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A comparison of the Krasnoselskii spectrum and the homotopy significant spectrum

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A comparison of the Krasnoselskii spectrum and the homotopy significant spectrum

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Abstract

Instead of defining eigenvalues only for linear maps, it is possible to define them for arbitrary continuous functions. First we however discuss in detail a proof of the Borsuk-Ulam theorem. We use the result to prove for quadratic forms that the set of eigenvalues found by the Courant-Fisher min-max principle equals the Krasnoselskii spectrum. Then we move on to the homotopy significant spectrum as defined by Gromov for functions defined on the real projective space. We prove that the Krasnoselskii spectrum for these functions is contained in the homotopy significant spectrum. In the case of the real projective line and plane we prove that the Krasnoselskii spectrum equals the homotopy significant spectrum. For the three-dimensional real projective space we prove that the Krasnoselskii spectrum forms a proper subset of the homotopy significant spectrum.
## Contents

1 Introduction 3

2 Preliminaries 4  
2.1 The Courant-Fisher min-max principle 4  
2.2 Homology groups and related concepts 4  
  2.2.1 Homology groups 4  
  2.2.2 Maps acting on homology groups 5  
  2.2.3 Homotopy 5  
  2.2.4 Degree 5  
  2.2.5 Simplicial cohomology 5  
2.3 The Krasnoselkii genus and the Lusternik-Schnirelmann category 6  
2.4 The homotopy significant spectrum 9

3 The Borsuk-Ulam theorem 10

4 The Krasnoselskii spectrum for quadratic forms 14

5 The homotopy significant spectrum 16  
  5.1 The homotopy significant spectrum in the case Φ = RP^3 20

6 Conclusion 24
1 Introduction

Oftentimes eigenvalues are only considered in the linear sense as properties of linear maps and matrices. However, it can be interesting to generalise the notion of eigenvalues to more general functions, the only constraint being for example continuity.

To see why we can call these generalisations still eigenvalues, we look into the case of quadratic forms. Consider therefore a quadratic form \( q : H \to \mathbb{R} \), where \( H \) a Hilbert space, given by \( q(u) = (Lu, u) \) for all \( u \in H \) with \( L : H \to H \) a symmetric linear map. Then we would like to find critical points of \( q \) on the sphere \( S \) in \( H \). To do this, we will use the Lagrange multiplier method. The points on the sphere are solutions of \( (u, u) = 1 \), so the corresponding Lagrange function \( \Lambda : H \times \mathbb{R} \to \mathbb{R} \) is given by \( \Lambda(u, \lambda) = (Lu, u) - \lambda((u, u) - 1) \). Then we know that critical points of our original function \( q \) on \( S \) are critical points of the Lagrange function. Setting all partial derivatives of \( \Lambda \) to zero results in the following system

\[
\begin{align*}
2Lu - 2\lambda u &= 0 \\
(u, u) - 1 &= 0.
\end{align*}
\]

The first equation implies \( Lu = \lambda u \), so the Lagrange multipliers \( \lambda \) which are solutions of the system are also eigenvalues of \( L \). Then taking the inner product with \( u \) and using the second equation results in \( q(u) = (Lu, u) = \lambda \). Hence the eigenvalues of the linear map \( L \) are also the values of the quadratic form \( q \) in its critical points on the sphere. Hence it would be natural to define the eigenvalues of a quadratic form as the values the quadratic form takes in its critical points on the sphere.

The Krasnosel’kii spectrum which defines eigenvalues for even and continuous functions generalizes this observation. It can namely be shown that the eigenvalues in the Krasnosel’skii spectrum for even and continuously differentiable functions equal the function’s values in its critical points on the sphere [5, II Lemma 5.6].

Eigenvalues of a linear map can be found using the Courant-Fisher min-max principle, so our first goal will be to directly prove for quadratic forms that this principle gives the same eigenvalues as those found in the Krasnosel’kii spectrum. Thereafter, we will consider a second definition for non-linear eigenvalues which is defined by Gromov as the homotopy significant spectrum [1, p. 12-13]. Our second goal is then to compare the homotopy significant spectrum to the Krasnosel’skii spectrum for continuous functions on the real projective space.

We discuss all necessary preliminaries in Section 2. In Section 3 we will prove the Borsuk-Ulam theorem, which we use in Section 4 to prove for quadratic forms that the eigenvalues found by the Courant-Fisher min-max principle equal the eigenvalues in the Krasnosel’skii spectrum. In Section 5 we compare the homotopy significant spectrum to the Krasnosel’skii spectrum.
2 Preliminaries

In this section we will introduce the concepts related to the eigenvalues we will study. The first definition of eigenvalues is to be found in Section 2.1. Then we will cover some more topological concepts related to homology groups in Section 2.2. We will start by introducing homology groups themselves and then continue with maps acting on those homology groups, homotopies between maps and a notion of degree for maps acting on homology groups. We will also discuss cohomology. Then we will discuss the Krasnoselskii genus and the closely related Lusternik-Schnirelmann category in Section 2.3 to introduce a second definition of eigenvalues, the Krasnoselskii spectrum. A third definition of eigenvalues, the homotopy significant spectrum, will then follow in Section 2.4.

2.1 The Courant-Fisher min-max principle

We will call a linear map \( L : H \to H \) symmetric if \( (Lx, y) = (x, Ly) \) for all \( x, y \) in the Hilbert space \( H \). The Courant-Fisher minmax principle then characterizes the eigenvalues of such symmetric linear maps.

**Definition 2.1.** Let \( H \) be a \( n \)-dimensional Hilbert space and \( L : H \to H \) a symmetric linear map, then we define for \( k \in \{1, \ldots, n\} \) the eigenvalues \( \lambda_k \) of \( L \) as

\[
\lambda_k = \inf_{V \subset H, \dim V = k} \sup_{u \in S(V)} (Lu, u)
\]

in which \( S(V) = \{ u \in V | \|u\| = 1 \} \).

2.2 Homology groups and related concepts

For notation concerning homology groups and related concepts we will follow the notation used by Allen Hatcher [2]. Throughout this discussion of homology groups we will let \( X, Y \) refer to topological spaces.

2.2.1 Homology groups

Before we can define (singular) homology groups, we need to define the chain groups consisting of singular simplices. We define a singular \( n \)-simplex in \( X \) as a map \( \sigma : \Delta^n \to X \) with \( \Delta^n \) the standard \( n \)-simplex given by

\[
\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} | \sum_{i=1}^{n} t_i = 1 \text{ and } t_i \geq 0 \quad \forall i \in \{1, \ldots, n\}\}.
\]

We then define the chain group \( C_n(X; G) \) for abelian groups \( G \) as the free abelian group consisting of \( n \)-chains, i.e. finite sums of the form \( \sum_{i} n_i \sigma_i \) with for all \( i, n_i \in G \) and \( \sigma_i \) a singular \( n \)-simplex. The group \( G \) is called the coefficient group. We often shorten the notation \( C_n(X; G) \) to \( C_n(X) \) when \( G \) equals \( \mathbb{Z} \) or when the result holds for all abelian coefficient groups. The boundary map \( \partial_n : C_n(X) \to C_{n-1}(X) \) do we define as

\[
\partial_n(\sigma) = \sum_{i=1}^{n} (-1)^i \sigma|[v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n]
\]

with \([v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n]\) the simplex spanned by the vertices \( \{v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\} \) of \( \Delta^n \). The simplex \([v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n]\) is in general not the standard \( n \)-simplex, however in this definition we canonically identify \([v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n]\) with \( \Delta^{n-1} \) while we preserve the order of the vertices. For \( \partial_n \) we will also write just \( \partial \). The combination of chain groups and boundary maps in the following sequence

\[
\cdots \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \cdots \to C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0
\]

is called the chain complex of \( X \), since \( \text{Im} \partial_{n+1} \subset \text{Ker} \partial_n \) [2, Lemma 2.1].
Then we can finally define homology groups. We denote the $n$th homology group of $X$ with coefficients in $G$ by $H_n(X; G)$ and define $H_n(X; G) = \text{Ker} \partial_n / \text{Im} \partial_{n+1}$. Analogous to chain groups, we often shorten $H_n(X; G)$ to $H_n(X)$.

To conclude this introduction of homology groups we will state the homology groups of the $n$-dimensional sphere, denoted $S^n$, since they are often encountered. If $G$ is an abelian group, then they are $H_i(S^n; G) \cong G$ for $i \in \{0, n\}$ and $H_i(S^n; G) \cong 0$ otherwise [2, p. 162].

### 2.2.2 Maps acting on homology groups

Let $f : X \to Y$ be a continuous map. Then we define the homomorphism $f_\# : C_n(X) \to C_n(Y)$ induced by $f$ as $f_\#(\sigma) = f \circ \sigma : \Delta^n \to Y$ and extend $f_\#$ linearly via $f_\#(\sum_{i=1}^n n_i \sigma_i) = \sum_{i=1}^n n_i f \circ \sigma_i$. The maps $f_\#$ form a chain map from the singular chain complex of $X$ to the one of $Y$, that is they satisfy $f_\# \partial = \partial f_\#$ [2, p. 111]. This implies that $f_\#$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$ [2, p. 111]. If needed we will use the notation $f_{*,n}$ to denote the homology group $f_*$ is working on. In the case of $f_{*,n}$ it would be $H_n(X)$.

### 2.2.3 Homotopy

We call two continuous maps $f_0, f_1 : X \to Y$ homotopic, and denote this as $f_0 \simeq f_1$, if there exists a homotopy between them. That is, there exists a continuous map $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for all $x \in X$. Two topological spaces $X, Y$ are homotopy equivalent if there exists a map $f : X \to Y$ and a map $g : Y \to X$ such that $gf \simeq 1_X$ and $fg \simeq 1_Y$, in which we denote by $1_X$ and $1_Y$ the identity on $X$, respectively $Y$. The maps $f, g$ are then called a homotopy equivalence. An important result we will mention here is that the homology groups are homotopy invariant, that is [2, Corollary 2.11]:

**Theorem 2.2.** The maps $f_* : H_n(X) \to H_n(Y)$ induced by a homotopy equivalence $f : X \to Y$ are isomorphisms.

### 2.2.4 Degree

The map $g_* : H_n(S^n) \to H_n(S^n)$ induced by a map $g : S^n \to S^n$ for $n > 0$ is an homomorphism of the form $g_* (x) = dx$ for some $d \in \mathbb{Z}$ and all $x \in H_n(S^n)$. We can see this because $H_n(S^n) \cong \mathbb{Z}$ and hence $g_*$ is an homomorphism from an infinite cyclic group to itself. We will name the integer $d$ the degree of $g$, and denote it as $\deg g$. This degree possesses some useful properties of which we will name a few [2, p. 134-135].

**Proposition 2.3.** Let $g, h : S^n \to S^n$ induce the homomorphisms $g_* , h_* : H_n(S^n) \to H_n(S^n)$, then

1. $\deg 1 = 1$,
2. $\deg g = 0$ if $g$ is not surjective,
3. $\deg g = \deg h$ if $g \simeq h$,
4. $\deg g \circ h = \deg g \deg h$.

### 2.2.5 Simplicial cohomology

Cohomology is the dual to homology, but where we discussed the singular variant of homology, we will now discuss the simplicial variant of cohomology. Therefore we will first need to define a $\Delta$-complex structure on a topological space $X$. We define the boundary of a simplex $\Delta^n$ as the union of its faces, denoted as $\partial \Delta^n$. The interior of $\Delta^n$, which we will denote as $\Delta^n$, we define as $\Delta^n - \partial \Delta^n$. Then we are able to define a $\Delta$-complex. [2, p. 103]

**Definition 2.4.** A $\Delta$-complex structure on a space $X$ is a collection of maps $\sigma_\alpha : \Delta^n \to X$, with $n$ depending on the index $\alpha$, such that:

- The restriction $\sigma_\alpha|\Delta^n$ is injective, and each point of $X$ is in the image of exactly one such restriction $\sigma_\alpha|\Delta^n$.

5
Our next step is to dualize the boundary map \( \delta \) define the cochain complex of \( \Delta^n \). Just as how we defined the homology group \( H_n \) or \( Z \) coefficents of both the homology as cohomology groups in \( X \), the same space \( \sigma \), the cup product has an identity \( 1 \in G \) induces a cup product on cohomology groups. We note that if \( \sigma \) and \( \tau \) is again a cocycle [2, Lemma 3.6], the cup product on cochain groups since the cup product of a coboundary and a cocycle is a coboundary, and the cup product of

\[
\Delta^n(X) \rightarrow \Delta^{n+1}(X).
\]

We define the coboundary map \( \delta : \Delta^n(X) \rightarrow \Delta^{n+1}(X) \) for \( \phi \in \Delta^n(X) \) as \( \delta \phi = \phi \partial \). Note that the coboundary maps work in the opposite direction of the boundary maps.

