BACHELOR

On a work load model with downward jumps

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On a work load model with downward jumps

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Abstract

This thesis will discuss first three existing queueing models: a specific class of reflected AR(1) processes, a M/G Lindley-Type model and lastly the INGAR$^+$ model. Whereas the first two are based on continuous random variables, the INGAR$^+$ model is constructed of discrete random variables. Several well known techniques in the field of mathematics, such as the use of Laplace-Stieltjes transforms, are used for solving the steady-state distributions that belong to those models.

Subsequently a new model is introduced: a work load model where new work can enter the system, but where it is also possible that work gets removed. Using Laplace-Stieltjes transforms the distribution of the amount of work just before and just after arrivals or removals can be determined. Furthermore we find implicit formulas for the moments of the amount of work at arbitrary instants.

Contents

Chapter 1

Introduction

The first part of this thesis consists of a literature study. We introduce three models originating from the field of queueing theory. The first chapter is devoted to a specific class of reflected AR(1) processes. Autoregressive (AR) processes are typically modelled as a random process, which varies over time. An AR(1) process means that the new state is only directly dependent on the previous one, i.e. it is of the first order. Reflected processes can never attain negative values (which holds for many real life applications) and throughout this thesis we will in fact only analyse reflected models. In this first chapter we will mainly be using Laplace-Stieltjes transforms (LST).

The second model is the M/G Lindley-Type model. It looks quite similar to the model in chapter 1. However, the assumptions will be quite different, making the use of LST's for analysing this model not suitable. Therefore a more direct approach is applied for finding the steady-state distribution in this mode.

The third and final model is a discrete process, the so called INGAR$^+$ model. Although the random variables are not continuous, it will be clear this model actually looks very similar to the first one. The analysis has great resemblances with the approach that is used chapter 1.

In the second part we will discuss a new model. It concerns a work load model where work can either be added to the system or be removed. So it is a queueing model where not only upward jumps can occur, but also downward jumps. There is one server to process the work and it serves with a rate dependent on the current amount of work. The mathematical techniques for analysing and discussing this model are to a large extent the same as the techniques that were used in the first part of this thesis.

Part I

Literature study

Chapter 2

Distribution of AR(1) processes

This chapter is devoted to a discussion of the paper "On a class of reflected AR(1) processes" by Boxma et al. [2]. In this paper, the following recursion is considered, which is a reflected autoregressive process of the first order:

$$Z_{n+1} = \max\{aZ_n + X_n, 0\}.$$

Moreover, we consider a process in which X_n can be written as $B_n - A_n$, with $(A_n)_n$ a sequence of random variables, all being independent and having the exponential distribution with parameter λ. $(B_n)_n$ is also a sequence of independent identically distributed random variables, but with an unspecified distribution. Furthermore we assume that $|a| < 1$ in order to make a sensible stationary analysis of the process.

This recursion is actually a model for a single server queue, where Z_n and B_n are respectively the waiting and service time of the n-th customer, and A_n is the time between the arrivals of the n-th and $(n+1)$-st customer.

2.1 Transient distribution

First we look at the transient distribution of Z_n for each $n \in \mathbb{N}$. This is done by evaluating a double transform: the generating function of the Laplace-Stieltjes transform of each Z_n. This way we define the following two functions, where $|r| < 1$:

$$Z_z(r,s) = \sum_{n=0}^{\infty} r^n \mathbb{E}[\mathrm{e}^{-sZ_n}|Z_0 = z], \quad U_z(r,s) = \sum_{n=0}^{\infty} r^n \mathbb{E}[\mathrm{e}^{-s\,\min\{aZ_n + B_n - A_n, 0\}}|Z_0 = z].$$

U_z will be useful for the derivation of Z_z since the property $1 + \mathrm{e}^x = \mathrm{e}^{x^+} + \mathrm{e}^{x^-}$ leads to $\mathrm{e}^{-sZ_{n+1}} = \mathrm{e}^{-s(aZ_n + B_n - A_n)} + 1 - \mathrm{e}^{-sW_n}$, where W_n is defined as $W_n :=\ \min\{aZ_n + B_n - A_n, 0\}$. Also, note that $Z_z(r,s)$ is analytic if the real part of s is greater than or equal to 0 and $U_z(r,s)$ is analytic if it is smaller than or equal to 0. Taking expectations on both sides of the last equation and taking into account that Z_n, A_n and B_n are all independent (so that the transform of the sum equals the product of the individual transforms), yields the following result:

$$\mathbb{E}[\mathrm{e}^{-sZ_{n+1}}|Z_0 = z] = \mathbb{E}[\mathrm{e}^{sA_n}]\mathbb{E}[\mathrm{e}^{-sB_n}]\mathbb{E}[\mathrm{e}^{-saZ_n}|Z_0 = z] + 1 - \mathbb{E}[\mathrm{e}^{-sW_n}|Z_0 = z]. \tag{2.1}$$

We use that the Laplace-Stieltjes transform (LST) of each A_n equals $\dfrac{\lambda}{\lambda + s}$ and hence $-A_n$ has LST $\dfrac{\lambda}{\lambda - s}$ (provided that $\mathrm{Re}s \leq 0$). Furthermore we denote the LST of B by $\varphi_B(s)$ (provided that

Re$s \geq 0$). Then multiplying equation (2.1) by r^n and taking the sum of $n = 0$ to infinity gives:

$$\frac{1}{r}(Z_z(r,s) - \mathrm{e}^{-sz}) = \frac{\lambda}{\lambda - s}\varphi_B(s)Z_z(r,as) + \frac{1}{1-r} - U_z(r,s), \quad \text{Re } s = 0. \tag{2.2}$$

If equation (2.2) is now multiplied with $r(\lambda - s)$ the following equality is reached:

$$(\lambda - s)(Z_z(r,s) - \mathrm{e}^{-sz}) - r\lambda\varphi_B(s)Z_z(r,as) = (\lambda - s)\left(\frac{r}{1-r} - rU_z(r,s)\right), \quad \text{Re } s = 0. \tag{2.3}$$

The main goal is now to find an expression for $Z_z(r,s)$. Observe, that the right hand side of equation (2.3) is analytic whenever Re $s < 0$ and the left hand side is analytic whenever Re $s > 0$. It can also be noted that $Z_z(r,s)$ and $U_z(r,s)$ are bounded by $(1-r)^{-1}$ in their half planes, Re $s > 0$ and Re $s < 0$ respectively. The Generalised Theorem of Liouville [4] now states that, in their respective half planes, both the left and right hand side can be rewritten as the same first order polynomial in s (dependent on r) for large s. In other words:

$$(\lambda - s)(Z_z(r,s) - \mathrm{e}^{-sz}) - r\lambda\varphi_B(s)Z_z(r,as) = C_{0,z}(r) + sC_{1,z}(r), \text{Re } s \geq 0. \tag{2.4}$$

$$(\lambda - s)\left(\frac{r}{1-r} - rU_z(r,s)\right) = C_{0,z}(r) + sC_{1,z}(r), \text{Re } s \leq 0. \tag{2.5}$$

Both equations (2.4) and (2.5) must hold when $s = 0$, which gives that the following equality holds (note that $\varphi_B(0) = 1$):

$$\lambda\left(\sum_{n=0}^{\infty} r^n - 1\right) - r\lambda\sum_{n=0}^{\infty} r^n = C_{0,z}(r) \implies \frac{\lambda}{1-r} - \lambda - \frac{r\lambda}{1-r} = 0 = C_{0,z}(r).$$

Substituting $s = \lambda$ in equation (2.4) yields that:

$$C_{1,z}(r) = -r\varphi_B(\lambda)Z_z(r,a\lambda). \tag{2.6}$$

Now $C_{1,z}(r)$ needs to be determined, using the last expression, which will lead to a formula for $Z_z(r,s)$. First, the terms in equation (2.4) will be rearranged, leading to the formula below:

$$Z_z(r,s) = \frac{sC_{1,z}(r)}{\lambda - s} + \mathrm{e}^{-sz} + r\frac{\lambda}{\lambda - s}\varphi_B(s)Z_z(r,as). \tag{2.7}$$

If iteration is then applied on $Z_z(r,a^i s)$ we get:

$$\begin{aligned}
Z_z(r,s) &= \frac{sC_{1,z}(r)}{\lambda - s} + \mathrm{e}^{-sz} + r\frac{\lambda}{\lambda - s}\varphi_B(s)\left(r\frac{\lambda}{\lambda - as}\varphi_B(as)Z_z(r,a^2 s) + \frac{asC_{1,z}(r)}{\lambda - as} + \mathrm{e}^{-asz}\right) \\
&= \frac{sC_{1,z}(r)}{\lambda - s} + \mathrm{e}^{-sz} + r\frac{\lambda}{\lambda - s}\varphi_B(s)\left(\frac{asC_{1,z}(r)}{\lambda - as} + \mathrm{e}^{-asz}\right) \\
&\quad + r^2\frac{\lambda^2}{(\lambda - s)(\lambda - as)}\varphi_B(s)\varphi_B(as)Z_z(r,a^2 s) \\
&= \text{...iteration continues...} \\
&= \sum_{n=0}^{\infty} r^n\left(\frac{a^n sC_{1,z}(r)}{\lambda - a^n s} + \mathrm{e}^{-a^n sz}\right)\prod_{j=0}^{n-1}\varphi_B(a^j s)\frac{\lambda}{\lambda - a^j s} + \lim_{n\to\infty} r^n\prod_{j=0}^{n}\frac{\lambda\varphi_B(a^j s)}{\lambda - a^j s}Z_z(r,a^n s) \\
&= C_{1,z}(r)\sum_{n=0}^{\infty} r^n\frac{a^n s}{\lambda - a^n s}\prod_{j=0}^{n-1}\varphi_B(a^j s)\frac{\lambda}{\lambda - a^j s} + \sum_{n=0}^{\infty} r^n \mathrm{e}^{-a^n sz}\prod_{j=0}^{n-1}\varphi_B(a^j s)\frac{\lambda}{\lambda - a^j s},
\end{aligned} \tag{2.8}$$

where the empty product equals one.

Remark 2.1.1. Note that in the computation above we have that: $\lim_{n\to\infty} r^n \prod_{j=0}^{n} \dfrac{\lambda\varphi_B(a^j s)}{\lambda - a^j s} Z_z(r, a^n s)$ equals 0, since the ratio of two consecutive terms of the series $a_n = r^n \prod_{j=0}^{n} \dfrac{\lambda\varphi_B(a^j s)}{\lambda - a^j s} Z_z(r, a^n s)$ converges to r and for r we have that $|r| < 1$, hence by the ratio test we have that $a_n \to 0$ as $n \to \infty$.

If $s = a\lambda$ is substituted in equation (2.8) and then equation (2.6) is used as well, the following formula for $C_{1,z}(r)$ is found:

$$C_{1,z}(r) = -\frac{\varphi_B(\lambda)\sum_{n=0}^{\infty} r^{n+1} \mathrm{e}^{-a^{n+1}\lambda z} g(a, n)}{1 + \varphi_B(\lambda)\sum_{n=0}^{\infty} \dfrac{(ar)^{n+1}}{1 - a^{n+1}}}, \tag{2.9}$$

where $g(a, n)$ is defined by: $g(a, n) = \prod_{j=0}^{n-1} \dfrac{\varphi_B(a^{j+1}\lambda)}{1 - a^{j+1}}$. So, finally we have an expression for $Z_z(r, s)$:

$$Z_z(r, s) = \sum_{n=0}^{\infty} r^n \left(\frac{a^n s C_{1,z}(r)}{\lambda - a^n s} + \mathrm{e}^{-a^n s z} \right) \prod_{j=0}^{n-1} \varphi_B(a^j s) \frac{\lambda}{\lambda - a^j s}, \tag{2.10}$$

where $C_{1,z}(r)$ is defined in equation (2.9).

