A Time-based Policy for Empty Container Management by Consignees

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Abstract

Despite the many advantages of containerization for goods transportation, the management of empty containers is a major disadvantage, driving costly repositioning operations. We investigate the potential for consignees to manage an inventory of empty containers at their location so as to enable direct reuse of these containers by consignors located in the surroundings. One difficulty is that the detention fees imposed by shipping companies under merchant haulage are nonlinear. Moreover, cleaning and related activities might be necessary if the consignee keeps some empties. These difficulties incentivize consignees to immediately return containers to the sea terminal. Contrary to this practice, we show that managing containers via time-based threshold policies can save costs. Without cleaning costs, we prove that a time-based policy with a timeout return time is optimal. We next develop a policy improvement approach to include the cleaning costs in the analysis. This results in a two-thresholds time based policy where the two time thresholds control the admission into and rejection out of the inventory. We next analyze the impact of this proactive management on the level of direct container reuse. This practice enables a high level of direct reuse. It also reduces container repositioning costs. Yet, the incentive to implement our policy varies a lot from one setting to another. In particular, low cleaning costs or high repositioning costs lead to significant costs improvement when implementing our policy. Finally, we further explore if the incentive could be made stronger by modifying the structure and/or purpose of the detention costs.

Keywords: Empty container management; Markov decision process; inventory policy; time-based threshold.

1 Introduction

Containerization is the trend towards the deployment of a standard and dedicated international transport system for containers from door-to-door. Containerization has shaped global supply chains by providing reliable, low cost and secure service for international trade. Container-based trade has expanded from 50 million Twenty-foot Equivalent Units (TEUs) in 1996 to 180 million TEUs in 2015, such that containerized cargo now represents more than half the value of all international seaborne trade (Trade and Development (2016)). As a result, many authors consider ocean container transport as critical for global supply chain performance (Fransoo and Lee, 2013).

Container transport has many advantages including standardization, ease of handling, protection against damage and security. However, empty container movements are considered as an important disadvantage of containerization. Once emptied at destination, the container often needs to be repositioned to be filled.
in again. For instance, the review of maritime transport published by the United Nations (2011) highlights that the costs of seaborne empty container repositioning was estimated to $20 billion in 2009, while the costs of empty container repositioning in the hinterland was around $10 billion for the same year. Globally, empty container repositioning accounted for 19% of the global industry income in 2009 (United Nations, 2011).

In this article, we focus on empty container management in the hinterland of a deep-sea port, i.e., at the regional level (Boile et al., 2008). Consider an import container emptied in the hinterland and assume that this one needs to be reloaded with export cargo. Although inefficient, the common practice consists of having first the empty container being shipped back to the port, before being sent to the consignor. Besides the financial impacts associated with this practice, unnecessary movements of empty containers lead to negative societal impacts such as pollution, congestion and accidents. In this article, we focus on street-turns; the most straightforward strategy to reduce the unnecessary movements of empty containers. The idea behind a street-turn is very basic and consists of shipping the empty container directly from the consignee to the consignor without passing by a terminal. There are two important advantages of the street-turn strategy. First, empty movements are reduced and accordingly repositioning costs decrease. For example, Hjortnaes et al. (2017) estimate that street-turns (referred to as direct repositioning in their article) result in cost savings of up to 17%. The reduction of empty movements may also imply a decrease in the number road accidents. Each street-turn also avoids two movements to and from the terminal, reducing congestion. Second, empty container demand from the consignors can be met sooner, increasing the container utilization rate which positively impacts container fleet sizing (Jula et al., 2006; Dong and Song, 2012).

Even if the advantages of the street-turn strategy appear to be important for all parties involved, Lei and Church (2011) highlight that street-turns are only used 10% of the time in the hinterland of Los Angeles/Long Beach. Similarly, Wolff et al. (2007) find that the share of street-turns was in a range of 5-10% in the hinterland of the port of Hamburg. Consequently, Braekers et al. (2011) estimate that empty containers account for 40% to 50% of the regional movements. There are multiple explanations for the limited use of street-turns. Among them, we highlight here that consignees are generally not involved in empty container management as containers are generally controlled by the shipping lines who own and/or lease a pool of containers. Therefore, empty containers that become available at a consignee’s site are shipped back to the shipping line before an export match could be identified. This practice is highlighted by Lee and Song (2017) who state that regional container movements are operated by actors beyond the control of ocean carriers. This article explores the demand for consignees to proactively manage empty containers at their location to enhance the feasibility of street-turns. From our knowledge, this article is the first one to investigate this option. The street-turn strategy is traditionally studied from a shipping line perspective (Jula et al., 2006; Deidda et al., 2008; Furió et al., 2013; Sánz Bernat et al., 2016).

We acknowledge that a centralized management of empty containers involving coordination between the consignee, the consignor and the shipping line would bring important benefits. This setting might make sense in a carrier haulage setting under which the shipping line is controlling the movement of containers (both full and empty) in the hinterland. This is the reason why many shipping lines attempt to increase the share of carrier haulage. However, based on many discussions with experts, we believe that merchant
haulage (that represents for instance 70% of the market share in Europe nowadays) will still be critical in the foreseeable future as freight forwarders, consignees and consignors have developed unique expertise in managing containers in the hinterland. In our article, we propose a new policy to increase the performance of empty container management in the hinterland taking the decentralized setting of merchant haulage for granted. Obviously, it could be argued that a decentralized setting is by essence sub-optimal, but our aim is to explore a tractable policy that requires little coordination between the consignee, the consignor and the shipping line. In the setting considered through the article, both the shipping line and the consignor are better off when a match is made, so we believe that we can reasonably assume to take solely the consignee’s perspective.

Under merchant haulage, the shipping lines charge so-called detention and demurrage fees. Demurrage fees are incurred when the container stays at the deep-sea terminal, and detention fees are charged when the container is in the hinterland, until being shipped back to the shipping line. From the perspective of the shipping company who owns the containers, containers are assets in the hinterland and therefore, they represent some opportunity costs. A shipping line executive clearly highlighted this fact during our preliminary interviews by stating that the container should make money in each single part of the chain. Shipping lines also use detention and demurrage fees as a way to indirectly control the containers in the hinterland in order to ensure timely return of their asset. As a result, detention and demurrage fees are relatively high in practice. However, consignees tend to negotiate detention and demurrage free periods. This leads to a complex structure in practice, with a free period, and several levels of fees. As an example the official detention fees for a 40 ft. dry container imported to the port of Rotterdam and transported further by truck include a free period of 3 days, then the rate is €55/day from day 4 to day 7 and finally the rate is €85/day after 7 days (Maersk, 2016). In our discussions with consignees across Europe (Netherlands, France and Sweden), detention fees are always mentioned as one of the main barriers against street-turn strategies as the consignees feel that they do not have enough time to identify an export match before incurring high detention costs. A secondary barrier may be the difficulty to manage containers at the location due to cleaning and handling costs.

In this article, we assess whether consignees might take a proactive role in empty container management by formulating an inventory model for empty container management under merchant haulage. Our setting involves nonlinear inventory holding costs inviting for controlling the time spent by the containers at the consignee’s location (control out). Moreover, we account for fixed cleaning costs that can be avoided with an immediate return. This invites for controlling whether the containers should be placed into inventory or sent back directly to the shipping line (control in). These two levels of control should be optimized such that a good trade-off between repositioning and managing costs can be obtained in a cost minimization perspective. The traditional decision variable for such a problem is the quantity of containers in inventory. This choice is known to be optimal for linear inventory holding costs (Li et al., 2004; Song and Zhang, 2010; Zhang et al., 2014). However, this decision variable is no longer optimal under non-constant detention fees per time unit. The ages of each container in inventory should also be considered. The derivation of the optimal policy based on the number of containers together with their ages is challenging and will most likely
lead to a very complex structure. Such a policy is unlikely to be implemented. Instead, based on the data of Maersk (2016) which suggest a strong time-dependency of the detention fees, we only consider the age of the containers as a decision variable. This choice also follows from the practice where the tracking of containers has significantly improved over the last years. Using a Markov decision process approach, we prove that a deterministic time based threshold policy is optimal for the control out. Following a Markov chain analysis, we derive closed-form expressions of the performance measures and the optimal cost per container. Based on proven monotonicity properties of the performance measures, we next develop an algorithm to compute efficiently the optimal time-threshold. In addition, an asymptotic analysis is proposed to get closed-form expressions of the threshold in particular regimes. The policy for the control in is obtained via a procedure of policy improvement from the optimal policy for the control out. To this end, we explicitly derive the relative value function under the optimal policy for the control out. The improvement step consists of improving an optimal Bernoulli policy for the control in using the expression of the relative value function. This results in a two-thresholds time based policy where the two thresholds control the admission in and the rejection out of the inventory. Using the numerical values of Maersk (2016), we show that the proactive management of empty containers by consignees enables reaching a high level of street-turns. Moreover, we show that the cost saving ranges from 16% to 69% as compared to the immediate return of all empty containers to the shipping line based on the data from the Netherlands. We also identify other conditions for which the incentive is relatively limited. Therefore, we further investigate if the incentive could be made stronger by increasing the detention free period or by changing the detention fees structure. Without modifying the rewards of the shipping company, these changes may reduce the intensity of empty containers transportation.

**Structure of the article.** We conclude this section with a short literature survey. Section 2 describes the optimization problem and the methodological approach. Section 3 develops a Markov decision process approach to derive time-based policies for container management. Section 4 evaluates the obtained policies. Section 5 presents the main insights of the study. Section 6 concludes the article. All proofs and a reminder of the notations are given in the Appendix at the end of the article.

**Literature review.** The literature on containerization is quite extensive. We refer to Levinson (2010) for a historical perspective on containerization. Empty container management has deserved a lot of attention from the transport and maritime economics communities. Many relevant and inspiring results have been generated. However, the transport and maritime economics literature does not take the inventory perspective into account, and hence their research paradigms cannot be used to tackle the problem we consider here. We refer to Dejax and Crainic (1987) for a review of early works of the Operations Management/Transportation Science community in this field and we refer to Braekers et al. (2011), Song and Dong (2015) and Lee and Song (2017) for recent overviews. Lee and Song (2017) classify the existing contributions into two categories. The first one applies network flow models to the empty container repositioning problem. The second category considers the empty container repositioning problem from an inventory theory perspective by considering empty containers as inventories that enable meeting customer demand. For instance, Lee and Song (2017) highlight that many shipping lines are using inventory-based policies to reposition empty containers. The
related contributions provide evidences of the optimality of control policies based on quantity-thresholds and sometimes provide closed-form solutions for these threshold levels. More specifically, Li et al. (2004), Song and Zhang (2010), Zhang et al. (2014) focus on a single empty depot located in a port and controlled by a shipping line. Song (2005, 2007), Lam et al. (2007), Shi and Xu (2011), Ng et al. (2012), Xie et al. (2017), Yu et al. (2018) focus on empty container (or equivalently vehicle) management for a two-depot system. Finally, we refer to Du and Hall (1997), Li et al. (2007), Song and Dong (2008), Yun et al. (2011), Dang et al. (2012) and Dang et al. (2013) for a focus on more general networks. Considering empty containers from an inventory control perspective also allows to make another link with the existing literature as this setting shares some similarities with remanufacturing/disposal models in the reverse logistics context (see e.g., Heyman (1977) or Teunter and Vlachos (2002)).

2 Problem formulation

Model and optimization problem. We model the containers’ inventory at the consignee’s location as a queueing system. Containers arrive over time at the consignee’s location. The arrival process of containers is assumed to be Poisson with parameter $\lambda$. At arrival, a container is emptied and the consignee may decide either to send it back directly to the sea terminal or to keep it for a certain amount of time in order to answer the future demand from a consignor in need of an empty container. The latter option is called a street-turn. We assume that the aggregated demand for empty containers from consignors is stochastic and follows an exponential distribution with rate $\mu$. In this article, we refer to a single consignor for simplicity. The analysis also holds in the case where multiple consignors are located in the same surrounding, each of them with a Poisson demand for empties. The management of empty containers at the consignee is very complex. At any point in time and for each container, the consignee has to decide whether to keep this container or to send it back to the sea terminal. The objective is to minimize the long-run expected cost per container; $E(C)$.

The motivation to keep an empty container is to operate a street-turn with a consignor. The advantage of street-turns for the consignee results from the reduction in transportation costs that can be obtained if sending the empty container to the consignor. Let the cost of sending an empty container to the shipping line (either to the deep-sea terminal or to an empty depot depending on the cheapest available option) be $c_{sl}$ monetary units. In case of a match, let the transportation cost per container be $c_{st}$ monetary units. We consequently define the repositioning costs as $c_s = c_{sl} - c_{st}$. Operating a match allows the consignee to save the cost $c_s$.

The motivation to send back an empty container to the sea terminal is to save the management cost of this container. When a empty container is placed in the inventory, the consignee incurs linear holding costs due to physical storage costs, detention costs and initial costs due to container handling and possible cleaning and preparation (such as maintenance). These initial costs are incurred only if the empty container is placed in the inventory. For simplicity, we refer to those initial inventory costs as cleaning costs. Also for simplicity, we aggregate holding costs, detention costs and cleaning costs and we refer to them as inventory holding costs. From the structure of the detention fees, we assume that inventory holding costs are increasing and convex in the time spent in inventory. Therefore, the oldest container in inventory is also the most costly
one. We denote by \( c(t) \) the inventory holding cost of a container which has stayed exactly \( t \) time units in the inventory. Note that the cleaning costs are counted at the arrival of a container in the inventory with \( c(0) > 0 \). If a container is directly sent back to the shipping line, then this cost is not counted.

The management of containers at the consignee’s location consists in implementing a policy which provides a good trade-off between management costs and repositioning costs in order to minimize the expected cost per container. In practice, the policy optimization considered in this article is often avoided. Instead, consignees apply a simple policy which consists of sending back directly all empty containers to the shipping line. This policy is referred as the immediate return policy or Policy IR. This policy is optimal if \( c_s \leq 0 \). Otherwise if \( c_s > 0 \), it may be beneficial to implement a more sophisticated policy.