Since we now have constructed the duals of the chain groups and the boundary maps, we can define the cochain complex of \( \Delta^n \) as

\[
\cdots \leftarrow \Delta_{n+1}(X) \overset{\delta_n}{\leftarrow} \Delta_n(X) \overset{\delta_{n-1}}{\leftarrow} \Delta_{n-1}(X) \leftarrow \cdots \overset{\delta_0}{\leftarrow} \Delta_0(X) \leftarrow 0.
\]

Just as how we defined the homology group \( H_n(X;G) \), we can define the cohomology group \( H^n(X;G) \) as

\[
H^n(X;G) = \operatorname{Ker} \delta_n / \operatorname{Im} \delta_{n-1}.
\]

The cohomology groups of a topological space \( X \) can be related to the homology groups of the same space by the universal coefficient theorem for cohomology. We will not discuss the theorem itself, but we will state a corollary of this theorem for the special case when we take the coefficients of both the homology as cohomology groups in \( \mathbb{Q} \) or \( \mathbb{Z}_p \) with \( p \) prime [2, p. 198-199].

**Corollary 2.5.** Let \( X \) be a topological space with homology groups \( H_n(X;F) \) and cohomology groups \( H^n(X;F) \). If \( F \) is taken to be \( \mathbb{Q} \) or \( \mathbb{Z}_p \) with \( p \) prime, we obtain

\[
H^n(X;F) \cong \text{Hom}(H_n(X;F),F).
\]

An interesting property of cohomology is that a ring structure on the cohomology groups can be defined if we take our coefficient group \( G \) to be a ring. For such a ring structure we need a notion of multiplication, which we first will introduce on the cochain group level. If we take cochains \( \phi \in \Delta^n(X;G) \) and \( \psi \in \Delta^m(X;G) \), we define the cup product \( \phi \circ \psi \in \Delta^{n+m}(X;G) \) as the cochain which takes the following values on a map \( \sigma : \Delta^{n+m} \rightarrow X \) in the cohomology complex on \( X \):

\[
(\phi \circ \psi)(\sigma) = \phi(\sigma|\{v_0,\ldots,v_n\})\psi(\sigma|\{v_n,\ldots,v_n+m\}).
\]

Since the cup product of a coboundary and a cocycle is a coboundary, and the cup product of a cocycle and a cocycle is again a cocycle [2, Lemma 3.6], the cup product on cochain groups induces a cup product on cohomology groups. We note that if \( G \) has a multiplicative identity 1, the cup product has an identity 1 \( \in H^0(X;G) \), namely the cocycle taking the value 1 on each vertex. Since the cup product is also distributive and associative, we have the possibility to define a ring structure on the cohomology groups using the cup product as our multiplication. Therefore, we define \( H^*(X;G) \) as the direct sum of the cohomology groups \( H^n(X;G) \). Then an element of \( H^*(X;G) \) is of the form \( \sum_{i \in \mathbb{N}} \alpha_i \), with \( \alpha_i \in H^i(X;G) \). If we take another element \( \sum_{j \in \mathbb{N}} \beta_j \in H^*(X;G) \), we can then define multiplication as \( (\sum_{i \in \mathbb{N}} \alpha_i)(\sum_{j \in \mathbb{N}} \beta_j) = \sum_{i,j \in \mathbb{N}} \alpha_i \beta_j \). Together with the normal addition on homology groups and the properties of the cup product, this establishes \( H^*(X;G) \) as a ring.

### 2.3 The Krasnoselkii genus and the Lusternik-Schnirelmann category

For a Hilbert space \( H \), let \( \mathcal{V}(H) \) denote

\[
\mathcal{V}(H) = \{ A \subset H | A \text{ closed and symmetric} \}
\]

in which a symmetric subset is a subset \( A \subset H \) which satisfies \( A = -A \). Then we define the Krasnoselkii genus as follows.

**Definition 2.6.** Let \( H \) be a Hilbert space and \( A \in \mathcal{V}(H) \). Then we define the Krasnoselkii genus of non-empty \( A \) as

\[
\gamma(A) = \begin{cases} 
\inf \{ m \in \mathbb{N} : \exists h \in C^m(A;\mathbb{R}^m \setminus \{0\}), h(-u) = h(u) \qquad \forall u \in A \}, \\
\infty, \text{ if } \{\ldots\} = \emptyset. 
\end{cases}
\]
If $A = \emptyset$, then we define $\gamma(A) = 0$.

For later use, we will list a few properties of the Krasnoselkii genus [5, p. 95-96].

**Proposition 2.7.** Let $H$ be a Hilbert space, $A, A_1, A_2 \in \mathcal{V}(H)$ and let $h : H \to H$ be an odd and continuous map. Then the following hold:

1. $\gamma(A) \geq 0$, $\gamma(A) = 0 \iff A = \emptyset$;
2. $\gamma(A_1) \leq \gamma(A_2)$ if $A_1 \subset A_2$;
3. $\gamma(A_1 \cup A_2) \leq \gamma(A_1) + \gamma(A_2)$;
4. $\gamma(A) \leq \gamma(h(A))$;
5. if $A$ is a finite collection of antipodal pairs $u_i, -u_i \in H \setminus \{0\}$, then $\gamma(A) = 1$;
6. if $A$ is compact and $0 \notin A$, then $\gamma(A) < \infty$ and a neighbourhood of $N$ of $A$ in $H$ exists such that $N \in \mathcal{V}(H)$ and $\gamma(A) = \gamma(N)$.

**Proof of Proposition 2.7.**

1. This first property of the Krasnoselkii genus follows directly from the definition.
2. If $\gamma(A_2) = \infty$ the claim follows immediately. Therefore, assume $\gamma(A_2) = m < \infty$. Then an odd function $h \in C^0(A_2, \mathbb{R}^m \setminus \{0\})$ exists. The restriction of $h$ to $A_1$ is also odd and continuous, and does not contain 0 in its image, so $\gamma(A_1) \leq m$.
3. Suppose $\gamma(A_1) = m_1 < \infty$ and $\gamma(A_2) = m_2 < \infty$. Then odd maps $g_1 \in C^0(A_1, \mathbb{R}^{m_1} \setminus \{0\})$ and $g_2 \in C^0(A_2, \mathbb{R}^{m_2} \setminus \{0\})$ exist. We can extend these maps to both be continuously defined on $A_1 \cup A_2$ by the Tietze extension theorem and we will denote the resulting maps as $h_1 \in C^0(A_1 \cup A_2, \mathbb{R}^{m_1})$ and $h_2 \in C^0(A_1 \cup A_2, \mathbb{R}^{m_2})$. We can assume these maps $h_1$ and $h_2$ to be odd, since if $h_1$ was not odd, we could define the odd function $\tilde{h}_1 \in C^0(A_1 \cup A_2, \mathbb{R}^{m_1})$ by $\tilde{h}_1 = \frac{1}{2}(h_1(u) - h_1(-u))$. Combining these maps $h_1$ and $h_2$ to a map $h \in C^0(A_1 \cup A_2, \mathbb{R}^{m_1 + m_2} \setminus \{0\})$ given by $h(u) = (h_1(u), h_2(u))$ for all $u \in A_1 \cup A_2$, gives us that $\gamma(A_1 \cup A_2) \leq m_1 + m_2$.
4. Again suppose $\gamma(h(A)) = m$ is finite. Then an odd function $g \in C^0(h(A), \mathbb{R}^m \setminus \{0\})$ exists. Consider the composition $g \circ h : A \to \mathbb{R}^m \setminus \{0\}$. A composition of continuous and odd functions is yet again continuous and odd, so $\gamma(A) \leq \gamma(h(A))$.
5. For each pair $\{u_i, -u_i\}$ send $u_i$ to 1 and $-u_i$ to $-1$. This forms an odd continuous function which maps $A$ onto $\mathbb{R} \setminus \{0\}$, hence $\gamma(A) = 1$.
6. For this proof we refer to [5, p. 96].

We say that a function $f$ is even is it satisfies $f(x) = f(-x)$ for all $x$ in the domain of $f$. Using the Krasnoselkii genus we can pose a definition for eigenvalues for even and continuous functions. We will call the set of all these eigenvalues the Krasnoselkii spectrum.

**Definition 2.8.** Let $H$ be a $n$-dimensional Hilbert space and $f$ an even and continuous function on $H$. Then we define for $k \in \{1, \ldots, n\}$ the eigenvalues

$$\beta_k = \inf_{A \in \mathcal{V}(S(H))} \sup_{u \in A} f(u) \quad (10)$$

in which $\mathcal{V}(S(H)) = \{A \subset S(H) \mid A \text{ closed and symmetric}\}$.

A concept related to the Krasnoselkii genus is the Lusternik-Schnirelmann subspace category (LS category).

**Definition 2.9.** Let $M$ be a topological space and $A \subset M$ a closed subset. The Lusternik-Schnirelmann subspace category of $A$ in $M$, denoted $\text{cat}_M A$, is the smallest integer $m$ such that $A$ is covered by closed sets $U_1, \ldots, U_m$ which are contractible in $M$. If no such covering exists we write $\text{cat}_M A = \infty$. 

7
proof of Proposition 2.10. Let M be a topological space, A, A_1, A_2 \subset M closed and let h : M \to M be a homeomorphism. Then the following hold:

1. \cat_M(A) \geq 0, \cat_M(A) = 0 \iff A = \emptyset
2. A_1 \subset A_2 \Rightarrow \cat_M(A_1) \leq \cat_M(A_2)
3. \cat_M(A_1 \cup A_2) \leq \cat_M(A_1) + \cat_M(A_2)
4. \cat_M(A) = \cat_M(h(A))

Proof of Proposition 2.10. 1. This first property follows directly from the definition.

2. If \cat_M(A_2) = \infty the claim follows directly, so suppose \cat_M(A_2) = m < \infty. Then we know a cover of A_2, \{U_1, \ldots, U_m\}, exists such that U_i is closed and contractible in M for all i \in \{1, \ldots, m\}. Then \{U_1, \ldots, U_m\} also covers A_1, which implies that \cat_M(A_1) \leq \cat_M(A_2).

3. Suppose \cat_M(A_1) = m_1 < \infty and \cat_M(A_2) = m_2 < \infty. Then a closed cover of A_1, \{U_1, \ldots, U_m\}, exists and a closed cover of A_2, \{V_1, \ldots, V_m\}, exists, where all U_i and V_i are contractible. Hence \{U_1, \ldots, U_m, V_1, \ldots, V_m\} forms a cover of A_1 \cup A_2 consisting of closed and contractible sets in M. Hence \cat_M(A_1 \cup A_2) \leq m_1 + m_2.

4. Consider again the case \cat_M(A) = m < \infty. Then a closed cover of X exists consisting of closed sets \{U_1, \ldots, U_m\} which are contractible in M. We prove that \{h(U_1), \ldots, h(U_m)\} is a cover of h(A) consisting of closed and contractible sets in M, implying that \cat_M(A) \leq \cat_M(h(A)).

Under h these sets \{U_1, \ldots, U_m\} stay closed by continuity of \inv{h}. Since \{U_1, \ldots, U_m\} form a cover of A, the sets \{h(U_1), \ldots, h(U_m)\} also cover h(A). Left is to prove these sets \{h(U_1), \ldots, h(U_m)\} are contractible in M. Fix therefore i \in \{1, \ldots, m\}. Since U_i is contractible, there exists a homotopy \tilde{G} : U_i \times [0, 1] \to M such that \tilde{G}(x, 0) = \mathbb{1}_{U_i}(x) and \tilde{G}(x, 1) = x_0 \in M. We claim that \tilde{G} : h(U_i) \times [0, 1] \to M given by \tilde{G}(y, t) = h(\inv{h}(y), t)) is a homotopy between \mathbb{1}_{h(U_i)} and the constant map h(x_0). Since \tilde{G} is a composition of continuous functions it is itself continuous. Further we have for all y \in h(U_i)

\begin{align*}
\tilde{G}(y, 0) &= h(\inv{h}(y), 0)) = h(\mathbb{1}_{U_i}(h^{-1}(y))) = h(h^{-1}(y)) = \mathbb{1}_{h(U_i)}(y) \\
\tilde{G}(y, 1) &= h(\inv{h}(y), 1)) = h(x_0).
\end{align*}

Hence \tilde{G} is a homotopy between \mathbb{1}_{h(U_i)} and the constant map h(x_0), so for all i \in \{1, \ldots, m\} is U_i contractible. We conclude \cat_M(A) \geq \cat_M(h(A)).

The inverse inequality \cat_M(A) \leq \cat_M(h(A)) can be proved similarly which implies that \cat_M(A) = \cat_M(h(A)).

The relation between the Krasnoselskii genus and the LS category can be made precise. Let B be a Banach space and p : B \setminus \{0\} \to (B \setminus \{0\})/\Z_2 the quotient map which identifies points \{u, -u\} for all u \in B \setminus \{0\}. For A \subset B \setminus \{0\} we then define A^* = p(A). Further, we define C^* for C \subset (B \setminus \{0\})^* = (B \setminus \{0\})/\Z_2 as C^* = p^{-1}(C). We state the following result which states the relation between the Krasnoselskii genus and the LS category with proof to be found in [4, Theorem 3.7].

Theorem 2.11. Let B be a real Banach space. If A \in \mathcal{V}(B) is compact and 0 \notin A then \gamma(A) = \cat_{(B \setminus \{0\})^*} A^*.
2.4 The homotopy significant spectrum

In his paper [1, p. 12-13] Gromov gives yet another definition for eigenvalues, purely based on sublevels of a function and the existence of homotopies between those sublevels. Later on, we would like to compare this definition of eigenvalues to the previously encountered definitions of eigenvalues.

