcal{L}_T^{-1})(\mathcal{L}_T u) = H u$ with lifted system $\mathcal{H} = \mathcal{L}_T H \mathcal{L}_T^{-1}$.

Lifted controller $C_d$ is given by Lemma 1 and obtained by lifting the LPTV state-space controller $C_d$ operating on sampling sequence $\Delta_{ne}$ in (2) over period $T\delta_b$, which corresponds to lifting over $r$ samples. For a proof see [17] Section 6.2.3.

**Lemma 1 (Lifting $C_d$).** The $r \in \mathbb{N}$ periodic state-space controller $C_d(z) = (A[k], B[k])$, $D[k]$, $k = 0, 1, 2, \ldots$, with $A[k+r] = A[k], B[k+r] = B[k], C[k+r] = C[k], D[k+r] = D[k]$, lifted over $r$ samples is given by $C_d(z) = \mathcal{L}_r C_d \mathcal{L}_r^{-1} \in \mathbb{C}^{nr \times r}$,

$$C_d(z) = \begin{bmatrix} \Psi \\ \Phi_{r,2} B[1] \\ \vdots \\ \cdots \\ \Phi_{r,1} \end{bmatrix} = \begin{bmatrix} C[1] \\ D[1] \\ 0 \\ \vdots \\ \cdots \\ 0 \end{bmatrix},$$

with transition matrix

$$\Phi_{k_2,k_1} = \begin{cases} I, & k_2 = k_1, \\ A[k_2 - 1]A[k_2 - 2] \ldots A[k_1], & k_2 > k_1, \end{cases}$$

and monodromy matrix $\Psi = \Phi_{r+1,1}$.

**Lemma 2** shows that LPTV systems lifted over their period are LTI and that stability is preserved under lifting, see also [22] Section 2. Both properties are used in the Nyquist stability test presented in the next section. In the remainder of this section, preliminary results related to lifting are presented.

**Lemma 2.** Let $H$ be an LPTV system with period $T$ (Definition 2) and let $\mathcal{H} = \mathcal{L}_T H \mathcal{L}_T^{-1}$, then

(i) $\mathcal{H}$ is LTI (Definition 3), and

(ii) $\mathcal{H}$ is stable if and only if $H$ is stable.

**Proof.** The LTI properties are evident from Lemma 1. LTI system $H = (A_H, B_H, C_H, D_H)$ is stable if and only if $|\lambda_i(A_H)| < 1$, for all $i$. By Lemma 1, $A_H = \Psi$ in (5), and hence the stability condition is identical to (6) \qed

The lifted system $G_{b,d}$ is given by Lemma 3 and obtained by lifting $G_{b,d}$ operating on sampling sequence $\Delta_{ne}$ in (1) over period $T\delta_b$, which corresponds to lifting over $T$ samples. For a proof, see [17] Section 6.2.1. Lemma 3 is expressed in terms of transfer functions to facilitate application to FRF measurements by replacing $z$ with $e^{i\omega T \delta_b}$.

**Lemma 3 (Lifting $G_{b,d}$).** The LTI transfer function $G_{b,d}(z)$ lifted over $T \in \mathbb{N}$ samples is given by $G_{b,d} = \mathcal{L}_T G_{b,d} \mathcal{L}_T^{-1} \in \mathbb{C}^{T \times r}$, with element $G_{b,d}(z)[i,j], i, j = 1, 2, \ldots, T$, given by

$$G_{b,d}(z)[i,j] = G_{b,d}(z)[i-j],$$

where

$$G_{b,d}(z) = z^T \sum_{k=0}^{T-1} G_{b,d}(z \Phi^k) \Phi^{kT}, \quad \Phi = e^{\frac{2\pi i}{T}}.$$

Next, the downsampler and upsampler are lifted. Let

$$\mu_T[i] := i - 1, \quad i = 1, 2, \ldots, T,$$

$$\mu_T[i] := \begin{cases} 0, & i = 1, \\ \sum_{j=1}^{i-1} \gamma_j, & i = 2, 3, \ldots, T + 1, \end{cases}$$
then lifting $D$ and $H$ over period time $T \delta_b$ yields the non-square systems given by Lemma 4 and Lemma 5 respectively. The results follow directly from (13) and (14), and Assumption 2. Note that the input and output are lifted over a different number of samples due to the different sampling sequences.

**Lemma 4** (Lifting $D$). Lifting downsampler $D$ in Figure 3 over period $T \delta_b$ yields $D = \mathcal{L}_r D \mathcal{L}_r^{-1} \in \mathbb{N}^{r \times T}$, with element $D[i, j]$, $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, T$, given by

$$D[i, j] = \begin{cases} 1, & \mu_r[i] = \mu_r[j], \\ 0, & \text{otherwise}, \end{cases}$$

(15)

with $\mu_r$, $\mu_l$ in (13) and (14) respectively.

**Lemma 5** (Lifting $H$). Lifting upsampler with zero-order-hold interpolation $H$ in Figure 3 over period $T \delta_b$ yields $H = \mathcal{L}_r H \mathcal{L}_r^{-1} \in \mathbb{N}^{T \times r}$, with element $H[i, j]$, $i = 1, 2, \ldots, T$, $j = 1, 2, \ldots, r$, given by

$$H[i, j] = \begin{cases} 1, & \mu_r[j] \leq \mu_r[i] < \mu_r[i + 1], \\ 0, & \text{otherwise}, \end{cases}$$

(16)

with $\mu_r$, $\mu_l$ in (13) and (14) respectively.

An important observation is that all lifted systems are LTI, see also Lemma 2, and hence any interconnection of lifted systems is LTI. The results in this section form the basis for the Nyquist stability test for LPTV systems presented in the next section.

**Remark 3.** Note that all lifted systems correspond to lifting over period $T \delta_b$, although the periodicities, and hence the dimensions, differ depending on the sampling sequence, i.e., $T$ for $\Delta_b$ and $r$ for $\Delta_{ne}$.

### 3.3 Nyquist stability test

The results of the previous sections are used in this section for the stability test of the closed-loop LPTV system in Figure 3. The presented Nyquist stability test constitutes Contribution (I).

The stability test makes use of the principle of the argument in Lemma 6 and the results of Lemma 7.

**Lemma 6.** Let $f(z) \in \mathcal{R}$ and let $C$ denote a closed contour in the complex plane. Assume that

(i) $f(z)$ is analytic on $C$, i.e., $f(z)$ has no poles on $C$,

(ii) $f(z)$ has $Z$ zeros inside $C$, and

(iii) $f(z)$ has $P$ poles inside $C$.

Then, the image $f(z)$ as $z$ traverses the contour $C$ once in a clockwise direction will make $N = Z - P$ clockwise encirclements of the origin.