Since the cleaning costs are not counted for containers which are directly sent back to the shipping line, it makes sense not to let all containers enter the inventory if \( c(0) > 0 \). Admission control at container arrival is referred to the control in. Specifically, the consignee has to determine at the arrival of a new container whether this container should be cleaned and kept in the inventory for an eventual future match or should be directly sent back to the shipping line. Once it has been decided to keep a container in the inventory, the consignee has to decide at any point of time and for each container whether to keep this container for a future match or to send it back to the shipping line. Since \( c(t) \) is increasing in \( t \), the oldest container is also the most costly. Therefore, it is optimal to apply a first-in-first-out policy for sending back containers from the inventory. This reduces the decision action to the Oldest Container at the Consignee (OCC). This level of decision is referred to the control out.

**Remark.** We consider the regional empty container management problem from the consignee’s perspective in isolation from the problem of managing the cargo inside the containers. The inventory management decisions traditionally focus on the cargo and exclude the transportation packaging units. As a first attempt in analyzing the management of empty containers from the consignee perspective, we focus on models that do not affect the management of the cargo itself. The supply of empty containers is consequently not considered as a decision variable in our model and we consider this one as stochastic.

**Methodological approach.** We propose the following approach to derive the policy for container management at the consignee. First, we assume that all containers are admitted in the inventory and we propose to optimize the control out. We develop a Markov decision process approach where the age of the oldest container is employed to define the system state. Using an induction step approach, we prove that the optimal time-based policy for this problem is a deterministic threshold policy, referred to as Policy TBP. Note that the decision to let all containers enter the inventory is optimal if the cleaning costs are negligible compared to the other costs (\( c(0) = 0 \)). This in turn makes Policy TBP optimal if \( c(0) = 0 \). Next, we explicitly determine the components of the optimal cost under Policy TBP as well as the optimal time threshold in some asymptotic regimes.

Second, we consider the control in. The optimal policy with non negligible cleaning costs is difficult to obtain due to the two levels of control. To overcome this difficulty, we propose to develop a one-step policy improvement approach to determine a close to optimal policy. We explicitly derive the relative value
function under Policy TBP. Next, we assume that the consignee applies a Bernoulli policy for the control in. This allows us to decouple the system into two independent subsystems where the relative value function and the expected cost can be explicitly computed. The randomizing parameter of the Bernoulli Policy and the optimal threshold for the control out are chosen such that the expected cost is minimized. This defines the *Optimal Bernoulli Policy*, referred to as Policy OB. Policy OB is blind in the sense that it does not take the state of the system in consideration for making a decision. Therefore, it can be improved. Using the relative value function computed for the Policy OB, we determine the best action between sending back a container or letting this container enter the system based on the age of the oldest container. This leads us to a two-thresholds deterministic policy, referred to as Policy TBP.

**Setting of the numerical analysis.** A numerical analysis is conducted to evaluate the quality of the proposed policies (Section 4) and to determine the stake of managing containers at the consignee’s location (Section 5). In particular, we are interested in the expected cost per container, the expected number of container at the location and the proportion of match. This proportion is computed as the ratio of the expected number of matches and the expected demand from the consignor on a given period of time. Note that we consider an import zone with $\lambda \geq \mu$ so the maximum number of matches is limited by the demand for empty containers by the consignor.

The cleaning cost, $c(0)$, and the repositioning cost, $c_s$, are important drivers of the decisions and may strongly vary from one consignee to the other. Therefore, the insights of our analysis are discussed as functions of these costs. We use the detention cost function of Maersk (2016). They include a Detention Free Period of 3 days, then the rate is €55/day from day 4 to day 7 and finally the rate is €85/day after 7 days. In addition, we assume that the physical storage cost is equal to €5/day. Thus, we have $c(t) = c(0) + 5t\mathbb{1}_{0<t\leq3} + (60t - 165)\mathbb{1}_{3<t\leq7} + (90t - 375)\mathbb{1}_{t>7}$. We denote by $a$ the ratio between the arrival and the demand (i.e., $a = \frac{\lambda}{\mu}$). The ratio $a$ represents the import/export balance. We choose an expected demand, $\mu$, of 1 container per day. We take $a = 1$, $a = 2$ and $a = 5$ as examples. These values are representative of imbalance in major trade lanes. For instance, Theofanis and Boile (2009) report that trade imbalance for the transpacific lane increased from 50% (i.e., $a = 2$) to 67% (i.e., $a = 3$) from 2000 to 2005.

### 3 Policy analysis

In Section 3.1, we assume that all containers are admitted at the consignee’s inventory. This allows us to prove the threshold form of the optimal policy for the control out and to determine the expression of the cost components under this policy. Next, in Section 3.2, we develop a policy improvement approach to derive a time-based policy for the control in.

#### 3.1 Optimal policy for the control out with $c(0) = 0$

Section 3.1.1 proves that the optimal control policy for the control out is a threshold policy on the age of the oldest container. Section 3.1.2 computes the performance measures and the optimal cost under this policy. Section 3.1.3 develops an asymptotic analysis which leads to explicit expressions of the optimal threshold.
The traditional state description based on the number of containers in the hinterland fails to evaluate the overall cost per time unit of a set of containers due to the nonlinear inventory holding cost structure. The information on the age of each container is needed for such an evaluation. Given the Poisson arrival process and the first-in-first-out discipline in the inventory, the information on the age of the oldest container at the consignee allows us to determine the age distribution of all containers at the consignee. Therefore, the age of the oldest container is chosen as a decision variable to determine the optimal policy.

The problem is formulated as a Markov decision process. Subsequently, we use the value iteration technique to prove the form of the optimal policy. We consider a non-traditional approach for the modeling of the queueing system, as proposed in Koole et al. (2012) and Legros et al. (2017). The idea is to discretize the waiting time of the Oldest Container at the Consignee (OCC) by an Erlang distribution with rate $\gamma$ per phase. As $\gamma$ tends to infinity, this approximate setup converges to the original one, which in turn leads to an exact analysis.

Let us denote by $x$ a state of the system, where $x \geq 0$. State $x = 0$ corresponds to an empty inventory. States with $x > 0$ correspond to a situation where the OCC has an age of $x$ phases. Lumping together the states representing the empty system and the time spent by the OCC at the consignee in one dimension can be done as the system cannot be empty while containers are waiting. We next describe the 3 possible transitions in the Markov process. When the OCC changes, because of a consignor’s demand (see transition Type 2), the age phase changes from $x > 0$ to $x - h$ with probability $q_{x,x-h}$, where $q_{x,x-h} = \left( \frac{\lambda}{\lambda + \gamma} \right) \left( \frac{\gamma}{\lambda + \gamma} \right)^h$ for $0 \leq h < x$ and $q_{x,0} = \left( \frac{\gamma}{\lambda + \gamma} \right)^x$ (Koole et al., 2012; Legros et al., 2017).

1. An arrival with rate $\lambda$ while the system is empty ($x = 0$), which changes the state to $x = 1$, then the OCC entity is created.

2. A consignor’s demand with rate $\mu q_{x,x-h}$ while the system is not empty ($x > 0$), which changes the state to $x - h$, that is, the new OCC is in waiting phase $x - h$.

3. A phase increase with rate $\gamma$ while the system is not empty ($x > 0$), which changes the state to $x + 1$. The waiting phase of the OCC is increased by 1.

Let us denote by $V_n(x)$ the expected cost over $n$ steps, for $n \geq 0$ and $x \geq 0$. We pay a cost of $c_s$ per container sent back to the shipping line and let $c(x)$ be the inventory holding cost function per container defined for $x > 0$. We assume that $c(x)$ is a increasing and convex function in $x$. The cost $c(x)$ can therefore be used to model both the physical storage cost and the detention cost which may include a Detention Free Period (DFP). Since the total event rate is bounded, our continuous-time model is uniformizable (Section 11.5.2. in Puterman (1994)). This allows us to consider the system only at transition instants and simplify the problem formulation. The uniformization is done using the maximal event rate $\lambda + \mu + \gamma$, that we assume equal to 1. We denote by $F$ the operator on the set of functions $f$ from $\mathbb{N}$ to $\mathbb{R}$ defined by
\[ F(f(x)) = \sum_{h=0}^{\infty} q_{x,x-h} f(x-h) \text{ for } x > 0, \text{ and } F(f(0)) = f(0) \text{ for } x = 0. \] For \( n \geq 0 \), we have

\[
V_{n+1}(0) = \lambda W_n(0) + (1 - \lambda) V_n(0), \quad \text{and,} \\
V_{n+1}(x) = \gamma W_n(x) + \mu (F(V_n(x)) + c(x)) + (1 - \gamma - \mu) V_n(x), \text{ for } x > 0,
\]

with \( W_n(x) = \min(F(V_n(x)) + c_s + c(x), V_n(x + 1)) \) if \( x \geq 0 \). We assume that \( V_0 = W_0 = 0 \). For each \( n > 0 \) and every state \( x (x \geq 0) \) there is a minimizing action: send an empty container to the shipping line or keep all containers in the inventory. As \( n \) tends to infinity, this policy converges to the average optimal policy (see Theorem 8.4.5 of Puterman (1994)).

In Theorem 1, we prove that the optimal policy is a Time-Based threshold Policy (Policy TBP). This means that all containers are allowed to join the inventory, regardless of the system state. However, there exists a time limit, referred to as the Threshold Detention Time (TDT), such that if a container has waited since exactly TDT time units, then it is sent back to the shipping line.

**Theorem 1** The optimal policy for the control out, referred to as Policy TBP, with \( c(0) = 0 \) is as follows:

- No rejection is exercised at a container arrival,
- A container is rejected to the sea terminal whenever its age has reached TDT time units.

### 3.1.2 Performance analysis

We evaluate here the performance of the system under Policy TBP. We approximate the deterministic duration before being sent back to the shipping line by an Erlang random variable with \( n \) phases and rate \( \gamma \) per phase. We choose \( n \) and \( \gamma \) such that \( \frac{n}{\gamma} \Delta = \text{TDT} \). The Laplace transform of the Erlang distribution with parameters \( n \) and \( \gamma \) is \( \left( \frac{\gamma}{\gamma + s} \right)^n \). We have

\[
\left( \frac{\gamma}{\gamma + s} \right)^n = e^{n \ln((1+s/\gamma)^{-1})} \sim_{\gamma \to \infty} e^{n \ln(1-s/\gamma)} \sim_{\gamma \to \infty} e^{-ns/\gamma} = e^{-s\text{TDT}},
\]

where we write \( f(a) \sim_{a \to a_0} g(a) \) to express that \( \lim_{a \to a_0} \frac{f(a)}{g(a)} = 1 \), for \( a_0 \in \mathbb{R} \). Applying the Levy continuity theorem for Laplace transforms, this result ensures that as \( n \) and \( \gamma \) go to infinity, the considered Erlang random variable converges in distribution to the deterministic duration before being sent back to the shipping line, TDT.

The transition structure is identical to the one in Section 3.1.1. The only difference is the transition from state \( x = n \); a transition from \( n \) to a state \( n - h \) can be caused not only by a \( \mu \)-transition but also by a \( \gamma \)-transition which represents then a container sent back to the shipping line. In Theorem 2, we give closed-form expressions for the probability of an empty system \( p_0 \), the proportion of containers sent back to the shipping line, \( P_s \), the expected time spent by a container in the inventory, \( E(T) \), the expected number of container in the inventory, \( E(N) \), and the proportion of empty containers which has spent more than \( t \) time units in the inventory, \( P(T > t) \) with \( 0 \leq t \leq \text{TDT} \).
Theorem 2 We have

\[ p_0 = \frac{1 - a}{1 - ae^{-TDT(\mu - \lambda)}}, \quad P_s = p_0 \cdot e^{-TDT(\mu - \lambda)}, \quad E(N) = a \cdot \frac{1 - e^{-TDT(\mu - \lambda)(1 + aTDT(\mu - \lambda))}}{(1 - a)(1 - ae^{-TDT(\mu - \lambda)})}, \]

\[ E(T) = \frac{E(N)}{\lambda}, \quad \text{and,} \quad P(T > t) = \mathbb{1}_{t < TDT} \frac{e^{-t(\mu - \lambda)} - ae^{-TDT(\mu - \lambda)}}{1 - ae^{-TDT(\mu - \lambda)}}, \]

where \( \mathbb{1}_{x \in A} \) is the indicator function of a subset \( A \).

Note that our system reduces to an M/M/1+D queue where the arrival process is generated by the arrival of empty containers and the service is ensured by the consignor. The metrics provided above were already derived. We refer the reader to Graves (1978) for the derivation of the first four metrics in a context of perishable inventory and to Baccelli and Hebuterne (1981) for the derivation of the waiting time distribution in an M/M/1+D queue.

Using the expression of \( P(T > t) \), one may obtain the probability density function of the time spent in the system by a matched container; \( \frac{\partial (1 - P(T > t))}{\partial t} \). This leads to the expression of the expected cost per container in Corollary 1.

Corollary 1 We have

\[ E(C) = (c_s + c(TDT)) \cdot \frac{(1 - a)e^{-TDT(\mu - \lambda)}}{1 - ae^{-TDT(\mu - \lambda)}} + \int_{t=0}^{TDT} \frac{(1 - a)e^{-t(\mu - \lambda)}}{1 - ae^{-TDT(\mu - \lambda)}} \cdot c(t) \, dt. \]  

(1)

In Proposition 1, we give the first and second order monotonicity properties of the main performance measures as a function of the control parameter TDT. These properties will be used to develop a method to compute numerically the optimal threshold for a general cost function. Moreover, these results will also be used in Section 5 to better explain the behavior of the expected cost.

Proposition 1 For \( a > 0 \) and \( TDT > 0 \), the following holds:

1. The proportion of containers sent back to the shipping line, \( P_s \), is strictly decreasing and strictly convex in TDT (the proportion of matches is thus strictly increasing and strictly concave in TDT).