For this, let \( \Phi \) be a topological space and \( E : \Phi \to \mathbb{R} \) a continuous real-valued function. Then we define the closed \( t \)-sublevels of \( E \) for \( t \in \mathbb{R} \) as

\[
\Phi_{\leq t} = \{ x \in \Phi | E(x) \leq t \}. \tag{11}
\]

The open \( t \)-sublevels \( \Phi_{< t} \) can we define analogously. Gromov then defines \( e \in \mathbb{R} \) as an eigenvalue if the homotopy type of \( \Phi_{< e} \) is significantly different from \( \Phi_{\leq e} \). He refines this to saying that \( e \) is an eigenvalue if there does not exists a homotopy from \( \Phi_{\leq e} \) to \( \Phi_{< e} \). This amounts to the following.

**Definition 2.12.** Let \( \Phi \) be a topological space and \( E : \Phi \to \mathbb{R} \) a continuous real-valued function. Then \( e \in \mathbb{R} \) is an eigenvalue in the homotopy significant spectrum of \( E \) if a homotopy \( H : \Phi_{\leq e} \times [0,1] \to \Phi \) such that \( H(x,0) = 1_{\Phi_{\leq e}}(x) \) for all \( x \in \Phi_{\leq e} \), and \( H(\Phi_{\leq e},1) \subset \Phi_{< e} \) does not exist. These eigenvalues form the homotopy significant spectrum of \( E \).
3 The Borsuk-Ulam theorem

In this section we will prove the Borsuk-Ulam theorem, following the proof as presented by Allen Hatcher [2, p. 174]. Later we will use this theorem to prove properties of the sphere. However, before we prove the Borsuk-Ulam theorem itself we must prove the following proposition of which the proof is also derived from Allen Hatcher [2, p. 174-176]. Noteworthy is that in the proof of the proposition we can easily derive the homotopy groups of the real projective space.

Proposition 3.1. An odd map \( f : S^n \to S^n \), that is a map satisfying \( f(-x) = -f(x) \) for all \( x \), has odd degree.

Proof of Proposition 3.1. The proof of this proposition will be rather lengthy and is for convenience of the reader divided in several parts.

Part 1: Constructing a long exact sequence

To prove the proposition we will need a long exact sequence of homotopy groups of the \( n \)-dimensional sphere and the \( n \)-dimensional real projective space, but we will start more general.

Let \( X \) be a topological space with two-sheeted covering space \( \tilde{X} \) and covering projection \( p : \tilde{X} \to X \). We will construct a short exact sequence of chain groups

\[
0 \to C_n(X;\mathbb{Z}_2) \xrightarrow{\partial} C_n(\tilde{X};\mathbb{Z}_2) \xrightarrow{p_\#} C_n(X;\mathbb{Z}_2) \to 0 \tag{12}
\]

with \( p_\# \) the homomorphism induced by \( p \), and \( \tau \) yet to be constructed. First of all, since every singular simplex \( \sigma : \Delta^n \to X \) is simply connected every singular simplex has a lift in \( \tilde{X} \) [2, Proposition 33]. Since \( \tilde{X} \) is a two-sheeted covering space, every singular simplex \( \sigma \) has exactly two lifts which we will denote by \( \tilde{\sigma}_1 \) and \( \tilde{\sigma}_2 \) [2, Proposition 34]. Hence \( p_\# \) is surjective.

We claim that the kernel of \( p_\# \) is generated by the sums \( \tilde{\sigma}_1 + \tilde{\sigma}_2 \) for all singular simplices \( \sigma \) in \( X \). Since \( p_\#(\tilde{\sigma}_1 + \tilde{\sigma}_2) = p_\#(\tilde{\sigma}_1) + p_\#(\tilde{\sigma}_2) = p\tilde{\sigma}_1 + p\tilde{\sigma}_2 = 2\sigma = 0 \), these sums form at least part of the kernel. To see that they generate the whole kernel assume \( \sum_{i=1}^m \tilde{\sigma}_i \in \text{Ker} p_\# \) with \( \tilde{\sigma}_i \neq \tilde{\sigma}_j \) for all \( i \neq j \). Define \( \sigma_j \) and \( \tilde{m} \leq m \) such that \( \{p\tilde{\sigma}_i \mid i \in \{1,\ldots,m\}\} = \{\sigma_j \mid j \in \{1,\ldots,\tilde{m}\}\} \) and \( I_j = \{i \in \{1,\ldots,m\} \mid p\tilde{\sigma}_i = \sigma_j\} \). Then \( 0 = \sum_{i=1}^m p\tilde{\sigma}_i = \sum_{j=1}^\tilde{m} |I_j| \mod 2 \sigma_j \). Hence we must conclude that \( |I_j| = 0 \mod 2 \), which proves our claim.

Since we wanted to construct \( \tau \) such that the short sequence (12) was exact, we define \( \tau(\sigma) = \tilde{\sigma}_1 + \tilde{\sigma}_2 \). Then we need to check whether \( \tau \) is injective. Assume therefore that \( \tau(\sum_{i=1}^m \sigma_i) = \tau(\sum_{i=1}^n \rho_i) \) with \( \sigma_i \neq \rho_j \) for all \( i \neq j \). Then \( \sum_{i=1}^m \tilde{\sigma}_1 + \tilde{\sigma}_2 + \sum_{i=1}^n \tilde{\rho}_1 + \tilde{\rho}_2 = 0 \). We define \( \tilde{\sigma}_1 \), \( \tilde{\sigma}_2 \) as the lifts of \( \sigma_i \), and \( \tilde{\rho}_1 \), \( \tilde{\rho}_2 \) likewise. Define then \( \nu_j \) and \( \tilde{m} \leq (m_1 + m_2) \) such that \( \{\tilde{\sigma}_i \mid i \in \{1,\ldots,m_1\}, k \in \{1,2\}\} \cup \{\tilde{\rho}_i \mid i \in \{1,\ldots,m_2\}, k \in \{1,2\}\} = \{\nu_j \mid j \in \{1,\ldots,\tilde{m}\}\} \). Further, define \( I_j = \# \{i \in \{1,\ldots,m_1\}, k \in \{1,2\} \mid \tilde{\sigma}_i = \nu_j\} + \# \{i \in \{1,\ldots,m_2\}, k \in \{1,2\} \mid \tilde{\rho}_i = \nu_j\}. \) Then \( \sum_{j=1}^\tilde{m} |I_j| \mod 2 = 0 \). Again we must conclude that \( |I_j| = 0 \mod 2 \) and combining this with the fact that every simplex has exactly two, different, lifts, this implies that \( \sum_{i=1}^m \tilde{\sigma}_i + \tilde{\sigma}_2 = \sum_{i=1}^{m_1} \tilde{\rho}_1 + \tilde{\rho}_2 \). Hence \( \sum_{i=1}^m \tilde{\sigma}_i = \sum_{i=1}^{m_1} \tilde{\rho}_1 \), so \( \tau \) is injective.

The map \( \tau \) commutes with the boundary maps, since

\[
\partial \tau(\sigma) = \partial(\tilde{\sigma}_1 + \tilde{\sigma}_2) = \partial \tilde{\sigma}_1 + \partial \tilde{\sigma}_2 = \tau(\partial\sigma) = \tau(0) = \tau \partial(\sigma) \tag{13}
\]

and likewise does \( p_\# \). This gives us a short exact sequence of chain complexes

\[
\begin{array}{c}
0 \\
\downarrow \\
\cdots \\
\downarrow \\
C_{n+1}(X;\mathbb{Z}_2) \xrightarrow{\partial} C_n(X;\mathbb{Z}_2) \xrightarrow{\partial} C_{n-1}(X;\mathbb{Z}_2) \to \cdots \\
\downarrow \tau \\
C_{n+1}(\tilde{X};\mathbb{Z}_2) \xrightarrow{\partial} C_n(\tilde{X};\mathbb{Z}_2) \xrightarrow{\partial} C_{n-1}(\tilde{X};\mathbb{Z}_2) \to \cdots \\
\downarrow p_\# \\
C_{n+1}(X;\mathbb{Z}_2) \xrightarrow{\partial} C_n(X;\mathbb{Z}_2) \xrightarrow{\partial} C_{n-1}(X;\mathbb{Z}_2) \to \cdots \\
\downarrow 0 \\
0 \\
\end{array}
\]

and hence the following long exact sequence of homotopy groups [2, Theorem 2.16]

\[
\cdots \to H_n(X;\mathbb{Z}_2) \xrightarrow{\partial} H_n(\tilde{X};\mathbb{Z}_2) \xrightarrow{p_\#} H_n(X;\mathbb{Z}_2) \xrightarrow{\partial} H_{n-1}(X;\mathbb{Z}_2) \to \cdots . \tag{15}
\]
in which $p_*$ and $\tau$ are the maps induced by $p_\#$ and $\tau$ respectively, as in Section 2.2.2. The map $\partial$ in the long exact sequence (15) is defined on basis of the maps in diagram (14). To construct the map $\partial : H_n(X; \mathbb{Z}_2) \to H_{n-1}(X; \mathbb{Z}_2)$ take a cycle $c \in C_n(X; \mathbb{Z}_2)$, which is an element of $C_n(X; \mathbb{Z}_2)$ that satisfies $\partial c = 0$. Since the vertical sequences in diagram (14) are exact, $p_\#$ is surjective. Hence $p_\#(b) = c$ for some $b \in C_n(X; \mathbb{Z}_2)$. Define $b' = \partial b$, then we know $p_\# b' = p_\# \partial b = \partial p_\# b = \partial c = 0$. By exactness an $a \in C_{n-1}(X; \mathbb{Z}_2)$ exists such that $\tau a = b'$. Then we define $\partial : H_n(X; \mathbb{Z}_2) \to H_{n-1}(X; \mathbb{Z}_2)$ by sending the homology class of $c$ to the homology class of $a$. One can check this map is then well-defined [2, p. 116].

**Part 2: The long exact sequence for $\mathbb{R}P^n$, $n > 1$**

Now that we have established a long exact sequence for arbitrary $X$ with a two-sheeted covering space $\tilde{X}$, we can apply this to the $n$-dimensional real projective space, $\mathbb{R}P^n$, and its two-sheeted covering space $S^n$. We will work with homology groups with coefficients in $\mathbb{Z}_2$. This results in the following long exact sequence for $n > 1$:

$$0 \to H_n(\mathbb{R}P^n; \mathbb{Z}_2) \xrightarrow{\tau_\ast} H_n(S^n; \mathbb{Z}_2) \xrightarrow{p_\#} H_n(\mathbb{R}P^n; \mathbb{Z}_2) \xrightarrow{\partial} H_{n-1}(\mathbb{R}P^n; \mathbb{Z}_2) \xrightarrow{\tau_\ast} 0 \to \ldots$$

$$\ldots \xrightarrow{\tau_\ast} 0 \xrightarrow{\tau_\ast} H_1(\mathbb{R}P^n; \mathbb{Z}_2) \xrightarrow{\partial} H_0(\mathbb{R}P^n; \mathbb{Z}_2) \xrightarrow{\tau_\ast} H_0(S^n; \mathbb{Z}_2) \xrightarrow{p_\#} H_0(\mathbb{R}P^n; \mathbb{Z}_2) \xrightarrow{\partial} 0. \quad (16)$$

Since $\mathbb{R}P^n$ is an $n$-dimensional CW complex, $H_i(\mathbb{R}P^n; \mathbb{Z}_2) = 0$ for all $i > n$. This gives us the first 0 in the long exact sequence. The other zero terms, except for the last one, are the homology groups of the $n$-dimensional sphere, since $H_i(S^n) = 0$ for $0 < i < n$.

**Part 3: The homology groups $H_i(\mathbb{R}P^n; \mathbb{Z}_2)$, $n > 1$**

From this sequence we can derive the homology groups $H_i(\mathbb{R}P^n; \mathbb{Z}_2)$ for $i \leq n$. We will start by determining $H_n(\mathbb{R}P^n; \mathbb{Z}_2)$. Recall that $\tau_{*,i}$ for $i \in \mathbb{N}$ is the map induced by $\tau$ working on the $i$th homology group of $\mathbb{R}P^n$, $H_i(\mathbb{R}P^n; \mathbb{Z}_2)$. The first line of sequence (16) gives the following information

$$\begin{cases}
0 = \text{Ker} \tau_{*,n} \\
\text{Im} \tau_{*,n} = \text{Ker} p_{*,n} \\
\text{Im} p_{*,n} = \text{Ker} \partial_n \\
\text{Im} \partial_n = H_{n-1}(\mathbb{R}P^n; \mathbb{Z}_2). 
\end{cases}$$

Since $\text{Ker} p_{*,n} \subseteq \mathbb{Z}_2$ there are two options: either $\text{Ker} p_{*,n} \cong 0$ or $\text{Ker} p_{*,n} \cong \mathbb{Z}_2$. Assume, in order to derive a contradiction later, that $\text{Ker} p_{*,n} \cong 0$. Then $\text{Im} \tau_{*,n} = 0$ by exactness. We already knew that $\text{Ker} \tau_{*,n} = 0$, so $H_n(\mathbb{R}P^n; \mathbb{Z}_2) \cong 0$. Since $\text{Im} \partial_n = H_{n-1}(\mathbb{R}P^n; \mathbb{Z}_2)$, also this group would be isomorphic to 0. From exactness we could then conclude that $\partial_n$ is an isomorphism for $1 < i < n$, which would give $H_i(\mathbb{R}P^n; \mathbb{Z}_2) \cong 0$ for $1 \leq i \leq n - 2$.