**Lemma 7.** Let $L_{b,d} = G_{b,d} H C_d D$ and $L_d = D G_{b,d} H C_d$ with $C_d$ in Lemma 1, $G_d$ in Lemma 3, $D$ in Lemma 4 and $H$ in Lemma 5, then

$$\det(I_T + L_{b,d}) = \det(I_T + L_d).$$

(17)

**Proof.** By properties of determinants and rank $\{D\} = r$ follows

$$\det(I_T + L_{b,d}) = \det(I_T + G_{b,d} H C_d D)$$

(18a)

$$= \det\left(I_T + \begin{bmatrix} D G_{b,d} H C_d & 0 \\ 0 & 0 \\ 0 & I_{T-r} \end{bmatrix} \right)$$

(18b)

$$= \det\left(I_T + \begin{bmatrix} D G_{b,d} H C_d & 0 \\ 0 & I_{T-r} \end{bmatrix} \right)$$

(18c)

$$= \det(I_T + D G_{b,d} H C_d)$$

(18d)

$$= \det(I_T + L_d).$$

(18e)
In Lemma 7, $I_T + L_{b,d}$ has dimensions $T \times T$ and $I_T + L_d$ has dimensions $\tau \times \tau$. Since $\tau \leq T$, the latter is preferred to calculate the determinant and used in the stability test. The stability test for LPTV systems is presented in Theorem 2.

**Theorem 2.** Given $C_d$ in Lemma 1, $G_d$ in Lemma 3, $D$ in Lemma 4 and $H$ in Lemma 5, the closed-loop transfer function $\rho_h \rightarrow \epsilon_h$ in Figure 3 is given by $S_{b,d}$ is stable if and only if the image of $\det(I_T + L_{b,d})$, with

$$L_{b,d}(e^{j\omega T \mathcal{D}_h}) = D G_{b,d}(e^{j\omega T \mathcal{D}_h}) H C_d(e^{j\omega T \mathcal{D}_h}),$$

(i) does not pass through the origin, and

(ii) makes $P$ anti-clockwise encirclements of the origin, with $P$ the number of unstable poles of $L_{b,d}$ counting multiplicities.

**Proof.** By Lemma 2, the state matrix of $L_T S_{b,d} L_T^{-1}$ is given by $\Psi$ in (5) hence the roots of $\phi(z)$ in (7) are the poles of $L_T S_{b,d} L_T^{-1} = L_T (1 + L_{b,d})^{-1} L_T^{-1} = (I_T + L_{b,d} L_T^{-1})^{-1} = (I_T + L_{b,d})^{-1}$, i.e., the roots of $\det(I_T + L_{b,d})$, which by Lemma 7 are the roots of $\det(I_T + L_d)$. Let $L_d \equiv (A, B, C, D)$, then the open-loop and closed-loop characteristic polynomials are $\phi_d(z) = \det(zI - A)$ and $\phi_d(z) = \det(zI - A_c)$, with $A_c = A - B(I_T + D)^{-1}C$. Using the Schur complement,

$$\det(I_T + L_d) = \det(I_T + C(zI - A)^{-1}B + D)$$

(20a)

$$= \frac{1}{\det(zI - A)} \det \begin{bmatrix} zI - A & B \\ -C & I_T + D \end{bmatrix}$$

(20b)

$$= \frac{1}{\det(zI - A)} \det(I_T + D) \det(zI - A + B(I_T + D)^{-1}C)$$

(20c)

$$= \frac{\det(zI - A_c)}{\det(zI - A)} \det(I_T + D)$$

(20d)

$$= \frac{\phi_d(z)}{\phi_d(z)} c,$$

(20e)

with constant $c = \det(I_T + D)$. Hence, the closed-loop poles are the roots of $\phi_d(z)$ and the closed-loop zeros are the roots of $\phi_d(z)$.

The stability conditions follow from applying Lemma 6 to (20e) with $C$ being the contour encircling the region outside the unit disk such that $Z$ is the number of unstable closed-loop poles (roots of $\phi_d(z)$ with $|z| > 1$) and $P$ is the number of unstable closed-loop zeros (roots of $\phi_d(z)$ with $|z| > 1$). The first condition ensures $\det(I_T + L_{b,d})$ is analytic on $C$. The second condition ensures closed-loop stability through $Z = 0$ as follows from the choice of contour $C$ and (6).

The number of unstable poles $P$ in Theorem 2 follows from the design of $C_d$ and the number of unstable poles of $G_{b,d}$.

Interestingly, in view of Theorem 2, Lemma 7 essentially shows that stability on the equidistant base rate $\Delta_b$ is equivalent to stability on the non-equidistant rate $\Delta_{ne}$. The result is explained by the fact that feedback is only applied on the non-equidistant rate, i.e., in between these sampling instances the system is in open-loop, and therefore it suffices to check stability on $\Delta_{ne}$.

Importantly, it does not suffice to check stability on $\Delta_{eq}$.

**Remark 4.** For multivariable systems, care has to be taken regarding indentations, since indentations outside the unit disc may lead to undesirable results for multivariable systems.

**Remark 5.** The sampling in Figure 3 should be non-pathological to preserve controllability and observability.

## 4 PERFORMANCE: FRFS FOR LPTV SYSTEMS

In this section, the performance of the system in Figure 3 is quantified, which addresses subproblem (B). The performance is quantified in terms of FRFs and constitutes Contribution (II). Importantly, Bode plots, used for performance characterization of LTI systems, are not directly applicable for performance characterization of LPTV systems since for LPTV systems a single input frequency generally yields multiple output frequencies. In this section, the periodicity is exploited to obtain equivalent Bode plots for performance characterization, extending the multirate approach in to non-equidistantly sampled systems. Indeed, the results in are recovered as a special case.
First, several preliminary results are presented. In Section 4.1, the conversion between equidistant rates based on multirate building blocks is presented. Based on the filter banks, frequency response functions (FRFs) are presented in Section 4.3. The FRFs provide a full characterization of the system but are not convenient for control design. The main result, performance functions for the system in Figure 3 based on FRFs, is presented in Section 4.4 and used for control design in Section 5.

4.1 | Multirate building blocks

Conversion between equidistant rates is described by the multirate operators in Figure 5. These operators are defined in Definitions 5 to 8 with $A, B$ the Fourier transforms of the signals $\alpha, \beta$, respectively.