2. The expected time spent at the consignee \( E(T) \) and the expected number of containers are strictly increasing in TDT.

3. The proportion \( P(T > t) \) is strictly increasing in TDT for \( t < TDT \).

4. Given that \( c(t) \) is a continuous piecewise linear increasing in \( t \), the expected cost per container, \( E(C) \), is first decreasing and next increasing in TDT.

The expected time spent at the consignee as well as the expected number of containers at the consignee are neither convex nor concave in TDT for \( TDT > 0 \). These monotonicity results apply for the M/M/1+D queue. The last statement of the proposition allows us to propose the following simple algorithm to obtain the optimal threshold.
Algorithm: Computation of the optimal threshold. We denote by $c_k$, the slope of the function $c(t)$ on the interval $[t_{k-1}, t_k)$ with the convention that $t_0 = 0$ for $k \geq 1$.

Start with $k = 1$. Go to line 1.

1. If $c_s \mu > c_k$, solve

\[ c_s \mu (1 - a)^2 = c_k (1 - 2a + a(1 - a) \mu \text{TDT} + a^2 e^{-\mu (1 - a) \text{TDT}}). \]  

Denote by $\text{TDT}_k$ the solution of this equation. If $\text{TDT}_k \geq t_k$, increase $k$ by 1 and go back to line 1. If $t_{k-1} \leq \text{TDT}_k < t_k$, go to line 2. If $\text{TDT}_k < t_{k-1}$, go to line 3. If $c_s \mu \leq c_k$, go to line 3.

2. The optimal solution is $\text{TDT}_k$.

3. The optimal solution is $t_{k-1}$.

Corollary 2 On each interval $[t_{k-1}, t_k)$, for $k > 0$, the optimal threshold $\text{TDT}$ is increasing and piecewise concave in $c_s$ and decreasing and piecewise convex in $c_k$. Moreover, the proportion of matches, the expected number of empty containers at the consignee and the optimal cost per container are increasing and piecewise concave in $c_s$ and decreasing and piecewise convex in $c_k$.

3.1.3 Asymptotic analysis

Even if Equation (2) in the algorithm can be easily solved numerically, it does not lead to an explicit expression of the optimal threshold. We therefore additionally provide explicit expressions of the optimal threshold in extreme cases of the import/export balance. These expressions are derived from Equation (2) using Taylor expansions and equivalent expressions. The following holds

- As $a$ tends to infinity (if the import/export balance is high), Equation (2) leads to

\[ \text{TDT} \overset{a \to \infty}{\sim} \frac{1}{\lambda} \ln \left( \frac{c_s \mu}{c_k} \right). \]

- As $a$ tends to zero (if the arrival of containers is low or if the demand is high), Equation (2) leads to

\[ \text{TDT} \overset{a \to 0}{\sim} \frac{1}{\lambda} \left( \frac{c_s \mu}{c_k} - 1 \right). \]

In Table 1, we evaluate the expected cost obtained using these two expressions of the threshold in comparison with the use of the optimal threshold for different values of the arrival rate in the case of a constant holding cost per time unit, $c_1$. In the second and the third column we give the optimal threshold and its related expected cost. We compute the relative difference between the expected cost obtained with the two approximations and the optimal expected cost by $\text{rd} = \frac{E(C)_{\text{approximation}} - E(C)_{\text{optimal}}}{E(C)_{\text{optimal}}}$. As expected, Table 1 reveals that the first approximation is the best for high arrival rate situations whereas the second one is the best for low arrival rate situations. However, it is interesting to observe that under low arrival rate
Table 1: Performance comparison ($\mu = 1$, $c_s = 80$, $c_1 = 5$)

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>Optimal policy</th>
<th>Approximation 1</th>
<th>Approximation 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$TDT_{opt}$</td>
<td>$E(C)_{opt}$</td>
<td>$TDT_1 = \frac{1}{\lambda} \ln \left( \frac{c_s}{c_u} \right)$</td>
</tr>
<tr>
<td>0.01</td>
<td>1485.00</td>
<td>0.05</td>
<td>277.26</td>
</tr>
<tr>
<td>0.1</td>
<td>135.10</td>
<td>0.56</td>
<td>27.73</td>
</tr>
<tr>
<td>0.25</td>
<td>45.35</td>
<td>1.67</td>
<td>11.09</td>
</tr>
<tr>
<td>0.5</td>
<td>16.00</td>
<td>5.00</td>
<td>5.55</td>
</tr>
<tr>
<td>0.75</td>
<td>7.55</td>
<td>13.29</td>
<td>3.70</td>
</tr>
<tr>
<td>1</td>
<td>4.57</td>
<td>27.84</td>
<td>2.77</td>
</tr>
<tr>
<td>1.5</td>
<td>2.51</td>
<td>63.80</td>
<td>1.85</td>
</tr>
<tr>
<td>2</td>
<td>1.73</td>
<td>102.25</td>
<td>1.30</td>
</tr>
<tr>
<td>3</td>
<td>1.06</td>
<td>180.94</td>
<td>0.92</td>
</tr>
<tr>
<td>5</td>
<td>0.60</td>
<td>340.03</td>
<td>0.55</td>
</tr>
<tr>
<td>10</td>
<td>0.29</td>
<td>739.42</td>
<td>0.28</td>
</tr>
<tr>
<td>100</td>
<td>0.03</td>
<td>7938.92</td>
<td>0.03</td>
</tr>
</tbody>
</table>

In situations the first approximation still performs well. The explanation is given by the result $\ln(1 + x) \sim x$. Hence, the second expression found in the case where $a$ tends to zero is equivalent to the first one found in the case where $a$ tends to infinity. Note also that the worst degradation of the performance is observed with the second approximation.

### 3.2 Policy improvement approach for the control in with $c(0) > 0$

**On the need to go beyond Policy TBP.** Under Policy TBP all containers are admitted at the consignees’ inventory. Therefore they all incur the initial cleaning cost $c(0)$. In Figure 1, we show how Policy TBP behaves in comparison with the immediate return policy for different intensities of the arrival parameter. As the arrival rate and the cleaning cost increase, the performance of Policy TBP decreases and may even be worse than the immediate return policy. This figure illustrates the need to operate a control in when significant cleaning costs are encountered. For this purpose, Section 3.2.1 derives the relative value function under the optimal policy for the control out. From this result, Section 3.2.2 obtains the Optimal Bernoulli policy and performs a policy improvement step.

To analyze this new level of control, we consider a simpler cost function for a container which stays at the consignee; $c(t) = c(0) + w \cdot t + d \cdot 1_{t > 7}$, where $c(0)$ is the fixed cost for cleaning/maintaining a container at the consignee (recall that this cost is not counted if a container is directly sent back to the shipping line),

Figure 1: Expected cost per container as a function of $c(0)$ under Policy TBP ($c_s = €80$/container, $\mu = 1$ arrival/day, $c(t) = c(0) + 5t \cdot 1_{0 < t \leq 3} + (60t - 165) \cdot 1_{3 < t \leq 7} + (90t - 375) \cdot 1_{t > 7}$)
$w$ is the cost per container and per time unit spent at the consignee, and $d$ is the penalty paid per container which has stayed more than $t^*$ at the consignee (i.e., $t^*$ is the end of the detention free period). This simpler cost function captures the main features of the general cost function and can be made more complex in the following analysis as required.

### 3.2.1 Relative value function

We consider the discrete model of Section 3.1.2 where all containers are admitted in the inventory. Again, a state of the system is defined by a phase time of the oldest container at the consignee. We choose $x$, $n$, and $\gamma$ such that $\frac{x}{\gamma} \leq t$ and $\frac{x}{\gamma} \leq TDT$. Moreover, we define the threshold $u$ (1 < $u$ ≤ $n$) such that $\frac{x}{\gamma} = t^*$. The Markov chain at match initiation or rejection epochs is considered. Service initiations occur at $\mu$-transitions from states $x > 0$. Rejection initiations only occur in state $n$ with a $\gamma$-transition. So, we associate the multiplicative constant $\frac{x}{\gamma}$ and $\frac{x}{\gamma}$ to the cost parameters. Therefore, the relative value function under the optimal policy for the control out, $V(x)$, is given by the following dynamic programming optimality equations:

\begin{align}
V(0) + E(C) &= \lambda V(1) + (1 - \lambda) V(0), \\
V(x) + E(C) &= \frac{c(0) \mu}{\lambda} + \frac{w \mu}{\gamma} x + \frac{d \mu}{\lambda} \mathbf{1}_{x \geq u} + \gamma V(x + 1) + \mu F(V(x)) + (1 - \gamma - \mu) V(x), \quad \text{for } 0 < x < n, \quad \text{and,} \\
V(n) + E(C) &= \frac{c(0) (\mu + \gamma)}{\lambda} + \frac{w \mu + \gamma}{\gamma} n + \frac{c x + d (\mu + \gamma)}{\lambda} \mathbf{1}_{n \geq u} + (\mu + \gamma) F(V(n)) + (1 - \gamma - \mu) V(n). 
\end{align}

Theorem 3 gives the solution of Equations (3)-(5) and Corollary 3 evaluates the relative value function and the average expected cost for the real system. The approach to solve the set of equations is as follows. We introduce the difference $\Delta(x) = V(x + 1) - V(x)$, for 0 ≤ $x$ ≤ $n - 1$. Using the empty system as a reference state, the initial value of $\Delta(x)$ can directly be expressed as a function of $E(C)$. Subtracting the expression of $V(x)$ to the expression of $V(x + 1)$ using the optimality equations allows us to derive a linear relation for $\Delta(x)$. Using this relation we obtain an expression of $\Delta(x)$ as a function of $E(C)$. Finally, from the boundary optimality equation at $x = n$, we obtain the unique expression of $E(C)$ which in turn leads to the expression $V(x)$ via $V(x) = \sum_{k \leq x - 1} \Delta(k)$. 

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Theorem 3 We have \( V(0) = 0, V(1) = \frac{E(C)}{\lambda}, \)

\[
V(x) = E(C) \frac{x}{\lambda} + E(C) \left( \frac{\mu + \gamma}{\mu - \lambda} \right)^{x-1} \left( \frac{\mu + \gamma}{\mu - \lambda} \right) - \frac{\lambda(x-1)}{\mu - \lambda} \left( \frac{\mu + \gamma}{\mu - \lambda} \right)
\]

\[
- \frac{\mu}{\lambda} \left( \frac{\mu + \gamma}{\mu - \lambda} \right) \left( \frac{\mu + \gamma}{\mu - \lambda} \right)^{x-1} \left( \frac{\mu + \gamma}{\mu - \lambda} \right) - \frac{\lambda(x-1)}{2(\mu - \lambda)} - \frac{\mu(\lambda + \gamma)}{(\mu - \lambda)^2} (x-1)
\]

\[
- \frac{d}{\gamma} \left[ x \geq u \right] \mu \left( \frac{\mu + \gamma}{\mu - \lambda} \right)^{x-u} \left( \frac{\mu + \gamma}{\mu - \lambda} \right) - \frac{\lambda(x-u)}{\mu - \lambda}
\]

\[- c(0) \mu \left( \frac{\mu + \gamma}{\mu - \lambda} \right)^{x-1} \left( \frac{\mu + \gamma}{\mu - \lambda} \right) - \frac{\lambda(x-1)}{\mu - \lambda} \], for \( 2 \leq x \leq n, \)

and \( E(C) = E(C)_w + E(C)_d + E(C)_{c(0)} + E(C)_{c_\ast}, \) where

\[
E(C)_w = w \cdot \frac{\gamma(\mu - \lambda)}{\gamma + \mu \lambda^2 - \lambda(\lambda + \gamma) \left( \frac{\lambda + \gamma}{\mu + \gamma} \right)^n} \left( \frac{n}{\gamma} \left( \frac{\lambda + \gamma}{\mu + \gamma} \right)^n + \frac{\mu + \gamma}{\gamma^2 \mu + \gamma} - \frac{(n+1)}{(1 - \frac{\lambda + \gamma}{\mu + \gamma})^2} \right),
\]

\[
E(C)_d = d \cdot \frac{\mu + \gamma}{\mu + \lambda^2 - \lambda(\lambda + \gamma) \left( \frac{\lambda + \gamma}{\mu + \gamma} \right)^n}, \quad E(C)_{c(0)} = c(0) \cdot \frac{\mu + \gamma}{\gamma + \lambda^2 - \lambda \left( \frac{\lambda + \gamma}{\mu + \gamma} \right)^n},
\]

\[
E(C)_{c_\ast} = c_{\ast} \cdot \frac{\gamma(\mu - \lambda)}{\gamma + \lambda^2 - \lambda(\lambda + \gamma) \left( \frac{\lambda + \gamma}{\mu + \gamma} \right)^n}.
\]

Corollary 3 We have \( V(0) = 0, \)

\[
V(t) = \frac{E(C)}{\lambda} \left[ \frac{e^{(\mu - \lambda)t} - 1}{\mu(a(1-a))} \right] - \frac{w}{\mu a(1-a)^2} \left( \frac{e^{(\mu - \lambda)t} - 1}{2(\mu - \lambda)} - t(\mu - \lambda) \right)
\]

\[- \frac{d}{\mu a(1-a)^2} \left( \frac{e^{(\mu - \lambda)(t-t\ast)} - 1}{\lambda(t-t\ast)} (1-a) \right)
\]

\[- c(0) \frac{1}{\mu a(1-a)^2} \left( \frac{e^{(\mu - \lambda)t} - 1}{\lambda(t-1-a)} \right), \] for \( 0 < t \leq TDT, \)

where \( E(C) = E(C)_w + E(C)_d + E(C)_{c(0)} + E(C)_{c_\ast}, \) with

\[
E(C)_w = w \cdot \frac{1}{\mu} \cdot \frac{1}{(1-a)(1-a) \mu} e^{-(\mu - \lambda)(1 + a TDT(\mu - \lambda))},
\]

\[
E(C)_d = d \cdot \frac{1}{\mu} \cdot \frac{1}{(1-a) \mu} e^{-(\mu - \lambda)(1 + a TDT(\mu - \lambda))} - \frac{1}{2(a - \mu)(\mu - \lambda) TDT} e^{-\frac{a(\mu - \lambda) t - 1}{(1-a) TDT}},
\]

\[
E(C)_{c(0)} = c(0), \quad E(C)_{c_\ast} = c_{\ast} \cdot \frac{1}{(1-a) \mu} e^{-\frac{\mu - \lambda}{\mu - \lambda TDT}},
\]

Note that \( \lim_{t \to 0} V(t) = \frac{E(C)}{\lambda} \neq 0. \) The reason is that the state \( x = 0 \) differs from the state \( t \to 0. \) In the first one the system is empty. In the second one, one container has just arrived in the inventory. Note also that the average costs \( (E(C)_w, E(C)_d) \) and \( (E(C)_d) \) are the performance measures already found in Section 3.1.2.
3.2.2 Computation of the improved policy

We first assume that the consignee applies a Bernoulli policy for the control in. A parameter \( r \) (\( 0 \leq r \leq 1 \)) is chosen such that an arriving container is accepted at the consignee with probability \( r \) and sent back to the shipping line. This decouples the system into two independent subsystems. In the first one, all containers are admitted at the consignee. In the second one, all containers are directly sent back to the shipping line. Hence, the arrival process at the consignee in the first subsystem is Poisson with parameter \( r \cdot \lambda \) and the optimal policy for the control out remains a time-based threshold policy with parameter TDT. The arrival process in the second subsystem is also Poisson with parameter \((1 - r) \cdot \lambda\).