By exactness the last line of sequence (16) contains the following information:

$$\begin{cases}
0 = \text{Ker} \tau_{*,0} \\
\text{Im} \tau_{*,0} = \text{Ker} p_{*,0} \\
\text{Im} p_{*,0} = H_0(\mathbb{R}P^n; \mathbb{Z}_2). 
\end{cases}$$

From $0 = \text{Ker} \tau_{*,0}$ would follow that $\tau_{*,0}$ is injective. This would imply that $H_0(\mathbb{R}P^n; \mathbb{Z}_2)$ is either isomorphic to 0 or $\mathbb{Z}_2$. Both cases cause now a contradiction: If $H_0(\mathbb{R}P^n; \mathbb{Z}_2) \cong 0$ then by exactness $0 = \text{Im} \tau_{*,0} = \text{Ker} p_{*,0} = \mathbb{Z}_2$, which is a contradiction. If $H_0(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$ then respectively injectivity of $\tau_{*,0}$ and surjectivity of $p_{*,0}$ forces both $\tau_{*,0}$ and $p_{*,0}$ to be isomorphisms. This also constitutes a contradiction since then $\text{Ker} p_{*,0} = 0$ by $p_{*,0}$ being an isomorphism, but also $\text{Ker} p_{*,0} = \text{Im} \tau_{*,0} = \mathbb{Z}_2$ by exactness.

Hence it cannot be the case that our first assumption $\text{Ker} p_{*,n} \cong 0$ was true, so $\text{Ker} p_{*,n} \cong \mathbb{Z}_2$. It now follows from exactness that $\text{Im} \tau_{*,n} = \mathbb{Z}_2$ and since we already knew that $\tau_{*,n}$ was injective, it must also be an isomorphism. We conclude that $H_n(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$. By exactness $\partial_n$ is also an isomorphism, so $H_{n-1}(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$. The isomorphism $\partial_n$ for $2 \leq i \leq n - 1$ extends this result to $H_i(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$ for $1 \leq i < n - 2$.

The remaining homology group to determine is $H_0(\mathbb{R}P^n; \mathbb{Z}_2)$. Exactness gives the injectivity of $\partial_1$ and the surjectivity of $p_{*,0}$, since $\text{Ker} \partial_1 = 0$ and $\text{Im} p_{*,0} = H_0(\mathbb{R}P^n; \mathbb{Z}_2)$. Injectivity of $\partial_1$ tells
us that \( H_0(\mathbb{RP}^n; \mathbb{Z}_2) \supseteq \mathbb{Z}_2 \); surjectivity of \( p_{\ast,0} \) tells us that \( H_0(\mathbb{RP}^n; \mathbb{Z}_2) \subseteq \mathbb{Z}_2 \). As a consequence, \( H_0(\mathbb{RP}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2 \).

Hence we can conclude

\[
H_i(\mathbb{RP}^n; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & \text{for } 0 \leq i \leq n \\ 0, & \text{otherwise.} \end{cases}
\]

**Part 4: Properties of the long exact sequence for \( n > 1 \)**

From the exactness of the sequence we can identify various isomorphisms and zero-maps. We already identified \( \partial_i \) for \( 1 < i \leq n \) and \( \tau_{\ast,n} \) as isomorphisms and \( p_{\ast,n} \) as a zero-map. Now we will identify the remaining homomorphisms \( \partial_1, \tau_{\ast,0} \) and \( p_{\ast,0} \). From exactness we know that

\[
\begin{align*}
0 &= \ker \partial_1 \\
\operatorname{Im} \partial_1 &= \ker \tau_{\ast,0} \\
\operatorname{Im} \tau_{\ast,0} &= \ker p_{\ast,0} \\
\operatorname{Im} p_{\ast,0} &= \mathbb{Z}_2.
\end{align*}
\]

Injectivity of \( \partial_1 \) follows from \( 0 = \ker \partial_1 \). Since both \( H_1(\mathbb{RP}^n; \mathbb{Z}_2) \) and \( H_0(\mathbb{RP}^n; \mathbb{Z}_2) \) are isomorphic to \( \mathbb{Z}_2 \), \( \partial_1 \) must be an isomorphism. Hence \( \ker \tau_{\ast,0} = \operatorname{Im} \partial_1 = \mathbb{Z}_2 \), making \( \tau_{\ast,0} \) a zero map. The map \( p_{\ast,0} \) was already surjective by \( \operatorname{Im} p_{\ast,0} = \mathbb{Z}_2 \), but now also turns out to be an isomorphism by \( 0 = \operatorname{Im} \tau_{\ast,0} = \ker p_{\ast,0} \).

We can collect all this information in the sequence (16), giving the following

\[
0 \to H_n(\mathbb{RP}^n; \mathbb{Z}_2) \xrightarrow{\tau_n} H_n(S^n; \mathbb{Z}_2) \xrightarrow{p_{\ast,n}} H_n(\mathbb{RP}^n; \mathbb{Z}_2) \xrightarrow{\partial_n} H_{n-1}(\mathbb{RP}^n; \mathbb{Z}_2) \xrightarrow{\tau_{n-1}} \cdots \xrightarrow{\tau_1} 0 \xrightarrow{p_{\ast,1}} H_1(\mathbb{RP}^n; \mathbb{Z}_2) \xrightarrow{\partial_1} H_0(\mathbb{RP}^n; \mathbb{Z}_2) \xrightarrow{\tau_0} H_0(S^n; \mathbb{Z}_2) \xrightarrow{p_{\ast,0}} H_0(\mathbb{RP}^n; \mathbb{Z}_2) \xrightarrow{\partial_0} 0. \tag{17}
\]

**Part 5: The long exact sequence for \( \mathbb{RP}^1 \)**

The case \( n = 1 \) results in a slightly different sequence, namely the following long exact sequence:

\[
0 \to H_1(\mathbb{RP}^1; \mathbb{Z}_2) \xrightarrow{\tau_1} H_1(S^1; \mathbb{Z}_2) \xrightarrow{p_{\ast,1}} H_1(\mathbb{RP}^1; \mathbb{Z}_2) \xrightarrow{\partial_1} H_0(\mathbb{RP}^1; \mathbb{Z}_2) \xrightarrow{\tau_0} H_0(S^1; \mathbb{Z}_2) \xrightarrow{p_{\ast,0}} H_0(\mathbb{RP}^1; \mathbb{Z}_2) \xrightarrow{\partial_0} 0. \tag{18}
\]

The first 0 is yet again because \( \mathbb{RP}^1 \) is an 1-dimensional CW complex. The homology groups \( H_1(\mathbb{RP}^1; \mathbb{Z}_2) \) and \( H_0(\mathbb{RP}^1; \mathbb{Z}_2) \) can be determined in the same way as for \( n > 1 \). This gives \( H_i(\mathbb{RP}^1; \mathbb{Z}_2) \cong \mathbb{Z}_2 \) for all \( i \in \{0, 1\} \). The isomorphisms and zero-maps follow in the same way from exactness as for the \( n > 1 \)-case.

**Part 6: Applying an odd map to the long exact sequence**

Let \( f : S^n \to S^n \) be an odd map. This map induces a map \( \overline{f} : \mathbb{RP}^n \to \mathbb{RP}^n \). To make sure these maps induce in turn a map from the long exact sequence (16) to itself, we need to check whether the corresponding diagram on the level of chain groups is commutative [2, p. 127]

\[
\begin{align*}
0 &\to C_i(\mathbb{RP}^n; \mathbb{Z}_2) \xrightarrow{\tau_i} C_i(S^n; \mathbb{Z}_2) \xrightarrow{p_{\ast,i}} C_i(\mathbb{RP}^n; \mathbb{Z}_2) \\
\downarrow \tau_i &\quad \downarrow f_\ast &\quad \downarrow \tau_i \\
0 &\to C_i(\mathbb{RP}^n; \mathbb{Z}_2) \xrightarrow{\tau_i} C_i(S^n; \mathbb{Z}_2) \xrightarrow{p_{\ast,i}} C_i(\mathbb{RP}^n; \mathbb{Z}_2) \to 0.
\end{align*}
\tag{19}
\]

By definition of \( p \) the relation \( pf = \overline{f}p \) holds, implying that the right-hand side commutes. For the left-hand side, let \( \sigma : \Delta^i \to \mathbb{RP}^n \) be a singular \( i \)-simplex with lifts \( \tilde{\sigma}_1 \) and \( \tilde{\sigma}_2 \). Then, since \( f \) maps antipodal points onto antipodal points, \( \tau(f(\sigma)) = \tau(f\tilde{\sigma}) = f\tilde{\sigma}_1 + f\tilde{\sigma}_2 = f\overline{f}(\tau\sigma) \). So this diagram is indeed commutative. Then it follows by naturality of the long exact sequence that the maps \( f_* \overline{f} \) induce maps \( f_* \overline{f} \) from the long exact sequence (16) to itself and that the corresponding diagram is commutative [2, p. 127].

We claim that these maps \( f_* \overline{f} \) are isomorphisms between the corresponding homology groups and will prove this by induction on dimension. We will start with arguing why \( f_{\ast,0} : \)
\( H_0(S^n; \mathbb{Z}_2) \) is an isomorphism. Therefore, let the homology class of \( H_0(S^n; \mathbb{Z}_2) \) be represented by \( x : \Delta^0 \to S^n \). Since \( f \) maps the sphere to itself, there is a map \( y : \Delta^0 \to S^n \) such that \((f \circ x)(v_0) = y(v_0)\). Path-connectedness of the sphere \( S^n \) then implies that \( x(v_0) \) and \( y(v_0) \) are path connected. Since \( H_0(S^n; \mathbb{Z}_2) \) is generated by the path-components of \( S^n \), this means that \( x \) and \( y \) belong to the same homology class. Hence, the map \( f^* \), 0 is an isomorphism. The same argument can be used to argue why \( f^*: H_0(\mathbb{RP}^n; \mathbb{Z}_2) \to H_0(\mathbb{RP}^n; \mathbb{Z}_2) \) is an isomorphism, since also \( \mathbb{RP}^n \) is path connected. When we now move upwards in the diagram of homology groups, we pass the following part of the commutative diagram

\[
\begin{array}{cccccc}
\cdots & \to & H_1(\mathbb{RP}^n; \mathbb{Z}_2) & \overset{\partial}{\to} & H_0(\mathbb{RP}^n; \mathbb{Z}_2) & \to & \cdots \\
\downarrow f^* & & \downarrow f^* & & \downarrow f^* & & \\
\cdots & \to & H_1(\mathbb{RP}^n; \mathbb{Z}_2) & \overset{\partial}{\to} & H_0(\mathbb{RP}^n; \mathbb{Z}_2) & \to & \cdots \\
\end{array}
\]

Of the right-hand side \( f^* \) we have just established that it is an isomorphism. Hence we have three isomorphisms in the commutative square, so the fourth, \( f^* \), must also be an isomorphism. A similar argument can be repeated up until we have proven that \( f^* : H_n(S^n; \mathbb{Z}_2) \to H_n(S^n; \mathbb{Z}_2) \) is an isomorphism. Then the degree of \( f \) must be odd, since \( f^* \) is multiplication by \( \deg f \mod 2 \) [2, Lemma 2.49]. Otherwise \( f \) would have been the zero-map and hence not an isomorphism.

The goal of this section, proving the Borsuk-Ulam theorem, follows now quite directly.

**Corollary 3.2** (The Borsuk-Ulam theorem). For every map \( g : S^n \to \mathbb{R}^n \) there exists a point \( x_0 \in S^n \) with \( g(x_0) = g(-x_0) \).

**Proof of Corollary 3.2.** Define the odd map \( f : S^n \to \mathbb{R}^n \) by \( f(x) = g(x) - g(-x) \) for all \( x \in S^n \). Assume a point \( x_0 \) with \( g(x_0) = g(-x_0) \) does not exist, then \( f \) does not vanish on \( S^n \). Hence define \( h : S^n \to S^n \) by \( h = \frac{f}{|f|} \). On the one hand, since \( h(S^n) = S^{n-1} \), \( h \) is not surjective and hence has degree zero. On the other hand however, \( h \) is an odd function so has odd degree by Proposition 3.1. This constitutes a contradiction. 

\[ 13 \]
4 The Krasnosel'skii spectrum for quadratic forms

The goal of this section will be to prove for a real Hilbert space of dimension $n < \infty$ that the set of eigenvalues as found by the Courant-Fisher min-max principle equals the Krasnosel'skii spectrum for a quadratic form $q$. We will use the following definition for a quadratic form.

**Definition 4.1.** Let $H$ be a Hilbert space. A function $q : H \to \mathbb{R}$ is a quadratic form if a symmetric linear map $L : H \to H$ exists such that

$$q(u) = (Lu, u)$$

(21)

for all $u \in H$.

The first of the two definitions of eigenvalues to be compared is the Courant-Fisher min-max principle from Definition 2.1.

