An operator theoretic approach to infinite-dimensional control systems

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ORIGINAL PAPER

An operator theoretic approach to infinite-dimensional control systems

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In this survey we use an operator theoretic approach to infinite-dimensional systems theory. As this research field is quite rich, we restrict ourselves to the class of infinite-dimensional linear port-Hamiltonian systems and we will focus on topics such as well-posedness, stability and stabilizability. We combine the abstract operator theoretic approach with the more physical approach based on Hamiltonians. This enables us to derive easy verifiable conditions for well-posedness and stability.

KEYWORDS

C_0-semigroup, infinite-dimensional systems theory, partial differential equation, port-Hamiltonian system, stability, stabilizability

1 | INTRODUCTION

Systems described by partial differential equations (PDEs) can be investigated either by operator theoretic or PDE methods. The PDE methods are specialized to specific classes of PDEs, and therefore lead to refined results. The operator theoretic methods formulate the main concepts and investigate their interconnections. The advantage of the operator theoretic approach is that it allows for a general abstract framework. In this survey, we combine the abstract operator theoretic approach with a more physical approach based on Hamiltonians in order to derive easy verifiable conditions for well-posedness and stability of port-Hamiltonian systems.

Many physical systems can be formulated using a Hamiltonian framework. This class of systems contains ordinary as well as PDEs. Each system in this class has a Hamiltonian, generally given by the energy function. In the study of Hamiltonian systems it is usually assumed that the system does not interact with its environment. However, for the purpose of control and for the interconnection of two or more Hamiltonian systems it is essential to take this interaction with the environment into account. This led to the class of port-Hamiltonian systems, see van der Schaft[1] and van der Schaft and Maschke.[2] The Hamiltonian/energy has been used to control a port-Hamiltonian system, see, for example, Baaiu et al.,[3] Cervera et al.,[4] Hamroun et al.[5] and Ortega et al.[6] For port-Hamiltonian systems described by ordinary differential equations this approach is very successful, see the references mentioned above. Port-Hamiltonian systems described by PDE is a subject of current research, see, for example, Eberard et al.,[7] Jeltsema and van der Schaft[8], Kurula et al.,[9] Macchelli and Macchelli.[10] As mentioned above, we concentrate on an operator-theoretic approach to port-Hamiltonian systems. There are other approaches, such as using differential geometry and/or Dirac structures, see, for example, van der Schaft.[1]

2 | EXAMPLE OF A PORT-HAMILTONIAN SYSTEM

Various control systems can be modeled by PDEs such as vibrating strings, flexible structures, the propagation of sound waves, and networks of strings or flexible structures. Here we consider the simple example of a transmission line with boundary control and observation.

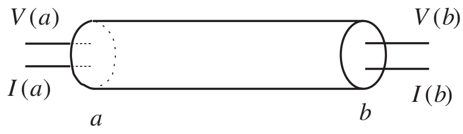

FIGURE 1 Schematic representation of the transmission line

Example 2.1. Transmission lines with boundary controls are used for electric power transmission.

The problem can be approximated by the 1D system of Figure 1 representing propagation of electric charges and magnetic fluxes.

The *lossless transmission line* on the spatial interval $[a, b]$ is described by the PDE:

$$\begin{aligned}\frac{\partial Q}{\partial t}(\zeta,t) &= -\frac{\partial}{\partial \zeta}\frac{\phi(\zeta,t)}{L(\zeta)} \\ \frac{\partial \phi}{\partial t}(\zeta,t) &= -\frac{\partial}{\partial \zeta}\frac{Q(\zeta,t)}{C(\zeta)}.\end{aligned} \tag{1}$$

Here $Q(\zeta, t)$ is the charge at position $\zeta \in [a, b]$ and time $t > 0$, and $\phi(\zeta, t)$ is the (magnetic) flux at position ζ and time t. C is the (distributed) capacity and L is the (distributed) inductance. The voltage and current are given by $V = Q/C$ and $I = \phi/L$, respectively. The energy of this system is given by

$$E(t) = \frac{1}{2}\int_a^b \frac{\phi(\zeta,t)^2}{L(\zeta)} + \frac{Q(\zeta,t)^2}{C(\zeta)}d\zeta.$$