Definition 5 (Forward shift). The forward shift operator in Figure 5(a) is defined as

$$\beta[k] = \alpha[k + 1], \quad B(e^{j\alpha}) = e^{j\alpha} A(e^{j\alpha}).$$

Definition 6 (Downsampler). The downsampling operator $S_{d,F}$ in Figure 5(b) with downsample factor $F \in \mathbb{N}$ is defined as

$$\beta[k] = \alpha[Fk], \quad B(e^{j\alpha}) = \frac{1}{F} \sum_{f=0}^{F-1} A \left( e^{j\frac{\alpha}{F}(2\pi f)} \right).$$

Definition 7 (Upsampler). The upsampling operator $S_{u,F}$ in Figure 5(c) with upsample factor $F \in \mathbb{N}$ is defined as

$$\beta[k] = \begin{cases} \alpha[k], & k \in \mathbb{Z}, \\ 0, & k \notin \mathbb{Z}, \end{cases} \quad B(e^{j\alpha}) = A(e^{j\alpha F}).$$

Definition 8 (Zero-order-hold interpolator). The zero-order-hold interpolator $I_{zoh,F}$ in Figure 5(d) with interpolation factor $F \in \mathbb{N}$ is defined as

$$\beta[k] = \alpha[F \lfloor \frac{k}{F} \rfloor], \quad B(e^{j\alpha}) = A(e^{j\alpha}) \sum_{f=0}^{F-1} e^{-j\alpha f}.$$
4.3 Frequency response functions of LPTV systems

In this section, frequency response functions (FRFs) of LPTV systems are presented. The FRF of $HC_dD$ is given by Theorem 3.

**Theorem 3.** Let $E_b(e^{j\omega_b})$ be the Fourier transform of $\epsilon_b$, then the Fourier transform of $v_b = HC_dD\epsilon_b$ in Figure 7 is given by

$$N_b(e^{j\omega_b}) = \sum_{i=1}^{\tau} \sum_{f=0}^{\gamma_i-1} e^{-j\omega_b\gamma_i} \left( \sum_{k=1}^{T} C_d[i, k](e^{j\omega_bT^2\delta_k}) \frac{1}{T} \sum_{f=0}^{T-1} \left( e^{j\bar{\gamma}_i\omega_b(2\pi f/T)} E_b(e^{j\omega_b(2\pi f/T)}) \right) \right),$$

with $\bar{\gamma}_i = \sum_{j=1}^{\gamma_i-1} \gamma_j$.

**Proof.** The dependency is given by Figure 7. The result follows from substitution of the Fourier transforms of the multirate building blocks given by Definitions 5 to 8. □

Importantly, Theorem 3 shows that the output $N_b(e^{j\omega_b})$ at frequency $\omega$ depends on the $T$ input frequencies $(\omega\delta_b - 2\pi f_i^T)$, $f_i = 0, 1, \ldots, T$, of $E_b$. Vice versa, a single frequency in $E_b$ contributes to $T$ frequencies in $N_b$. The result directly leads to the frequency response matrix $HC_dD$ satisfying $N_b = HC_dDE_b$. Since the system is LPTV with period $T$, the FRM has the structure

$$HC_dD : \begin{bmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

(26)
FIGURE 7 Filter bank of the transfer function $\varepsilon_b \mapsto \nu_b$ in Figure 3, i.e., $HC_dD$, with base rate (-----) defined by $\Delta_b$ in (1) and non-equidistant rate (-----) defined by $\Delta_{ne}$ in (2). The left side decomposes the equidistantly sampled signal $\varepsilon_b$ into $\tau$ subband signals with periodicity $T$ which are the input to the lifted controller $C_d$. The right side constructs the equidistantly sampled signal $\nu_b$ through upsampling with zero-order-hold interpolation.

consisting of $T \times T$ diagonal submatrices. Since $G_{b,d}$ is LTI, the Fourier transform of $\psi_b$ is given by $Y_b(e^{j\omega_b}) = \overline{G_{b,d}} N_b(e^{j\omega_b})$, where $\overline{\frac{\psi_b}{G_{b,d}}}$ has the diagonal structure

\begin{equation}
\overline{G_{b,d}} = \begin{bmatrix}
\end{equation}

Let $R_b(e^{j\omega_b}), H_b(e^{j\omega_b})$ be the Fourier transforms of $\rho_b, \eta_b$ in Figure 3 then

\begin{equation}
E_b(e^{j\omega_b}) = \overline{S_{b,d}} R_b(e^{j\omega_b}) - \overline{S_{b,d}} H_b(e^{j\omega_b}),
\end{equation}

with $\overline{S_{b,d}} = \left( I + \overline{G_{b,d}}HC_dD \right)^{-1}$ which has the same structure as $\overline{HC_dD}$, see (26). In the next section, the structure of the FRM is exploited for performance evaluation.

Remark 6. For equidistant control on $\Delta_b$, it follows that $\Gamma_{ne} = 1$, $T = 1$, and hence the controller is LTI where (25) reduces to

\begin{equation}
N_b(e^{j\omega_b}) = C_d(e^{j\omega_b}) E_b(e^{j\omega_b}),
\end{equation}

and the FRM in (26) has the structure

\begin{equation}
\overline{HC_dD}_{|_{\Gamma_{ne} = 1}} = \begin{bmatrix}
\end{equation}

4.4 Frequency-domain performance of LPTV systems

In traditional loop-shaping control design for LTI systems, Bode plots are used to quantify the performance and based on the frequency separation principle. As shown by Theorem 3, the frequency separation principle does not hold for the LPTV system in Figure 3. Aimed at loop-shaping control design for LPTV systems, the interest is in performance functions that only depend on the input frequency, similar as in Bode plots for LTI systems. In this section, two such performance functions are presented. The functions show those aspects of the FRFs most relevant for controller design.

The two functions are generalizations of the fundamental transfer function (FTF) $F$ and the performance frequency gain (PFG) $P$ as used for multirate and sampled-data systems, see [25,27] and references therein. Generalizations for LPTV systems in terms of the FRM are given by Definition 9 and Definition 10 and follow from the multirate definitions in [25] and the structure of the FRM.

Definition 9 (Fundamental transfer function (FTF)). Given a frequency response matrix $\overline{G}$ with elements $\overline{G}[i, j]$ corresponding to the $i$th output frequency and the $j$th input frequency, the fundamental transfer function (FTF) for the $k$th input frequency is defined by

\begin{equation}
F_k = \overline{G}[k, k] \in C.
\end{equation}
Definition 10 (Performance frequency gain (PFG)). Given a frequency response matrix $\mathcal{G}$ with elements $\mathcal{G}[i, j]$ corresponding to the $i$th output frequency and the $j$th input frequency, the performance frequency gain (PFG) for the $k$th input frequency is defined by

$$P_k = \sqrt{\sum_i \| \mathcal{G}[i, k] \|^2_2} \in \mathbb{R}.$$ \hspace{1cm} (32)

Note that both the FTF and the PFG are defined in terms of the input frequency. The FTF corresponds to the diagonal of the FRM and hence only takes into account the fundamental frequency component. The PFG takes into account the full intersample behavior and relates the root-mean-square (rms) value of the input to that of the output. This is particularly relevant to quantify control performance as also shown in Section 6 and Section 7.

In the next section, the stability test presented in Section 3 and the performance functions presented in this section are used for LPTV loop-shaping controller design.