**Definition 4.2.** Let $H$ be a $n$-dimensional Hilbert space and $q : H \to \mathbb{R}$ a quadratic form on $H$, then we define for $k \in \{1, \ldots, n\}$

$$\lambda_k = \inf_{V \subset H} \sup_{\dim V = k} \sup_{u \in S(V)} q(u).$$

(22)

Note that from this definition follows that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

As a specific implementation of the Krasnosel'skii spectrum of Definition 2.8, a second definition for eigenvalues of quadratic forms can be posed.

**Definition 4.3.** Let $H$ be a $n$-dimensional Hilbert space and $q : H \to \mathbb{R}$ a quadratic form on $H$, then we define for $k \in \{1, \ldots, n\}$

$$\beta_k = \inf_{A \in V(S(H))} \sup_{\gamma(A) \geq k} q(u).$$

(23)

The goal of this section can now be formulated as to prove the following theorem.

**Theorem 4.4.** Let $H$ be a $n$-dimensional real Hilbert space and $q : H \to \mathbb{R}$ a quadratic form, then $\lambda_k = \beta_k$ for all $k \in \{1, \ldots, n\}$.

To prove Theorem 4.4 we will need the following lemma about the genus of a sphere.

**Lemma 4.5.** $\gamma(S^{k-1}) = k$.

**Proof of Lemma 4.5.** First we note that $\gamma(S^{k-1}) \leq k$ by taking $h$ from the definition to be the identity map $\mathbb{I}$.

Now we assume $\gamma(S^{k-1}) = l < k$. By definition of $\gamma$ there exists an odd continuous function $h : S^{k-1} \to \mathbb{R}^l \setminus \{0\} \subset \mathbb{R}^{k-1}$. From the Borsuk-Ulam theorem (Corollary 3.2) follows that a point $x_0 \in S^{k-1}$ exists such that $h(x_0) = h(-x_0)$. Contradiction follows since $h(x_0) = h(-x_0) = -h(x_0)$ implies $h(x_0) = 0$. \qed

Now we can prove Theorem 4.4.

**Proof of Theorem 4.4.** First we note that for $k = 1$, $\lambda_1$ equals $\inf_{u \in S(H)} q(u)$. The eigenvalue $\beta_1$ also equals $\inf_{u \in S(H)} q(u)$ since a finite collection of antipodal points has genus 1 as in property 5 of Proposition 2.7. Hence $\lambda_1 = \beta_1$.

The remainder of the proof consist of two parts; proving the inequalities $\lambda_k \geq \beta_k$ and $\lambda_k \leq \beta_k$ respectively for $k \geq 2$. The first of the two inequalities follows from Lemma 4.5, since if $V$ is a $k$-dimensional linear subspace of $H$, $\gamma(S(V)) = k$.

For the second inequality fix $k \in \{2, \ldots, n\}$. Since $q$ is a quadratic form, the associated linear map $L : H \to H$ is symmetric. Hence $L$ has $n$ orthonormal eigenvectors $\{u_i : 1 \leq i \leq n\}$ each corresponding to an eigenvalue $\lambda_i$. Take $u \in E = S(\text{span}\{u_k, \ldots, u_n\})$, then $u$ can be written as
\[ u = \sum_{i=k}^{n} \alpha_i u_i \text{ with } \alpha_i \in \mathbb{R} \text{ for } i = 1, \ldots, n \text{ and } \sum_{i=k}^{n} \alpha_i^2 = 1. \]

Hence

\[ q(u) = (Lu, u) = (E \sum_{i=k}^{n} \alpha_i u_i, \sum_{j=k}^{n} \alpha_j u_j) \]

\[ = (\sum_{i=k}^{n} \alpha_i L u_i, \sum_{j=k}^{n} \alpha_j u_j) \]

\[ = (\sum_{i=k}^{n} \alpha_i \lambda_i u_i, \sum_{j=k}^{n} \alpha_j u_j) \]

\[ = \sum_{i=k}^{n} \sum_{j=k}^{n} \alpha_i \alpha_j \lambda_i (u_i, u_j) \]

\[ \geq \lambda_k \]

(24)

where the last inequality follows from the earlier made observation that \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \). Now take a symmetric and closed subset \( A \) of \( S(H) \) with \( \gamma(A) \geq k \), then we will prove by contradiction that \( \sup_{u \in A} q(u) \geq \lambda_k \). Hence assume \( \sup_{u \in A} q(u) < \lambda_k \). Then by inequality (24) we know that \( A \cap E = \emptyset \).

Therefore we can define \( F = S(\text{span}\{u_1, \ldots, u_{k-1}\}) \subset S^n \) and the projection \( p: S^n \setminus E \to F \) by

\[ p(u) = \frac{\sum_{i=1}^{k-1} (u, u_i) u_i}{\sum_{i=1}^{k-1} (u, u_i)^2} \text{ for all } u \in S^n \setminus E. \]

(25)

Note that for all \( u \) in \( A \), \( p(u) \) is well defined, since \( A \cap E = \emptyset \). The function \( p \) is odd and continuous, and \( p(A) \subset F \), so \( \gamma(A) \leq \gamma(p(A)) \leq \gamma(F) \) by properties 2 and 4 of Proposition 2.7. However, \( \gamma(F) \leq \gamma(S^{k-2}) = k - 1 \) by Lemma 4.5. A contradiction arises, implying for all closed and symmetric subsets \( A \subset H \) with \( \gamma(A) \geq k \) that \( \sup q(u) \geq \lambda_k \). Hence \( \lambda_k \leq \beta_k \).

\[ \square \]

In the proof we noted that \( \lambda_1 = \beta_1 = \inf_{u \in S(H)} q(u) \). It is easy to see from the definition of \( \lambda_n \) that \( \lambda_n = \sup_{u \in S(H)} q(u) \), which implies by the theorem we have just proven that also \( \beta_n = \sup_{u \in S(H)} q(u) \). We could have proven this result in a more ad hoc way using Lemma 4.6 which tells us that \( S^{n-1} \) is the only subset of \( S(H) \) with genus \( n \). Since we have already proven Theorem 4.4 this may now seem unnecessary. However, we will still mention Lemma 4.6 since this lemma has other implications which we will later use, amongst others that \( \beta_n = \sup_{u \in S(H)} E(u) \) for any function \( E: H \to \mathbb{R} \).

**Lemma 4.6.** If \( H \) is an \( n \)-dimensional Hilbert space, then \( S^{n-1} \) is the only subset of \( S(H) \) with genus \( n \).

**Proof of Lemma 4.6.** Assume there exists a proper subset \( A \) of \( S(H) \) with \( \gamma(A) = n \). Assume without loss of generality that \( A \) does not contain the points \((0, \ldots, 0, \pm 1)\). Define \( p: A \to \mathbb{R}^{n-1} \) as \( p(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}) \). Then we have a contradiction since \( p \) is odd, continuous and \( 0 \notin p(A) \).

\[ \square \]
The homotopy significant spectrum

As announced in Section 2.4, we would like to compare the homotopy significant spectrum as defined by Gromov to the previously studied definitions of eigenvalues. More specifically we will compare Gromov’s eigenvalues to the Krasnoselskii spectrum from Definition 2.8. Hence it will be useful to understand the relation between the existence of a homotopy between two sublevels and their Krasnoselskii genus. We will be working in $\Phi = \mathbb{R}^n$ and we will consider continuous functions $E : \Phi \to \mathbb{R}$. We will start with the following theorem which relates the existence of a homotopy between two subsets and their LS category for general $\Phi$.

**Theorem 5.1.** Let $\Phi$ be a topological space and $Y \subset X \subset \Phi$. If there exists a homotopy $H : X \times [0, 1] \to \Phi$ with $H(x, 0) = \mathbb{1}_Y(x)$ for all $x \in X$ and $H(X, 1) \subset Y$ then $X$ and $Y$ have the same LS category, i.e. $\text{cat}_\Phi(X) = \text{cat}_\Phi(Y)$.

**Proof of Theorem 5.1.** First we note that since $Y \subset X$, $\text{cat}_\Phi(Y) \leq \text{cat}_\Phi(X)$. For the other inequality assume $\text{cat}_\Phi(Y) = m$. Let $\{C_1, \ldots, C_m\}$ be a collection of closed and contractible sets in $\Phi$ which cover $Y$. Define $f : X \to Y$ by $f(x) = H(x, 1)$ for all $x \in X$ and $A_i = f^{-1}(C_i)$ for $1 \leq i \leq m$. Then we claim that all $A_i$ are closed and contractible in $\Phi$, and $\{A_1, \ldots, A_m\}$ forms a cover of $X$. Take $i \in \{1, \ldots, m\}$. Then $A_i$ is closed by continuity of $f$.

Since $C_i$ is contractable a homotopy $\tilde{H} : C_i \times [0, 1] \to \Phi$ exists such that $\tilde{H}(x, 0) = \mathbb{1}_{C_i}(x)$ and $\tilde{H}(x, 1) = x_0 \in \Phi$ for all $x \in C_i$. Consider the function $G : A_i \times [0, 1] \to \Phi$ given by

$$G(x, t) = \begin{cases} H(x, 2t), & \text{for } 0 \leq t < 1/2 \\ \tilde{H}(x, 1, 2t - 1), & \text{for } 1/2 \leq t \leq 1 \end{cases}$$

for all $x \in A_i$. This is a continuous function since $H$ and $\tilde{H}$ are continuous and for all $x \in A_i$,

$$\lim_{t \uparrow 1/2} G(x, t) = H(x, 1) = f(x) = \mathbb{1}_{C_i}(f(x)) = \tilde{H}(x, 1, 0) = G(x, 1/2).$$

Since $G(x, 0) = \mathbb{1}_{A_i}(x)$ and $G(x, 1) = x_0$ for all $x \in A_i$, $G$ forms a homotopy between $\mathbb{1}_{A_i}$ and $x_0$. Hence $A_i$ is contractible.

The last thing to check is whether $\{A_1, \ldots, A_m\}$ forms a cover of $X$. Assume not, then there exists such that $\mathbb{1}_X \notin \bigcup_{i=1}^m A_i$. Specifically $x \notin f^{-1}(C_j)$ for all $j \in \{1, \ldots, m\}$. Since $\{C_1, \ldots, C_m\}$ forms a cover of $Y$, $f(x) \in f(X) \subset \bigcup_{j=1}^m C_j$. This gives a contradiction, so for all $x \in X$ there exists such that $f(x) \in C_j$ and hence $\{A_1, \ldots, A_m\}$ forms a cover of $X$. It follows that $\text{cat}_\Phi(Y) \leq \text{cat}_\Phi(X)$.

We conclude $\text{cat}_\Phi(X) = \text{cat}_\Phi(Y)$.

Because of this theorem we know that if a homotopy exists which brings one sublevel to a smaller sublevel, these sublevels have the same LS category. More important for the remainder is that we can also conclude from this theorem that if two sublevels have a different LS category, a homotopy between these sublevels does not exist. Hence the values where the category of the sublevels increases constitute a set in the homotopy significant spectrum.

**Definition 5.2.** Let $\Phi = \mathbb{R}^n$ and let $\Phi_{\leq t}$ be the closed $t$-sublevels of a continuous function $E : \Phi \to \mathbb{R}$. Then we define the eigenvalues $\mu_k \in \mathbb{R}$ of $E$ as

$$\mu_k = \inf \{ t \in \mathbb{R} | \text{cat}_\Phi(\Phi_{\leq t}) \geq k \}. $$

These eigenvalues $\mu_k$ are eigenvalues by Gromov’s definition for the homotopy significant spectrum, but may not be all the values in the homotopy significant spectrum. It could namely be the case that there are values where there does not exist a homotopy which brings one sublevel to another sublevel, but where the category of the sublevels is the same. We will come back to this later. However we can prove that $\mu_k$ is equal to $\beta_k$, which implies that the Krasnoselskii spectrum is contained in the homotopy significant spectrum. To prove this we would like to convert Definition 5.2 to a definition concerning the Krasnoselskii genus of a sublevel. For this we use the following lemma. Remember that we have defined $A^*$ as the image of $A \subset \mathbb{R}^n \setminus \{0\}$ under the quotient map $p : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}/\mathbb{Z}_2$.

**Lemma 5.3.** Let $A \subset S(\mathbb{R}^n)$ be closed and symmetric. Then $\text{cat}_{\mathbb{R}^n \setminus \{0\}/\mathbb{Z}_2}(A^*) = \text{cat}_{S(\mathbb{R}^n)/\mathbb{Z}_2}(A^*)$. 

16
Proof of Lemma 5.3. First we will prove \( \text{cat}_{S^{n-1}/Z_2}(A^*) \leq \text{cat}_{S^n}(A^*) \), then we will prove the reverse inequality. Denote \( S(S^n) = S^{n-1} \).

Suppose \( \text{cat}_{S^{n-1}/Z_2}(A^*) = m \). Let \( \{U_1,\ldots,U_m\} \) be a covering of \( A^* \) consisting of closed and contractible sets in \( S^{n-1}/Z_2 \). Then all these \( U_i \) are closed in \( S^n \setminus \{0\}/Z_2 \) since \( S^{n-1}/Z_2 \) is closed in \( \mathbb{R}^n \setminus \{0\}/Z_2 \). Since \( S^{n-1}/Z_2 \subset S^n \setminus \{0\}/Z_2 \) these sets \( U_i \) are also contractible in \( \mathbb{R}^n \setminus \{0\}/Z_2 \).