Now, we would like to compute the LST of Z_n, $\mathbb{E}[\mathrm{e}^{-sZ_n}|Z_0 = z]$, for $n = 0, 1, 2, \ldots$. The main difficulty that comes along with that, is the fact that r does not only appear in the formula for $Z_z(r, s)$ by the r^n factor, but also in $C_{1,z}(r)$. Therefore, a different, more appropriate expression for $C_{1,z}(r)$ will be used. First note, that as $s \to \infty$: $\mathbb{E}[\mathrm{e}^{-sZ_n}|Z_0 = z] = \mathbb{P}(Z_n = 0|Z_0 = z)$. From the definition of $Z_z(r, s)$ then follows that:

$$Z_z(r, s) - \mathrm{e}^{-sz} \xrightarrow{s\to\infty} \sum_{n=1}^{\infty} r^n \mathbb{P}(Z_n = 0|Z_0 = z).$$

If this property is combined with equation (2.4), and one keeps in mind that $\varphi_B(s) \xrightarrow{s\to\infty} 0$, one gets this definition for $C_{1,z}(r)$:

$$C_{1,z}(r) = -\sum_{n=1}^{\infty} r^n \mathbb{P}(Z_n = 0|Z_0 = z). \tag{2.11}$$

To derive an expression for $\mathbb{E}[\mathrm{e}^{-sZ_n}|Z_0 = z]$, we need to take the coefficients of r^n for fixed n in equation (2.10) using (2.11). The first part of the sum in (2.10) would then be as follows (the product at the end is not taken into account for this derivation):

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} r^{k+m+1} \mathbb{P}(Z_{k+1} = 0|Z_0 = z) \frac{a^m s}{\lambda - a^m s}.$$

The coefficient of r^n is then found by the summation over $k + m + 1 = n$:

$$\sum_{k+m+1=n} r^{k+m+1} \mathbb{P}(Z_{k+1} = 0|Z_0 = z) \frac{a^m s}{\lambda - a^m s} = \sum_{m=0}^{n-1} r^m \mathbb{P}(Z_{n-m} = 0|Z_0 = z) \frac{a^m s}{\lambda - a^m s},$$

which leads to this formula for $\mathbb{E}[\mathrm{e}^{-sZ_n}|Z_0 = z]$:

$$\mathbb{E}[\mathrm{e}^{-sZ_n}|Z_0 = z] = -\sum_{m=0}^{n-1} \mathbb{P}(Z_{n-m} = 0|Z_0 = z) \frac{a^m s}{\lambda - a^m s} \prod_{j=0}^{m-1} \frac{\lambda\varphi_B(a^j s)}{\lambda - a^j s} + \mathrm{e}^{-a^n s z} \prod_{j=0}^{n-1} \frac{\lambda\varphi_B(a^j s)}{\lambda - a^j s}. \tag{2.12}$$

Remark 2.1.2. In (2.10), the $\lambda - a^j s$ factors seem to be somewhat troublesome, since these give rise to singularities in $s = \lambda/a^j, j = 0, 1, ...$ However, it can be shown that these singularities are in fact removable. First, look at $s = \lambda$. Earlier on, two formulas for $C_{1,z}(r)$ have been derived, namely (2.6) and (2.11). The first one has clearly no singularities, so if the second one is indeed in agreement with the first one, we know that the singularity in $Z_z(r, as)$ for $s = \lambda$ is removable. For $n = 1, 2, 3, ...$ it holds that:

$$\mathbb{P}(Z_n = 0|Z_0 = z) = \mathbb{P}(aZ_{n-1} + B_{n-1} < A_{n-1}|Z_0 = z).$$

Since A_{n-1} is exponentially distributed, it is memoryless, hence it holds that: $\mathbb{P}(A > B + C) = \mathbb{P}(A > B)\mathbb{P}(A > C)$. Above equation can then be rewritten:

$$\begin{aligned}
\mathbb{P}(Z_n = 0|Z_0 = z) &= \mathbb{P}(aZ_{n-1} < A_{n-1}|Z_0 = z)\mathbb{P}(B_{n-1} < A_{n-1}) \\
&= \left(\int_0^\infty F_{aZ_{n-1}}(y|Z_0 = z)\mathrm{e}^{-\lambda y}\mathrm{d}y\right)\left(\int_0^\infty F_{B_{n-1}}(y)\mathrm{e}^{-\lambda y}\mathrm{d}y\right) \\
&= \left(\int_0^\infty\int_0^x f_{aZ_{n-1}}(x|Z_0 = z)\mathrm{d}x\lambda\mathrm{e}^{-\lambda y}\mathrm{d}y\right)\left(\int_0^\infty\int_0^x f_{B_{n-1}}(x)\mathrm{d}x\lambda\mathrm{e}^{-\lambda y}\mathrm{d}y\right) \\
&= \left(\int_x^\infty\int_0^\infty f_{aZ_{n-1}}(x|Z_0 = z)\mathrm{d}x\lambda\mathrm{e}^{-\lambda y}\mathrm{d}y\right)\left(\int_x^\infty\int_0^\infty f_{B_{n-1}}(x)\mathrm{d}x\lambda\mathrm{e}^{-\lambda y}\mathrm{d}y\right) \\
&= \left(\int_0^\infty f_{aZ_{n-1}}(x|Z_0 = z)\mathrm{e}^{-\lambda x}\mathrm{d}x\right)\left(\int_0^\infty f_{B_{n-1}}(x)\mathrm{e}^{-\lambda x}\mathrm{d}x\right) \\
&= \mathbb{E}[e^{-a\lambda Z_{n-1}}|Z_0 = z]\varphi_B(\lambda).
\end{aligned}$$

So, indeed (2.6) and (2.11) are the same. Hence the singularity in $s = \lambda$ is removable. Using this result and (2.4), we immediately see that $Z_z(r, a\lambda)$ is also well defined, so no problematic singularity in $s = \lambda/a$ either. This procedure can then be repeated to show that there are also no poles in $s = \lambda/a^i$ for $i = 2, 3, 4....$

2.2 Stationary distribution

Next, we look at the stationary distribution of Z_n for $n \to \infty$: $Z_\infty \stackrel{d}{=} \max\{aZ_\infty + B - A, 0\}$. This distribution exists whenever $|a| < 1$ [2]. Again, we are interested in the Laplace-Stieltjes transform: $\mathbb{E}[\mathrm{e}^{-sZ_\infty}]$. In the following derivation we use that $\mathbb{E}[A\mathbb{1}_B] = \mathbb{P}(B)\mathbb{E}[A|B]$:

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^{-sZ_\infty}] &= \mathbb{E}[\mathrm{e}^{-s\max\{aZ_\infty+B-A,0\}}] \\
&= \mathbb{E}[\mathrm{e}^{-s\max\{aZ_\infty+B-A,0\}}\mathbb{1}_{aZ_\infty+B-A\geq 0}] + \mathbb{E}[\mathrm{e}^{-s\max\{aZ_\infty+B-A,0\}}\mathbb{1}_{aZ_\infty+B-A<0}] \\
&= \mathbb{E}[\mathrm{e}^{-s(aZ_\infty+B-A)}\mathbb{1}_{aZ_\infty+B-A\geq 0}] \\
&\quad + \mathbb{P}(aZ_\infty + B - A < 0)\mathbb{E}[\mathrm{e}^{-s\max\{aZ_\infty+B-A,0\}}|aZ_\infty + B - A < 0] \\
&= \mathbb{E}[\mathrm{e}^{-s(aZ_\infty+B-A)}] - \mathbb{E}[\mathrm{e}^{-s(aZ_\infty+B-A)}\mathbb{1}_{aZ_\infty+B-A<0}] + \mathbb{P}(aZ_\infty + B - A < 0)\cdot 1 \\
&=_* \mathbb{E}[\mathrm{e}^{-saZ_\infty}]\mathbb{E}[\mathrm{e}^{-sB}]\mathbb{E}[\mathrm{e}^{sA}] - \mathbb{P}(aZ_\infty + B - A < 0)\mathbb{E}[\mathrm{e}^{-s(aZ_\infty+B-A)}|aZ_\infty + B - A < 0] \\
&\quad + \mathbb{P}(aZ_\infty + B - A < 0) \\
&=_\triangle \mathbb{E}[\mathrm{e}^{-saZ_\infty}]\mathbb{E}[\mathrm{e}^{-sB}]\frac{\lambda}{\lambda - s} + \mathbb{P}(aZ_\infty + B - A < 0)\left(1 - \frac{\lambda}{\lambda - s}\right).
\end{aligned} \tag{2.13}$$

At $*$ we use that Z, A and B are independent and at $\triangle$ we use that $A \sim \exp(\lambda)$ and also the memoryless property of the exponential distribution for the factor $\mathbb{E}[\mathrm{e}^{-s(aZ_\infty+B-A)}|aZ_\infty + B - A <$

0], since this is the LST of the probability distribution characterized by:

$$\begin{aligned}\mathbb{P}(aZ_\infty + B - A \le x | aZ_\infty + B - A < 0) &= \mathbb{P}(A \ge aZ_\infty + B - x | A > aZ_\infty + B)\\ &= \mathbb{P}(A \ge -x) = \mathbb{P}(-A \le x).\end{aligned}$$

So $\mathbb{E}[\mathrm{e}^{-s(aZ_\infty+B-A)}|aZ_\infty + B - A < 0]$ equals $\lambda/(\lambda - s)$.

Just as for the transient distribution, a formula can be found for the stationary distribution by iterating. Say $\varphi_Z(s) = \mathbb{E}[\mathrm{e}^{-sZ_\infty}]$. Furthermore denote $\mathbb{P}(aZ_\infty + B - A < 0)$ as P and $\mathbb{E}[\mathrm{e}^{-sB}]$ as $\varphi_B(s)$. Then we get:

$$\begin{aligned}\varphi_Z(s) &= \varphi_Z(as)\varphi_B(s)\frac{\lambda}{\lambda - s} - \frac{s}{\lambda - s}P\\ &= -\frac{Ps}{\lambda - s} + \varphi_B(s)\frac{\lambda}{\lambda - s}\left(-\frac{Pas}{\lambda - as} + \varphi_B(as)\frac{\lambda}{\lambda - as}\varphi_Z(a^2 s)\right)\\ &= \ldots\\ &= -\sum_{n=0}^{\infty}\frac{Pa^n s}{\lambda - a^n s}\prod_{j=0}^{n-1}\varphi_B(a^j s)\frac{\lambda}{\lambda - a^j s} + \lim_{n\to\infty}\varphi_Z(a^n s)\prod_{j=0}^{n-1}\frac{\varphi_B(a^j s)\lambda}{\lambda - a^j s}\\ &=_\circ -P\sum_{n=0}^{\infty}\frac{a^n s}{\lambda - a^n s}\prod_{j=0}^{n-1}\frac{\varphi_B(a^j s)\lambda}{\lambda - a^j s} + \prod_{j=0}^{\infty}\frac{\varphi_B(a^j s)\lambda}{\lambda - a^j s}.\end{aligned} \tag{2.14}$$

At $\circ$ we use the fact that $\lim_{n\to\infty}\varphi_Z(a^n s) = \varphi_Z(0) = 1$. What is still unknown, is the probability P. Note that $P = \mathbb{P}(aZ_\infty + B - A < 0) = \mathbb{P}(Z_\infty = 0)$. This P can be found by subsequently multiplying equation (2.14) by $\lambda - s$ and substituting $s = \lambda$. We then find the following equality:

$$P = \varphi_B(\lambda)\left(\prod_{j=0}^{\infty}\frac{\varphi_B(a^{j+1}\lambda)}{1 - a^{j+1}}\right) - P\varphi_B(\lambda)\sum_{m=0}^{\infty}\frac{a^{m+1}}{1 - a^{m+1}}\prod_{j=0}^{m-1}\frac{\varphi_B(a^{j+1}\lambda)}{1 - a^{j+1}},$$

which then obviously leads to the formula for P:

$$P = \frac{\varphi_B(\lambda)\left(\prod_{j=0}^{\infty}\frac{\varphi_B(a^{j+1}\lambda)}{1 - a^{j+1}}\right)}{1 + \varphi_B(\lambda)\sum_{m=0}^{\infty}\frac{a^{m+1}}{1 - a^{m+1}}\prod_{j=0}^{m-1}\frac{\varphi_B(a^{j+1}\lambda)}{1 - a^{j+1}}}. \tag{2.15}$$

Remark 2.2.1. Formula (2.14) can also be found by letting $n \to \infty$ in equation (2.12) or by applying Abel's Theorem on power series to (2.10). This theorem states the following:

$$\lim_{r\to 1}(1 - r)\sum_{n=0}^{\infty} g_n r^n = g_\infty, \text{ if } g_n \to g_\infty.$$

So, in our case we have: $\mathbb{E}[\mathrm{e}^{-sZ_\infty}] = \lim_{r\to 1}(1 - r)Z_z(r, s)$. Now we split the sum in (2.10) into two parts, S_1 and S_2, in this way:

$$S_1 = \sum_{n=0}^{\infty} r^n\left(\frac{a^n s C_{1,z}(r)}{\lambda - a^n s}\right)\prod_{j=0}^{n-1}\varphi_B(a^j s)\frac{\lambda}{\lambda - a^j s}, \quad S_2 = \sum_{n=0}^{\infty} r^n\left(\mathrm{e}^{-a^n s z}\right)\prod_{j=0}^{n-1}\varphi_B(a^j s)\frac{\lambda}{\lambda - a^j s}.$$

S_1 consists of two factors containing r and from (2.11) we then have that $\lim_{r\to 1}(1 - r)C_{1,z}(r) = -\mathbb{P}(Z_\infty = 0) = P$. We conclude thus that the first part of the sum becomes:

$$\lim_{r\to 1}(1 - r)S_1 = -P\sum_{n=0}^{\infty}\frac{a^n s}{\lambda - a^n s}\prod_{j=0}^{n-1}\frac{\varphi_B(a^j s)\lambda}{\lambda - a^j s}. \tag{2.16}$$

For S_2 we have also by Abel's theorem that:

$$\lim_{r \to 1}(1-r)S_2 = \prod_{j=0}^{\infty} \frac{\varphi_B(a^j s)\lambda}{\lambda - a^j s}. \tag{2.17}$$

Adding the limits we have in (2.16) and (2.17) yields then indeed the same expression for $\varphi_Z(s)$ as in (2.14).