Remark 7. For LTI systems, the FRM is diagonal and hence the output frequencies equal the input frequencies, the FTF (Definition 9) equals the FRF, and the PFG (Definition 10) equals the magnitude response of the FRF. See also Remark 6.

5 | LOOP-SHAPING CONTROL DESIGN

In the previous two sections, the stability and performance aspects of the main problem in Section 2.4 are addressed. In this section, the loop-shaping control design based on FRF measurements is presented, which constitutes Contribution (III).

First, different approaches for loop-shaping control design for LTI systems are evaluated. Second, a loop-shaping design procedure for LTI systems is presented. Finally, loop-shaping design procedures for LPTV systems are presented.

5.1 | Control design approaches for LTI systems

In this section, the design of a discrete-time LTI controller $C_d(z)$ using loop-shaping techniques is considered. The starting point is an identification experiment from which a continuous-time FRF measurement $G_c(j\omega)$ or a discrete-time FRF measurement $G_d(e^{j\omega})$ with sampling time $\delta$ can be obtained.

There are two main requirements for loop-shaping design for LPTV systems. First, the frequency response behavior should be asymptotic with respect to the frequency since stability and performance specifications are defined in terms of cut-off frequencies and asymptotes. Second, the discretization should be exact. Discretization methods such as zero-order-hold and Tustin introduce approximation errors close to the Nyquist frequency, as illustrated in Appendix B. For most LTI control designs this does not pose problems since the designs do not include features near the Nyquist frequency. However, for LPTV controller designs, features near the Nyquist frequency are relevant, as also shown in Section 7.

The three main design approaches are visualized in Figure 8, see also and evaluated in the subsequent sections.

5.1.1 | Discrete-time design

The first approach is a discrete-time design based on the discrete-time FRF $G_d(e^{j\omega})$. Since the FRF is non-rational in frequency $\omega$, the asymptotic behavior of the frequency response with respect to the frequency is lost and therefore the approach is unsuited for loop-shaping design.

5.1.2 | Continuous-time design

The second approach is based on the continuous-time FRF $G_c(j\omega)$ obtained from the identification experiment. FRF $G_c(j\omega)$ is rational in $\omega$ and hence suited for continuous-time loop-shaping control design. However, the approach is not suited for discrete-time control design since (i) $G_c(j\omega)$ does not capture discrete-time aspects; and (ii) the discretization of $C_c(s)$ to $C_d(z)$ is approximate rather than exact.
FIGURE 8 Discrete-time controller $C_d(z)$ can be designed using the continuous-time $s$-domain (---), the discrete-time $z$-domain (---), or the auxiliary $w$-domain (----), see also [30]. Design via the $w$-domain is preferred since it facilitates loop-shaping design and provides an exact transformation.

5.1.3 $w$-plane design

The third approach is based on transforming the FRF $G_d(e^{j\omega})$ to $G_a(j\nu)$ in the auxiliary $w$-domain and combines the advantages of the previous two approaches. The approach, which is detailed below, enables loop-shaping design and provides exact discretization.

The transformation from the discrete-time $z$-domain to the auxiliary $w$-domain and vice versa is performed using bilinear Tustin transformations, a special case of linear fractional or Möbius transformations [41], given by

$$w = \frac{2(z - 1)}{\delta(z + 1)}, \quad z = \frac{1 + \frac{\omega\delta}{2}}{1 - \frac{\omega\delta}{2}}.$$  \hfill (33)

Let controller $C_a(w)$ have state-space realization $C_a = (A_a, B_a, C_a, D_a)$, then the discrete-time controller $C_d$ with sampling time $\delta$ is given by $C_d = (A_d, B_d, C_d, D_d)$, with

$$A_d = (I - \frac{\delta}{2} A_a)^{-1}(I + \frac{\delta}{2} A_a),$$ \hfill (34a)
$$B_d = \delta(I - \frac{\delta}{2} A_a)^{-1}B_a,$$ \hfill (34b)
$$C_d = C_a(I - \frac{\delta}{2} A_a)^{-1},$$ \hfill (34c)
$$D_d = D_a + \frac{1}{2}\delta C_a(I - \frac{\delta}{2} A_a)^{-1}B_a.$$ \hfill (34d)

Transformation (33) preserves all magnitude and phase characteristics, but introduces frequency warping, i.e., $G_d(e^{j\omega\delta}) = G_a(j\nu)$ where frequency axis $z = e^{j\omega\delta}$ is mapped to frequency axis $w = j\nu$ with fictitious frequency

$$\nu = \frac{2}{\delta} \tan \left( \frac{\omega\delta}{2} \right).$$ \hfill (35)

The frequency warping is compensated by implementing a characteristic at discrete-time frequency $\omega$ at the fictitious frequency $\nu$ given by (35) An example of this pre-warping is presented in Appendix B.

Importantly, the auxiliary $w$-plane has the same characteristics as the continuous-time $s$-plane [30] and thus enables loop-shaping of $C_a(w)$ based on $G_a(j\nu)$ similar to continuous-time approaches. Since the $w$-plane approach has asymptotic behavior of the frequency response with respect to the frequency and yields exact discretization, the approach is used in the loop-shaping design procedures presented in the next sections.

5.2 Loop-shaping for LTI systems

In this section, a loop-shaping design procedure is presented for the design of a discrete-time LTI controller $C_d(z)$ based on an FRF measurement $G_d(e^{j\omega\delta})$. The loop-shaping design procedure for LPTV systems is presented in Section 5.3.
The LTI control design is performed in the $w$-plane, with the bandwidth (gain crossover frequency $\omega_{bw}$) defined as the first 0 dB crossing of the discrete-time open-loop system $L_d(z) = G_d(z)C_d(z)$, i.e., $\omega_{bw} := \min \omega |L_d(e^{j\omega})| = 1$. The procedure provides general guidelines for the design, which may need adjustment to the specific situation. The rationale behind the procedure can be found in [28]. The procedure is given by Procedure 1.