In the first subsystem we obtain the system performance by replacing \( \lambda \) by \( r \lambda \) and \( a \) by \( a_r = \frac{r \lambda}{\mu} \) and multiplying the cost components by \( r \) in Corollary 3. In the second subsystem all containers are directly sent back to the shipping line, the expected cost per container is then \( c_s \). Hence, the overall system cost per container, \( E(C)_r = E(C)_1^0 + E(C)_2^r \), is given by

\[
E(C)_r^1 = r \left( c(0) + w \cdot \frac{1 - e^{-TDT(\mu - r \lambda)}(1 + a_r TDT(\mu - r \lambda))}{(1 - a_r)(1 - a_r e^{-TDT(\mu - r \lambda)})} + d \cdot \frac{e^{-(\mu - r \lambda) TDT} - a_r e^{-(\mu - r \lambda) TDT}}{1 - a_r e^{-(\mu - r \lambda) TDT}} + c_s \cdot \frac{(1 - a_r) e^{-(\mu - r \lambda) TDT}}{1 - a_r e^{-(\mu - r \lambda) TDT}} \right)
\]

\[
E(C)_r^2 = (1 - r)c_s.
\]

The parameters \( r \) and TDT are chosen such that the long run expected cost of the system is minimized. This defines the optimal Bernoulli policy within the class of Bernoulli policies. The complexity of the expected cost does not allow us to obtain an explicit expression of the optimal parameters \( r \) and TDT in general. In some cases however it is possible to obtain them explicitly.

Case 1: \( c(0) > c_s \). In this case, clearly it is optimal to have \( r = 0 \) and the choice of TDT is irrelevant.

Case 2: \( c(0) < c_s \), \( d = 0 \), and \( \lambda < \mu \). In this case, we may have TDT=\( \infty \) and \( r = \frac{1}{\lambda} \left( 1 + \sqrt{\frac{w}{\mu(c_s - c(0))}} \right) \) and the optimal cost is \( E(C) = \frac{1}{\lambda} \left( \mu(c(0) - c_s(1 - a)) - w - 2\sqrt{\mu w(c_s - c(0))} \right) \).

Under the optimal Bernoulli Policy, Equations (3)-(5) can be adjusted to obtain the value function of the system. By replacing \( \lambda \) by \( r \cdot \lambda \) and multiplying each cost components by \( r \), we obtain the value function for the first subsystem composed by containers which enter the inventory. The relative value function of this subsystem is denoted by \( V_1(t) \). By replacing \( \lambda \) by \( (1 - r) \cdot \lambda \), multiplying each cost components by \( 1 - r \), and choosing \( w = c(0) = TDT = d = 0 \), we get the value function for the subsystem composed by containers which are directly sent back to the shipping line. The relative value function of this subsystem is denoted by \( V_2(t) \). The value function \( V_2(t) \) is defined as \( t \) tends to 0 by \( \lim_{t \to 0} V_2(t) = \frac{(1 - r)c_s}{\lambda} \). Given a Bernoulli policy, this value represents the cost of sending a container to the sea terminal. It is a constant since no other future costs can be expected after taking the decision to send back a container. Given the age of the oldest container at the consignee (i.e., in the first subsystem), we propose a policy improvement approach from the optimal Bernoulli policy by choosing the minimizing action between keeping or rejecting a container at arrival. This method has been introduced by Norman (1972) and Ott and Krishnan (1992) and further developed by Hwang et al. (2000) and Bhulai and Koole (2003). The one-step policy improvement works
for models for which the value function of a certain policy can be obtained. One-step policy improvement consists of doing the improvement step on the basis of this initial policy. The motivation for considering this method is that policy improvement method gives the biggest improvement during the first step and leads to nearly optimal policies.

For each \( t > 0 \), we improve the initial Bernoulli policy from the minimizing action between keeping a container at the consignee or sending back this container to the shipping line. More precisely, at a container arrival, a container is kept at the consignee if and only if \( V_1(t) < \lim_{t \to 0} V_2(t) = (1-r)c_s \). Since \( V_1(t) \) is increasing in \( t \), the improved policy has a threshold structure; an arriving container is kept at the consignee if and only if the age of the oldest container is lower than a time threshold.

The two-thresholds policy. The above study allows us to define a time based 2-thresholds policy, denoted as Policy TBP. There exists 2 time thresholds, the Time Acceptance Threshold (TAT) and the time detention threshold (TDT, as for Policy TBP) such that, if the age of the oldest container at the consignee is \( t \geq 0 \),

- An arriving container is kept at the consignee if \( t < \text{TAT} \). Otherwise, this container is directly sent back to the shipping line.
- A container at the consignee is kept in the inventory until \( t = \text{TDT} \). At \( t = \text{TDT} \), a container is sent back to the shipping line.

Note that with \( \text{TAT} \geq \text{TDT} \), Policy TBP reduces to Policy TBP.

4 Policy evaluation

We now evaluate Policy TBP and Policy TBP. With \( c(0) = 0 \), we have shown that Policy TBP is optimal among the set of time-based policies. Since quantity-based policies are more common in the academic literature, we propose in Section 4.1 to compare the two policy classes. With \( c(0) > 0 \), it is not possible to derive the optimal policy. Instead, we have build a new policy, Policy TBP as an improvement of Policy TBP. We propose in Section 4.2 to evaluate the cost improvement obtained by implementing two levels of control at the consignee.

4.1 The value of time-based decisions

We want to compare the following policies:

- **Policy TBP**: The time-based threshold policy with parameter TDT. All arriving empty containers are admitted in the inventory. The oldest container in the inventory is rejected from the system when a match is realized or when its age reaches TDT.

- **Policy QBP**: The quantity-based threshold policy with parameter \( N \). All arriving empty containers are admitted in the inventory. The oldest container in the inventory is rejected from the system when a match is realized or when an arrival occurs and \( N \) containers are already in the inventory.

The optimal threshold and the optimal cost per container under TBP are determined using the results of Section 3 and are computed using simulation under Policy QBP. Since the quantity-based threshold is an
integer and the time-based threshold is a real, we allow for randomization between two adjacent thresholds for QBP so as to give a fair comparison. In Figure 2, we provide the comparison between the expected cost per container under the two policies. We observe that for high values of $c_s$, or equivalently when the optimal thresholds are high, Policy TBP leads to better performance. Policy TBP is indeed better at controlling the time spent in inventory so as to avoid detention fees. In a situation with a higher arrival rate, Policy TBP seems to be always better than Policy QBP. These two observations justify the preference for the time-based threshold policy in our context since i) our study consider import zones where the imbalance is strong and ii) even when the quantity-based threshold policy is cheaper, then gap between the two policies is small. However, one critic that may be formulated against Policy TBP is that the quantity of containers in the inventory is not directly controlled. This aspect is studied in Section 5.1.2. At the end of the appendix, we provide analytical results for the comparison between the two policies.

4.2 The value of exercising two levels of control

We evaluate here the improvement gained by implementing Policy $\text{TBP}$ instead of Policy TBP. For this purpose, Table 2 provides the costs and the optimal parameters of Policy TBP, the Optimal Bernoulli policy, called Policy OB, and Policy $\text{TBP}$. Corollary 3 is used to determine the expected cost per container of each policy except Policy $\text{TBP}$. This one is obtained via simulations. In addition, we compute the relative cost differences $rd_i$ for $i = 1, 2$ defined by $rd_1 = \frac{E(C)_{\text{OB}} - E(C)_{\text{TBP}}}{E(C)_{\text{TBP}}}$ and $rd_2 = \frac{E(C)_{\text{TBP}} - E(C)_{\text{OB}}}{E(C)_{\text{OB}}}$. The idea is to evaluate the steps of the policy improvement from Policy TBP to Policy TBP. We observe that $|rd_1| < |rd_2|$. This translates that the first step of policy improvement from Policy TBP to Policy OB has a stronger impact than the second one from Policy OB to Policy TBP. In the first step of improvement, the cleaning cost which was initially ignored is considered in the control decision. Therefore, as expected from Figure 1 and measured by $rd_1$, this first step of improvement has a strong positive effect when Policy TBP fails to well manage the consignee’s inventory. This is particularly the case when the cleaning cost and the arrival rate are high (see Figure 1).

The second step of policy improvement consists of considering the system state for accepting or rejecting a new container at the consignee’s location instead of taking blind decisions. As for the first step of improvement, with a high cleaning cost and a high arrival rate this step of improvement has a strong impact.
However, if the optimal parameter of Policy OB is close to 0% or 100%, then the impact of this second step reduces. In such cases either Policy TBP or the immediate return policy are close to optimal. These extreme policies for the control in cannot be highly improved. The second step of improvement also consists of replacing a randomized Bernoulli policy by a deterministic threshold one. Therefore, the negative impact of the variability of Policy OB is removed in this second step of improvement. As a confirmation, we observe that the value of \( r_d \) is significant when \( r^* \) is close to 50% (i.e., when the variability of Policy OB is the highest).

**Remark.** It is also interesting to note that the optimal rejection threshold after staying at the consignee for the optimal Bernoulli policy and Policy TBP is equal to the end of the detention free period. This means that when it is decided to accept a container at the consignee, this container is kept either until a match can be operated or until the end of the detention free period. The impact of the detention free period is further investigated in Section 5.2.

## 5 Managing containers at the location; Analysis and Insights

We are now interested in conducting a numerical analysis to better understand the stake of managing containers at the consignee’s location. In particular, we are interested in exploring the potential of street-turns and cost savings resulting from this implementation. Next, the purpose of the detention fee structure will be discussed.

### 5.1 Values of managing containers of the consignee’s location

We investigate here the potential that managing containers at the consignee’s location may have on street-turns and on the control of the containers’ inventory. We also analyze the impact on costs that this implementation may have in import zones.

#### 5.1.1 A high potential of street-turns

In Figure 3, we evaluate the proportion of matches under the optimal policy as a function of \( c_s \) (Figure 3(a)) and as a function of \( c(0) \) (Figure 3(b)). Figure 3(a) illustrates the result of Corollary 2 which states that the proportion of matches is increasing and piecewise concave in \( c_s \). Moreover, we observe that the proportion of matches is very high as soon as \( c_s > c_1 \). This means that empty container management by consignees
Figure 3: Proportion of matches \( \mu = 1 \) arrival/day, \( c(t) = c(0) + 5t I_{t<1} + (60t - 165) I_{3<t<7} + (90t - 375) I_{t>7} \)

shows a very high potential for reducing unnecessary movements of empty containers in the hinterland. For reference, within the Netherlands, the cost \( c_s \) is about €100. This leads to a proportion of matches exceeding 70% even for \( \lambda = 1 \). Figure 3(b) indicates that the proportion of matches may however reduce significantly when the cleaning cost increases. With a high cleaning cost, it is preferable to return more containers to the sea terminal at arrival instead of keeping them for a future match. We thus get the following insight:

**Insight 1** The proactive management of empty containers by the consignees enables reaching a high level of street-turns in import zones if cleaning costs remain low.

Street-turns are beneficial not only for consignees (as a way to decrease their costs) but also for shipping lines (by increasing the utilization rate of the containers and reducing the cost of sending back empty containers to consignors). Additionally, the literature on street-turns highlighted in Section 1 shows that street-turns enable reducing congestion, accidents and pollution. Insight 1 highlights that the new management practice studied in this article may strongly help in optimizing container flows in the hinterland.

### 5.1.2 A good control of the expected number of containers

Policies TBP and TBP may be criticized by their indirect control of the quantity in the inventory. This critic is particularly relevant if the consignee’s capacity is limited and if the variations in the arrival rate are important. Therefore in Figure 4, we evaluate the expected number of containers in the inventory. As proven in Corollary 2, the expected number of container in inventory is increasing and piecewise concave in \( c_s \) (see Figure 4(a)). Moreover, it is interesting to observe that the sensitivity of the expected number of containers is decreasing in \( \lambda \). This means that for high arrival rates, the variation of the arrival rates have a little impact on the number of containers at the consignee. This leads to the following insight:

**Insight 2** In import zones, the variations of the arrival rate have a little impact on expected number of containers at the consignee.