Chapter 3

Lindley-Type recursions

This chapter is based on the PhD thesis “Lindley-Type recursions” by Vlasiou [5].

We consider again a single server queue, where the server switches between two service points once it has completed a service. Each customer entering a service point, has to complete two phases: the preparation phase, in which the server is not involved, and the service phase. After completing a service at one point, the server always goes to the other point. Furthermore, it is assumed that at both service points there is an infinite queue of waiting customers, meaning that the server will never have to wait for a customer to get the preparation phase started. Whereas in the previous chapter we were interested in the waiting time of customers, we are now focusing on the waiting time of the server itself. See Figure 3.1 for a graphical representation of the queueing model.

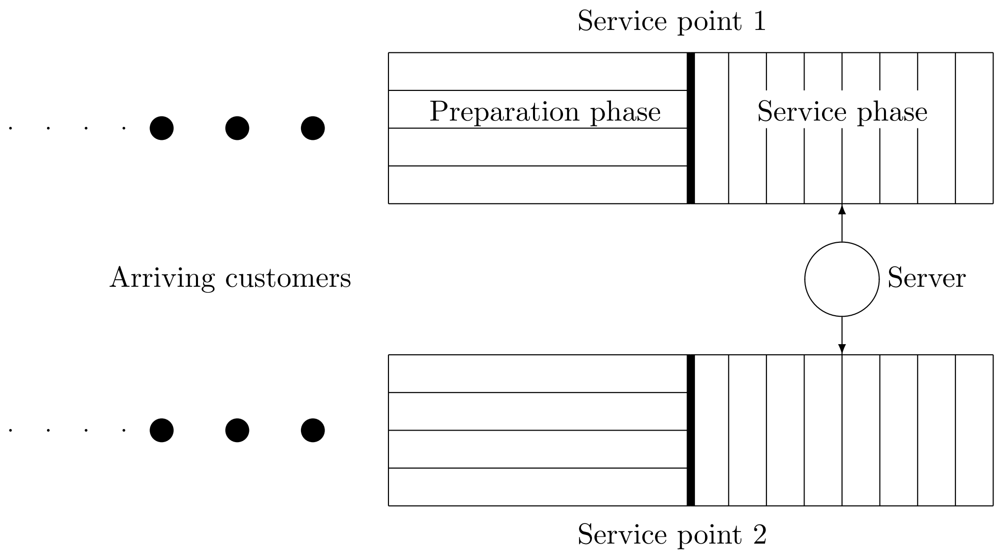

Figure 3.1: The model

Denote the service time of the n-th customer by A_n, the time the server spends waiting for the n-th customer to complete the preparation phase by W_n and let B_n be the preparation time of the n-th customer. Both $(A_n)_n$ and $(B_n)_n$ are sequences of non-negative, independent, identically distributed random variables. Furthermore, both sequences are jointly independent. W_{n+1} can never be greater than the preparation time of the awaited customer, which is B_{n+1}. However, it might be the case that the $(n+1)$-st customer began the preparation phase while the server was still serving or waiting for the previous customer. So we need to subtract $W_n + A_n$ from B_{n+1}. Taking into account that the waiting can never be negative, we arrive at the following recursive model for the waiting time W_{n+1} for $n \geq 0$:

$$W_{n+1} = \max\{B_{n+1} - A_n - W_n, 0\}.$$

The stationary distribution for the server's waiting time will be the topic of interest.

We will look at the so called M/G model, meaning that the preparation time B has a general distribution and A, the service time, follows the exponential distribution with parameter λ. So we have: $f_A(x) = \lambda \mathrm{e}^{-\lambda x}$. We will see that, although the recursion equation looks very similar to the one studied in Chapter 2, the distribution of W_{n+1} cannot be solved using the same techniques. The reason for this is the difference in sign before W, which makes solving the model using Laplace transforms not possible.

3.1 The M/G model

The stationary distribution is denoted in the following way:

$$W \stackrel{d}{=} \max\{B - A - W, 0\},$$

where A, B and W are as defined earlier. This distribution exists whenever $\mathbb{P}(B < A) > 0$, so that there is a strictly positive probability that the service time of a customer is longer than the preparation time of the next customer, resulting in a waiting time of zero for the server. We omit the mathematical proof for this stability condition, but it can be found in the thesis by Vlasiou. As a result, there is a strictly positive probability that the waiting time is 0, which is denoted as follows: $\mathbb{P}(W = 0) = 1 - \int_0^\infty f_W(y)\mathrm{d}y = \pi_0$. Note that π_0 needs still to be determined. We refer to the literature [5] (page 104) that W indeed has a density, f_W. In the following derivation is used that for two mutually independent and non-negative random variables X and Y holds that the convolution $Z = X + Y$ has the distribution function: $F_Z(z) = \int_0^z F_X(z-y)f_y(y)\mathrm{d}y$. From now on, the distribution function of the waiting time, F_W, can systematically be derived.

$$\begin{aligned}
F_W(x) &= \mathbb{P}(W \leq x) = \mathbb{P}(B - A - W \leq x) \\
&= \int_0^\infty \int_0^\infty F_B(x+z+y) f_A(z) f_W(y)\mathrm{d}z\mathrm{d}y \\
&= \pi_0 \int_0^\infty F_B(x+z) f_A(z)\mathrm{d}z + \int_{0+}^\infty \int_0^\infty F_B(x+z+y) f_A(z) f_W(y)\mathrm{d}z\mathrm{d}y \\
&= \pi_0 \int_0^\infty F_B(x+z) \lambda \mathrm{e}^{-\lambda z}\mathrm{d}z + \int_{0+}^\infty \int_0^\infty F_B(x+z+y) \lambda \mathrm{e}^{-\lambda z} f_W(y)\mathrm{d}z\mathrm{d}y \\
&= \lambda \pi_0 \int_{0+}^\infty F_B(x+z) \mathrm{e}^{\lambda x} \mathrm{e}^{-\lambda(x+z)}\mathrm{d}z \\
&\quad + \int_{0+}^\infty \int_0^\infty F_B(x+z+y) \lambda \mathrm{e}^{\lambda(x+y)} \mathrm{e}^{-\lambda(x+z+y)} f_W(y)\mathrm{d}z\mathrm{d}y.
\end{aligned}$$

If now appropriate substitution is applied and the integral boundaries are adjusted accordingly the following equation is obtained:

$$F_W(x) = \lambda \mathrm{e}^{\lambda x} \pi_0 \int_x^\infty F_B(u) \mathrm{e}^{-\lambda u}\mathrm{d}u + \int_{0+}^\infty \lambda \mathrm{e}^{\lambda(x+y)} f_W(y) \int_{x+y}^\infty F_B(u) \mathrm{e}^{-\lambda u}\mathrm{d}u\mathrm{d}y. \tag{3.1}$$

For deriving the density f_W we need to have that F_B is continuous and has a continuous first derivative, $\frac{\mathrm{d}}{\mathrm{d}x}F_B(x)$, then in the relevant domain we can by the Leibniz integral rule [3] interchange the differentiation operator and the integral, resulting in this expression for the density:

$$\begin{aligned}
f_W(x) &= \lambda^2 \pi_0 \mathrm{e}^{\lambda x} \int_x^\infty F_B(u) \mathrm{e}^{-\lambda u}\mathrm{d}u - \lambda \pi_0 F_B(x) + \\
&\quad + \lambda \int_{0+}^\infty \lambda \mathrm{e}^{\lambda(x+y)} f_W(y) \int_{x+y}^\infty F_B(u) \mathrm{e}^{-\lambda u}\mathrm{d}u\mathrm{d}y - \lambda \int_0^\infty F_B(x+y) f_W(y)\mathrm{d}y.
\end{aligned} \tag{3.2}$$

It can be seen that (3.1) appears with an extra λ factor in (3.2). Therefore the density could be rewritten in the more concise form:

$$f_W(x) = -\lambda\pi_0 F_B(x) - \lambda \int_0^\infty F_B(x+y) f_W(y)\mathrm{d}y + \lambda F_W(x). \tag{3.3}$$

As mentioned before, it is not straightforward to solve equation (3.3), because of the fact that $F_B(x+y)$ is not a convolution (it would be if there were a $-$ sign). Furthermore it is somewhat unclear if there actually exists a solution, since there is yet little known about F_B. Apparently, if B has a distribution that belongs to the so called $\mathcal{M}$ class, then for F_W an explicit expression can be derived ([5], page 106/107). This $\mathcal{M}$ class is the set of distributions for which the tail, ($\bar{F}$), can be decomposed in the following way:

$$\bar{F}(x+y) = 1 - F(x+y) = \sum_{i=1}^n g_i(x) h_i(y), \quad \forall x, y \geq 0.$$

It can be shown that all distributions which have a rational Laplace transform belong to the $\mathcal{M}$ class. For the remainder of this chapter we assume that F_B is in $\mathcal{M}$ and that for every $i = 1, ..., n$ both h_i and $\int_0^\infty |g_i(x)|\mathrm{d}x$ are bounded. So (3.3) becomes now:

$$\begin{aligned} f_W(x) &= -\lambda\pi_0(1 - \bar{F}_B(x)) - \lambda \int_0^\infty (1 - \bar{F}_B(x+y)) f_W(y)\mathrm{d}y + \lambda F_W(x) \\ &= -\lambda\pi_0 + \lambda\pi_0 \bar{F}_B(x) - \lambda \int_0^\infty f_W(y)\mathrm{d}y + \lambda \int_0^\infty \bar{F}_B(x+y) f_W(y)\mathrm{d}y + \lambda F_W(x). \end{aligned}$$

Using the expression for the $\bar{F}(x+y)$ term and defining $c_i = \int_0^\infty h_i(y) f_W(y)\mathrm{d}y$ we arrive at:

$$f_W(x) = \lambda F_W(x) + \lambda \left(\pi_0 \bar{F}_B(x) - 1 + \sum_{i=1}^n c_i g_i(x) \right). \tag{3.4}$$

(3.4) is in fact a first order linear differential equation (with initial value π_0) that can be solved for example by the method of integrating factors, which leads to the following solution:

$$F_W(x) = \pi_0 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} \int_0^x \lambda \mathrm{e}^{-\lambda\tau} \left(\pi_0 \bar{F}_B(\tau) - 1 + \sum_{i=1}^n c_i g_i(\tau) \right) \mathrm{d}\tau. \tag{3.5}$$

If the Laplace transform of $\bar{F}_B$ and g_i are respectively denoted by $\hat{\beta}$ and γ_i, then (3.5) reduces to:

$$\begin{aligned} F_W(x) = \mathrm{e}^{\lambda x} &\left(\pi_0 + \lambda\pi_0\hat{\beta}(\lambda) + \lambda \sum_{i=1}^n c_i \gamma_i(\lambda) - 1 \right) \\ &- \mathrm{e}^{\lambda x} \int_x^\infty \lambda \mathrm{e}^{-\lambda\tau} \left(\pi_0 \bar{F}_B(\tau) + \sum_{i=1}^n c_i g_i(\tau) \right) \mathrm{d}\tau + 1. \end{aligned} \tag{3.6}$$

As $x \to \infty$ we know that $F_W(x) \to 1$, which gives that the following needs to hold:

$$\pi_0 + \lambda\pi_0\hat{\beta}(\lambda) + \lambda \sum_{i=1}^n c_i \gamma_i(\lambda) = 1. \tag{3.7}$$

Simplifying (3.6) by using (3.7) and then substituting this result in (3.4) gives an expression for f_W that can be used in order to determine the coefficients c_i for $i = 1, .., n$. More precisely, the c_i's are solutions of the linear system that is given by:

$$\begin{aligned} c_i = \lambda\pi_0 &\int_0^\infty h_i(x) \left(\bar{F}_B(x) - \lambda \int_x^\infty \mathrm{e}^{-\lambda(\tau - x)} \bar{F}_B(\tau)\mathrm{d}\tau \right) \mathrm{d}x \\ &+ \lambda \sum_{j=1}^n c_j \int_0^\infty h_i(x) \left(g_j(x) - \lambda \int_x^\infty \mathrm{e}^{-\lambda(\tau - x)} g_j(\tau)\mathrm{d}\tau \right) \mathrm{d}x. \end{aligned} \tag{3.8}$$

And then the distribution of the waiting time is given by the reduced form of (3.6):

$$F_W(x) = 1 - \lambda \int_x^\infty \mathrm{e}^{-\lambda(\tau - x)} \left(\pi_0 \bar{F}_B(\tau) + \sum_{i=1}^{n} c_i g_i(\tau) \right) \mathrm{d}\tau. \tag{3.9}$$

Although we imposed the criterion that $F_W(x) \to 1$ as $x \to \infty$, this is not completely clear from the formula in (3.9), since there is a factor $e^{\lambda x}$ that tends to infinity. However, it can be proven that the way F_W has been derived in combination with some conditions, leads in fact to the unique limiting distribution for W. For specific functions F_B (3.9) can be evaluated further, but the computation can be quite tedious dependent on the characteristics of F_B.