**Procedure 1 (LTI loop-shaping via the $w$-plane).** Let FRF measurement $G_d(e^{j\omega})$ be given.

1. Transform $G_d(e^{j\omega})$ to $G_d(j\nu)$ by warping the frequency axis using (35).
2. Define a desired bandwidth $\omega_{bw}$ and determine $\nu_{bw}$ using (35).
3. Stabilize the system.
   3.a Create phase lead at the bandwidth by adding a lead filter $\frac{\omega}{\omega+1}$ (e.g., $\nu_{11} = \frac{1}{3}\nu_{bw}, \nu_{12} = 3\nu_{bw}$).
   3.b Adjust gain such that $|L_d(j\nu_{bw})| = 1$, with $L_d(w) = G_d(w)C_d(w)$.
   3.c Use the Nyquist plot of $1 + L_d(w)$ to check closed-loop stability and the phase margin $\angle L_d(j\nu_{bw}) + 180^\circ$ (typically $30^\circ - 60^\circ$). If the closed-loop system is unstable or the phase margin is unsatisfactory, go back to step 2 and lower the bandwidth $\omega_{bw}$, or go back to step 3.a and retune $\nu_{11}, \nu_{12}$.
4. Increase performance. Check stability after each step. If the system is unstable, retune the parameters or go back to step 2 and lower the bandwidth $\omega_{bw}$.
   4.a Remove resonances to improve stability margins or shape closed-loop transfer functions in specific frequency ranges using (skewed) notch filters $\frac{\omega^2 + 2\nu \omega + 1}{\omega^2 + 2\nu \omega + 1}$ (typically: modulus margin $\exp(\nu_i) = 6$ dB).
   4.b Improve steady-state behavior by adding integrators with cut-off $\frac{\nu + \nu_i}{\nu}$ (e.g., $\nu_i = \frac{1}{3}\nu_{bw}$).
   4.c Cut-off high frequent controller gain by adding a first-order low-pass filter $\frac{\omega}{\omega + 1}$ (e.g., $\nu_c = 6\nu_{bw}$) or a second-order low-pass filter $\frac{\omega^2 + 2\nu \omega + 1}{\omega^2 + 2\nu \omega + 1}$ (e.g., $\nu_c = 6\nu_{bw}, \beta = 0.5$).
   4.d Check performance by evaluating the Bode plot of the relevant transfer functions. If unsatisfactory, retune the parameters.
5. Transform $C_d(w)$ to $C_d(z)$ using (34).

Note that closed-loop stability and performance with controller $C_d(z)$ is guaranteed by virtue of the bilinear transformation in (33) and the design of $C_d(w)$ since the only difference is the frequency warping. The procedure for LTI systems presented in this section forms the basis of the LPTV loop-shaping design procedures presented in the next section.

### 5.3 Loop-shaping for LPTV systems

In this section, three loop-shaping control designs for the LPTV system in Figure 3 are presented. The first procedure, Procedure 2, provides an LTI controller design for the equidistant sampling sequence $\Delta_{eq}$ in (3).

**Procedure 2 (LTI control design).** Let Assumption 1 and Assumption 2 be satisfied and $G_{b,d}(e^{j\omega_{bw}})$ be given.

1. Design an LTI controller $C_d$ in the $w$-plane based on $G_{b,d}(e^{j\omega_{bw}})$ using Procedure 1.
2. Determine sampling sequence $\Delta_{eq}$ in (3).
3. Transform $C_d$ to $C_a$ using (34) with $\delta = \Delta_{eq}$.
4. Check closed-loop stability for $\Delta_{eq}$ using Theorem 2. If the closed-loop system is unstable, go back to step 1 and redesign $C_a$.
5. Check performance by evaluating $P$ and/or $P$ in Definitions 9 and 10, respectively. If unsatisfactory, go back to step 1 and adjust $C_a$ accordingly.
TABLE 1 Overview of proposed controller design procedures for LPTV systems in terms of the \(w\)-plane controller design.

<table>
<thead>
<tr>
<th>Description</th>
<th>Reference</th>
<th>Control sequence (w)-plane</th>
</tr>
</thead>
<tbody>
<tr>
<td>LTI control design</td>
<td>Procedure 2</td>
<td>((C_a, C_a, \ldots)) on (\Delta_{eq})</td>
</tr>
<tr>
<td>LPTV control: single design</td>
<td>Procedure 3</td>
<td>((C_a, C_a, \ldots)) on (\Delta_{ne})</td>
</tr>
<tr>
<td>LPTV control: multiple design</td>
<td>Procedure 4</td>
<td>((C_a[1], \ldots, C_a[r], C_a[1], \ldots)) on (\Delta_{ne})</td>
</tr>
</tbody>
</table>

Importantly, although the controller design in Procedure 2 is LTI, stability and performance should be checked through Theorem 2 and \(F/P\) since \(\Delta_{eq}\) differs from \(\Delta\) and hence the closed-loop system is LPTV, see also [23].

To exploit the potential of non-equidistant sampling, two design procedures for the non-equidistant sampling sequence \(\Delta_{ne}\) are presented. Procedure 3 provides an LPTV control design based on a single \(w\)-plane LTI control design.

**Procedure 3 (LPTV control: single design).** Let Assumption 1 and Assumption 2 be satisfied and \(G_{b,d}(e^{j\omega \delta_i})\) be given.

1. Design an LTI controller \(C_a\) in the \(w\)-plane based on \(G_{b,d}(e^{j\omega \delta_i})\) using Procedure 1.
2. Determine the state-space representation of LPTV controller \(C_d[i]\) during interval \(i, i = 1, 2, \ldots, \tau\), by transforming \(C_a\) using (34) with \(\delta = \delta_i\).
3. Check closed-loop stability for \(\Delta\) using Theorem 2. If the closed-loop system is unstable, go back to step 1 and redesign \(C_a\).
4. Check performance by evaluating \(F\) and/or \(P\) in Definition 9 and Definition 10 respectively. If unsatisfactory, go back to step 1 and adjust \(C_a\) accordingly.

Procedure 3 is based on the same LTI controller design for each time interval \(\delta_i\). To exploit the full potential of non-equidistant sampling, Procedure 4 provides separate control designs for each time interval \(\delta_i\).

**Procedure 4 (LPTV control: multiple designs).** Let Assumption 1 and Assumption 2 be satisfied and \(G_{b,d}(e^{j\omega \delta_i})\) be given.

1. For each time interval \(\delta_i, i = 1, 2, \ldots, \tau\), design an LTI controller \(C_a[i]\) in the \(w\)-plane based on \(G_{b,d}(e^{j\omega \delta_i})\) using Procedure 1.
2. Determine the state-space representation of LPTV controller \(C_d[i]\) during interval \(i, i = 1, 2, \ldots, \tau\), by transforming \(C_a[i]\) using (34) with \(\delta = \delta_i\).
3. Check closed-loop stability for \(\Delta\) using Theorem 2. If the closed-loop system is unstable, go back to step 1 and redesign \(C_a[i]\).
4. Check performance by evaluating \(F\) and/or \(P\) in Definition 9 and Definition 10 respectively. If unsatisfactory, go back to step 1 and adjust \(C_a[i]\) accordingly.

The \(w\)-plane controller designs for the three design procedures are summarized in Table 1. Importantly, for all three design procedures closed-loop stability should be checked for the discrete-time controller \(C_d\) since there is no guarantee closed-loop stability is preserved under equidistant sampling (Procedure 2), non-equidistant sampling (Procedure 3), or concatenating controllers (Procedure 4).

In the next section, the three control design procedures are used for controller design.