The control, boundary conditions and observation associated to this problem are

$$V(a,t) = u(t) \qquad V(b,t) = RI(b,t) \qquad I(a,t) = y(t)$$

corresponding to a controlled voltage at point a, a resistive charge at point b and an observed current at point a. Here $u(t)$ is the control, $y(t)$ the observation and $R \geq 0$ is the resistor. For the change of energy we obtain

$$\begin{aligned}\frac{dE}{dt}(t) &= \int_a^b \frac{\phi(\zeta,t)}{L(\zeta)}\frac{\partial\phi}{\partial t}(\zeta,t) + \frac{Q(\zeta,t)}{C(\zeta)}\frac{\partial Q}{\partial t}(\zeta,t)d\zeta \\ &= \int_a^b -\frac{\phi(\zeta,t)}{L(\zeta)}\frac{\partial}{\partial\zeta}\frac{Q(\zeta,t)}{C(\zeta)} - \frac{Q(\zeta,t)}{C(\zeta)}\frac{\partial}{\partial\zeta}\frac{\phi(\zeta,t)}{L(\zeta)}d\zeta \\ &= -\int_a^b \frac{\partial}{\partial\zeta}\left(\frac{\phi(\zeta,t)}{L(\zeta)}\frac{Q(\zeta,t)}{C(\zeta)}\right)d\zeta \\ &= \frac{\phi(a,t)}{L(a)}\frac{Q(a,t)}{C(a)} - \frac{\phi(b,t)}{L(b)}\frac{Q(b,t)}{C(b)} \\ &= V(a,t)I(a,t) - V(b,t)I(b,t) \qquad (2) \\ &= u(t)y(t) - RI(b,t)^2, \qquad (3)\end{aligned}$$

where we used the boundary conditions. Equation (2) shows that the change of energy can only occur via the boundary. Since voltage times current equals power and the change of energy is also power, this equation represents a power balance.

3 | CLASS OF PORT-HAMILTONIAN SYSTEMS

Many physical systems can be modelled by the following equation

$$\begin{aligned}\frac{\partial x}{\partial t}(\zeta,t) &= P_1\frac{\partial}{\partial\zeta}(\mathcal{H}(\zeta)x(\zeta,t)) + P_0(\mathcal{H}(\zeta)x(\zeta,t)), \quad \zeta\in(a,b), t>0, \\ x(\zeta,0) &= x_0(\zeta), \qquad \zeta\in(a,b), \\ u(t) &= W_{B,1}\begin{bmatrix}\mathcal{H}(b)x(b,t)\\ \mathcal{H}(a)x(a,t)\end{bmatrix}, \qquad t>0, \\ 0 &= W_{B,2}\begin{bmatrix}\mathcal{H}(b)x(b,t)\\ \mathcal{H}(a)x(a,t)\end{bmatrix}, \qquad t>0, \\ y(t) &= W_C\begin{bmatrix}\mathcal{H}(b)x(b,t)\\ \mathcal{H}(a)x(a,t)\end{bmatrix}, \qquad t>0.\end{aligned} \tag{4}$$

Here $P_1 \in \mathbb{C}^{n\times n}$ is invertible and self-adjoint, that is, $P_1^* = P_1$, and $P_0 \in \mathbb{C}^{n\times n}$ is *skew-adjoint*, that is, $P_0^* = -P_0$. We assume that $\mathcal{H} \in L^\infty((a,b);\mathbb{C}^{n\times n})$, for every $\zeta \in [a,b]$, $\mathcal{H}(\zeta)$ a is self-adjoint matrix and there exist constants $c, C > 0$, such that $cI \leq \mathcal{H}(\zeta) \leq CI$ for almost every $\zeta \in [a,b]$. Finally, $W_{B,1}$ is a $m \times 2n$-matrix, $W_{B,2}$ is a $(n-m)\times 2n$-matrix, and W_C a $k \times 2n$-matrix. Here $u(t) \in \mathbb{C}^m$ denotes the input and $y(t) \in \mathbb{C}^k$ the output at time t. We call (4) a *port-Hamiltonian system*.

The energy or Hamiltonian can be expressed by using x and $\mathcal{H}$. That is

$$E(x(\cdot,t)) = \frac{1}{2}\int_a^b x(\zeta,t)^*\mathcal{H}(\zeta)x(\zeta,t)d\zeta. \tag{5}$$

In Example 2.1, the change of energy (power) of the system was only possible via the boundary of its spatial domain. In general, for the Hamiltonian given by (5) the following balance equation holds for all (classical) solutions of (4)

$$\frac{dE}{dt}(x(\cdot,t)) = \frac{1}{2}\left[(\mathcal{H}x)^*(\zeta,t)P_1\left(\mathcal{H}x\right)(\zeta,t)\right]_a^b. \tag{6}$$

This balance equation will prove to be very important and will be useful in many problems, such as the existence of solutions and stability.