Furthermore, Figure 4(b) reveals that the imbalance between the arrival of containers and the demand from the consignor has an insignificant impact on the expected number of containers in the inventory as soon as the cleaning cost is above €20 per container.
In Figure 5, we compute the optimal cost per container. As stated in the algorithm of Section 3.2, if $c_s \mu \leq c_1$ or if $c_s \leq c(0)$, then Policy IR is optimal, i.e., empty container management is not interesting for the consignee. Otherwise, as $c_s$ increases (respectively as $c(0)$ increases), the absolute and relative profit obtained when applying a time-based policy as compared to Policy IR increases (respectively decreases). For instance, with $c_s = \€100$ and $c(0) = \€0$, the potential savings per container are from 16% for $\lambda = 5$ to 69% for $\lambda = 1$. This shows that managing containers at the consignee’s location may significantly reduce consignees’ transportation costs. However, if $c_s$ is small or if $c(0)$ is high, we can notice that the cost saving is strongly reduced. Note also that the benefit of implementing the optimal policy decreases as the arrival
rate of containers increases. Recall that the proportion of matches increases as the arrival rate of empty containers increases. This means that the incentive to manage empties is lower in the case this practice could be the most impactful. This leads to a third insight:

**Insight 3** In import zones, empty container management by the consignee can significantly reduce inbound transportation costs. However, the savings are lower under the most favorable conditions for street-turns, i.e., in case of strong imbalance.

This third insight may be quite disappointing at first glance as we have highlighted that many hinterlands are subject to strong trade imbalance. Combined with Insight 1, we face a very effective management practice to handle the problem of reducing empty movements of containers in the hinterland, but we show that this practice may not necessarily be applied under the conditions that are the most favorable for street-turns.

5.2 Incentives for managing containers at location

The results highlighted above show that containers’ management by consignees leads to cost savings that are strongly dependent on the conditions faced by consignees. Therefore, we investigate here if the incentives could be made stronger. This leads us to consider the possibilities of implementing linear detention cost or extending the duration of the DFP. These remedies could be used against the negative effect that detention fees have on the willingness consignees have to implement empty container management policies.

5.2.1 On the usefulness of detention fees

In Figure 6, we compute the optimal timeout threshold (TDT) under Policy TBP (Figure 6(a)) and Policy TBP (Figure 6(b)). The idea is to relate this time threshold which represents the maximal time spent at the consignee’s location by a container to the DFP (3 days in our illustration). As proven in Corollary 2 and illustrated in Figure 6(a), the optimal threshold is increasing and piecewise concave in \( c_s \). In the two figures, it never exceeds the DFP. Exceeding the DFP could only occur in extreme cases of \( c_s \) and \( c(0) \). For instance with \( \lambda = 1 \) and \( c(0) = 0 \), we need to have \( c_s > 654 \) to obtain a timeout threshold exceeding the DFP.

![Figure 6](image)

**Figure 6**: Optimal threshold out \((\mu = 1 \text{ arrival/day}, c(t) = c(0) + 5t \mathbb{I}_{0 < t \leq 3} + (60t - 165) \mathbb{I}_{3 < t \leq 7} + (90t - 375) \mathbb{I}_{t > 7})\)

observation corresponds to the behavior of the consignees we interacted with in several European countries as they highlight that they always try to avoid paying detention fees. This may lead us to conclude that
detentions fees are efficient in controlling the time spent by containers in the hinterland, in line with one of the main purpose of these fees.

Assume that we omit the potential revenue that shipping lines obtain from their containers in the hinterland and assume that we omit detention fees. Many shipping line companies would be reluctant to implement such type of tariff as they would be afraid of losing control of their containers, by incentivizing consignees to keep empties for a long time. However, we can notice from Figure 6(a) that the proposed policies often leads to send back empties to the shipping line before the end of the DFP. We conclude that empty containers will be shipped back to the consigners quite quickly, independently of detention fees as the physical costs of keeping empties, even if they are small, are sufficient to ensure a low value for the optimal threshold. We thus derive the following insight.

**Insight 4** Detention fees are not necessary to ensure that containers do not spend too much time in the hinterland. The physical cost of storing empties is enough to deter consignees to keep empties for a long time before shipping them back to the shipping line.

Insight 4 questions the real objective of the detention fees. Detention fees are often claimed to help shipping lines to control their containers in the hinterland, but they may also be a source of additional revenue for shipping lines. In what follows, we further investigate two scenarios.

### 5.2.2 Proposing linear detention fee structure

First, we show how the detention fee structure can be modified to encourage consignees to manage empty containers at their location, while protecting revenues generated from detention fees. We investigate the option of proposing a linear detention fee structure, by changing the aim of these fees. Detention fees are nowadays mainly perceived by consignors as penalty in case of late delivery of empty containers to the shipping line. We could have them to be perceived as renting fees, for the equipment shipping lines provide to the consignee in case of merchant haulage. Assume, for instance, that the shipping line proposes a new detention tariff which consists of a single rate of €15/day (independently of the time spent by the container in the hinterland). Assume that this rate has been estimated to generate the same revenue as under the complex detention fee structure exposed in Section 1. This leads to a linear inventory holding costs $c_1$ of €20 in our model (€15 detention fee + €5 physical storage costs). The results appear in Figure 7. Figure 7(a) shows that the proportion of match is high, so this linear detention fee structure can be helpful in improving the direct reuse of containers in the hinterland. Of course, this may not work if the rate chosen is too high, but we expect that the linear rates proposed by shipping lines would be low if they aim at equalizing with the revenues currently generated. To better investigate this statement, we illustrate a decomposition of the costs incurred by the consignee in Figure 7(b). As proven in Corollary 2, the detention cost per container is concave in the linear detention rate, $c_1$. So, when increasing $c_1$, more revenue is generated per container per time unit but the containers stay at the consignee for a shorter period of time. This analysis enables us to highlight that there exists a rate that maximizes the profit from detention fees for the shipping line. This rate is equal to €25/day in Figure 7. For this detention rate, the proportion of matches is higher than 50% according to Figure 7(a). This enables us to derive the following insight.
Insight 5: Linear detention fees may help increasing the proportion of street-turns compared to the current cost structure.

Linear detention fees would at first be difficult to implement as consignees are used to DFP, but we expect that relabeling them into renting fees would help. Overall, our results show that the detention fee structure used by shipping lines may be overly complex as compared to the targeted result which is primarily the control of containers.

5.2.3 Extending the duration of the DFP

In Figure 8, we present the impact of the duration of the DFP. We vary the DFP from 0 to 4 days. We consider the cost structure defined in Section 2 after the end of the DFP. We observe that empty container management by the consignees has a very high potential for reducing the unnecessary movements of empty containers in the hinterland as soon as the residual DFP at the consignee is high enough (Figure 8(d)). This enables us to investigate the following option. The shipping lines could propose to increase the DFP for consignees who accept to hold few empties at their location to proceed to street-turns. This solution will not strongly affect the time spent by containers in the hinterland (as shown in Insight 4) and may be perceived by consignees as a strong incentive to proactively manage empties at their location. Moreover, the TBP will enable consignees to make sure not to keep empties for too long. Also note that this solution may help shipping lines to reduce their repositioning costs incurred when sending empty containers to the consignors and may also help shipping lines to increase the time spent by containers generating revenue in deep-sea vessels. Especially, Figure 8(d) shows that in case the trade imbalance is not very high, an increase in the DFP will have a strong effect on the level on street-turns.

Insight 6: In order to increase the proportion of street turns in the hinterland, shipping lines may propose an increase in the detention free period for consignees who accept to hold empties at their location.

Insight 6 may help shipping lines to identify commercial solutions to entice consignees to proactively manage empty containers. This practice is already in place to entice consignees to use intermodal transportation in the hinterland. As an example, we stated in Section 1 that the DFP for a 40ft. dry container...
imported to the port of Rotterdam and transported by truck was 3 days. Maersk (2016) additionally states that this DFP is extended to 5 days in case of barge or train transportation. The same type of agreement may therefore be put in place to entice consignees to manage empties at their location.

**Conclusion.** The results of this section that are particularly relevant for import zones are that (i) the time-based policy has a high potential of street-turns, (ii) it allows the consignee a good control of the number of containers at location, (iii) cost savings for the consignee are strongly dependent on the conditions. In addition, this study allows us to highlight that the detention fees often do not impact the optimal control since consignees may always choose to send back their containers before the end of the DFP. This leads us to investigate the possibility for shipping lines to implement linear detention fees or to extend the duration of the DFP. In both cases, we find that such decisions may strongly increase the proportion of street-turns.

### 6 Conclusion

The use of street-turns is considered as one of the most efficient strategies for empty container management in the hinterland. We investigate if the proactive management of empty containers by consignees could improve the proportion of street-turns in the hinterland. For this purpose, we propose a model of empty container management at the consignee’s location. Our model accounts for repositioning costs, convex inventory holding costs and fixed cleaning costs incurred only for containers placed in inventory.

Using a Markov decision process, we prove the threshold form of the optimal policy for the control out
when cleaning costs are negligible. A timeout threshold can be set by the consignee to return containers to the sea terminal. Under this policy, we compute the optimal cost and the relative value function. The quality of this policy is evaluated in comparison with a quantity based one. We show in particular that our time based policy performs well with strongly convex detention costs. When cleaning costs are non-negligible, we develop a policy improvement approach for the control in based on the relative value function of an initial Bernoulli policy. The improvement step results in a two-thresholds policy. The two thresholds control the admission in and rejection out of the inventory. The improvement from the one-threshold to the two-thresholds policy is shown to be significant when the cleaning cost and the arrival rate are high.

From a level of street-turns of around 10% reported in the literature, our results show that much higher levels could be achieved if the consignees were proactive in managing empty containers. However, the difference between the total costs per container incurred by the consignee as compared to the costs incurred under the immediate return policy is low. This practice also enables the consignee to reduce container repositioning costs but the incentive varies a lot from one setting to another. So, we further explore if the incentive could be made stronger by modifying the structure and/or purpose of the detention costs.

Although we focus solely on the consignee’s perspective in our model, the implications for shipping lines are as follows. First, note that in case of a match, the shipping line is saving transportation costs as the consignee incurs empty container repositioning costs in lieu of the shipping line. This explains why we believe that shipping lines might be willing to explore the option of letting consignees handling empty containers in the hinterland. However, according to our discussions with executives from several shipping companies worldwide, some of them are reluctant to allow for this type of practice as they feel that the loss of control of their asset might lead to a loss of opportunity. For instance, the revenue of maritime transportation from Europe to Asia is often much less than the revenue from Asia to Europe. Therefore, shipping lines might be willing to reposition empties to Asia as quickly as possible. This argument about opportunity costs is difficult to assess as-is. First, we highlight in the introduction that the demand from the consignor is met sooner in case of a match and that this might increase the utilization rate of the container. Second, the opportunity cost is also related to container fleet sizing decisions made by shipping lines. We believe that the cost of adding more containers in the system is quite low (much lower than the detention rate) and therefore, the opportunity cost highlighted by shipping lines is often overestimated. The features we highlight here are only preliminary comments. The implications for the shipping lines require further exploration and deserve future research.

This article explores an effective strategy to reduce unnecessary movements of empty containers in the hinterland. Many other ones may be investigated, such as the sharing of transportation costs between the consignee and the consignor in case of a street-turn. Additionally, other actors in the hinterland such as terminal operators, port authorities and local policy makers may entice consignees to investigate this option. We believe that this article will draw attention to empty container management by the consignees. This may be one of the most powerful and simple options for tackling the problem of empty container repositioning in the hinterland.
References


Maersk (2016). Demurrage and detention tariff, port of Rotterdam.


Appendix

Notations

Table 3: Notations

<table>
<thead>
<tr>
<th>Exogenous parameters</th>
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<td>Arrival rate of empty containers</td>
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<tr>
<td>$\mu$</td>
<td>Demand rate from the consignor</td>
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<tr>
<td>$a = \lambda/\mu$</td>
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<td>$c_s$</td>
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<td>$c(t)$</td>
<td>Inventory holding cost of a container which has stayed exactly $t$ time units in the inventory</td>
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<td>$c_k$</td>
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<table>
<thead>
<tr>
<th>Acronym</th>
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<td>DFP</td>
<td>Detention Free Period</td>
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<tr>
<td>IR</td>
<td>Immediate Return Policy</td>
</tr>
<tr>
<td>TBP</td>
<td>Time-Based threshold Policy for the control out</td>
</tr>
<tr>
<td>TBP*</td>
<td>Time-Based two-thresholds Policy</td>
</tr>
<tr>
<td>OB</td>
<td>Optimal Bernoulli Policy</td>
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<td>Threshold Detention Time (control of Policy TBP)</td>
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<td>TAT</td>
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<td>$r$</td>
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<td>Probability of an empty system</td>
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<tr>
<td>$P_s$</td>
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<td>$E(N)$</td>
<td>Expected number of containers in the inventory</td>
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<tr>
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<td>Probability that a container spends more than $t$ time units at the consignee</td>
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<tr>
<td>$E(C)$</td>
<td>Expected cost per container given by Equation (1)</td>
</tr>
</tbody>
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Proof of Theorem 1

We prove by induction that the optimal policy for sending back containers is of threshold type. We thus need to show that $V_n(x+1) - F(V_n(x)) - c_s - c(x)$ is increasing in $x$, for $x \geq 0$ and $n \geq 0$. In other words, we need to show that

$$F(V_n(x)) + V_n(x+2) + c(x) \geq F(V_n(x+1)) + V_n(x+1) + c(x+1),$$

for $x \geq 0$ and $n \geq 0$. This relation is refereed to as generalized convexity (gcv). In this proof, we also have to show that $V_n$ is increasing in $x$, for $x \geq 0$ and $n \geq 0$. Since $V_0(x) = 0$, then $V_0$ is increasing and generally convex (igcv). Note that the arguments $c(x)$ increasing and convex and $c(0) = 0$ are essential to show this property. Although $c(0) = 0$, we still write $c(0)$ in the developed expressions to better show the use of the convex property of $c$ when it is needed.

First, we assume that $V_n$ is igcv for a given $n \geq 0$, and we want to show that the same property holds for $W_n$.