**Remark 8.** Note that the states of controllers \(C_d[i], i = 1, 2, \ldots, \tau,\) in Procedure 4 should match.

6 | APPLICATION TO A MOTION SYSTEM

In this section, the LPTV control design procedures presented in Section 5 are used in controller design for a motion system. The designs show the advantages of non-equidistant sampling over equidistant sampling and constitute Contribution (IV). In Section 7, the presented controller designs are validated in experiments.
The two rotating masses are connected via a flexible shaft. The collocated motor and encoder on the right-hand side are used as input and output, respectively.

**FIGURE 9** Experimental setup: a two-mass-spring-damper system.

### 6.1 Experimental motion system

The experimental setup is shown in Figure 9. The system in Figure 9(a) consists of two rotating masses that are connected via a flexible shaft. The FRF measurement $G_{b,d}$ is shown in Figure 9(b) and obtained through a dedicated identification procedure [42, Chapter 2] for sampling sequence $\Delta_{b}$, with sampling time $\delta_b = 0.25$ ms.

Analysis of the system reveals that there are two rigid body modes in $G_{b,d}$ and no unstable poles. In the remainder, only stable controllers are considered, hence $P = 2$ in Theorem 2. Consequently, by application of Theorem 2, the closed-loop system is stable if and only if the Nyquist plot does not pass through the origin and has two anti-clockwise encirclements of the origin, see also Remark 4.

### 6.2 Case study

The control diagram in Figure 3 is considered where $\Delta_{nc}$ in (3) is given by $\Gamma_{nc} = [2 2 4]$. By Definition 4 $\Delta_{eq}$ is given by $\Gamma_{eq} = 4$, i.e., $\delta_{eq} = 1$ ms, and has Nyquist frequency $f_{eq,n} = \frac{\omega}{\delta_{eq}} = 500$ Hz.

The controller designs are evaluated for non-aliased and aliased disturbances [13,15] by setting

$$\eta_b[k] = 0.04\sqrt{2}\sin(2\pi f_1 k) + 0.015\sqrt{2}\sin(2\pi f_2 k),$$

with $f_1 = 10$ Hz and $f_2 = 890$ Hz such that $f_1 < f_{eq,n}$ and $f_2 > f_{eq,n}$. The reference trajectory is set to zero, i.e., $\rho_b[k] = 0$, for all $k$. The relevant transfer function $\eta_b \mapsto \epsilon_b$ is given by $-S_{b,d}$, with $S_{b,d} = (1 + G_{b,d}HC_{d}D)^{-1}$. For a fair comparison, the desired bandwidth in the $\omega$-plane is fixed at $\omega_{b,w} = 25 \cdot 2\pi$ rad/s for all controller designs. Note that $\omega_{b,w}$ in the $z$-domain does not provide a fair comparison since it depends on $\delta$.

Next, five controller designs based on the procedures in Section 5 are presented. An overview is presented in Table 2. Experimental validation of the designs is presented in Section 7.

### 6.3 Design 1: Equidistant control for stability

The first controller is designed for $\Delta_{eq}$ and based on Procedure 2 with the following steps.

1. In Procedure 1, the bandwidth of $C_{a1}$ is set to $\nu_{b,w} = 25 \cdot 2\pi$ rad/s. A lead filter with $\nu_{1} = \frac{1}{3} \nu_{b,w}$ and $\nu_{2} = 3\nu_{b,w}$ is used to create phase margin near the bandwidth $\nu_{b,w}$. An integrator with cut-off at $\frac{1}{3} \nu_{b,w}$ is used to overcome friction.

2. $\Delta_{eq}$ is equidistant with sampling interval $\delta_{eq} = 1$ ms, see Section 6.2.

3. $C_{d1}$ is obtained from $C_{a1}$ using (34) with $\delta = \delta_{eq}$.
FIGURE 10 Nyquist plot of $\det(L_c + L_d)$ for designs $C_{d1}$ (○), $C_{d2}$ (●), and $C_{d3}$ (■). By application of the LPTV Nyquist test of Theorem 2, all three controllers stabilize the system.

![Nyquist plot](image1)

(a) Nyquist plot.

![Zoom view around the origin](image2)

(b) Zoom view around the origin.

FIGURE 11 Bode magnitude diagrams for LTI controller designs $C_{a1}$, $C_{a2}$, and $C_{a3}$ in the $\omega$-plane.

![Controller designs](image3)

(a) Controller designs $C_{a1}$ (—), $C_{a2}$ (—), and $C_{a3}$ (—).

![Sensitivity functions](image4)

(b) Sensitivity functions in the $\omega$-plane. Compared to design $S_{a1}$ (●), design $S_{a2}$ (○) suppresses frequency $\nu_1 = 10 \cdot 2\pi$ rad/s. Design $S_{a3}$ (□) also suppresses frequency $\nu_{\omega} = 110 \cdot 2\pi$ rad/s.

4. Closed-loop stability for $\Delta_{ne}$ is verified using Theorem 2 based on the Nyquist plot in Figure 10.

5. The performance functions are not shown since the design is not aimed at performance.

Design $C_{d1}$ stabilizes the system, but achieves moderate performance since step 4 in Procedure 1 is omitted. Next, the performance is improved by also designing for performance.

6.4 | Design 2: Equidistant control for performance

Controller design $C_{d2}$ is an extension of controller design $C_{d1}$ in which performance is taken into account by suppressing disturbance frequency $f_1$ using step 4 in Procedure 1. The steps in Procedure 2 are as follows.

1. $C_{2a}$ is obtained by adding an inverse notch filter to design $C_{1a}$, with $\nu_{n1} = \nu_{n2} = \nu_1$, $\beta_1 = 0.1$, $\beta_2 = 0.01$, where $\nu_1 = 10 \cdot 2\pi$ rad/s follows from $f_1$ through (35). The sensitivity function $S_{a2}$ in Figure 11(b) shows the additional suppression at $\nu_1$.

2. $\Delta_{eq}$ is equidistant with sampling interval $\delta_{eq} = 1$ ms.

3. $C_{d2}$ is obtained from $C_{a2}$ using (34) with $\delta = \delta_{eq}$.

4. Closed-loop stability for $\Delta_{ne}$ is verified using Theorem 2 based on the Nyquist plot in Figure 10 similar as for $C_{d1}$.

5. The PFG of $S_{b,d2}$ is shown in Figure 12 and shows suppression at $f_1$. 
All designs suppress $f_1 = 10$ Hz. Design $S_{b,d5}$ yields the smallest amplification at $f_2 = 890$ Hz, and hence the best performance, due to a dedicated design.

The FRM, see Section 4.3, of the LPTV sensitivity function $S_{b,d2}$ (not shown) reveals that the frequencies most dominantly contributing to $\epsilon_b$ are $f_2 = 890$ Hz and the aliased frequency $f_{2a} = \frac{1}{\delta_{eq}} - f_2 = 110$ Hz, where $\frac{1}{\delta_{eq}} = 1000$ Hz corresponds to the sampling periodicity of sequence $\Delta_{eq}$. Aliasing of $f_2$ also yields contributions at other output frequencies, but these contributions are negligible compared to those at $f_2, f_{2a}$. Next, controller design for $\Delta_{ne}$ is presented.