Hence \( \text{cat}_{S^n}(A^*) \leq \text{cat}_{S^n}(u) \).

Now suppose \( \text{cat}_{S^n}(A^*) = m \). Let \( \{U_1,\ldots,U_m\} \) be a covering of \( A^* \) consisting of closed and contractible sets in \( \mathbb{R}^n \setminus \{0\}/Z_2 \). Define \( V_i = U_i \cap S^{n-1}/Z_2 \) for each \( i \in \{1,\ldots,m\} \). Then \( V_i \) is closed in \( S^{n-1}/Z_2 \), since \( U_i \) is closed in \( \mathbb{R}^n \setminus \{0\}/Z_2 \) and \( S^{n-1}/Z_2 \) has the subspace topology induced from \( \mathbb{R}^n \setminus \{0\}/Z_2 \). The collection \( \{V_1,\ldots,V_m\} \) also forms a covering of \( A^* \) because both \( A^* \subset \bigcup_{i=1}^m U_i \) and \( A^* \subset S^{n-1}/Z_2 \). The remaining question is whether the sets \( V_i \) are contractible in \( S^{n-1}/Z_2 \) for all \( i \in \{1,\ldots,m\} \).

Fix \( i \in \{1,\ldots,m\} \). The set \( U_i \) is contractible, so a homotopy \( H : U_i \times [0,1] \rightarrow \mathbb{R}^n \setminus \{0\}/Z_2 \) exists such that \( H(x,0) = x \) and \( H(x,1) = x_0 \) for each \( x \in S^{n-1}/Z_2 \). Define the continuous function \( \tilde{H} = \frac{H}{\|H\|} \). Since \( H(x,0) = 1 \) and \( H(x,1) = \frac{x_0}{\|x_0\|} \) in \( S^{n-1}/Z_2 \), the map \( \tilde{H} \) forms a homotopy between \( 1 \) and \( x_0 \) in \( S^{n-1}/Z_2 \). Hence all sets \( V_i \) are contractible in \( S^{n-1}/Z_2 \). It follows that \( \text{cat}_{S^n}(A^*) \leq \text{cat}_{S^n}(u) \).

Since we work with \( \Phi = \mathbb{RP}^n = S(\mathbb{RP}^{n+1})/Z_2 \), we can use Lemma 5.3 together with Theorem 2.11 to formulate a definition for \( \mu_k \) equivalent to Definition 5.2 in terms of the Krasnoselskii genus.

Definition 5.4 ( Equivalent to Definition 5.2). Let \( \Phi = \mathbb{RP}^n \) and let \( \Phi_{\leq t} \) be the closed t-sublevels of a continuous function \( E : \Phi \rightarrow \mathbb{R} \). Then we define the eigenvalues \( \mu_k \in \mathbb{R} \) of \( E \) as

\[
\mu_k = \inf \{ t \in \mathbb{R} | \gamma(\Phi_{\leq t}) \geq k \}. \tag{28}
\]

Proof of the equivalence of Definition 5.2 and 5.4. Theorem 2.11 tells us that \( \gamma(\Phi_{\leq t}) \geq k \) is equivalent to \( \text{cat}_{S^n}(\Phi_{\leq t}) \geq k \). From Lemma 5.3 follows that the latter expression is equivalent to \( \text{cat}_{S^n}(\Phi_{\leq t}) \geq k \).\( \square \)

Since we now have formulated \( \mu_k \) in terms of the Krasnoselskii genus, we are ready to prove that \( \mu_k \) is equal to \( \beta_k \). This implies in particular that the Krasnoselskii spectrum is contained in the homotopy significant spectrum. Recall that we defined \( \beta_k \) in Definition 2.8 for even and continuous functions \( f : H \rightarrow \mathbb{R} \) on a Hilbert space \( H \) as

\[
\beta_k = \inf_{A \in \mathcal{V}(S(H))} \sup_{u \in A} \frac{\inf_{\gamma(A) \geq k} f(u)}{\gamma(\Phi_{\leq t}) \geq k} \tag{29}
\]

in which \( \mathcal{V}(S(H)) = \{A \subset S(H) | A \text{ closed and symmetric} \} \). This definition can be generalised to continuous functions \( E : \mathbb{RP}^n \rightarrow \mathbb{R} \), since such functions induce an even and continuous function on the sphere \( S^n \).

Theorem 5.5. Let \( \Phi = \mathbb{RP}^n \) and \( E : \Phi \rightarrow \mathbb{R} \) a continuous function, then \( \mu_k = \beta_k \) for all \( k \in \{1,\ldots,n+1\} \).

Proof of Theorem 5.5. Fix \( k \in \{1,\ldots,n+1\} \). We will prove the equality yet again by first proving \( \beta_k \leq \mu_k \) and then \( \beta_k \geq \mu_k \).

The sublevel \( \Phi_{\leq t} \) for \( t \in \mathbb{R} \) is closed and symmetric by definition, so \( \Phi_{\leq t} \in \mathcal{V}(S^n) \). In particular we have that the sublevels form a subset of \( \mathcal{V}(S^n) \), which implies that

\[
\beta_k = \inf_{\gamma(\Phi_{\leq t}) \geq k} \sup_{\gamma(A) \geq k} E(u) \leq \inf_{t \in \mathbb{R}} \sup_{u \in \Phi_{\leq t}} E(u). \tag{30}
\]

Using this inequality we can derive the first inequality \( \beta_k \leq \mu_k \) as follows. By definition of the sublevels \( \Phi_{\leq t} \) and the fact that the function \( E \) is continuous, it follows that \( \sup_{u \in \Phi_{\leq t}} E(u) = t \) for \( t \leq \beta_k \). That is to be restricted to \( t \leq \beta_k \), since \( \beta_k = \inf_{u \in S^n} E(u) \) and \( \beta_k = \sup_{u \in S^n} E(u) \). Then we recognise in the resulting expression our definition for \( \mu_k \) from Definition 5.4. More precisely, we can construct the first inequality, \( \beta_k \leq \mu_k \), as follows:

\[
\beta_k \leq \inf_{t \in \mathbb{R}} E(u) = \inf_{t \in \mathbb{R}} t = \mu_k. \tag{31}
\]
For the second inequality, \( \beta_k \geq \mu_k \), we define \( t_A = \sup_{u \in A} E(u) \) for all \( A \in \mathcal{V}(S^n) \). This definition allows us to express \( \beta_k \) in terms of the infimum over \( t_A \) by

\[
\beta_k = \inf_{A \in \mathcal{V}(S^n)} \sup_{\gamma(A) \geq k} E(u) = \inf_{A \in \mathcal{V}(S^n)} t_A. \tag{32}
\]

Considering a set \( A \in \mathcal{V}(S^n) \) with \( \gamma(A) \geq k \), we then want to find a relation between \( t_A \) and \( \mu_k \). As we discussed for the previous inequality we have \( t_A = \sup_{u \in \Phi^*_{<t_A}} E(u) \). We also know that \( A \subset \Phi^*_{\leq t_A} \) by definition of \( \Phi^*_{\leq t_A} \), which implies in turn \( \gamma(\Phi^*_{\leq t_A}) \geq \gamma(A) = k \) by property 2 of Proposition 2.7. The sublevel \( \Phi^*_{\leq t_A} \) is therefore just one of the sublevels \( \Phi_{<k} \) that satisfies \( \gamma(\Phi_{<k}) \geq k \). Hence we know for all \( A \in \mathcal{V}(S^n) \) such that \( \gamma(A) \geq k \) that \( \mu_k \) is a lower bound for \( t_A \) because

\[
t_A = \sup_{u \in \Phi^*_{<t_A}} E(u) \geq \inf_{t \in \mathbb{R}} \sup_{u \in \Phi^*_{<t}} E(u) = \inf_{t \in \mathbb{R}} t = \mu_k. \tag{33}
\]

Since the inequality \( t_A \geq \mu_k \) holds for all \( A \in \mathcal{V}(S^n) \) such that \( \gamma(A) \geq k \), the second inequality follows:

\[
\beta_k = \inf_{A \in \mathcal{V}(S^n)} t_A \geq \mu_k. \tag{34}
\]

Hence \( \mu_k = \beta_k \) for all \( k \in \{1, \ldots, n\} \).

As discussed earlier, these \( \mu_k \) may not be all eigenvalues that Gromov describes in his definition for eigenvalues. There could be values in the homotopy significant spectrum which are not contained in the Krasnosel’skii spectrum but where there does not exist a homotopy which brings one sublevel to another. We can however prove for the cases \( \Phi = \mathbb{R}P^3 \) and \( \Phi = \mathbb{R}P^2 \), that the values where the category, or equivalently genus, increases, form the entire homotopy significant spectrum. In other words, we can prove that in these cases the Krasnosel’skii spectrum equals the homotopy significant spectrum. This means in particular that for every value of \( t \in \mathbb{R} \) which does not equal one of the eigenvalues where the category increases, the sublevel \( \Phi_{<t} \) can be brought by an homotopy to the sublevel \( \Phi_{<t} \).

Note that any potential candidate for the homotopy significant spectrum \( t \in \mathbb{R} \setminus \{\mu_1, \ldots, \mu_n\} \) must satisfy at least \( \mu_1 < t < \mu_n \) since \( \mu_1 = \inf_{u \in \Phi} E(u) \) and \( \mu_n = \sup_{u \in \Phi} E(u) \). Instead of directly proving that the Krasnosel’skii spectrum equals the homotopy significant spectrum in the case of \( \mathbb{R}P^3 \), we can prove a more general result about the existence of eigenvalues in the homotopy significant spectrum below \( \mu_2 \). From this theorem the desired result in \( \mathbb{R}P^3 \) will follow.

**Theorem 5.6.** Let \( \Phi = \mathbb{R}P^n \) and \( E : \Phi \to \mathbb{R} \) a continuous function. There is nothing in the homotopy significant spectrum of \( E \) besides \( \mu_1 \) below \( \mu_2 \).

**Proof of Theorem 5.6.** Assume without loss of generality that \( \mu_1 \neq \mu_2 \), otherwise the claim follows immediately, and take \( t \in (\mu_1, \mu_2) \). We will prove that there exists a homotopy \( H : \Phi_{<t} \times [0, 1] \to \Phi \) with \( H(x, 0) = 1_{\Phi_{<t}}(x) \) and \( H(\Phi_{<t}, 1) \subset \Phi_{<t} \).

By definition of both \( \mu_1 \) and \( \mu_2 \) we have that \( \text{cat}_{\Phi} (\Phi_{<t}) = 1 \). Hence a set \( A \subset \Phi \) exists such that \( A \) is closed, contractible and \( \Phi_{<t} \subset A \). Since \( A \) is contractible, a homotopy \( G : \Phi_{<t} \times [0, 1] \to \Phi \) exists such that \( G(x, 0) = 1_A(x) \) and \( G(x, 1) = x_0 \in \Phi \). Define \( T : [0, 1] \to \Phi \) as the path connecting \( x_0 \) and a certain \( x_1 \in \Phi_{<t} \), which is possible since \( \Phi \) is path connected. Then we define \( H : \Phi_{<t} \times [0, 1] \to \Phi \) by

\[
H(x, s) = \begin{cases} 
G(x, 2s), & \text{for } 0 \leq s < \frac{1}{2}, \\
T(2s - 1), & \text{for } \frac{1}{2} \leq s \leq 1.
\end{cases} \tag{35}
\]

The map \( H \) forms a homotopy between \( H(x, 0) = 1_{\Phi_{<t}}(x) \) and \( H(x, 1) = x_1 \in \Phi_{<t} \) for all \( x \in \Phi_{<t} \), so in particular \( H(\Phi_{<t}, 1) = x_1 \in \Phi_{<t} \). Hence \( t \) cannot be an eigenvalue, so \( \mu_1 \) is the only eigenvalue in the homotopy significant spectrum belows \( \mu_2 \).

That the homotopy significant spectrum only contains \( \mu_1 \) and \( \mu_2 \) if \( \Phi = \mathbb{R}P^3 \), follows immediately.

**Corollary 5.7.** Let \( \Phi = \mathbb{R}P^3 \) and \( E : \Phi \to \mathbb{R} \) a continuous function. The eigenvalues \( \mu_1 \) and \( \mu_2 \) form the homotopy significant spectrum of \( E \).

18
The case $\Phi = \mathbb{RP}^2$ we need to prove separately.

**Theorem 5.8.** Let $\Phi = \mathbb{RP}^2$ and $E : \Phi \to \mathbb{R}$ a continuous function. The eigenvalues $\mu_1, \mu_2$ and $\mu_3$ form the homotopy significant spectrum of $E$.

**Proof of Theorem 5.8.** Assume without loss of generality that $\mu_1 \neq \mu_2 \neq \mu_3$. If $t \in \mathbb{R}$ is an eigenvalue by Gromov's definition it must satisfy either $\mu_1 < t < \mu_2$ or $\mu_2 < t < \mu_3$. It follows from Theorem 5.6 that $\mu_1 < t < \mu_2$ cannot be the case, so we will consider $t$ such that $\mu_2 < t < \mu_3$.