$V_n$ increasing in $x$ $\implies$ $W_n$ increasing in $x$. First, we show that if $V_n$ is increasing in $x$, then $F(V_n)$ is
also increasing in \( x \). To simplify the notations, we denote by \( q \) the ratio \( \frac{x}{x+y} \). We have for \( x \geq 0 \),

\[
F(V_n(x+1)) - F(V_n(x)) = \sum_{h=0}^{x+1} q_{x+1,x+1-h} V_n(x+1-h) - \sum_{h=0}^{x} q_{x,x-h} V_n(x-h)
\]

\[
= \sum_{h=0}^{x-1} q(1-q)^h (V_n(x+1-h) - V_n(x-h)) - (1-q)^x V_n(0) + (1-q)^{x+1} V_n(0) + q(1-q)^x V_n(1)
\]

\[
= \sum_{h=0}^{x-1} q(1-q)^h (V_n(x+1-h) - V_n(x-h)) + q(1-q)^x (V_n(1) - V_n(0)) \geq 0.
\]

Therefore \( F(V_n(x)) \) is increasing in \( x \).

Using the definition of \( W_n \), one may write,

\[
W_n(x) \leq V_n(x+1), \text{ and } W_n(x) \leq F(V_n(x)) + c_s + c(x). \tag{9}
\]

If \( W_n(x+1) = V_n(x+2) \), then the first inequality in (9) proves that \( W_n \) is increasing. If \( W_n(x+1) = F(V_n(x+1)) + c_s + c(x+1) \), then the second inequality in (9) proves that \( W_n \) is increasing because \( F(V_n(x)) \leq F(V_n(x+1)) \) and \( c(x) \leq c(x+1) \).

\( V_n \) is gcv \( \iff \) \( W_n \) is gcv. We prove the convexity property of \( W_n \). Using the definition of \( W_n \) and \( q_{x,x-h} = q_{x+1,x+1-h} \), for \( 0 \leq h < x \), we may write

\[
W_n(x+1) + F(W_n(x+1)) + c(x+1) \leq c(x+1) + V_n(x+2) + \sum_{h=0}^{x+1} q_{x+1,x+1-h} V_n(x+2-h) \tag{10}
\]

\[
= c(x+1) + V_n(x+2) + F(V_n(x+2)) + \left( \frac{\gamma}{\gamma + \lambda} \right)^{x+2} (V_n(1) - V_n(0)) ,
\]

and,

\[
W_n(x+1) + F(W_n(x+1)) + c(x+1) \leq c(x+1) + F(V_n(x+1)) + c_s + c(x+1) \tag{11}
\]

\[
+ \sum_{h=0}^{k} q_{x+1,x+1-h} V_n(x+2-h) + \sum_{h=k+1}^{x+1} q_{x+1,x+1-h} (F(V_n(x+1-h)) + c_s + c(x+1-h)),
\]

for \( 0 \leq k \leq x \). The right hand side in Equation (10) corresponds to a case where containers should not be sent back if their age is lower than or equal to \( x+1 \). The right hand side in Equation (11) corresponds to a case where containers should be sent back if and only if their age is strictly higher than \( k \), for \( 0 \leq k \leq x \).

We distinguish two cases.

**Case 1: No rejection.** \( W_n(h) = V_n(h+1) \) for \( 0 \leq h \leq x+2 \) and \( x \geq 0 \).

\[
W_n(x+2) + F(W_n(x)) + c(x) = V_n(x+3) + \sum_{h=0}^{x} q_{x,x-h} V_n(x+1-h) + c(x)
\]

\[
= V_n(x+3) + F(V_n(x+1)) + \left( \frac{\gamma}{\gamma + \lambda} \right)^{x+1} (V_n(1) - V_n(0)) + c(x).
\]

Note that the second equality is a consequence of \( q_{x,x-h} = q_{x+1,x+1-h} \), for \( 0 \leq h < x \). Since \( V_n \) is gcv, we
have
\[ V_n(x + 3) + F(V_n(x + 1)) + c(x + 1) \geq V_n(x + 2) + F(V_n(x + 2)) + c(x + 2). \]

Therefore,
\[
W_n(x + 2) + F(W_n(x)) + c(x) - W_n(x + 1) - F(W_n(x + 1)) - c(x + 1) \\
\geq V_n(x + 3) + F(V_n(x + 1)) + \left(\frac{\gamma}{\gamma + \lambda}\right)^{x+1} (V_n(1) - V_n(0)) + c(x) \\
- c(x + 1) - V_n(x + 2) - F(V_n(x + 2)) - \left(\frac{\gamma}{\gamma + \lambda}\right)^{x+2} (V_n(1) - V_n(0)) \\
\geq V_n(x + 3) + F(V_n(x + 1)) - V_n(x + 2) - F(V_n(x + 2)) + c(x + 1) - c(x + 2) \\
+ c(x + 2) + c(x) - 2c(x + 1) + q(1 - q)^{x+1}(V_n(1) - V_n(0)) \geq 0,
\]
because \( V_n \) is gcv, \( c(x) \) is convex in \( x \) and \( V_n \) is increasing in \( x \). This proves that \( W_n \) is also gcv in this case.

- **Case 2: Rejection above a given waiting phase.** Assume that for a given \( k \) (\( 0 \leq k \leq x + 1 \)), we have \( W_n(h) = F(V_n(h)) + c_s + c(h) \) for \( k < h \leq x + 2 \) and \( W_n(h) = V_n(h + 1) \) for \( 0 \leq h \leq k \).

\[
W_n(x + 2) + F(W_n(x)) + c(x) = F(V_n(x + 2)) + c_s + c(x + 2) + \sum_{h=0}^{k} q_{x,x-h} V_n(x + 1 - h) \\
+ \sum_{h=k+1}^{x} q_{x,x-h} (F(V_n(x - h)) + c_s + c(x - h)) + c(x).
\]

Let us subtract this expression to the right hand side of Equation (11). Using \( q_{x,x-h} = q_{x+1,x+1-h} \), for
0 \leq h < x$, we get

\[
F(V_n(x + 2)) + \varphi c + c(x + 2) + \sum_{h=0}^{k} q_{x,x-h} V_n(x + 1 - h) + \sum_{h=k+1}^{x} q_{x,x-h} (F(V_n(x-h)) + c_s + c(x-h)) + c(x)
- 2c(x + 1) - F(V_n(x + 1)) - \varphi c - \sum_{h=0}^{k} q_{x+1,x+1-h} V_n(x + 2 - h) - \sum_{h=k+1}^{x+1} q_{x+1,x+1-h} (F(V_n(x + 1 - h)) + c_s + c(x + 1 - h))
= c(x + 2) + c(x) - 2c(x + 1) - F(\varphi V_n(x + 1)) - \sum_{h=k+1}^{x+2} q_{x+2,x+2-h} V_n(x + 2 - h) - F(\varphi V_n(x + 2)) + \sum_{h=k+1}^{x} q_{x,x-h} (F(V_n(x-h)) + c_s + c(x-h))
\sum_{h=k+1}^{x+1} q_{x+1,x+1-h} (F(V_n(x + 1 - h)) + c_s + c(x + 1 - h))
= c(x + 2) + c(x) + \sum_{h=k+1}^{x+2} q_{x+2,x+2-h} V_n(x + 2 - h) + (1-q)^{x+1} V_n(0) - q(1-q)^{x} V_n(1) + (1-q)^{x+2} V_n(0) + q(1-q)^{x+1} V_n(1) + q(1-q)^{x} V_n(2)
+ (1-q)^{x}(V_n(0) + c_s + c(0)) - (1-q)^{x+1}(V_n(0) + c_s + c(0)) - q(1-q)^{x}(qV_n(1) + (1-q)V_n(0) + c_s + c(1)).
\]

The first term of the expression is positive since $c(x)$ is convex in $x$. The second term is also positive since $V_n$ is $gcv$. The last two lines of the expression can be simplified into

\[
q(1-q)^{x}(V_n(2) - 2qV_n(1) + (2q - 1)V_n(0) + c(0) - c(1)).
\]

We can further decompose this expression into

\[
V_n(2) - 2qV_n(1) + (2q - 1)V_n(0) + c(0) - c(1)
= V_n(2) - (1+q)V_n(1) + qV_n(0) + c(0) - c(1)
+ (1-q)(V_n(1) - V_n(0)).
\]

The first line is positive since $V_n$ is $gcv$ and the second one is also positive since $V_n$ is increasing in $x$. This proves that $W_n$ is $gcv$.

From cases 1 and 2, we also deduce that

\[
W_n(x + 2) + F(W_n(x)) + c(x) - W_n(x + 1) - F(W_n(x + 1)) - c(x + 1) \geq q(1-q)^{x+1}(V_n(1) - V_n(0)).
\]

(12)
This inequality will be used to prove the next induction step.

\( V_n, W_n \) increasing in \( x \) \( \implies V_{n+1} \) increasing in \( x \). We now assume that \( W_n \) and \( V_n \) are igcv and we prove that \( V_{n+1} \) is also igcv. We first prove that \( V_{n+1} \) is increasing in \( x \). For \( x = 0 \), we have

\[
V_{n+1}(1) - V_{n+1}(0) = \gamma W_n(1) - \lambda W_n(0) + \mu (F(V_n(1)) + c(1) - V_n(0)) + (1 - \lambda - \mu)(V_n(1) - V_n(0)) \\
+ (\lambda - \gamma) V_n(1) \\
= \gamma(W_n(1) - W_n(0)) + \gamma(W_n(0) - V_n(0)) + \lambda(V_n(1) - W_n(0)) \\
+ \muq (V_n(1) - V_n(0)) + \mu c(1).
\]

The first term proportional with \( \gamma \) is positive since \( W_n \) is increasing in \( x \), the second term in \( \gamma \) is positive because either \( W_n(0) = V_n(1) \) and \( W_n(0) - V_n(0) = V_n(1) - V_n(0) \geq 0 \) or \( W_n(0) = V_n(0) + c_n + c(0) \) and \( W_n(0) - V_n(0) = c_n + c(0) \geq 0 \), the term in \( \lambda \) is also positive because \( W_n(0) = \min(V_n(1), V_n(0) + c_n + c(0)) \leq V_n(1) \), the other terms are also clearly positive. Therefore, \( V_{n+1}(1) \geq V_{n+1}(0) \). For \( x > 0 \), we have

\[
V_{n+1}(x + 1) - V_{n+1}(x) = \gamma(W_n(x + 1) - W_n(x)) + \mu (F(V_n(x + 1)) - F(V_n(x)) + c(x + 1) - c(x)) \\
+ (1 - \lambda - \mu)(V_n(x + 1) - V_n(x)) \geq 0.
\]

Therefore \( V_{n+1} \) is increasing in \( x \) for \( x \geq 0 \).

\( V_n, W_n \) gcv \( \implies V_{n+1} \) gcv. We now prove that \( V_{n+1} \) is generally convex. For \( x = 0 \), we may write

\[
V_{n+1}(2) + qV_{n+1}(0) - (1 + q)V_{n+1}(1) + c(0) - c(1) \\
= \gamma(W_n(2) - (1 + q)W_n(1) + qW_n(0) + c(0) - c(1)) \\
+ (1 - \gamma - \mu)(V_n(2) - (1 + q)V_n(1) + qV_n(0) + c(0) - c(1)) \\
+ q\mu(V_n(2) - (1 + q)V_n(1) + qV_n(0) + c(0) - c(1)) \\
+ q(1 - q)\mu(V_n(1) - V_n(0)) \\
+ \mu(c(2) + c(0) - 2c(1)) \\
- q\mu c(0) \\
- (\gamma - \lambda) q(W_n(0) - V_n(0)).
\]

The first three lines after the equality are positive since both \( W_n \) and \( V_n \) are gcv. The fourth line is also positive since \( V_n \) is increasing in \( x \). The fifth line is positive since \( c \) is convex. The sixth line is equal to zero since \( c(0) = 0 \). The last line is negative. This however can be compensated by the first and the fourth lines.

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Using Equation (12) and $W_n(0) \leq V_n(1)$, one may write

$$
\gamma(W_n(2) - (1 + q)W_n(1) + qW_n(0) + c(0) - c(1))
+ q(1 - q)\mu(V_n(1) - V_n(0)) - (\gamma - \lambda)q(W_n(0) - V_n(0))
\geq \gamma q(1 - q)(V_n(1) - V_n(0)) + \mu q(1 - q)(V_n(1) - V_n(0)) - q(\gamma - \lambda)(V_n(1) - V_n(0))
= q(V_n(1) - V_n(0))\frac{\mu \gamma + \lambda^2}{\lambda + \gamma} \geq 0.
$$

This proves that $V_{n+1}$ is gcv.

For $x > 0$, we have

$$
F(V_{n+1}(x)) = \sum_{h=0}^{x} q_{x-h} V_{n+1}(x-h)
= \gamma F(W_n(x)) + (1 - \gamma - \mu)F(V_n(x)) + \mu \sum_{h=0}^{x} q_{x-h}(F(V_n(x-h)) + c(x-h))
+ (1 - q)^x(\gamma - \lambda)(V_n(0) - W_n(0)).
$$

So,

$$
V_{n+1}(x + 2) + F(V_{n+1}(x)) + c(x) - V_{n+1}(x + 1) - F(V_{n+1}(x + 1)) - c(x + 1)
= \gamma(W_n(x + 2) + F(W_n(x)) - W_n(x + 1) - F(W_n(x + 1)) + c(x) - c(x + 1))
+ (1 - \gamma - \mu)(V_n(x + 2) + F(V_n(x)) - V_n(x + 1) - F(V_n(x + 1)) + c(x) - c(x + 1))
+ \mu \sum_{h=0}^{x-1} q(1 - q)^h (V_n(x + 2 - h) + F(V_n(x - h)) - V_n(x + 1 - h) - F(V_n(x + 1 - h))
+ c(x - h) - c(x + 1 - h))
+ \mu(c(x + 2) + c(x) - 2c(x + 1))
+ \mu q(1 - q)^x(V_n(2) + qV_n(1) - (1 + q)V_n(0) + c(0) - c(1))
+ \mu q(1 - q)^{x+1}(V_n(1) - V_n(0))
- q(1 - q)^x(\gamma - \lambda)(W_n(0) - V_n(0)).
$$

The first three terms of the expression are positive since $V_n$ and $W_n$ are gcv. The fourth term is positive since $c(x)$ is convex in $x$. The fifth term again is positive since $V_n$ is gcv. The sixth term is positive since $V_n$ is increasing in $x$. Only the last term is negative. In what follows we show that this can be compensated.
by the first term in $\gamma$ and the sixth term. Using Equation (12) and $W_n(0) \leq V_n(1)$, one may write
\[
\gamma (W_n(x+2) + F(W_n(x)) - W_n(x+1) - F(W_n(x+1)) + c(x) - c(x+1))
+ \mu q(1 - q)^{x+1}(V_n(1) - V_n(0)) - q(1 - q)^x(\gamma - \lambda)(W_n(0) - V_n(0))
\geq \gamma q(1 - q)^{x+1}(V_n(1) - V_n(0)) + \mu q(1 - q)^{x+1}(V_n(1) - V_n(0)) - q(1 - q)^x(\gamma - \lambda)(V_n(1) - V_n(0))
\]
\[
= q(1 - q)^x(V_n(1) - V_n(0))\frac{\mu \gamma + \lambda^2}{\lambda + \gamma} \geq 0.
\]
This proves that $V_{n+1}$ is gcv.