### 6.5 Design 3: Equidistant control suppressing aliased components

An important observation for design $C_{d2}$ is that the component at $f_2$ cannot be suppressed since $f_2 > f_{eq,a}$. To suppress the component at $f_{2a}$, a notch filter is used in design $C_{d3}$. The steps for designing $C_{d3}$ using Procedure 2 are as follows.

1. $C_{3a}$ is obtained by adding an inverse notch filter to design $C_{2a}$, with $v_{n1} = v_{n2} = v_{2a}$, $\beta_1 = -0.015$, $\beta_2 = 0.001$, where $v_{2a} = 110.27 \cdot 2\pi$ rad/s follows from $f_{2a}$ through (35). The sensitivity function $S_{n3}$ in Figure 11(b) shows the additional suppression at $v_{2a}$.

2. $\Delta_{eq}$ is equidistant with sampling interval $\delta_{eq} = 1$ ms.

3. $C_{d2}$ is obtained from $C_{d2}$ using (34) with $\delta = \delta_{eq}$.

4. Closed-loop stability for $\Delta_{ne}$ is verified using Theorem 2 based on the Nyquist plot in Figure 10 similar as for $C_{d1}$ and $C_{d2}$.

5. The PFG of $S_{b,d3}$ is shown in Figure 12.

The PFG in Figure 12 shows that design $C_{d3}$ yields a performance degradation, instead of a performance improvement, for frequency $f_{2a}$ compared to design $C_{d2}$. The performance degrades since $f_{2a}$ results from aliasing and is not present in $\eta_b$, see (36).

The equidistant controller designs $C_{d1}, C_{d2}, C_{d3}$ show that disturbances below the Nyquist frequency can be effectively suppressed. The aliased components of disturbances above the Nyquist frequency can be compensated, which improves the on-sample behavior, but degrades the intersample behavior, see $C_{d3}$. For these reasons, design $C_{d2}$ is expected to yield the best performance among the equidistant controller designs. The observations are corroborated by the experiments in Section 7.

### 6.6 Design 4: Non-equidistant control, single design

The non-equidistant sampling sequence $\Delta_{ne}$ has periods smaller than $\frac{1}{2f_2} = 0.56$ ms and hence there is potential to suppress frequency $f_2$. Note that this potential is absent with $\Delta_{eq}$. Design $C_{a2}$ successfully suppresses $f_1$ and is used as starting point for design $C_{d4}$. The steps in Procedure 3 are as follows.

1. $C_{a4} = C_{a2}$.

2. $C_{d4}$ is obtained by transforming $C_{a4}$ using (34) with $\delta = \delta_i, i = 1, 2, 3$. 

FIGURE 12 Performance frequency gain $P$ of $S_{b,d2}$, $S_{b,d3}$, and $S_{b,d5}$. All designs suppress $f_1 = 10$ Hz. Design $S_{b,d5}$ yields the smallest amplification at $f_2 = 890$ Hz, and hence the best performance, due to a dedicated design.
TABLE 2 Overview of the different control designs. Designs $C_{d^4}$ - $C_{d^5}$ for the non-equidistant sequence $\Delta_{ne}$ outperform the designs $C_{d^1}$, $C_{d^2}$, $C_{d^3}$ for the equidistant sequence $\Delta_{eq}$ in terms of minimizing the rms value of $\varepsilon_b$. Suppressing $f_{2a}$ in $C_{d^3}$ improves on-sample behavior $\varepsilon$, but degrades intersample behavior in terms of minimizing the rms value of $\varepsilon_b$. Design $C_{d^5}$ for the non-equidistantly sampled sequence achieves the best performance.

<table>
<thead>
<tr>
<th>Label</th>
<th>Sampling sequence</th>
<th>Targeted frequencies</th>
<th>On-sample $\varepsilon_{rms}$ [mrad]</th>
<th>Intersample $\varepsilon_{b,rms}$ [mrad]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{d^1}$</td>
<td>$\Delta_{eq}$</td>
<td>x x x</td>
<td>24.03</td>
<td>19.84</td>
</tr>
<tr>
<td>$C_{d^2}$</td>
<td>$\Delta_{eq}$</td>
<td>✓ x x</td>
<td>22.09</td>
<td>17.48</td>
</tr>
<tr>
<td>$C_{d^3}$</td>
<td>$\Delta_{eq}$</td>
<td>✓ ✓ x</td>
<td>3.56</td>
<td>19.51</td>
</tr>
<tr>
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<td>$\Delta_{ne}$</td>
<td>✓ x x</td>
<td>19.39</td>
<td>16.53</td>
</tr>
<tr>
<td>$C_{d^5}$</td>
<td>$\Delta_{ne}$</td>
<td>✓ ✓ ✓</td>
<td>13.54</td>
<td>12.65</td>
</tr>
</tbody>
</table>

3. Closed-loop stability for $\Delta_{ne}$ is verified using Theorem 2 in Appendix D.

4. The performance functions are not shown since the design is not aimed at performance.

Design $C_{d^4}$ only addresses the $f_1$ component. Next, frequency $f_2$ is also addressed.

6.7 Design 5: Non-equidistant control, multiple designs

To suppress the $f_2$ component, an additional inverse notch filter is used during the first two intervals only to avoid aliasing during the third interval. This leads to the LPTV control design $C_{d^5}$ consisting of multiple designs, which is designed using Procedure 4 as follows.

1. $C_{d^5}[1] = C_{d^5}[2] = C_{d^2}$. $C_{d^5}[3]$ is obtained by extending $C_{d^2}$ with an inverse notch filter, with $\nu_{n1} = \nu_{n2} = \nu_2$, $\beta_1 = -0.07$, $\beta_2 = 0.005$, where $\nu_2 = 1070 \cdot 2\pi$ rad/s follows from $f_2$ using (35). The additional suppression at $f_2$ is shown in Appendix C.

2. $C_{d^5}[i]$ is obtained by transforming $C_{d^5}[i]$ using (34) with $\delta = \delta_i$, $i = 1, 2, 3$.

3. Closed-loop stability for $\Delta_{ne}$ is verified using Theorem 2 in Appendix D.

4. The PFG of $S_{b,d^5}$ is shown in Figure 12 and shows additional suppression at $f_2$ as desired.

Design 5 suppresses $f_1$ and $f_2$ and avoids aliasing. For these reasons, it is expected that $C_{d^5}$ yields the best performance among the non-equidistant controller designs. In the next section, the observations are corroborated by experiments.