First we will prove that every sublevel $\Phi_{\leq \mu_2 + \varepsilon}$ for $0 < \varepsilon < \mu_3 - \mu_2$ contains a loop which is not null-homotopic in $\Phi$.

Take $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon < \mu_3 - \mu_2$. Then by definition of $\mu_2$ and $\mu_3$, we have that $\gamma(\Phi_{\leq \mu_2 + \varepsilon}) = 2$. Let $A_i \subset S^2$ for $i \in I \subset \mathbb{N}$ be the connected components of $\Phi_{\leq \mu_2 + \varepsilon}$ such that $\Phi_{\leq \mu_2 + \varepsilon} = \bigsqcup_{i \in I} A_i$, where $\bigsqcup$ denotes the disjoint union.

We would like to prove that there is an index $i \in I$ such that $A_i = -A_i$. To prove this assume that $A_i \neq -A_i$ for all $i \in I$, then we will derive a contradiction. Since $\Phi_{\leq \mu_2 + \varepsilon}$ is symmetric, there exist $J_1, J_2 \subset I$ such that $I = J_1 \cup J_2$ and such that for all $i \in J_1$, $j \in J_2$ exists such that $A_i = -A_j$.

To see this, assume that a $l \in I$ exists such that $A_l \neq -A_k$ for all $k \in I$. Take $x_0 \in A_l$, then since $\Phi_{\leq \mu_2 + \varepsilon}$ is symmetric, there is a $k \in I$ such that $-x_0 \in A_k$ by assumption. Then there are two options. The first option is that there exists a $x_1 \in A_l$ such that $-x_1 \notin A_k$. This constitutes a contradiction, since this implies that $x_0$ and $-x_1$ are not connected by a path in $\Phi_{\leq \mu_2 + \varepsilon}$.

We conclude that there is an index $i \in I$ such that $A_i = -A_i$. Take $x_0 \in A_l$, then $-x_0 \in A_l$. Since $A_l$ is path-connected, we know by Lemma 4.6 that at least two antipodes on the sphere are not null-homotopic in $\Phi$. Moreover, since the fundamental group of $\Phi$ is isomorphic to $\mathbb{Z}_2$, $\tilde{\Phi}$ is homotopic to any non-contractible loop in $\Phi$ [2, Example 1.43].

We have now proven that for every $\varepsilon$ such that $0 < \varepsilon < \mu_3 - \mu_2$, $\Phi_{\leq \mu_2 + \varepsilon}$ contains a path $\tilde{\Phi}$ which is homotopic to any non-contractible loop in $\Phi$. Fix now $t \in (\mu_2, \mu_3)$ and subsequently $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon < (t - \mu_3)/2$. We will show that we can bring the sublevel $\Phi_{\leq t}$ to the path $\tilde{\Phi}$ in $\Phi_{\leq \mu_2 + \varepsilon}$.

Since $\gamma(\Phi_{\leq t}) < 3$, we know by Lemma 4.6 that at least two antipodes on the sphere are not contained in $\Phi_{\leq t}$, which implies that we can contract $\Phi_{\leq t}$ to an equator of $S^2$ through $x_0$ and $-x_0$. We can parametrise this equator by $R : [0, 2] \to S^2$ such that $R(0) = R(2) = x_0$ and $R(s) = -R(1 + s)$ for all $s \in [0, 1]$. Then we know in other words that a homotopy $G_1 : \Phi_{\leq t} \times [0, 1] \to S^2$ exists such that $G_1(x, 0) = 1_{\Phi_{\leq t}}(x)$ for all $x \in \Phi_{\leq t}$ and $G_1(\Phi_{\leq t}, 1) = R([0, 2])$. Since we can assume $G_1$ to be odd in $x$, that is $G_1(-x, s) = -G_1(x, s)$ for all $x \in \Phi_{\leq t}$ and $s \in [0, 1]$, and know that $R([0, 1])$ induces a loop $\tilde{R}$ which is not null-homotopic in $\Phi$, we know that $G_1$ induces a homotopy $\tilde{G}_1 : \Phi_{\leq t} \times [0, 1] \to \Phi$. This homotopy $\tilde{G}_1$ is such that $\tilde{G}_1(x, 0) = 1_{\Phi_{\leq t}}$, and $\tilde{G}_1(\Phi_{\leq t}, 1) = \tilde{R}([0, 1])$.

Since $\tilde{\Phi}$ is homotopic to any loop in $\Phi$ it is in particular homotopic to $\tilde{R}$. Hence a homotopy $G_2 : [\tilde{R}([0, 1]) \times [0, 1] \to \Phi$ exists such that $G_2(x, 0) = 1_{\tilde{R}([0, 1])}(x)$ and $G_2(\tilde{R}([0, 1]), 1) = P([0, 1])$. When we construct the map $G : \Phi_{\leq t} \times [0, 1] \to \Phi$ as

$$G(x, s) = \begin{cases} G_1(x, 2s), & \text{for } 0 \leq s < \frac{1}{2}, \\
G_2(G_1(x, 1), 2s - 1), & \text{for } \frac{1}{2} \leq s \leq 1 \end{cases}$$

for all $x \in \Phi_{\leq t}$, we have a homotopy such that $G(x, 0) = 1_{\Phi_{\leq t}}(x)$ and $G(\Phi_{\leq t}, 1) = \tilde{P}([0, 1]) \subset \Phi_{\leq \mu_2 + \varepsilon}$.

Hence all $t \in \mathbb{R}$ such that $\mu_2 < t < \mu_3$ cannot be eigenvalues by Gromov’s definition, which implies that $\mu_1, \mu_2$ and $\mu_3$ form the homotopy significant spectrum of $E$. 

\qed
We now know that the Krasnoselskii spectrum equals the homotopy significant spectrum in the cases where $\Phi = \mathbb{R}^3$ and $\Phi = \mathbb{R}P^2$. It is however not possible to extend this result to higher dimensions. We will discuss an example of this for the case $\Phi = \mathbb{R}P^3$ in the following section.

5.1 The homotopy significant spectrum in the case $\Phi = \mathbb{R}P^3$

In this section we will construct an example which shows that the homotopy significant spectrum can be strictly larger than the Krasnoselskii spectrum for functions defined on $\mathbb{R}P^3$. We will therefore first introduce a subset of $\mathbb{R}P^3$ and prove this subset has category 3. Then we will show we cannot bring $\mathbb{R}P^2$ via a homotopy to this subset. We will use this property to construct a function on $\mathbb{R}P^3$ whose homotopy significant spectrum consists of more than only $\mu_1, \mu_2, \mu_3$ and $\mu_4$ from Definition 5.2 and 5.4.

The subset of $\mathbb{R}P^3$ we will use is the connected sum of $\mathbb{R}P^2$ and the torus $T$, which we will denote as $\mathbb{R}P^2 \# T$. The $\Delta$-complex of this space is shown in Figure 1. Our first goal is to prove that $\mathbb{R}P^2 \# T$ has LS category 3 in $\mathbb{R}P^3$. In the proof we will use the standard distance on $\mathbb{R}P^3$, denoted as $\text{dist}(x, y)$ for $x, y \in \mathbb{R}P^3$, which is induced by the standard distance on $S^3$.

**Lemma 5.9.** $\text{cat}_{\mathbb{R}P^3}(\mathbb{R}P^2 \# T) = 3$.

**Proof of Lemma 5.9.** By property 6 of Proposition 2.7 there exists a neighbourhood $N \subset D^3$ of $(\mathbb{R}P^2 \# T)^-$ such that $\gamma(\overline{N}) = \gamma((\mathbb{R}P^2 \# T)^-)$, which implies that $\text{cat}_{\mathbb{R}P^3}(\overline{N}) = \text{cat}_{\mathbb{R}P^3}(\mathbb{R}P^2 \# T)$. By compactness of $\mathbb{R}P^2 \# T$ we choose this neighbourhood to be a tubular neighbourhood, where we define a tubular neighbourhood of $\mathbb{R}P^2 \# T$ as $T_x(\mathbb{R}P^2 \# T) = \{x \in \mathbb{R}P^3 : \text{dist}(x, \mathbb{R}P^2 \# T) < \varepsilon\}$ for $\varepsilon > 0$. Hence an $\varepsilon > 0$ exists such that $\text{cat}_{\mathbb{R}P^3}(T_x(\mathbb{R}P^2 \# T)) = \text{cat}_{\mathbb{R}P^3}(\mathbb{R}P^2 \# T)$. Property 3 of Proposition 2.10 then tells us that

$$\text{cat}_{\mathbb{R}P^3}(\mathbb{R}P^3) \leq \text{cat}_{\mathbb{R}P^3}(T_x(\mathbb{R}P^2 \# T)) + \text{cat}_{\mathbb{R}P^3}(\mathbb{R}P^3 \setminus T_x(\mathbb{R}P^2 \# T)), \quad (37)$$

because $\mathbb{R}P^3 = T_x(\mathbb{R}P^2 \# T) \cup \mathbb{R}P^3 \setminus T_x(\mathbb{R}P^2 \# T)$. Therefore we want to prove that $\text{cat}_{\mathbb{R}P^3}(\mathbb{R}P^3 \setminus T_x(\mathbb{R}P^2 \# T)) = 1$, in other words that $\mathbb{R}P^3 \setminus T_x(\mathbb{R}P^2 \# T)$ is contractible. The reason for this is that the contractibility of $\mathbb{R}P^3 \setminus T_x(\mathbb{R}P^2 \# T)$ together with the fact that $\text{cat}_{\mathbb{R}P^3}(\mathbb{R}P^3) = 4$ implies that $\text{cat}_{\mathbb{R}P^3}(T_x(\mathbb{R}P^2 \# T)) \geq 3$.

Consider therefore $\mathbb{R}P^3$ as $D^3$ with antipodes on the boundary $S^2$ identified, denoted as $D/\sim$. Then, if we choose $\varepsilon$ small enough, $D^3 \setminus T_x(\mathbb{R}P^2 \# T)^-$ consists of two connected components,
Lemma 5.10. We can compute the homology groups of $D^3 \setminus T_\epsilon(\mathbb{RP}^2 \# T)^- \times U$ which is an open space consisting of two path components. We will denote the two connected components of $D^3 \setminus T_\epsilon(\mathbb{RP}^2 \# T)^- \times U$ as $U$ and $L$, where $U$ is contained in the upper hemisphere of $D^3$ and $L$ is the other connected component as to be seen in Figure 2. In $D^3$ we can contract $U$ to the north pole $p_N$ and $L$ to the south pole $p_S$, since $D^3$ is itself contractible. More precisely there exists a homotopy $G_1 : U \times [0, 1] \to D^3$ such that $G_1(x, 0) = 1_U(x)$ and $G_1(x, 1) = p_N$ for all $x \in U$, and a homotopy $G_2 : L \times [0, 1] \to D^3$ such that $G_2(x, 0) = 1_L(x)$ and $G_2(x, 1) = p_S$ for all $x \in L$. We can combine these two homotopies to a homotopy $G(x, s) : D \setminus T_\epsilon(\mathbb{RP}^2 \# T)^- \times [0, 1] \to D^3$ given by

$$G(x, s) = \begin{cases} G_1(x, s), &\text{for } x \in U, s \in [0, 1], \\ G_2(x, s), &\text{for } x \in L, s \in [0, 1]. \end{cases}$$

(38)

Since we can assume that $G$ is symmetric on the boundary, this homotopy $G$ induces a homotopy $H : \mathbb{R}^3 \setminus T_\epsilon(\mathbb{RP}^2 \# T) \to \mathbb{R}^3$ such that $H(x, 0) = 1_{\mathbb{R}^3 \setminus T_\epsilon(\mathbb{RP}^2 \# T)}(x)$ and $H(x, 1) = p_N = p_S$. Hence $\mathbb{R}^3 \setminus T_\epsilon(\mathbb{RP}^2 \# T)$ is contractible, implying that $\text{cat}_{\mathbb{R}^3}(\mathbb{RP}^2 \# T) = \text{cat}_{\mathbb{R}^3}(T_\epsilon(\mathbb{RP}^2 \# T)) \geq 3$. Note that since $\mathbb{RP}^2 \# T$ is a proper subset of $\mathbb{RP}^3$, it has by Lemma 4.6 a category strictly smaller than 4. If we combine these two results we can conclude that $\text{cat}_{\mathbb{RP}^3}(\mathbb{RP}^2 \# T) = 3$.

![Figure 2: $D^3/\sim$ with the center $\mathbb{RP}^2 \# T$](image)

To be able to use the space $\mathbb{RP}^2 \# T$ as an example to why in the case of $\Phi = \mathbb{RP}^3$ the homotopy significant spectrum is larger than only $\mu_1, \mu_2$ and $\mu_3$, we want to prove a homotopy from $\mathbb{RP}^2$ to $\mathbb{RP}^2 \# T$ does not exist. However to prove this, we will first need to compute the cohomology groups of $\mathbb{RP}^2 \# T$.