The result is proven for the approximated model under the condition $\gamma > \lambda$. Since our discretized model converges to the continuous one as $\gamma$ tends to infinity, the condition $\gamma > \lambda$ is automatically satisfied for the exact model. This finishes the proof of the Theorem.

Proof of Theorem 2

Stationary probabilities. Observing that
\[
\left(\frac{\gamma}{\lambda + \gamma}\right)^x + \sum_{l=0}^{x-1} \left(\frac{\lambda}{\lambda + \gamma}\right)^l \left(\frac{\gamma}{\lambda + \gamma}\right)^h = \left(\frac{\gamma}{\lambda + \gamma}\right)^h,
\]
we deduce that the cumulative transition rate from state $x$ to states $0, 1, \ldots, x - h$ is $\mu \left(\frac{\gamma}{\lambda + \gamma}\right)^h$, for $0 \leq h < x < n$; and that from state $n$ to states $0, 1, \ldots, n - h$ is $(\mu + \gamma) \left(\frac{\gamma}{\lambda + \gamma}\right)^h$, for $0 \leq h < n$. We now give the steady-state probability to be in state $x$, denoted by $p_x$, for $0 \leq x \leq n$.

Lemma 1 We have
\[
p_0 = \frac{\gamma(\mu - \lambda)}{\gamma \mu + \lambda^2 - \lambda(\lambda + \gamma)\left(\frac{\lambda + \gamma}{\mu + \gamma}\right)^n},
\]
\[
p_x = \frac{\lambda}{\gamma} \left(\frac{\lambda + \gamma}{\mu + \gamma}\right)^x p_0, \text{ for } 0 < x \leq n.
\]

Proof. We first prove by induction on $x$ that
\[
p_{n-x} = \left(\frac{\mu + \gamma}{\lambda + \gamma}\right)^x p_n,
\]
and
for $0 \leq x < n$. For $x = 0$, Equation (15) is straightforward. Assume now that Equation (15) is true for any rank $r$ such that $0 \leq l \leq x$ and $0 \leq x < n - 1$. Using Equation (14), we may write

$$\gamma p_{n-(x+1)} = \mu \sum_{l=1}^{x} \left( \frac{\gamma}{\lambda + \gamma} \right)^{x+1-l} p_{r-l} + (\mu + \gamma) \left( \frac{\gamma}{\lambda + \gamma} \right)^{x+1} p_{n}$$

$$= \mu \sum_{l=1}^{x} \left( \frac{\gamma + \mu}{\lambda + \gamma} \right)^{x+1-l} \left( \frac{\mu + \gamma}{\lambda + \gamma} \right)^{l} p_{n} + (\mu + \gamma) \left( \frac{\gamma}{\lambda + \gamma} \right)^{x+1} p_{n}$$

$$= (\mu + \gamma) \left( \frac{\gamma}{\lambda + \gamma} \right)^{x+1} \left( \frac{\mu + \gamma}{\lambda + \gamma} \right)^{x} p_{n}.$$

This leads to $p_{n-(x+1)} = \left( \frac{\mu + \gamma}{\lambda + \gamma} \right)^{x+1} p_{n}$ and finishes the proof of Equation (15). Note now that for $p_{0}$, the transition rate from state 0 to 1 is $\lambda$ instead of $\gamma$. Thus

$$p_{0} = \frac{\gamma}{\lambda} \left( \frac{\mu + \gamma}{\lambda + \gamma} \right)^{n} p_{n}. \quad (16)$$

Since all probabilities sum up to one, we obtain $p_{0}$. This finishes the proof of the lemma.

**Probability of an empty system.** If $\lambda \neq \mu$, we have $\frac{\lambda + \gamma}{\mu + \gamma} = \frac{\lambda + n/TDT}{\mu + n/TDT} = \left( 1 + \frac{\lambda}{n/TDT} \right) 1 + o(1/n)$. As $n$ tends to $\infty$, $1 + \frac{\lambda}{n/TDT} = 1 - \frac{\mu}{n/TDT} + o(1/n)$. Thus as $n$ tends to $\infty$, $\frac{\lambda + \gamma}{\mu + \gamma} = 1 + \frac{\lambda - \mu}{n/TDT} + o(1/n)$. We also have as $n$ tends to $\infty$, $\ln \left( \frac{\lambda + \gamma}{\mu + \gamma} \right) = -TDT(\mu - \lambda)$, which implies $
lim_{n \to \infty} \left( \frac{\lambda + \gamma}{\mu + \gamma} \right)^{n} = e^{-TDT(\mu - \lambda)}$. If $\lambda = \mu$, we have $\frac{\lambda + \gamma}{\mu + \gamma} = 1$. Then, $\left( \frac{\lambda + \gamma}{\mu + \gamma} \right)^{n} = 1$, for $r \geq 1$ and we also have $\lim_{n \to \infty} \left( \frac{\lambda + \gamma}{\mu + \gamma} \right)^{n} = e^{-TDT(\mu - \lambda)}$. We therefore deduce the result of the proposition by letting $n$ and $\gamma$ go to infinity.

**Other performance measures.** We prove the equations for the case $\lambda \neq \mu$. The proofs for the case $\lambda = \mu$ follow from those of the case $\lambda \neq \mu$ by continuity. We consider the embedded Markov chain at matching initiation or sent back to shipping line epochs. Matching initiations occur at $\mu$-transitions from states $x > 0$. Sent back to the shipping line initiations only occur in state $n$ with a $\gamma$-transition. The state probability just before a match or a send back to the shipping line is denoted by $p(x)$. From flow conservation, we may write, $\alpha(x) = \frac{\mu p_{x}}{\lambda}$ for $0 < x < 0$, and $\alpha(n) = \frac{(\gamma + \mu) p_{x}}{\lambda}$ for $x = n$.

**Proportion of containers sent back to the shipping line:** It is given by $P_{s} = \lim_{n \to \infty} \left( \frac{\gamma p_{n}}{\lambda} \right)$.

**Expected time spent by container at the consignee:** A matched container waits $x$ $\gamma$-phases with probability $p_{x} \frac{\gamma}{\lambda}$, for $0 < x \leq n$. We denote by $T_{S}$ the time spent by a container which is sent to the consignor at the
the strict convexity of the performance measures in the case as to evaluate their sign. Since all these measures are infinitely continuously derivable in TDT for TDT > 0,

Proof of Proposition 1

We therefore obtain (1 − Pₚ) · E(Tₛ) = \( \frac{1 - e^{-TDT(\mu - \lambda)}}{\mu(1 - \alpha)(1 - ae^{-TDT(\mu - \lambda)})} \). Finally, E(T) = (1 − Pₚ) · E(Tₛ) + Pₛ · TDT.

Waiting time distribution: A matched container can wait \( \gamma \)-phases with probability \( \frac{\gamma}{x}p_x \), for 0 < x ≤ n.

For a container that matches from state \( x > 0 \), its waiting time is an Erlang random variable with \( x \) phases and a rate \( \gamma \) per phase. We thus have

\[
(1 - Pₚ) · P(Tₛ > t) = \lim_{n \to \infty} \sum_{x=1}^{n} \frac{\mu}{\lambda} \left( \frac{\lambda + \gamma}{\mu + \gamma} \right)^{x} p_0 \sum_{h=0}^{x-1} \frac{(\gamma t)^h}{h!} e^{-\gamma t}.
\]

We also may write

\[
\sum_{x=1}^{n} \sum_{h=0}^{x-1} \frac{(\gamma t)^h}{h!} \left( \frac{\lambda + \gamma}{\mu + \gamma} \right)^{x} = \frac{\lambda + \gamma}{\mu + \gamma} \sum_{h=0}^{n-1} \frac{(\gamma t)^h}{h!} \sum_{x=h}^{n-1} \left( \frac{\lambda + \gamma}{\mu + \gamma} \right)^{x}.
\]

\[
= \frac{\lambda + \gamma}{\mu + \gamma} \sum_{h=0}^{n-1} \frac{(\gamma t)^h}{h!} \left( \frac{\lambda + \gamma}{\mu + \gamma} \right)^{h} \frac{1 - \left( \frac{\lambda + \gamma}{\mu + \gamma} \right)^{n - (h - 1)}}{1 - \frac{\lambda + \gamma}{\mu + \gamma}}.
\]

\[
= \frac{\lambda + \gamma}{\mu - \lambda} \sum_{h=0}^{n-1} \frac{(\gamma t)^h}{h!} \left( \frac{\lambda + \gamma}{\mu + \gamma} \right)^{h} - \sum_{h=0}^{n-1} \frac{(\gamma t)^h}{h!} \left( \frac{\lambda + \gamma}{\mu + \gamma} \right)^{n}.
\]

Next, we get

\[
\lim_{n \to \infty} e^{-\gamma t} \sum_{h=0}^{n-1} \frac{(\gamma t)^h}{h!} \left( \frac{\lambda + \gamma}{\mu + \gamma} \right)^{h} = e^{-TDT(\mu - \lambda)}, \quad \text{and} \quad \lim_{n \to \infty} e^{-\gamma t} \sum_{h=0}^{n-1} \frac{(\gamma t)^h}{h!} \left( \frac{\lambda + \gamma}{\mu + \gamma} \right)^{h} = e^{-t(\mu - \lambda)}.
\]

Therefore (1 − Pₚ) · P(Tₛ > t) = \( \frac{e^{-t(\mu - \lambda)} - e^{-TDT(\mu - \lambda)}}{1 - ae^{-TDT(\mu - \lambda)}} \). Using P(T > t) = (1 − Pₚ) · P(Tₛ > t) + Pₛ, we next obtain the result.

Proof of Proposition 1 and Corollary 2

Proof of Proposition 1. We compute the first and second order derivatives of the performance measures so as to evaluate their sign. Since all these measures are infinitely continuously derivable in TDT for TDT > 0, the strict convexity of the performance measures in the case \( \lambda \neq \mu \) implies the convexity in the case \( \lambda = \mu \).
Moreover, note that the strict convexity for $\lambda = \mu$ holds also by following exactly the same approach as that for $\lambda \neq \mu$. In what follows, we focus on the case $\lambda \neq \mu$.

Let us first prove that $p_0(\infty)$ is decreasing and convex in TDT. In what follows we write $p_0$ instead of $p_0(\infty)$. We may write

$$\frac{\partial p_0}{\partial \text{TDT}} = -\lambda(1-a)^2 e^{-\text{TDT}(\mu-\lambda)} \left(1 - ae^{-\text{TDT}(\mu-\lambda)}\right)^2 < 0.$$ 

Thus $p_0$ is strictly decreasing in TDT. One may write

$$\frac{\partial^2 p_0}{\partial \text{TDT}^2} = \frac{\mu(1-a)^3 e^{-\text{TDT}(\mu-\lambda)} (1 + ae^{-\text{TDT}(\mu-\lambda)})}{(1 - ae^{-\text{TDT}(\mu-\lambda)})^3}.$$ 

The sign of $\frac{\partial^2 p_0}{\partial \text{TDT}^2}$ only depends on $\frac{\lambda}{1 - ae^{-\text{TDT}(\mu-\lambda)}}$ because the other parts of the expression are positive. If $1 - a > 0$, then TDT$(\mu - \lambda) > 0$. So $e^{-\text{TDT}(\mu-\lambda)} < 1$ and $1 - ae^{-\text{TDT}(\mu-\lambda)} > 0$. Similarly, if $1 - a < 0$, then $1 - ae^{-\text{TDT}(\mu-\lambda)} < 0$. In both cases the sign of the ratio is positive. Therefore, $\frac{\partial^2 p_0}{\partial \text{TDT}^2} > 0$ and $p_0$ is strictly decreasing and strictly convex in TDT.

We next focus on the proportion of containers sent back to the shipping line, $P_s$. We have

$$\frac{\partial P_s}{\partial \text{TDT}} = e^{-\text{TDT}(\mu-\lambda)} \left(- (\mu - \lambda) p_0 + \frac{\partial p_0}{\partial \text{TDT}}\right) = -\frac{\mu(1-a)^2 e^{-\text{TDT}(\mu-\lambda)}}{(1 - ae^{-\text{TDT}(\mu-\lambda)})^2} < 0.$$ 

Therefore $\frac{\partial P_s}{\partial \text{TDT}} < 0$, and $P_s$ is strictly decreasing in TDT. Observe that

$$\frac{\partial P_s}{\partial \text{TDT}} = a^{-1} \frac{\partial p_0}{\partial \text{TDT}}.$$ 

Thus, $\frac{\partial P_s}{\partial \text{TDT}}$ is positively proportional to $\frac{\partial p_0}{\partial \text{TDT}}$. Since $p_0$ is strictly convex in TDT, $P_s$ is also strictly convex in TDT.