7 EXPERIMENTAL VALIDATION

In this section, the five control designs of Section 6 are validated on the experimental setup presented in Section 6.1 and shown in Figure 9 which constitutes Contribution (V). An overview of the different control designs is presented in Table 2. As expected based on Section 6 in terms of intersample behavior, design $C_{d^2}$ yields the best performance among the equidistant controllers and design $C_{d^5}$ yields the best performance among the non-equidistant controllers. Most importantly, the non-equidistant controller designs are superior to the equidistant controller designs.

7.1 Equidistant control designs 1, 2, and 3

The error signals $\varepsilon_b$ for the equidistant controller designs $C_{d^1}$, $C_{d^2}$, $C_{d^3}$ are shown in Figure 13(a) and confirm closed-loop stability. For design $C_{d^1}$, the frequency components $f_1$, $f_2$ in (36) and aliased component $f_{2a}$ are clearly visible. The corresponding cumulative power spectra (CPS) of $\varepsilon_b$ in Figure 13(b) show that $C_{d^2}$ almost completely suppresses $f_1$ as desired.

The results in Figure 13(b) confirm the performance deterioration for design $C_{d^3}$ as suggested by the PFG in Figure 12. By Section 4.4 the PFG relates the root-mean-square (rms) values to the CPS values. Indeed, the PFG of $S_{b,d^3}$ in Figure 12 relates...
Equidistant control designs 4 and 5

Controller $C_{d_4}$ is based on the same $\nu$-plane design as controller $C_{d_2}$, with the key difference that it is implemented for the non-equidistant sampling sequence $\Delta_{ne}$, rather than the equidistant sampling sequence $\Delta_{eq}$. Due to the additional control variable in each period, it is expected that design $C_{d_4}$ outperforms design $C_{d_2}$, see also Section 6. The experiments indeed show that $C_{d_4}$ outperforms $C_{d_2}$, see the CPS of $\varepsilon_b$ in Figure 14(b) which corroborates the analysis in Section 6.5 and the reasoning in Section 2.2.

Design $C_{d_5}$ is based on the LPTV control design approach with multiple $\nu$-plane control designs. The CPS of $\varepsilon_b$ for design $C_{d_5}$ is shown in Figure 14(b) which shows that the addition of an inverse notch filter during the first two intervals results in a
smaller increase at $f_2$ as desired. At the same time, there is no aliasing since the notch filter is absent during the third interval. The results validate the reasoning in Section 6, i.e., the non-equidistant controller design $C_{d5}$ outperforms the non-equidistant controller design $C_{d4}$, and the non-equidistant controller designs are superior to the equidistant controller designs.

7.3 | Summary

The application and experimental validation of the proposed control design framework, presented in the previous and current section, show the following aspects: (i) loop-shaping design of non-equidistantly sampled controllers; (ii) application of the Nyquist stability criterion for both equidistantly and non-equidistantly sampled controller designs; (iii) application of the performance frequency gain for performance assessment; and (iv) superior performance with control design for the non-equidistant rate.

8 | CONCLUSION

An intuitive design framework for loop-shaping control design for non-equidistantly sampled systems is presented. The framework facilitates non-equidistant controller design, which enables a substantial performance improvement and cost reduction for control applications compared to conventional LTI designs. The stability of the time-varying closed-loop system is evaluated using a Nyquist stability test and the performance is quantified using performance functions. Both are based on non-parametric models and frequency response functions.

The LPTV loop-shaping design procedure is based on intuitive loop-shaping techniques, similar to those for LTI systems. Application of the design framework to a motion system and the experimental validation demonstrate the potential of non-equidistant sampling and the proposed control design framework.

Ongoing research focuses on extending the presented loop-shaping design guidelines for non-equidistantly sampled systems. Future research focuses on design of the (non-equidistant) sampling sequence to further optimize the performance/cost trade-off. See for early results in this direction. Future research also focuses on feedforward control for flexible sampled systems of which initial results can be found in. From a broader perspective, future research focuses on control design approaches for systems with flexible sampling.

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References


APPENDIX

A EXAMPLE FILTER BANK $HC_d D$

The filter bank of $HC_d D$ in Figure 7 consists of multiple steps. Figure A1 illustrates the different steps for a simple example.

B FREQUENCY DISTORTION

Many discretizations yield approximation errors close to the Nyquist frequency as shown in Figure B2. To avoid these effects, controllers are designed in the $\omega$-plane. The frequency warping is eliminated by implementing a characteristic at discrete-time frequency $\nu$ at the fictitious frequency $\nu$ given by (35).

C NON-EQUIDISTANT CONTROL TO SUPPRESS $\nu_2$

To suppress frequency $\nu_2$ in design $C_{d5}$, a notch filter is used during the first two intervals, i.e., in $S_{d5}[1], S_{d5}[2]$. The suppression is shown in Figure C3.

D NYQUIST STABILITY NON-EQUIDISTANT SAMPLING

Closed-loop stability for $\Delta_{ne}$ for designs $C_{d4}, C_{d5}$ is verified using Theorem 2 based on the Nyquist plot in Figure D4. Stability follows along similar lines as for the equidistant controller designs in Figure 10.
(a) The error is limited at low frequencies but significant at high frequencies.

FIGURE B2 The discretization of the continuous-time filter (——) with sampling frequency 1000 Hz yields approximation errors with the zero-order-hold (•••) and Tustin (-----) method. Through design in the $w$-plane (-----) the characteristics are preserved.

(b) Detailed view at high frequencies. There is an approximation error for the characteristics at high frequencies introduced by the discretization, except for the $w$-plane design.

FIGURE C3 Bode magnitude diagram of sensitivity functions $S_a$ in the $w$-plane near $v_2 = 1070 \cdot 2\pi$ rad/s. The suppression at $v_2$ (•••) is larger for $S_{a5,1} = S_{a5,2}$ (○) than for $S_{a2}$ (○).

E ON-SAMPLE PERFORMANCE $C_{d3}$

Design $C_{d3}$ improves the on-sample behavior compared to $C_{d2}$ as shown by Figure E5. However, the intersample behavior is poor, see Figure E6. In fact, the intersample behavior deteriorates compared to $C_{d2}$ as shown in Figure 13 and Table 2.
FIGURE D4 Nyquist plot of $\text{det}(I_r + L_d)$ for designs $C_{d2}$ (●), $C_{d4}$ (○), and $C_{d5}$ (●). By application of the LPTV Nyquist test of Theorem 2, all controllers stabilize the system.

FIGURE E5 Cumulative power spectrum of $\epsilon$ for the equidistant designs $C_{d1}$ (——), $C_{d2}$ (---) and $C_{d3}$ (-----). Suppression of $f_{2a} = 110$ Hz with $C_{d3}$ improves the on-sample behavior $\epsilon$, but degrades the intersample behavior $\epsilon_h$ in Figure 13(b).

FIGURE E6 Design $C_{d3}$ yields good on-sample behavior (●), but poor intersample behavior (○) by attenuating the aliased disturbance frequency $f_{2a}$, rather than the true disturbance frequency $f_2$. 