Chapter 4

On two Classes of Reflected Autoregressive Processes

This chapter is based on the unpublished work "On two Classes of Reflected Autoregressive Processes" by Boxma et al. [1]. The article addresses two models, named INGAR$^+$ and GAR$^+$. The first one is an integer valued process and the second a continuous one. Furthermore, both models describe the waiting time of customers, just like in Chapter 2. The GAR$^+$ model is actually a generalized version of the AR(1) process and therefore we will only discuss the INGAR$^+$ model in this thesis.

4.1 The INGAR$^+$ model

The INGAR$^+$ model describes a process $\{A_n\}$ with $n \in \mathbb{N}^+$ that has non-negative integer values and this process is defined by the recursion:

$$A_{n+1} = \max\{0, U(A_n) + C_n - W_n\},$$

where both (C_n) and (W_n) are independent identically distributed integer-valued and non-negative random variables. Furthermore W_n follows the geometric distribution with success probability $p \in (0, 1]$. So the probability mass function of W_n is: $\mathbb{P}(W_n = k) = p(1-p)^{k-1}$ for $k = 1, 2, ...$ and 0 otherwise. $U(n)$ is defined as the sum of n independent, identically distributed, integer-valued and non-negative random variables U_k. So $U(A_n) = \sum_{k=1}^{A_n} U_k$. It is also assumed that the sequences (U_n), (C_n) and (W_n) are mutually independent. The recursive process is then called an integer-valued generalized autoregressive process, which is reflected at zero. The abbreviated notation for this model is INGAR$^+$.

Discussing the stationary distribution of the process or other stationary behavior would only make sense if stability is guaranteed. For that, we have the following condition:

$$\mathbb{E}[U] < 1 \quad \text{or} \quad (\{\mathbb{E}[U] = 1\} \quad \text{and} \quad \{p\mathbb{E}[C] < 1\}).$$

For deriving transient and stationary distributions for the INGAR$^+$ model, we will use the 'swapped generating function' (SGF) for non-negative random variables:

$$G_X(s) = \mathbb{E}[(1-s)^X] = \sum_{j=0}^{\infty}(1-s)^j \mathbb{P}(X = j), \quad s \in [0, 1].$$

We will now derive the transient and stationary behavior of the model in terms of the SGF.

4.1.1 Transient distribution

In order to determine the transient behavior, we introduce a new generating function (some kind of double transform similar to the one introduced for AR(1) processes):

$$G(r,s)=\sum_{n=0}^{\infty} r^n G_{A_n}(s), \quad |r|<1. \tag{4.1}$$

Furthermore, we propose the function Ψ of s as follows:

$$\Psi(s)=1-G_U(s), \quad \Psi^{(0)}(s)=s, \quad \Psi^{(k)}(s)=\Psi(\Psi^{(k-1)}(s)), \quad k=1,2,....$$

Then:

$$\begin{aligned}
G_{U(X)}(s) &= \mathbb{E}[(1-s)^{\sum_{k=1}^{X} U_k}] = \mathbb{E}[\prod_{k=1}^{X}(1-s)^{U_k}] = \sum_{j=0}^{\infty}\mathbb{P}(X=j)\mathbb{E}[\prod_{k=1}^{X}(1-s)^{U_k}|X=j] \\
&= \sum_{j=0}^{\infty}\mathbb{P}(X=j)\mathbb{E}[\prod_{k=1}^{j}(1-s)^{U_k}] =_{\triangle} \sum_{j=0}^{\infty}\mathbb{P}(X=j)\prod_{k=1}^{j}\mathbb{E}[(1-s)^{U_k}] \\
&= \sum_{j=0}^{\infty}\mathbb{P}(X=j)G_U(s)^j = \sum_{j=0}^{\infty}\mathbb{P}(X=j)(1-\Psi(s))^j = G_X(\Psi(s)),
\end{aligned} \tag{4.2}$$

here we used at $\triangle$ that the U_k's are independent.

We will use the following lemma for computing $G(r,s)$:

Lemma 4.1.1. If X is a non-negative and integer-valued random variable and W an independent geometric random variable with success probability p, then:

$$G_{\max\{0,X-W_p\}}(s)=\frac{p}{p-s}G_X(s)-\frac{s}{p-s}G_X(p).$$

The proof can be found in [1, p.15-16].

As a result of Lemma 2.3.1, and that $G_{U(X)}(s)=G_X(\Psi(s))$ we can rewrite $G_{A_{n+1}}(s)$:

$$\begin{aligned}
G_{A_{n+1}}(s) = G_{\max\{0,U(A_n)+C_n-W_n\}}(s) &= \frac{p}{p-s}G_{U(A_n)+C_n}(s)-\frac{s}{p-s}G_{U(A_n)+C_n}(p) \\
&= \frac{p}{p-s}G_{U(A_n)}(s)G_C(s)-\frac{s}{p-s}G_{U(A_n)}(p)G_C(p) \\
&= \frac{pG_C(s)}{p-s}G_{A_n}(\Psi(s))-\frac{sG_C(p)}{p-s}G_{A_n}(\Psi(p)).
\end{aligned} \tag{4.3}$$

The initial condition that is imposed, is the following: $A_0=a$, hence $G_{A_0}=(1-s)^a$. The procedure is now similar to the computation of $Z_z(r,s)$ in Section 2.1. So we multiply (4.3) with r^{n+1} and then taking the sum of $n=0$ to infinity gives:

$$\sum_{n=0}^{\infty} r^{n+1}G_{A_{n+1}}(s)=\sum_{n=0}^{\infty} r^{n+1}\left(\frac{pG_C(s)}{p-s}G_{A_n}(\Psi(s))-\frac{sG_C(p)}{p-s}G_{A_n}(\Psi(p))\right),$$

and then using (4.1) we get:

$$G(r,s)-(1-s)^a=\frac{r}{p-s}\left(pG_C(s)G(r,\Psi(s))-sG_C(p)G(r,\Psi(p))\right). \tag{4.4}$$

So for $G(r,\Psi(s))$ we get now the following expression:

$$G(r,\Psi(s))-(1-\Psi(s))^a=\frac{r}{p-\Psi(s)}\left(pG_C(\Psi(s))G(r,\Psi^{(2)}(s))-\Psi(s)G_C(p)G(r,\Psi(p))\right).$$

$G(r, s)$ seems thus to be best found by iterating. To make the computation more well-organized, we introduce the following formulas:

$$k(r, s) = \frac{rp}{p - s} G_C(s), \quad l(r, s) = (1 - s)^a - \frac{rs}{p - s} G_C(p) G(r, \Psi(p)).$$

Hence $G(r, s)$ can be rewritten in the form

$$G(r, s) = l(r, s) + k(r, s) G(r, \Psi(s)). \tag{4.5}$$

Then iterating (4.5) $N - 1$ times gives the following expression for $G(r, s)$:

$$\begin{aligned} G(r, s) &= l(r, s) + k(r, s)(l(r, \Psi(s)) + k(r, \Psi(s)) G(r, \Psi^{(2)}(s))) \\ &= l(r, s) + k(r, s) l(r, \Psi(s)) + k(r, s) k(r, \Psi(s)) G(r, \Psi^{(2)}(s)) \\ &= \text{...iteration continues...} \\ &= \sum_{n=0}^{N-1} l(r, \Psi^{(n)}(s)) \prod_{j=0}^{n-1} k(r, \Psi^{(j)}(s)) + G(r, \Psi^{(N)}(s)) \prod_{j=0}^{N-1} k(r, \Psi^{(j)}(s)) \end{aligned} \tag{4.6}$$

Naturally, we are now interested in what happens in (4.6) if $N \to \infty$. First, consider $\Psi^{(N)}(s)$. Using the definition of $G_U(s)$ this can be rewritten as:

$$\begin{aligned} \Psi^{(N)}(s) &= 1 - \mathbb{E}[(1 - \Psi^{(N-1)}(s))^U] \\ &= 1 - \sum_{j=0}^{\infty} (1 - \Psi^{(N-1)}(s))^j \mathbb{P}(U = j) \\ &= \sum_{j=0}^{\infty} (1 - (1 - \Psi^{(N-1)}(s))^j) \mathbb{P}(U = j). \end{aligned} \tag{4.7}$$

Now, since it holds true that $(1 - (1 - x)^j) \leq jx$ for $x \geq 0$ (this can easily be verified by looking at the derivatives with respect to x), we have an upper bound for (4.7) as follows:

$$\Psi^{(N)}(s) \leq \Psi^{(N-1)}(s) \sum_{j=0}^{\infty} j \mathbb{P}(U = j) = \Psi^{(N-1)}(s) \mathbb{E}[U]. \tag{4.8}$$

Having the stability condition $\mathbb{E}[U] \leq 1$ imposed, we now conclude that $\Psi^{(N)}(s) \leq s$, therefore $G(r, \Psi^{(N)}(s))$ and $G_C(\Psi^{(N)}(s))$ are well defined in the sense that they do not diverge as $N \to \infty$. An even stronger conclusion can be drawn, namely that if $\mathbb{E}[U] < 1$ holds true, then $\Psi^{(N)}(s)$ tends to zero, for large N. Next we investigate whether the second product in (4.6) converges. (Note that SGF's are bounded by 1):

$$\prod_{j=0}^{N-1} k(r, \Psi^{(j)}(s)) = \prod_{j=0}^{N-1} \frac{rp}{p - \Psi^{(j)}(s)} G_C(\Psi^{(j)}(s)) \leq \prod_{j=0}^{N-1} \frac{rp}{p - \Psi^{(j)}(s)}. \tag{4.9}$$

As $N \to \infty$, it is easy to see that the coefficients in the product tend to 0 if $\mathbb{E}[U] < 1$, hence the products equals 0. If $\mathbb{E}[U] = 1$, then choosing r sufficiently small will also give rise to the desired result. Lastly, we need to check convergence of the first term in (4.6). Using the ratio test and the imposed conditions it can be verified that this sum indeed converges as $N \to \infty$. Hence, (4.6) becomes:

$$G(r, s) = \sum_{n=0}^{\infty} l(r, \Psi^{(n)}(s)) \prod_{j=0}^{n-1} k(r, \Psi^{(j)}(s)). \tag{4.10}$$

Still, in $l(r,s)$ we have the unknown $G(r,\Psi(p))$. This can however easily be found by substituting $s=\Psi(p)$ into (4.10), so that we get:

$$\begin{aligned}G(r,\Psi(p)) &= \sum_{n=0}^{\infty} l(r,\Psi^{(n+1)}(p)) \prod_{j=0}^{n-1} k(r,\Psi^{(j+1)}(p)) \\ &= \sum_{n=0}^{\infty} \left((1-\Psi^{(n+1)}(p))^a - \frac{r\Psi^{(n+1)}(p)}{p-\Psi^{(n+1)}(p)} G_C(p) G(r,\Psi(p)) \right) \prod_{j=0}^{n-1} k(r,\Psi^{(j+1)}(p)),\end{aligned}$$

and hence:

$$G(r,\Psi(p)) = \frac{\sum_{n=0}^{\infty}(1-\Psi^{(n+1)}(p))^a \prod_{j=0}^{n-1} k(r,\Psi^{(j+1)}(p))}{1+\sum_{n=0}^{\infty}\left(\frac{r\Psi^{(n+1)}(p)}{p-\Psi^{(n+1)}(p)} G_C(p)\right) \prod_{j=0}^{n-1} k(r,\Psi^{(j+1)}(p))}. \tag{4.11}$$

Remark 4.1.1. Similar to what we have seen for the AR(1) processes, one might think that poles can occur in $G(r,s)$ for $p=\Psi^{(i)}(s)$ for $i=1,2,\ldots$. However, also in this case we are not dealing with poles but with removable singularities. More precisely, for the same values of s for which the denominator has roots, the numerator becomes zero as well.