We next focus on $E(T)$ and $E(N)$. We have $E(T) = \frac{1 - e^{-X(1 + aX)}}{\mu(1-a)(1 - ae^{-X})}$, with $X = \text{TDT}(\mu - \lambda)$. This allows us to derive $E(T)$. We have

$$\frac{\partial E(T)}{\partial \text{TDT}} = \frac{\partial E(T)}{\partial X} \cdot \frac{\partial X}{\partial \text{TDT}}.$$ 

After some algebra, we get

$$\frac{\partial E(T)}{\partial \text{TDT}} = e^{-X} (1 - 2a + aX + a^2 e^{-X}) \frac{1}{(1 - ae^{-X})^2}.$$ 

The sign of this expression depends on the sign of $1 - 2a + aX + a^2 e^{-X}$. We have $e^{-X} = \sum_{k=2}^{\infty} \frac{(-\mu \text{TDT}(1-a))^k}{k!} + 1 - \mu \text{TDT}(1-a)$. So, $1 - 2a + aX + a^2 e^{-X} = (1-a)^2 (1 + a\mu \text{TDT}) + a^2 \sum_{k=2}^{\infty} \frac{(-X)^k}{k!}$. This expression is clearly
positive because $\sum_{k=2}^{\infty} \frac{(-X)^k}{k!} = e^{-X} - 1 + X \geq 0$ for $X \in \mathbb{R}$.

It remains to prove that $P(T > t)$ is strictly increasing in TDT for $t < \text{TDT}$. One may write

$$\frac{\partial P(T > \text{TDT})}{\partial \text{TDT}} = \frac{\lambda e^{-\mu \text{TDT}(1-a)}(1-a)(1-e^{-\mu(1-a)})}{(1-ae^{-\mu \text{TDT}(1-a)})^2}. $$

This sign of this expression depends on the sign of $(1-a)(1-e^{-\mu(1-a)})$. With the same approach as for $p_0$ by distinguishing the cases $a > 1$ and $a < 1$, one can show that this expression is positive. So, $P(T > \text{TDT})$ is strictly increasing in TDT for $t < \text{TDT}$.

Let us now consider the expected cost per container, $E(C)$. We assume that $c(t)$ is a continuous piecewise linear increasing in $t$. We denote by $c_k$ and $d_k$, the slope and the intercept of the function $c(t)$ on the interval $[t_{k-1}, t_k)$ with the convention that $t_0 = 0$ for $k \geq 1$. The cost function is convex, therefore $c_k$ is increasing in $k$. The parameters $d_k$ are adjusted in order to make $c(t)$ a continuous function. As a function of the time spent at the consignee, the cost function, $c(t)$, can be written as: $c(t) = c_k \cdot t + d_k$ for $t \in [t_{k-1}, t_k)$ and $k \geq 1$. For $\text{TDT} \in [t_{k-1}, t_k)$, using the result of Corollary 1, $E(C)$ is given by

$$E(C) = (c_k + c_k \text{TDT} + d_k) \left[ (1-a)e^{-\text{TDT}(\mu-\lambda)} \right] 1 - a e^{-\text{TDT}(\mu-\lambda)} + \int_{t=0}^{\text{TDT}} \mu \left( 1-a \right) e^{-t(\mu-\lambda)} \left( c_k t + d_k \right) dt. $$

After some algebra, this leads to

$$E(C) = \frac{c_k + d_k \mu(1-a) + e^{-\mu \text{TDT}(1-a)} \left( (1-a)^2(c_k + c_k \text{TDT} + d_k) - (1-a)\mu(c_k \text{TDT} + d_k) - c_k \right)}{\mu(1-a)(1-ae^{-\mu \text{TDT}(1-a)})},$$

for $\text{TDT} \in [t_{k-1}, t_k)$. We now compute the derivative of this function in TDT:

$$\frac{\partial E(C)}{\partial \text{TDT}} = \frac{e^{-\mu \text{TDT}(1-a)} \left( (1-a)^2 c_k \mu - c_k (1-2a + \mu \text{TDT} a(1-a) + a^2 e^{-\mu \text{TDT}(1-a)}) \right)}{(1-ae^{-\mu \text{TDT}(1-a)})^2},$$

for $\text{TDT} \in [t_{k-1}, t_k)$. Note that the parameter $d_k$ is not present in this expression therefore it has no influence on the optimal threshold. The sign of $\frac{\partial E(C)}{\partial \text{TDT}}$ depends on the sign of the function in TDT, $f'(\text{TDT}) = (1-a)^2 c_k \mu - c_k (1-2a + \mu \text{TDT} a(1-a) + a^2 e^{-\mu \text{TDT}(1-a)})$. This function is defined for $\text{TDT} \geq 0$. By deriving again the function $f$, we can prove that this function is strictly decreasing in TDT. Thus if $c_k \mu \leq c_k$, $\frac{\partial E(C)}{\partial \text{TDT}} \geq 0$ for $\text{TDT} \in [t_{k-1}, t_k)$ and the optimal threshold on the interval $[t_{k-1}, t_k)$ is $\text{TDT} = t_{k-1}$. Otherwise, $\frac{\partial E(C)}{\partial \text{TDT}}$ is first negative, next positive. Let us denote by $\text{TDT}_k$ the unique solution of $f'(\text{TDT}) = 0$ for $\text{TDT} \geq 0$. If $\text{TDT}_k < t_{k-1}$, then the optimal threshold on the interval $[t_{k-1}, t_k)$ is $\text{TDT} = t_{k-1}$. If $t_{k-1} \leq \text{TDT}_k < t_{k-1}$, then the optimal threshold on the interval $[t_{k-1}, t_k)$ is $\text{TDT} = \text{TDT}_k$. Finally, if $\text{TDT}_k \geq t_k$, then the optimal threshold on the interval $[t_{k-1}, t_k)$ is $\text{TDT} = t_k$. This allows us to find the optimal threshold on the interval $[t_{k-1}, t_k)$.

We are now interested in obtaining the global optimal threshold on the interval $[0, \infty)$. We now show that $\text{TDT}_k$ is decreasing in $k$. We use the equality which allows to obtain $\text{TDT}$ in order to find $\frac{\partial \text{TDT}}{\partial c_k}$. One
may write
\[
\frac{\partial \text{TDT}}{\partial c_k} = -\frac{c_s(1 - a)}{c_k^2a(1 - ae^{-\mu(1-a)}\text{TDT})} < 0.
\]
Therefore, TDT is decreasing and convex in \(c_k\). Since \(c_k\) is increasing in \(k\) then \(\text{TDT}_k\) is decreasing in \(k\). Otherwise the two conditions \(\text{TDT}_k \in [t_{k-1}, t_k]\) and \(f(\text{TDT}_k) = 0\) cannot be met together, therefore there exists an index \(k\) such that \(E(C)\) is decreasing for \(t \leq t_{k-1}\) and increasing otherwise. The optimal threshold is hence \(t_{k-1}\).

**Proof of Corollary 2.** Using proposition 1, we use Relation 2 to find \(\frac{\partial \text{TDT}}{\partial c_s}\) and \(\frac{\partial \text{TDT}}{\partial c_k}\). One may write
\[
\frac{\partial \text{TDT}}{\partial c_s} = 1 - a c_k a \left(1 - a e^{-\mu(1-a)}\text{TDT}\right) > 0,
\]
\[
\frac{\partial^2 \text{TDT}}{\partial c_s^2} = -\frac{\partial \text{TDT}}{\partial c_s} a \mu (1 - a)^2 e^{-\mu(1-a)}\text{TDT} < 0,
\]
\[
\frac{\partial^2 \text{TDT}}{\partial c_k^2} = c_s 2 c_k a (1 - a) \left(1 - ae^{-\mu(1-a)}\text{TDT}\right) + c_k^2 a^2 \mu (1 - a)^2 e^{-\mu(1-a)}\text{TDT} > 0.
\]
Therefore, TDT is increasing and piecewise concave in \(c_s\) and decreasing and piecewise convex in \(c_k\). The other results in Corollary 2 follow directly from combining Proposition 1 and the monotonicity results of TDT in \(c_s\) and \(c_k\).

**Proof of Theorem 3**

By subtracting Equation (3) to Equation (4) for \(x = 1\), we get
\[
\Delta(1) = \frac{\lambda + p\mu}{\gamma} \Delta(0) - \frac{c(0)\mu}{\lambda\gamma} - \frac{w\mu}{\lambda\gamma^2}.
\]
(17)

Observing that
\[
\sum_{h=0}^{x+1} q_{x+1,x+1-h} V(x+1-h) - \sum_{h=0}^{x} q_{x,x-h} V(x-h)
= (1 - q)^{x+1} V(0) + q(1-q)^x V(1) - (1-q)^x V(0) + \sum_{h=0}^{x-1} q(1-q)^h (V(x+1-h) - V(x-h))
= \sum_{h=0}^{x} q(1-q)^h \Delta(x-h),
\]
Equation (4) can be rewritten as
\[
\Delta(x+1) = -\frac{\mu}{\lambda\gamma^2} w - \frac{d\mu}{\lambda\gamma} \mathbb{1}_{x=1} + \left(1 + \frac{\mu}{\gamma}\right) \Delta(x) - \frac{\mu}{\gamma} \sum_{h=0}^{x} (1-q)^h \Delta(x-h),
\]
(18)
for \(1 \leq x \leq n - 2\). This expression allows us to prove by induction on \(x\) that

\[
\Delta(x) = F(x) + G_w(x) + G_d(x) + G_{c(0)}(x),
\]

(19)

where

\[
F(x) = \frac{\lambda}{\gamma} \Delta(0) \left( \left( \frac{\mu + \gamma}{\lambda + \gamma} \right)^{x-1} \left( \frac{\mu \gamma}{\lambda (\lambda + \gamma)} + \frac{\mu}{\mu - \lambda} \right) - \frac{\lambda}{\mu - \lambda} \right),
\]

\[
G_w(x) = -\frac{\mu}{\lambda \gamma^2} w \left( \frac{\mu (\mu + \gamma)}{(\mu - \lambda)^2} \left( \frac{\mu + \gamma}{\lambda + \gamma} \right)^{x-1} - \frac{\lambda x}{\mu - \lambda} - \frac{\mu (\lambda + \gamma)}{(\mu - \lambda)^2} \right),
\]

\[
G_d(x) = -\frac{d \mu}{\gamma} \frac{\lambda^2}{x \geq u} \left( \frac{\mu + \gamma}{\lambda + \gamma} \right)^{x-u} - \lambda, \quad \text{and},
\]

\[
G_{c(0)}(x) = -\frac{c(0) \mu}{\gamma} \frac{\lambda^2}{x \geq u} \left( \frac{\mu + \gamma}{\lambda + \gamma} \right)^{x-1} - \lambda
\]

for \(1 \leq x \leq n - 1\), with \(F(0) = \Delta(0)\), and \(G_w(0) = G_d(0) = 0\). The induction step consists in sums or derivatives of geometric progressions.

Since \(V(0) = 0\) (reference state), we can obtain \(V(x)\) via \(V(x) = \Delta(0) + \Delta(1) + \cdots + \Delta(x-1)\), for \(0 \leq x \leq n\) as a function of \(\Delta(0)\) and \(E(C)\). By replacing the expressions of \(V(x)\) into Equation (5), one can obtain the expressions of \(E(C)\) and \(\Delta(0)\).

\[\square\]

**Further development for comparing time-based and quantity based policies**

Deriving closed-form expression for the time spent at the consignee under QBP with nonlinear inventory holding costs appears to be difficult. In order to further investigate the comparison between the two policy classes, we compare in Proposition 2 some of the performance measures which may be involved in the expected cost per container.

**Proposition 2** For a given proportion of containers sent back to the shipping line,

- QBP achieves a lower value for \(E(T)\) than TBP.
- TBP achieves a lower value for \(P(T > \text{Detention free period})\) than QBP if \(\text{Detention free period} > TDT\).

Roughly speaking, QBP allows for a better control on the size of the inventory (or equivalently a better control on the expected time spent at the consignee due to Little’s law). TBP allows for a better control of excessive time spent at the consignee. The preference between the two policies depends on the nature of the cost structure. Intuitively, as the convexity of the cost function increases the relative value of TBP compared to QBP increases.

**Proof of Proposition 2.** In what follows, we give the index 1 or 2 to specify if a performance measure is evaluated under TBP or QBP. From a simple Markov chain chain analysis, some of the performance measures
under Policy 2 can be explicitly derived. For instance, the expected time spent in the system, \( E(T_2) \) and the proportion of containers sent back to the shipping line, \( P_{S2} \), are given by

\[
E(T_2) = \frac{1}{\mu} \frac{1 - a^N(1 + N(1 - a))}{(1 - a)(1 - a^{N+1})},
\]

and

\[
P_{S2} = \frac{(1 - a)a^N}{1 - a^{N+1}}.
\]

The equality \( P_{S1} = P_{S2} \) relates the thresholds \( TDT \) and \( N \). This leads to \( TDT = -N \frac{\ln(a)}{\mu(1 - a)} \). From this equality, one may compare between \( E(T_1) \) and \( E(T_2) \). We have

\[
E(T_2) - E(T_1) = \frac{Na^N}{\mu} \cdot \frac{(-1 + a - a \ln(a))}{(1 - a)(1 - a^{N+1})}
\]

The sign of \( E(T_2) - E(T_1) \) depends on the sign of \( f(a) = -1 + a - a \ln(a) \). We have \( f'(a) = -\ln(a) \), so \( f \) is increasing for \( a \in (0, 1] \) and decreasing for \( a \in [1, \infty) \). Since \( f(1) = 0 \), we have \( E(T_2) - E(T_1) \leq 0 \). This proves the first statement of the proposition. Clearly, TBP clearly achieves a lower value for \( P(T > t) \) if \( t > TDT \) because \( P(T > t) = 0 \) in this case. This proves the second statement of the proposition. \( \square \)