Lemma 5.10. $H^\ast(\mathbb{RP}^2 \# T; \mathbb{Z}_2) \cong \mathbb{Z}_2[x, y, z]/(xy, xz, y^2, z^2, x^2 + yz, x^3)\mathbb{Z}_2[x, y, z]$

Proof of Lemma 5.10. We can compute the homology groups of $\mathbb{RP}^2 \# T$ using simplicial homology as

$$H_i(\mathbb{RP}^2 \# T; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, &\text{for } i = 0, 2, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, &\text{for } i = 1, \\ 0, &\text{for } i \geq 3, \end{cases}$$

(39)

with the generators of the homology groups represented by: $v_0$ for $H_0(\mathbb{RP}^2 \# T; \mathbb{Z}_2)$, $e_1$, $e_2 + e_4$ and $e_{14} + e_{15}$ for $H_1(\mathbb{RP}^2 \# T; \mathbb{Z}_2)$ and $\sum_{i=1}^{10} f_i$ for $H_2(\mathbb{RP}^2 \# T; \mathbb{Z}_2)$. Since $\text{Hom}(\bigoplus_{i=1}^m \mathbb{Z}_2, \mathbb{Z}_2) = \bigoplus_{i=1}^m \mathbb{Z}_2$ for $m \in \mathbb{N}$ we know by Theorem 2.5 that

$$H^i(\mathbb{RP}^2 \# T; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, &\text{for } i = 0, 2, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, &\text{for } i = 1, \\ 0, &\text{for } i \geq 3. \end{cases}$$

(40)

The generator of $H^0(\mathbb{RP}^2 \# T; \mathbb{Z}_2)$ is the identity 1 which takes the value 1 on all the vertices. We can find representatives of the generators of the first cohomology group by drawing curves in $\mathbb{RP}^2 \# T$ as shown in Figure 1. The representative of the generator dual to $e_1$, which we will call $\phi_1$, should satisfy $\delta \phi_1 = \phi_1 \partial = 0$. Since we are working with $\mathbb{Z}_2$ coefficients, this means that the
number of times $\phi_1$ takes the value 1 on the edges of a 2-simplex should be even. Hence we can associate with $\phi_1$ a closed curve $C_1$ such that the number of intersections $C_1$ has with an edge, is the value $\phi_1$ takes on that edge. Since $\phi_1$ is dual to $e_1$, it should take 1 as value on $e_1$ and 0 on the other generators $e_2 + e_4$ and $e_1 + e_5$. Therefore the curve $C_1$ should pass through $e_1$ once and never through the other two generators. The curve $C_1$ is drawn in Figure 1 and we can read off $\phi_1$ as to be given by

$$\phi_1(e_i) = \begin{cases} 
1, & \text{for } i = 1, 6, 7, 10, 11, \\
0, & \text{otherwise.}
\end{cases} \quad (41)$$

Similarly we find $\phi_2$ and $\phi_3$, which represent the two other generators of $H^1(\RP^2 \# T, \Z_2)$, given by

$$\phi_2(e_i) = \begin{cases} 
1, & \text{for } i = 2, 3, 10, 11, 16, \\
0, & \text{otherwise.}
\end{cases} \quad (42)$$

and

$$\phi_3(e_i) = \begin{cases} 
1, & \text{for } i = 15, 16, 17, 18, \\
0, & \text{otherwise.}
\end{cases} \quad (43)$$

We know that $H_2(\RP^2 \# T; \Z_2)$ is generated by $\sum_{i=1}^{10} f_i$, so to completely determine the cup product $H^1(\RP^2 \# T; \Z_2) \times H^1(\RP^2 \# T; \Z_2) \to H^2(\RP^2 \# T; \Z_2)$ we compute $(\phi_k \sim \phi_l)(\sum_{i=1}^{10} f_i)$ and find

$$\phi_k \sim \phi_l \left( \sum_{i=1}^{i=10} f_i \right) = \begin{cases} 
1, & \text{for } (k,l) = (1,1), (2,3), \\
0, & \text{otherwise.}
\end{cases} \quad (44)$$

Hence the homology class represented by $\phi_1 \sim \phi_1$ is a generator of $H^2(\RP^2 \# T; \Z_2)$ and since $H^i(\RP^2 \# T; \Z_2) = 0$ for $i \geq 3$ we have now fully determined the cup product. This implies that the cohomology ring $H^*(\RP^2 \# T; \Z_2)$ is isomorphic to $\Z_2[x, y, z]/(xy, xz, y^2, z^2, x^2 + yz, x^3)\Z_2[x, y, z]$.

Now we can prove that we cannot bring $\RP^2$ via a homotopy inside $\RP^2 \# T$.

**Lemma 5.11.** A homotopy $H : \RP^2 \times [0, 1] \to \RP^3$ such that $H(x, 0) = 1_{\RP^2}(x)$ for all $x \in \RP^2$ and $H(\RP^2, 1) \subset \RP^2 \# T$ does not exist.

**Proof of Lemma 5.11.** Let $i : \RP^2 \hookrightarrow \RP^3$ denote the inclusion of $\RP^2$ into $\RP^3$, and $j : \RP^2 \# T \hookrightarrow \RP^3$ the inclusion of $\RP^2 \# T$ into $\RP^3$. Further, let $[\RP^2]$ denote the fundamental class of $\RP^2$, which is the homology class which generates $H_2(\RP^2; \Z_2) \cong \Z_2$, and let likewise $[\RP^2 \# T]$ be the fundamental class of $\RP^2 \# T$. In the case of cohomology we will use the notation $[\RP^2]^*$ for the generator of $H^2(\RP^2; \Z_2)$.

Assume a homotopy $F : \RP^2 \times [0, 1] \to \RP^3$ such that $F(x, 0) = 1_{\RP^2}(x)$ for all $x \in \RP^2$ and $F(\RP^2, 1) \subset \RP^2 \# T$ does exist, then we will derive a contradiction. We denote the homotopy here with $F$ to avoid confusion with the (co)homology groups.

Therefore define $F_0 : \RP^2 \to \RP^2$ as $F_0(x) = F(x, 0)$ for all $x \in \RP^2$ and $F_1 : \RP^2 \to \RP^2 \# T$ as $F_1(x) = F(x, 1)$ for all $x \in \RP^2$. Then since $i \circ F_0$ and $j \circ F_1$ are homotopic, it holds that $i_*F_0 = j_*F_1$ [2, Theorem 2.10]. We also know that $i_*[\RP^2]$ is the generator of $H_2(\RP^3; \Z_2)$ by cellular homology. If we combine these two, they imply that $j_*F_1_*[\RP^2] \neq 0$ since

$$j_*F_1_*[\RP^2] = i_*F_0_*[\RP^2] = i_*[\RP^2] 
eq 0. \quad (45)$$

Hence $F_1_*[\RP^2]$ itself is not zero in $H_2(\RP^2 \# T; \Z_2)$.

The map $F_1$ induces not only maps on homology classes, but also on cohomology classes, which we will denote as $F_1^*$. The map $F_1^* : H^1(\RP^2 \# T; \Z_2) \to H^1(\RP^2; \Z_2)$ is for instance defined as $F_1^* \circ \alpha = \alpha \circ F_1$ for all $\alpha \in H^1(\RP^2 \# T; \Z_2)$.

By definition we have $F_1^*([\RP^2 \# T]^*) = [\RP^2]^*F_1_*([\RP^2])$ of which the right-hand side is not equal to 0 since both $[\RP^2 \# T]^* \neq 0$ and $F_1_*[\RP^2] \neq 0$. Hence $F_1^*([\RP^2 \# T]^*)$ on the left-hand side is the generator of $H^1(\RP^2 \# T; \Z_2)$.

Since $H^1(\RP^2 \# T; \Z_2) \cong \Z_2 \oplus \Z_2 \oplus \Z_2$ and $H^1(\RP^2; \Z_2) \cong \Z_2$, there is as least one $\alpha \in H^1(\RP^2 \# T; \Z_2)$ such that $\alpha \neq 0$ and $F_1^*(\alpha) = 0$. The rank-nullity theorem from linear algebra namely tells us that the dimension of the kernel and the image of $F_1^*$ should add up to the dimension of
of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, which is 3. By (the proof of) Theorem 5.10 there is a $\beta \in H^1(\mathbb{RP}^2 \# T; \mathbb{Z}_2)$ such that $\alpha \sim \beta \neq 0$. Hence $\alpha \sim \beta$ equals $[\mathbb{RP}^2 \# T]^*$, since $H^2(\mathbb{RP}^2 \# T; \mathbb{Z}_2) \cong \mathbb{Z}_2$.

Combining these observations results in

$$[\mathbb{RP}^2]^* = F_1^\ast([\mathbb{RP}^2 \# T]^*) = F_1^\ast(\alpha \sim \beta) = (F_1^\ast(\alpha)) \sim (F_1^\ast(\beta)) = 0,$$

where we have used the naturality of the cup product in the second to last equality. However, it cannot be the case that $[\mathbb{RP}^2]^* = 0$ since $H^2(\mathbb{RP}^2; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Hence $F$ cannot exist.

Then we can now use what we have proved so far to show that there exists a function on $\mathbb{RP}^3$ of which the homotopy significant spectrum is larger than only the Krasnoselskii spectrum.

**Theorem 5.12.** A function $f : \mathbb{RP}^3 \to \mathbb{R}$ exists such that its homotopy significant spectrum is strictly larger than its Krasnoselskii spectrum.

**Proof of 5.12.** We claim that if we take a suitable version of $\mathbb{RP}^2 \# T$ and define the function $f : \mathbb{RP}^3 \to \mathbb{R}$ as $f(x) = \text{dist}(x, \mathbb{RP}^2 \# T)$ for all $x \in \mathbb{RP}^3$, then the homotopy significant spectrum of $f$ contains an eigenvalue that is not contained in the Krasnoselskii spectrum of $f$. In the remainder of this proof we will formalise what a suitable version of $\mathbb{RP}^2 \# T$ would be and prove our claim.

Therefore we will look at the closed $t$-sublevels $\Phi_{\leq t}$ of $f$ for $t \in \mathbb{R}$. The closed 0-sublevel, $\Phi_{\leq 0}$, is given by $\Phi_{\leq 0} = \{x \in \mathbb{RP}^2 \# T\}$. Let $r > 0$ denote the radius of the tube in the torus $T$ and let $R > r$ denote the radius from the center of the torus $T$ to the center of the torus tube, as indicated in Figure 3. Then we can choose $R$ such that $\Phi_{\leq t}$ for all $t \in [0, r)$ deformation retracts to $\mathbb{RP}^2 \# T = \Phi_{\leq 0}$. The $r$-sublevel $\Phi_{< r}$ does not, since it contains $\mathbb{RP}^2$. Moreover, since $\Phi_{< r}$ contains $\mathbb{RP}^2$, Theorem 5.11 tells us that a homotopy $G : \Phi_{< r} \times [0, 1] \to \mathbb{RP}^3$ such that $G(x, 0) = \mathbb{1}_{\Phi_{< r}}(x)$ for all $x \in \mathbb{RP}^2$ and $G(\Phi_{< r}, 1) \subset \Phi_{\leq 0}$ does not exist. Since $\Phi_{\leq 0}$ is a deformation retract of $\Phi_{< r}$, $\Phi_{\leq 0}$ and $\Phi_{< r}$ are homotopy equivalent. Hence also a homotopy $H : \Phi_{\leq r} \times [0, 1] \to \mathbb{RP}^3$ such that $H(x, 0) = \mathbb{1}_{\Phi_{\leq r}}(x)$ for all $x \in \mathbb{RP}^2$ and $H(\Phi_{\leq r}, 1) \subset \Phi_{< r}$ does not exist. Hence $r$ lies in the homotopy significant spectrum. However since $\Phi_{\leq 0} = \emptyset$ and $\Phi_{\leq 0} = \mathbb{RP}^2 \# T$ of which by Theorem 5.9 we know that $\text{cat}_{\mathbb{RP}^3}(\mathbb{RP}^2 \# T) = 3$, it follows that $\mu_3 = 0$. This also implies that $\text{cat}_{\mathbb{RP}^3}(\Phi_{\leq t}) = 3$ for all $t \in [0, \sup_{x \in \mathbb{RP}^3} \text{dist}(x, \mathbb{RP}^2 \# T))$, which implies that $r$ is not contained in the Krasnoselskii spectrum.

![Figure 3: Zoom-in view of where the torus T is attached to \mathbb{RP}^2 in \mathbb{RP}^2 \# T, with the radii r and R of the torus indicated](image-url)

\[\square\]
6 Conclusion

In this paper we have looked at definitions for non-linear eigenvalues. First we have proved the Borsuk-Ulam theorem. We have used this theorem to prove that the eigenvalues of a quadratic form given by the Courant-Fisher min-max principe equal those in the Krasnoselskii spectrum.

Our next goal was to relate the homotopy significant spectrum of a continuous function on the real projective space to the Krasnoselskii spectrum. We proved that a subset of the homotopy significant spectrum of such a function is given by the values where the Krasnoselskii genus, or Lusternik-Schnirelmann category, of the corresponding sublevels strictly increases. Moreover, we have showed that this subset of the homotopy significant spectrum equals the Krasnoselskii spectrum. Hence the Krasnoselskii spectrum is contained in the homotopy significant spectrum. In the one and two-dimensional case of the real projective space, we have proved that the Krasnoselskii spectrum forms the whole homotopy significant spectrum. This result can not be extended to higher dimensions, since we have constructed an example for the three-dimensional case where the homotopy significant spectrum is strictly larger than the Krasnoselskii spectrum.
References