4.1.2 Stationary distribution

Next, we consider the stationary distribution, so we look at G_{A_n} as $n\to\infty$. Assuming this distribution indeed exists, it must hold for large n that $G_{A_{n+1}}{=}G_{A_n}$. Now denote the SGF of A_∞ by $G_A(s)$. Then it follows from (4.3) and repeatedly iterating that:

$$\begin{aligned}G_A(s) &= \frac{pG_C(s)}{p-s} G_A(\Psi(s)) - \frac{sG_C(p)}{p-s} G_A(\Psi(p)) \\ &= \frac{pG_C(s)}{p-s}\frac{pG_C(\Psi(s))}{p-\Psi(s)} G_A(\Psi^{(2)}(s)) - \frac{sG_C(p)}{p-s} G_A(\Psi(p)) - \frac{pG_C(s)}{p-s}\frac{\Psi(s)}{p-\Psi(s)} G_C(p) G_A(\Psi(p)) \\ &= \ldots \\ &= \prod_{j=0}^{\infty} \frac{pG_C(\Psi^{(j)})(s)}{p-\Psi^{(j)}(s)} - G_C(p)G_A(\Psi(p)) \sum_{n=0}^{\infty} \frac{\Psi^{(n)}(s)}{p-\Psi^{(n)}(s)} \prod_{j=0}^{n-1} \frac{pG_C(\Psi^{(j)}(s))}{p-\Psi^{(j)}(s)}.\end{aligned} \tag{4.12}$$

$G_A(\Psi(p))$ can be found by substituting $s=\Psi(p)$ into (4.12), yielding the following formula:

$$G_A(\Psi(p)) = \frac{\prod_{j=0}^{\infty} \frac{pG_C(\Psi^{(j+1)})(p)}{p-\Psi^{(j+1)}(p)}}{1+G_C(p)\sum_{n=0}^{\infty} \frac{\Psi^{(n+1)}(p)}{p-\Psi^{(n+1)}(p)} \prod_{j=0}^{n-1} \frac{pG_C(\Psi^{(j+1)}(p))}{p-\Psi^{(j+1)}(p)}} \tag{4.13}$$

Together, (4.12) and (4.13) define now the swapped generating function for the stationary distribution of A in the INGAR$^+$ model.

Alternatively, (4.12) could also be obtained from (4.6) by applying an Abel theorem, similar to what was done in Chapter 2.

Part II

Research

Chapter 5

Work load model with downward jumps

In this chapter we will look at the work load (amount of work) in a system, where work arrives and is removed according to a Poisson process. First, a description of the model is given and then the steady state value of the work load will be analysed in term of the Laplace-Stieltjes transform.

Work is brought into the system or removed from it according to a Poisson process with rate λ. This arrival rate can be written as $\lambda = \lambda_+ + \lambda_-$, where λ_+ corresponds to the arrival intensity of extra work and λ_- is the rate of work removals. The amount of work that is added in case of a work arrival is denoted by the random variable B_+, and the amount of work that is removed is denoted by B_-. It is assumed that all random variables are independent. For now, we say that B_- is exponentially distributed with parameter μ and for B_+ we take a general distribution with $b(x)$ and $\beta(\vartheta)$ respectively denoting the density and the Laplace-Stieltjes transform. Furthermore, if there is now an amount x of work in the system, then the server serves with a rate rx, where $r > 0$.

Let $Q(t)$ denote the amount of work in the system at time $t > 0$. Then Q is the random variable that denotes the steady state of $Q(t)$, having density $q(x)$, cumulative distribution function $F_Q(x)$ and Laplace-Stieltjes transform (LST) $\varphi(\vartheta)$. Since the work rate of the server is proportional with the amount of work in the system and upward jumps only occur with finite intensity and have a finite magnitude, the system will never "explode", hence the model will always be stable in this setting.

We will derive an expression for $\varphi(\vartheta)$ by formulating a so called level crossings identity and solving the resulting differential equation.

5.1 Level crossings identity

Assuming that the stability condition is satisfied, we will formulate the level crossings identity. This is an equality that describes the rate at which a certain level, say x, is crossed. In steady state, each level is of course equally often crossed from above as from below. See figure 5.1 for an illustration of the model. The rate at which a level x is crossed from above is the sum of two components: during service or with a jump downwards. The first term is simply $rxq(x)$ and the second one is $\lambda_-\mathbb{P}(B_- > Q - x)$. Using basic probability theory and the fact that B_- is exponentially distributed, this term equals:

$$\lambda_- \int_x^\infty q(y)\mathrm{e}^{-\mu(y-x)}\mathrm{d}y.$$

A level can only be crossed from below when work is added to the system, which happens at a

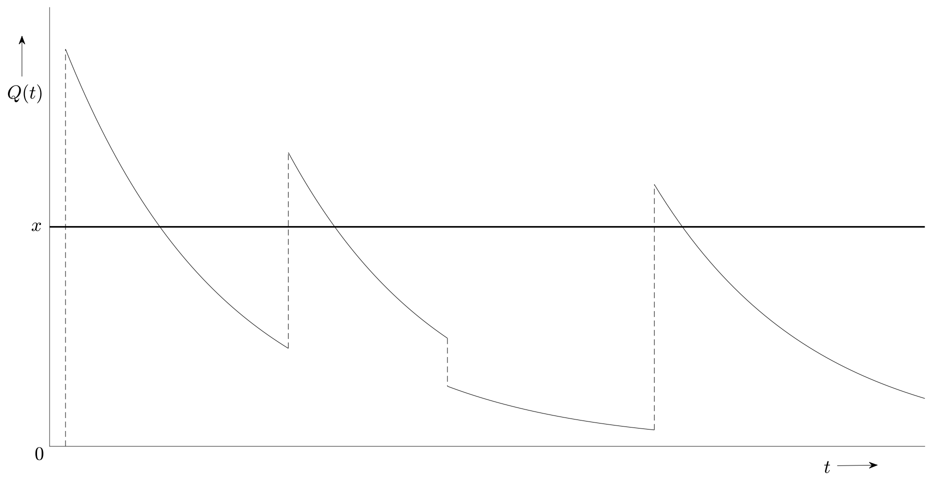

Figure 5.1: The work load in time

rate λ_+. Level x is then crossed with probability $p_x = \mathbb{P}(B_+ > x - Q) = \int_0^x \mathbb{P}(B_+ > x - y)\mathrm{d}F_Q(y)$. Similar to what we saw in Chapter 3, there is a strictly positive probability (π_0) that the system is empty (because of the reflection at zero), and hence p_x can be written as follows:

$$p_x = \lambda_+ \int_{0^+}^{x} q(y)\mathbb{P}(B_+ > x - y)\mathrm{d}y + \lambda_+\pi_0\mathbb{P}(B_+ > x),$$

where π_0 still needs to be determined. The level crossing identity is now the following:

$$rxq(x) + \lambda_- \int_{x}^{\infty} q(y)\mathrm{e}^{-\mu(y-x)}\mathrm{d}y = \lambda_+ \int_{0^+}^{x} q(y)\mathbb{P}(B_+ > x - y)\mathrm{d}y + \lambda_+\pi_0\mathbb{P}(B_+ > x). \tag{5.1}$$

We now multiply both sides of equation (5.1) with $\mathrm{e}^{-\vartheta x}$ and then integrate over $x > 0$. For the left hand side, say $L(x)$, this gives:

$$\begin{aligned}\int_0^\infty L(x)\mathrm{e}^{-\vartheta x}\mathrm{d}x &= \int_0^\infty rxq(x)\mathrm{e}^{-\vartheta x}\mathrm{d}x + \lambda_- \int_0^\infty \int_x^\infty q(y)\mathrm{e}^{-\mu(y-x)}\mathrm{d}y\mathrm{e}^{-\vartheta x}\mathrm{d}x \\ &= -r\int_0^\infty q(x)\frac{d}{d\vartheta}\mathrm{e}^{-\vartheta x}\mathrm{d}x + \lambda_- \int_0^\infty q(y)\mathrm{e}^{-\mu y}\int_0^y \mathrm{e}^{x(\mu-\vartheta)}\mathrm{d}x\mathrm{d}y \\ &= -r\frac{d}{d\vartheta}\varphi(\vartheta) - \lambda_-\left(\frac{1}{\vartheta-\mu}\right)\int_0^\infty q(y)\mathrm{e}^{-\mu y}(\mathrm{e}^{\mu y-\theta y} - 1)\mathrm{d}y \\ &= -r\varphi'(\vartheta) - \lambda_-\frac{\varphi(\vartheta)-\varphi(\mu)}{\vartheta-\mu}.\end{aligned} \tag{5.2}$$

Remark 5.1.1. For the computation of equation (5.2) we used that since $\mathrm{e}^{-\vartheta x}$ has continuous first partial derivatives it is allowed to change the order of differentiation and integration [3].

Remark 5.1.2. In this model we assumed that downward jumps follow the exponential distribution with parameter μ. If these would however follow a general distribution with density $g(x)$ and LST $\gamma(\vartheta)$, then the level crossing identity would of course be different as well. The second term in (5.1) (call it $c(x)$) would be different, whereas the other terms remain the same. Then multiplying $c(x)$ with $\mathrm{e}^{-\vartheta x}$ and integrating over x, this gives:

$$\begin{aligned}\int_0^\infty c(x)\mathrm{e}^{-\vartheta x}\mathrm{d}x =& \lambda_- \int_0^\infty \int_x^\infty q(y)\mathrm{e}^{-\vartheta x}\left(1 - \int_0^{y-x} g(t)\mathrm{d}t\right)\mathrm{d}y\mathrm{d}x \\ &= \lambda_-\frac{1-\varphi(\vartheta)}{\vartheta} - \lambda_- \int_0^\infty \int_0^y q(y)g(t)\int_0^{y-t} \mathrm{e}^{-\vartheta x}\mathrm{d}x\mathrm{d}t\mathrm{d}y.\end{aligned} \tag{5.3}$$

This integral seems troublesome for general distributions because we get a factor $e^{\vartheta t}$. Possibly, distributions which are part of the exponential family, i.e. hyper-exponential distributions, will give rise to definite integrals.

For the right hand side of equation (5.1), say $R(x)$, we do the same:

$$\begin{aligned}\int_0^\infty R(x)e^{-\vartheta x}dx &= \int_0^\infty \lambda_+ \int_{0+}^x q(y)\mathbb{P}(B_+ > x-y)dy e^{-\vartheta x}dx + \lambda_+\pi_0 \int_0^\infty \mathbb{P}(B_+ > x)e^{-\vartheta x}dx \\ &= \lambda_+ \int_0^\infty \int_{0+}^x q(y)\left(1 - \int_0^{x-y} b(t)dt\right)e^{-\vartheta x}dydx \\ &+ \lambda_+\pi_0 \int_0^\infty \left(1 - \int_0^x b(t)dt\right)e^{-\vartheta x}dx \\ &= \lambda_+ \int_{0+}^\infty \int_y^\infty q(y)e^{-\vartheta x}dxdy - \lambda_+ \int_0^\infty \int_{0+}^\infty q(y)b(t)\int_{t+y}^\infty e^{-\vartheta x}dxdydt \\ &+ \lambda_+\pi_0\left(\frac{1}{\vartheta} - \int_0^\infty \int_t^\infty b(t)e^{-\vartheta x}ddt\right) \\ &= \lambda_+\varphi(\vartheta)\frac{1}{\vartheta} - \lambda_+\frac{1}{\vartheta}\int_0^\infty q(y)e^{-\vartheta y}dy \int_0^\infty b(t)e^{-\vartheta t}dt + \lambda_+\pi_0\frac{1-\beta(\vartheta)}{\vartheta} \\ &= \lambda_+\varphi(\vartheta)\frac{1-\beta(\vartheta)}{\vartheta} + \lambda_+\pi_0\frac{1-\beta(\vartheta)}{\vartheta}.\end{aligned} \tag{5.4}$$

Since $L(x) = R(x)$, combining (5.2) with (5.4) gives the following first-order differential equation:

$$-r\varphi'(\vartheta) - \lambda_-\frac{\varphi(\vartheta)-\varphi(\mu)}{\vartheta-\mu} = \lambda_+\varphi(\vartheta)\frac{1-\beta(\vartheta)}{\vartheta} + \lambda_+\pi_0\frac{1-\beta(\vartheta)}{\vartheta}. \tag{5.5}$$

Taking into account that for any LST $\psi(x)$ it holds that, as $x \to \infty$, $\psi(x) \to 0$, then after multiplying equation (5.5) with ϑ and sending ϑ to ∞, we end up with this relation:

$$\lambda_-\varphi(\mu) = \lambda_+\pi_0. \tag{5.6}$$

$\varphi(\mu)$ is in fact the probability that a jump downwards will result in an empty system, i.e. a system with no work. The larger μ, the smaller the downward jumps on average, and hence also $\varphi(\mu)$ will be smaller. We see thus that increasing λ_- and decreasing λ_+ and μ will yield a higher probability of having an empty system.

We can use the relation (5.6) to derive from equation (5.5) the following equality, which will be used in the next section to get a formula for $\varphi(\vartheta)$:

$$\varphi'(\vartheta) + \frac{1}{r}\left(\frac{\lambda_-}{\vartheta-\mu} + \frac{\lambda_+(1-\beta(\vartheta))}{\vartheta}\right)\varphi(\vartheta) = -\frac{\lambda_+\pi_0}{r}\left(\frac{1-\beta(\vartheta)}{\vartheta} - \frac{1}{\vartheta-\mu}\right). \tag{5.7}$$

5.2 Laplace-Stieltjes transform of the steady-state work load

The goal is now to find a solution for this differential equation (5.7) so that we have the distribution of Q in terms of its LST. Note that, since $\mathbb{P}(Q=0) = \pi_0$, we cannot simply conclude that $\varphi(0) = 1$, but we have that: $\varphi(0) = 1-\pi_0$. Furthermore, we will use that in general the solution of a first-order differential equation of the form $y'(x) + p(x)y(x) = q(x), x > 0, y(0) = y_0$ is given by:

$$y(x) = y_0 e^{-\rho(x)} + e^{-\rho(x)}\int_0^x q(\tau)e^{\rho(\tau)}d\tau,$$

where $\rho(x) = \int_0^x p(\tau)d\tau$.

If we apply this formula on the equation in (5.7), then $\rho(\vartheta)$ is:

$$\begin{aligned}\rho(\vartheta) &= \int_0^{\vartheta}\left[\frac{\lambda_-}{r(\tau-\mu)}+\frac{\lambda_+}{r}\frac{1-\beta(\tau)}{\tau}\right]\mathrm{d}\tau = \frac{\lambda_-}{r}\left(\log(\vartheta-\mu)-\log(-\mu)\right)+\frac{\lambda_+}{r}\int_0^{\vartheta}\frac{1-\beta(\tau)}{\tau}\mathrm{d}\tau \\ &= \frac{\lambda_-}{r}\log\left(1-\frac{\vartheta}{\mu}\right)+\frac{\lambda_+}{r}\int_0^{\vartheta}\frac{1-\beta(\tau)}{\tau}\mathrm{d}\tau.\end{aligned} \tag{5.8}$$

So then we can write the solution for $\varphi(\vartheta)$:

$$\begin{aligned}\varphi(\vartheta) &= \left(1-\frac{\vartheta}{\mu}\right)^{-\frac{\lambda_-}{r}}\mathrm{e}^{-\frac{\lambda_+}{r}\int_0^{\vartheta}\frac{1-\beta(\tau)}{\tau}\mathrm{d}\tau}\times \\ &\times\left[1-\pi_0-\frac{\lambda_+\pi_0}{r}\int_0^{\vartheta}\left(\frac{1-\beta(t)}{t}-\frac{1}{t-\mu}\right)\left(1-\frac{t}{\mu}\right)^{\frac{\lambda_-}{r}}\mathrm{e}^{\frac{\lambda_+}{r}\int_0^{t}\frac{1-\beta(\tau)}{\tau}\mathrm{d}\tau}\mathrm{d}t\right].\end{aligned} \tag{5.9}$$

Clearly, from equation (5.9) we can conclude that the factor $(1-\vartheta/\mu)^{(-\lambda_-/r)}$ gives rise to a singularity in $\vartheta=\mu$. However, we do know from relation (5.6) that $\varphi(\mu)$ is finite, and hence the singularity is removable. In order to have that pole indeed removed, we take $1-\pi_0$ as follows:

$$1-\pi_0 = \frac{\lambda_+\pi_0}{r}\int_0^{\mu}\left(\frac{1-\beta(t)}{t}-\frac{1}{t-\mu}\right)\left(1-\frac{t}{\mu}\right)^{\frac{\lambda_-}{r}}\mathrm{e}^{\frac{\lambda_+}{r}\int_0^{t}\frac{1-\beta(\tau)}{\tau}\mathrm{d}\tau}\mathrm{d}t,$$

which gives then ultimately this expression for π_0 :

$$\pi_0 = \frac{1}{1+\frac{\lambda_+}{r}\int_0^{\mu}\left(\frac{1-\beta(t)}{t}-\frac{1}{t-\mu}\right)\left(1-\frac{t}{\mu}\right)^{\frac{\lambda_-}{r}}\mathrm{e}^{\frac{\lambda_+}{r}\int_0^{t}\frac{1-\beta(\tau)}{\tau}\mathrm{d}\tau}\mathrm{d}t}. \tag{5.10}$$

If we substitute the expression for $1-\pi_0$ into the formula derived for $\varphi(\vartheta)$, we end up with the solution in which all parameters are now defined (for reasons of conciseness we keep π_0 in the formula):

$$\begin{aligned}\varphi(\vartheta) &= \left(1-\frac{\vartheta}{\mu}\right)^{-\frac{\lambda_-}{r}}\mathrm{e}^{-\frac{\lambda_+}{r}\int_0^{\vartheta}\frac{1-\beta(\tau)}{\tau}\mathrm{d}\tau}\times\frac{\lambda_+\pi_0}{r}\int_{\vartheta}^{\mu}\left(\frac{1-\beta(t)}{t}-\frac{1}{t-\mu}\right)\left(1-\frac{t}{\mu}\right)^{\frac{\lambda_-}{r}}\mathrm{e}^{\frac{\lambda_+}{r}\int_0^{t}\frac{1-\beta(\tau)}{\tau}\mathrm{d}\tau}\mathrm{d}t \\ &= -\frac{\frac{\lambda_+\pi_0}{r}\int_{\mu}^{\vartheta}\left(\frac{1-\beta(t)}{t}-\frac{1}{t-\mu}\right)\left(1-\frac{t}{\mu}\right)^{\frac{\lambda_-}{r}}\mathrm{e}^{\frac{\lambda_+}{r}\int_0^{t}\frac{1-\beta(\tau)}{\tau}\mathrm{d}\tau}\mathrm{d}t}{\left(1-\frac{\vartheta}{\mu}\right)^{\frac{\lambda_-}{r}}\mathrm{e}^{\frac{\lambda_+}{r}\int_0^{\vartheta}\frac{1-\beta(\tau)}{\tau}\mathrm{d}\tau}} \\ &= -\frac{\lambda_+\pi_0}{r}\int_{\mu}^{\vartheta}\left(\frac{1-\beta(t)}{t}-\frac{1}{t-\mu}\right)\left(\frac{\mu-t}{\mu-\vartheta}\right)^{\frac{\lambda_-}{r}}\mathrm{e}^{\frac{\lambda_+}{r}\int_{\vartheta}^{t}\frac{1-\beta(\tau)}{\tau}\mathrm{d}\tau}\mathrm{d}t.\end{aligned} \tag{5.11}$$

To verify that if we let $\vartheta\to\mu$ we get indeed the same as in (5.6), we compute the following:

$$\lim_{\vartheta\to\mu}\varphi(\vartheta) = \lim_{\vartheta\to\mu}-\frac{\lambda_+\pi_0}{r}\int_{\mu}^{\vartheta}\left(\frac{1-\beta(t)}{t}-\frac{1}{t-\mu}\right)\left(\frac{\mu-t}{\mu-\vartheta}\right)^{\frac{\lambda_-}{r}}\mathrm{e}^{\frac{\lambda_+}{r}\int_{\vartheta}^{t}\frac{1-\beta(\tau)}{\tau}\mathrm{d}\tau}\mathrm{d}t.$$

The term containing $(1-\beta(t)/t)$ will vanish in the limit as $\vartheta\to\mu$, since this part is continuous in the entire integration domain. So we need to check the following:

$$\lim_{\vartheta\to\mu}-\frac{\lambda_+\pi_0}{r}\int_{\mu}^{\vartheta}\left(-\frac{1}{t-\mu}\right)\left(\frac{\mu-t}{\mu-\vartheta}\right)^{\frac{\lambda_-}{r}}\mathrm{e}^{\frac{\lambda_+}{r}\int_{\vartheta}^{t}\frac{1-\beta(\tau)}{\tau}\mathrm{d}\tau}\mathrm{d}t = \lim_{\vartheta\to\mu}\frac{\int_{\mu}^{\vartheta}h(t)(\mu-t)^{\frac{\lambda_-}{r}-1}\mathrm{d}t}{(\mu-\vartheta)^{\frac{\lambda_-}{r}}}, \tag{5.12}$$

where $h(t) = -\frac{\lambda_+ \pi_0}{r} \mathrm{e}^{\frac{\lambda_+}{r}\int_\vartheta^t \frac{1-\beta(\tau)}{\tau}\mathrm{d}\tau}$. We apply then L'Hopital rule on (5.12) to get:

$$\lim_{\vartheta\to\mu} \frac{\int_\mu^\vartheta h(t)(\mu-t)^{\frac{\lambda_-}{r}-1}\mathrm{d}t}{(\mu-\vartheta)^{\frac{\lambda_-}{r}}} = \lim_{\vartheta\to\mu} \frac{h(t)(\mu-t)^{\frac{\lambda_-}{r}-1}\Big|_\mu^\vartheta}{-\frac{\lambda_-}{r}(\mu-\vartheta)^{\frac{\lambda_-}{r}-1}} = -\frac{rh(\vartheta)}{\lambda_-}. \tag{5.13}$$

Clearly, $h(\vartheta) = -\frac{\lambda_+\pi_0}{r}$, so we can indeed conclude that $\varphi(\mu) = (\lambda_+\pi_0)/\lambda_-$, confirming (5.6).

Completely evaluating the integral that appears in (5.11), even in case the distribution of B_+ is known, would still yield a quite tedious formula, leaving us with very little insight in the distribution function (or density) of the amount of work in the system. To get a better grasp of how the distribution would look like, we will derive expressions for the expectation and other moments of Q.

5.3 Moments of the work load

In this section we will derive expressions for the moments of the work load in the model, so $\mathbb{E}[Q^n]$ for $n = 1, 2, 3, \ldots$. This could be done by differentiating the expression for the Laplace-Stieltjes transform, which has been derived in the previous section. An other way, which seems more convenient, is to work with equation (5.1) directly. The first moment, i.e. the expected value of Q, could then be found by integrating both sides of said equation over x from $x = 0$ to ∞. This leads to the following equality:

$$r\mathbb{E}[Q] + \frac{\lambda_-}{\mu}(1 - \pi_0 - \varphi(\mu)) = \lambda_+\mathbb{E}[B_+],$$

where we used that:$\int_0^\infty \mathbb{P}(X > x)\mathrm{d}x = \mathbb{E}[X]$. Hence we see that $\varphi(\mu)$ becomes an interesting quantity in the model regarding the moments of Q. Evaluating the above equation a bit further and using (5.6), we end up with the following explicit expression for the expectation of Q:

$$\mathbb{E}[Q] = \frac{\lambda_+\mathbb{E}[B_+] + \frac{1}{\mu}(\lambda_+\pi_0 + \lambda_-\pi_0 - \lambda_-)}{r}. \tag{5.14}$$

It is clear - and also something that one would expect - that increasing the mean value of B_+ and the value λ_+ and decreasing the values of λ_-, r and μ will lead to a higher expectation for Q.

For finding the n-th moment of Q we need to multiply (5.1) with x^{n-1} and then integrate over x from $x = 0$ to ∞, so that we end up with the equation:

$$r\int_0^\infty x^n q(x)\mathrm{d}x + \lambda_-\int_0^\infty\int_x^\infty q(y)x^{n-1}\mathrm{e}^{-\mu(y-x)}\mathrm{d}y\mathrm{d}x = \lambda_+\int_0^\infty\int_{0+}^x q(y)\mathbb{P}(B_+ > x-y)x^{n-1}\mathrm{d}y\mathrm{d}x$$
$$+ \lambda_+\int_0^\infty \pi_0\mathbb{P}(B_+ > x)x^{n-1}\mathrm{d}x.$$

We will now split this equation into four parts and evaluate them separately:

$$L_1(n) = r\int_0^\infty x^n q(x)\mathrm{d}x, \quad L_2(n) = \lambda_-\int_0^\infty\int_x^\infty q(y)x^{n-1}\mathrm{e}^{-\mu(y-x)}\mathrm{d}y\mathrm{d}x,$$
$$R_1(n) = \lambda_+\int_0^\infty\int_{0+}^x q(y)\mathbb{P}(B_+ > x-y)x^{n-1}\mathrm{d}y\mathrm{d}x, \quad R_2(n) = \lambda_+\int_0^\infty \pi_0\mathbb{P}(B_+ > x)x^{n-1}\mathrm{d}x.$$

$L_1(n)$ surely is the easiest term, since it describes the $n-$th moment of Q:

$$L_1(n) = r\mathbb{E}[Q^n]. \tag{5.15}$$

For $L_2(n)$ we will show how it looks like for $n = 2$ and then derive an expression for general n. Integration by parts will yields that for $n = 2$ we get:

$$\begin{aligned}
\frac{1}{\lambda_-}L_2(2) &= \int_0^\infty \int_x^\infty q(y)x\mathrm{e}^{-\mu(y-x)}\mathrm{d}y\mathrm{d}x = \int_0^\infty \int_0^y q(y)x\mathrm{e}^{-\mu(y-x)}\mathrm{d}x\mathrm{d}y \\
&= \int_0^\infty \left[\int_0^y x\mathrm{e}^{\mu x}\mathrm{d}x\right] q(y)\mathrm{e}^{-\mu y}\mathrm{d}y = \int_0^\infty \left[\frac{1}{\mu}x\mathrm{e}^{\mu x}\Big|_0^y - \frac{1}{\mu}\int_0^y \mathrm{e}^{\mu x}\mathrm{d}x\right] q(y)\mathrm{e}^{-\mu y}\mathrm{d}y \\
&= \int_0^\infty \left[\frac{1}{\mu}y\mathrm{e}^{\mu y} - \frac{1}{\mu^2}\left(\mathrm{e}^{\mu y} - 1\right)\right] q(y)\mathrm{e}^{-\mu y}\mathrm{d}y = \frac{1}{\mu}\left(\mathbb{E}[Q] - \frac{1}{\mu}(1-\pi_0) + \frac{1}{\mu}\varphi(\mu)\right).
\end{aligned}$$

For general $n \geq 2$ we have that:

$$\begin{aligned}
\frac{1}{\lambda_-}L_2(n) &= \int_0^\infty \int_x^\infty q(y)x^{n-1}\mathrm{e}^{-\mu(y-x)}\mathrm{d}y\mathrm{d}x = \int_0^\infty \left[\int_0^y x^{n-1}\mathrm{e}^{\mu x}\mathrm{d}x\right] q(y)\mathrm{e}^{-\mu y}\mathrm{d}y \\
&= \int_0^\infty \left[\frac{1}{\mu}x^{n-1}\mathrm{e}^{\mu x}\Big|_0^y - \frac{n-1}{\mu}\int_0^y x^{n-2}\mathrm{e}^{\mu x}\mathrm{d}x\right] q(y)\mathrm{e}^{-\mu y}\mathrm{d}y.
\end{aligned}$$

Now note that $\int_0^\infty \int_0^y x^{n-2}\mathrm{e}^{\mu x}\mathrm{d}xq(y)\mathrm{e}^{-\mu y}\mathrm{d}y$ is actually $(1/\lambda_-)L_2(n-1)$. Proceeding these calculations ultimately results in the following expression for $L_2(n)$:

$$L_2(n) = \lambda_-(n-1)!\left(\sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{\mu^k(n-k)!}\mathbb{E}[Q^{n-k}] - \frac{(-1)^n}{\mu^n}(1-\pi_0-\varphi(\mu))\right), \tag{5.16}$$

which holds for $n \geq 2$.

For the computation of $R_1(n)$ we use that for two independent continuous random variables X and Y with densities f_X and f_Y respectively and for some function $g(\cdot,\cdot)$ it holds that:

$$\mathbb{E}[g(X,Y)] = \int_{-\infty}^\infty \int_{-\infty}^\infty g(x,y)f_X(x)f_Y(y)\mathrm{d}x\mathrm{d}y.$$

$$\begin{aligned}
\frac{1}{\lambda_+}R_1(n) &= \int_0^\infty \int_{0+}^x q(y)\mathbb{P}(B_+ > x-y)x^{n-1}\mathrm{d}y\mathrm{d}x = \int_0^\infty \int_{0+}^x q(y)\left[\int_{x-y}^\infty b(t)\mathrm{d}t\right] x^{n-1}\mathrm{d}y\mathrm{d}x \\
&= \int_0^\infty \int_0^\infty q(y)b(t)\int_y^{y+t} x^{n-1}\mathrm{d}x\mathrm{d}t\mathrm{d}y = \int_0^\infty \int_0^\infty q(y)b(t)\frac{1}{n}\left[(y+t)^n - y^n\right]\mathrm{d}t\mathrm{d}y \qquad (5.17) \\
&= \frac{1}{n}(\mathbb{E}[(Q+B_+)^n] - \mathbb{E}[Q^n]).
\end{aligned}$$

Finally, in order to determine $R_2(n)$ we will use the following lemma.

Lemma 5.3.1. For any non-negative random variable X with density function $f(\cdot)$ and for any $r \geq 1$, it holds that:

$$\begin{aligned}
\int_0^\infty rx^{r-1}\mathbb{P}(X > x)\mathrm{d}x &= r\int_0^\infty x^{r-1}\left[\int_x^\infty f(y)\mathrm{d}y\right]\mathrm{d}x = r\int_0^\infty f(y)\left[\int_0^y x^{r-1}\mathrm{d}x\right]\mathrm{d}y \\
&= \int_0^\infty y^r f(y)\mathrm{d}y = \mathbb{E}[X^r].
\end{aligned}$$

Following this lemma the calculation of $R_2(n)$ is now trivial and the result is given by:

$$R_2(n) = \frac{\pi_0\lambda_+}{n}\mathbb{E}[B_+^n]. \tag{5.18}$$

Combining the expressions for $L_1(n)$, $L_2(n)$, $R_1(n)$ and $R_2(n)$ yields the following equality:

$$\begin{aligned} r\mathbb{E}[Q^n] + \lambda_-(n-1)! \left(\sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{\mu^k (n-k)!} \mathbb{E}[Q^{n-k}] - \frac{(-1)^n}{\mu^n}(1 - \pi_0 - \varphi(\mu)) \right) \\ = \frac{\lambda_+}{n} \left(\mathbb{E}[(Q + B_+)^n] - \mathbb{E}[Q^n] + \pi_0 \mathbb{E}[B_+^n] \right). \end{aligned} \tag{5.19}$$

Using (5.19), the moments of Q can be determined recursively. It is not possible however to give a closed expression for $\mathbb{E}[Q^n]$, since the $n-$th moment depends on all previous moments.

Chapter 6

Distribution of the amount of work just after arrival instants

In this chapter we will look at the same model as in the previous chapter. However, instead of studying the steady state distribution of the amount of work at arbitrary instants, we will now examine the steady state distribution of the amount of work just before and just after arrival instants, respectively named V and W. In particular, we are interested in the relation between those two distributions.

By the PASTA-property (PASTA is short for Poisson Arrivals See Time Averages) of a queueing model where customers arrive according to a Poisson process, we know that the distribution of the amount of work in the system as seen by an outside observer is the same as for an arriving customer. Hence, the distribution of V is the same as the distribution at arbitrary instants, which was derived in Chapter 5 in terms of the Laplace-Stieltjes transform (LST). It remains to analyse the distribution of W, which will be done in the next section, but first we will introduce a recursive formula for W_i, the amount of work just after the i-th arrival in terms of the other random variables in the model.

Let A_i denote the arrival time of the i-th customer and B_i its corresponding amount of work (which can be negative). Then the amount of work just after the arrival of the i-th customer is the amount of work which was in the system after the previous arrival plus B_i and minus the work which has been processed by the server, which happens at a rate rx, when there is x in the system. If we take furthermore into account that W can never be negative, we arrive at the following recursive formula for W_{i+1}:

$$W_{i+1} = \max\{\mathrm{e}^{-r(A_{i+1}-A_i)}W_i + B_{i+1}, 0\}. \tag{6.1}$$

The inter arrival times are exponentially distributed with parameter λ. For B as described in the previous chapter we could write:

$$B = \begin{cases} B_+ & \text{with probability } \lambda_+/\lambda \\ -B_- & \text{with probability } \lambda_-/\lambda, \end{cases}$$

where B_- is exponentially distributed with parameter μ. B_+ has a general distribution with LST $\beta(\cdot)$. In the next section we will derive a formula for the LST of W.

6.1 Distribution in terms of Laplace-Stieltjes transform

Let $\omega(\cdot)$ denote the LST of W and the random variable A the inter arrival times. The computation of $\psi(\cdot)$ will now be very similar to the derivation of the LST of the steady-state distribution of Z_n in Chapter 2.

$$\begin{aligned}\omega(s) = \mathbb{E}[\mathrm{e}^{-sW}] &= \mathbb{E}[\mathrm{e}^{-s\max\{\mathrm{e}^{-rA}W+B,0\}}]\\ &= \mathbb{E}[\mathrm{e}^{-s\max\{\mathrm{e}^{-rA}W+B,0\}}\mathbb{1}_{\mathrm{e}^{-rA}W+B\geq 0}] + \mathbb{E}[\mathrm{e}^{-s\max\{\mathrm{e}^{-rA}W+B,0\}}\mathbb{1}_{\mathrm{e}^{-rA}W+B<0}]\\ &= \mathbb{E}[\mathrm{e}^{-s(\mathrm{e}^{-rA}W+B)}] - \mathbb{E}[\mathrm{e}^{-s(\mathrm{e}^{-rA}W+B)}\mathbb{1}_{\mathrm{e}^{-rA}W+B<0}] + \mathbb{P}(\mathrm{e}^{-rA}W+B<0)\\ &= \mathbb{E}[\mathrm{e}^{-s\mathrm{e}^{-rA}W}]\mathbb{E}[\mathrm{e}^{-sB}] - \mathbb{P}(\mathrm{e}^{-rA}W+B<0)\mathbb{E}[\mathrm{e}^{-s(\mathrm{e}^{-rA}W+B)}|\mathrm{e}^{-rA}W+B<0]\\ &\quad + \mathbb{P}(\mathrm{e}^{-rA}W+B<0)\\ &=_* \left(\frac{\lambda_+}{\lambda}\beta(s) + \frac{\lambda_-}{\lambda}\frac{\mu}{\mu-s}\right)\int_0^\infty \lambda \mathrm{e}^{-\lambda t}\omega(s\mathrm{e}^{-rt})\mathrm{d}t + P\frac{s}{\mu-s},\end{aligned}$$

where $P = \mathbb{P}(\mathrm{e}^{-rA}W + B < 0)$. At $*$ we used that the condition $\mathrm{e}^{-rA}W + B < 0$ implies that B is in fact $-B_-$ and not B_+. Because of the memoryless property of B_-, which is exponentially distributed, we then get:

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{-s(\mathrm{e}^{-rA}W+B)}|\mathrm{e}^{-rA}W+B<0] &= \int_0^\infty \mathrm{e}^{-sx}\mathrm{d}_x\mathbb{P}(\mathrm{e}^{-rA}W+B\leq x|\mathrm{e}^{-rA}W+B<0)\\ &= \int_0^\infty \mathrm{e}^{-sx}\mathrm{d}_x\mathbb{P}(B_- > \mathrm{e}^{-rA}W - x|B_- > \mathrm{e}^{-rA}W)\\ &= \int_0^\infty \mathrm{e}^{-sx}\mathrm{d}_x\mathbb{P}(-B_-\leq x) = \mathbb{E}[\mathrm{e}^{sB_-}] = \frac{\mu}{\mu-s}.\end{aligned}$$

So we need to solve the equation:

$$\omega(s) = \left(\frac{\lambda_+}{\lambda}\beta(s) + \frac{\lambda_-}{\lambda}\frac{\mu}{\mu-s}\right)\int_0^\infty \lambda \mathrm{e}^{-\lambda t}\omega(s\mathrm{e}^{-rt})\mathrm{d}t + P\frac{s}{\mu-s}, \tag{6.2}$$

Differentiating both sides of the equality in (6.2) yields:

$$\begin{aligned}\omega'(s) &= \left(\frac{\lambda_+}{\lambda}\beta'(s) + \frac{\lambda_-}{\lambda}\frac{\mu}{(\mu-s)^2}\right)\int_0^\infty \lambda \mathrm{e}^{-\lambda t}\omega(s\mathrm{e}^{-rt})\mathrm{d}t\\ &\quad + \left(\frac{\lambda_+}{\lambda}\beta(s) + \frac{\lambda_-}{\lambda}\frac{\mu}{\mu-s}\right)\int_0^\infty \lambda \mathrm{e}^{-\lambda t}\frac{d}{ds}\omega(s\mathrm{e}^{-rt})\mathrm{d}t + P\frac{(\mu-s)+s}{(\mu-s)^2}\\ &= \left(\frac{\lambda_+}{\lambda}\beta'(s) + \frac{\lambda_-}{\lambda}\frac{\mu}{(\mu-s)^2}\right)\int_0^\infty \lambda \mathrm{e}^{-\lambda t}\omega(s\mathrm{e}^{-rt})\mathrm{d}t\\ &\quad - \left(\frac{\lambda_+}{\lambda}\beta(s) + \frac{\lambda_-}{\lambda}\frac{\mu}{\mu-s}\right)\int_0^\infty \lambda \mathrm{e}^{-\lambda t}\frac{1}{rs}\frac{d}{dt}\omega(s\mathrm{e}^{-rt})\mathrm{d}t + P\frac{\mu}{(\mu-s)^2},\end{aligned} \tag{6.3}$$

where we used that:

$$\frac{d}{ds}\omega(s\mathrm{e}^{-rt}) = \omega'(s\mathrm{e}^{-rt})\mathrm{e}^{-rt} = -\frac{1}{rs}\frac{d}{dt}\omega(s\mathrm{e}^{-rt}).$$

Let us now first introduce the following formula (for reasons of conciseness):

$$\psi(s) = \frac{\lambda_+}{\lambda}\beta(s) + \frac{\lambda_-}{\lambda}\frac{\mu}{\mu-s}.$$

Next we use integration by parts to solve the integral $\int_0^\infty \lambda e^{-\lambda t}\frac{1}{rs}\frac{d}{dt}\omega(se^{-rt})dt$ and express it in terms of $\omega(s)$:

$$\begin{aligned}\int_0^\infty \lambda e^{-\lambda t}\frac{1}{rs}\frac{d}{dt}\omega(se^{-rt})dt &= \frac{1}{rs}\left(\omega(se^{-rt})\lambda e^{-\lambda t}\Big|_{t=0}^{\infty} + \lambda\int_0^\infty \lambda e^{-\lambda t}\omega(se^{-rt})dt\right)\\ &= \frac{1}{rs}\left(-\omega(s)\lambda + \lambda\int_0^\infty \lambda e^{-\lambda t}\omega(se^{-rt})dt\right)\\ &= \frac{\lambda}{rs}\left(-\omega(s) + \frac{\omega(s)}{\psi(s)} - P\frac{s}{\mu-s}\frac{1}{\psi(s)}\right).\end{aligned}$$

We are now set to rewrite (6.3) as follows:

$$\begin{aligned}\omega'(s) &= \frac{\psi'(s)}{\psi(s)}\left(\omega(s) - P\frac{s}{\mu-s}\right) - \psi(s)\frac{\lambda}{rs}\left(-\omega(s)+\frac{\omega(s)}{\psi(s)} - P\frac{s}{\mu-s}\frac{1}{\psi(s)}\right) + P\frac{\mu}{(\mu-s)^2}\\ &= \omega(s)\left(\frac{\psi'(s)}{\psi(s)} - \frac{\lambda}{r}\frac{1-\psi(s)}{s}\right) + P\frac{1}{\mu-s}\left(\frac{\lambda}{r} - \frac{s\psi'(s)}{\psi(s)} + \frac{\mu}{\mu-s}\right).\end{aligned} \tag{6.4}$$

This differential equation can then be solved in the same fashion as we did in the previous chapter regarding the stationary distribution at arbitrary instants. The solution is of the form:

$$\omega(s) = e^{A(s)}\left(1 + P\int_0^s \frac{1}{\mu-\tau}\left(\frac{\lambda}{r} - \frac{\tau\psi'(\tau)}{\psi(\tau)} + \frac{\mu}{\mu-\tau}\right)e^{-A(\tau)}d\tau\right), \tag{6.5}$$

where $A(s)$ is defined as follows:

$$\begin{aligned}A(s) &= \int_0^s \frac{\psi'(t)}{\psi(t)} - \frac{\lambda}{r}\frac{1-\psi(t)}{t}dt = \log(\psi(s)) - \log(\psi(0)) - \frac{\lambda}{r}\int_0^s \frac{1-\psi(t)}{t}dt\\ &= \log(\psi(s)) - \frac{\lambda}{r}\int_0^s \frac{1-\psi(t)}{t}dt = \log(\psi(s)) - \frac{\lambda}{r}\int_0^s \frac{1-(\lambda_+/\lambda)\beta(t) - (\lambda_-/\lambda)\frac{\mu}{\mu-t}}{t}dt\\ &= \log(\psi(s)) - \frac{\lambda_+}{r}\int_0^s \frac{1-\beta(t)}{t}dt + \frac{\lambda_-}{r}\int_0^s \frac{1}{(\mu-t)}dt\\ &= \log(\psi(s)) - \frac{\lambda_+}{r}\int_0^s \frac{1-\beta(t)}{t}dt - \frac{\lambda_-}{r}\log\left(1-\frac{s}{\mu}\right).\end{aligned}$$

Taking into account that $\omega(0) = 1$, the homogeneous part of the solution now equals:

$$\omega_{\text{hom}}(s) = e^{A(s)} = \left(\frac{\lambda_+}{\lambda}\beta(s) + \frac{\lambda_-}{\lambda}\frac{\mu}{\mu-s}\right)\left(1-\frac{s}{\mu}\right)^{-\frac{\lambda_-}{r}}\exp\left(-\frac{\lambda_+}{r}\int_0^s\frac{1-\beta(t)}{t}dt\right) \tag{6.6}$$

6.2 Comparison between distribution just before and just after arrivals

If we compare $\omega(s)$ with the LST of Q just before arrival instants, which is given by (5.11), we see that

$$\left(1-\frac{s}{\mu}\right)^{-\frac{\lambda_-}{r}}\exp\left(-\frac{\lambda_+}{r}\int_0^s\frac{1-\beta(t)}{t}dt\right)$$

appears in $\varphi(s)$ under the integral as well. So we want to check the relation between $\omega(s)$ and $\varphi(s)$. Since equation (6.5) seems rather tedious, we take a direct approach to investigate the relation. First note that $e^{-r(A_{i+1}-A_i)}W_i$ is in fact the random variable Q, which we analysed in the previous chapter. So we could write alternatively:

$$W = \max\{Q+B, 0\}.$$

Then the following computation will yield an expression for $\omega(s)$ in terms of $\varphi(s)$:

$$\begin{aligned}\omega(s) = \mathbb{E}[\mathrm{e}^{-sW}] = \mathbb{E}[e^{-s\max\{Q+B,0\}}] &= \frac{\lambda_+}{\lambda}\mathbb{E}[e^{-s\max\{Q+B_+,0\}}] + \frac{\lambda_-}{\lambda}\mathbb{E}[e^{-s\max\{Q-B_-,0\}}] \\ &= \frac{\lambda_+}{\lambda}\mathbb{E}[e^{-sQ}]\mathbb{E}[e^{-sB_+}] + \frac{\lambda_-}{\lambda}\mathbb{E}[e^{-s\max\{Q-B_-,0\}}] \\ &= \frac{\lambda_+}{\lambda}\varphi(s)\beta(s) + \frac{\lambda_-}{\lambda}\left(\int_0^\infty \mathrm{e}^{-st}\mathrm{d}\mathbb{P}(Q-B_- < t) + \mathbb{P}(Q < B_-)\right).\end{aligned} \tag{6.7}$$

In the previous chapter we already saw that $\mathbb{P}(Q < B_-) = \varphi(\mu)$, so we will now focus on the computation of the integral:

$$\int_0^\infty \mathrm{e}^{-st}\mathrm{d}\mathbb{P}(Q-B_- < t) = \int_0^\infty\int_0^\infty \mathrm{e}^{-st}\mu\mathrm{e}^{-\mu x}\mathrm{d}\mathbb{P}(Q < t+x).$$

Now we apply the transform $t+x=y$:

$$\begin{aligned}\int_0^\infty\int_0^\infty \mathrm{e}^{-st}\mu\mathrm{e}^{-\mu x}\mathrm{d}\mathbb{P}(Q < t+x) &= \int_0^\infty\int_0^y \mathrm{e}^{-sy}\mu\mathrm{e}^{-(\mu-s)u}\mathrm{d}u\mathrm{d}\mathbb{P}(Q<y) \\ &= -\int_0^\infty \mathrm{e}^{-sy}q(y)\frac{\mu}{\mu-s}\left(\mathrm{e}^{(s-\mu)y}-1\right)\mathrm{d}y \\ &= \frac{\mu}{\mu-s}\left(\varphi(s)-\varphi(\mu)\right),\end{aligned}$$

plugging this into (6.7) then yields that:

$$\omega(s) = \left(\frac{\lambda_+}{\lambda}\beta(s) + \frac{\lambda_-}{\lambda}\frac{\mu}{\mu-s}\right)\varphi(s) - \frac{\lambda_-}{\lambda}\frac{s}{\mu-s}\varphi(\mu) = \psi(s)\varphi(s) - \frac{\lambda_-}{\lambda}\frac{s}{\mu-s}\varphi(\mu). \tag{6.8}$$

So ultimately we have a quite nice relation between the random variables W and Q in terms of their Laplace-Stieltjes transforms.

It is now also possible to express the moment of W in terms of the moments of Q, for which we derived implicit formulas in Section 5.3. We will show the result for just the first moment (we use that for LST's it holds that $\varphi(0)=1$).

$$\begin{aligned}\mathbb{E}[W] = -\frac{d}{ds}\omega(s)\Big|_{s=0} &= -\left(\varphi(s)\psi'(s) + \varphi'(s)\psi(s) - \frac{\lambda_-}{\lambda}\frac{\mu}{(\mu-s)^2}\right)\varphi(\mu)\Big|_{s=0} \\ &= \frac{\lambda_+}{\lambda}\mathbb{E}[B_+] - \frac{\lambda_-}{\lambda}\frac{1}{\mu} + \mathbb{E}[Q] + \frac{\lambda_-}{\lambda}\frac{1}{\mu}\varphi(\mu).\end{aligned} \tag{6.9}$$

For higher moments the calculation is not very hard, but the expressions become quite tedious.

6.3 Conclusion and discussion

We have seen that the work load model can be analysed in more or less the same fashion as the models we discussed in the literature study of this thesis. Although we obtained closed formulas for the Laplace-Stieltjes transforms of the relevant random variables in the model, namely the amount of work just after arrivals (W) and the amount of work at arbitrary instants (Q), they give not much insight in the behaviour of the distribution itself. More analysis, perhaps in the form of numerical simulation, would help gaining more insight into the model.

Furthermore we have been able to derive a relation between the two variables W and Q. However, finding this mathematical relation by comparing the LST's we derived for them separately seemed despite much effort rather difficult, which was quite unsatisfactory. Again, numerical simulation could possibly help us out.

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