Example 3.1 (Lossless transmission line)**.**
If we introduce in Example 2.1 the variables $x_1 = Q$ and $x_2 = \phi$, Equation 1 can be written as

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t)\end{bmatrix} = -\frac{\partial}{\partial \zeta}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix} \frac{1}{C(\zeta)} & 0 \\ 0 & \frac{1}{L(\zeta)}\end{bmatrix}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t)\end{bmatrix}\right)$$

which is of the form of Equation 4 with

$$P_1 = \begin{bmatrix} 0 & -1 \\ -1 & 0\end{bmatrix}, \quad P_0 = 0, \quad \mathcal{H}(\zeta) = \begin{bmatrix} \frac{1}{C(\zeta)} & 0 \\ 0 & \frac{1}{L(\zeta)}\end{bmatrix}. \tag{7}$$

The boundary condition, control and observation can be rewritten as

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & -R & 0 & 0\end{bmatrix}\begin{bmatrix} \frac{x_1}{C}(b,t) \\ \frac{x_2}{L}(b,t) \\ \frac{x_1}{C}(a,t) \\ \frac{x_2}{L}(a,t)\end{bmatrix} = \begin{bmatrix} u(t) \\ 0\end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix} \frac{x_1}{C}(b,t) \\ \frac{x_2}{L}(b,t) \\ \frac{x_1}{C}(a,t) \\ \frac{x_2}{L}(a,t)\end{bmatrix} = y(t), \tag{8}$$

that is, $W_{B,1} = \begin{bmatrix} 0 & 0 & 1 & 0\end{bmatrix}$, $W_{B,2} = \begin{bmatrix} 1 & -R & 0 & 0\end{bmatrix}$, $W_C = \begin{bmatrix} 0 & 0 & 0 & 1\end{bmatrix}$, $n = 2$, and $m = k = 1$. Finally, the Hamiltonian is written as

$$E(x(\cdot,t)) = \frac{1}{2}\int_a^b \frac{x_1(\zeta,t)^2}{C(\zeta)} + \frac{x_2(\zeta,t)^2}{L(\zeta)}d\zeta.$$

4 | OPERATOR THEORETIC APPROACH TO PORT-HAMILTONIAN SYSTEMS

In this section we rewrite our port-Hamiltonian system (4) as an abstract differential equation, which enables us to use operator theoretic methods. Motivated by the fact that the solution of a system $\dot{x}(t) = Ax(t)$ of ordinary differential equations, that is, A is a $n \times n$-matrix, with initial condition $x(0) = x_0$ is given by

$$x(t) = e^{tA}x_0,$$

we aim to write the port-Hamiltonian system (4) in the form

$$\dot{x}(t) = \mathfrak{A}x(t), \qquad u(t) = \mathfrak{B}x(t), \qquad y(t) = \mathfrak{C}x(t), \quad t \geq 0, \qquad x(0) = x_0. \tag{9}$$

Thus, we do not regard the solution $x(\cdot,\cdot)$ of (4) as a function of space and time, but as a function of time, which takes values in a function space, that is, we see $x(\zeta,t)$ as the function $x(\cdot,t)$ evaluated at ζ. With a slight abuse of notation, we write $x(\cdot,t) = (x(t))(\cdot)$. We "forget" the spatial dependence, and we write the PDE

$$\frac{\partial x}{\partial t}(\zeta,t) = P_1\frac{\partial}{\partial \zeta}\left(\mathcal{H}(\zeta)x(\zeta,t)\right) + P_0\left(\mathcal{H}(\zeta)x(\zeta,t)\right)$$

as the (abstract) ordinary differential equation

$$\frac{dx}{dt}(t) = P_1\frac{\partial}{\partial \zeta}\left(\mathcal{H}x(t)\right) + P_0\left(\mathcal{H}x(t)\right).$$

Hence, we consider the operator

$$\mathfrak{A}x := P_1 \frac{d}{d\zeta}(\mathcal{H}x) + P_0(\mathcal{H}x) \tag{10}$$

on a domain which includes the boundary conditions. The domain should be a part of the *state space* X, which we identify next. For our class of PDEs we have a natural energy function, see (5). Hence, it is quite natural to consider only states which have a finite energy. That is we take as our state space all functions for which $\int_a^b x(\zeta)^*\mathcal{H}(\zeta)x(\zeta)d\zeta$ is finite. Thanks to our assumptions on $\mathcal{H}(\zeta)$ the integral $\int_a^b x(\zeta)^*\mathcal{H}(\zeta)x(\zeta)d\zeta$ is finite if only if x is square integrable over (a,b) and so we choose the state space

$$X = L^2((a,b);\mathbb{C}^n)$$

with inner product

$$\langle f, g\rangle_{\mathcal{H}} = \frac{1}{2}\int_a^b f(\zeta)^*\mathcal{H}(\zeta)g(\zeta)d\zeta.$$

Thus, the squared norm of a state x equals the energy of this state. In order to achieve uniqueness of solutions we need to impose boundary conditions. For example, we equipped the lossless transmission line (Example 2.1) with the boundary condition, control and observation (8). In general, we consider boundary conditions, controls and observations as in (4). As $\mathfrak{A}x$ is not well-defined for every function x in X, we equip the operator $\mathfrak{A}$ with the domain $D(\mathfrak{A})$, which is given by

$$D(\mathfrak{A}) = \left\{ x \in L^2((a,b);\mathbb{C}^n) \mid \mathcal{H}x \in H^1((a,b);\mathbb{C}^n),\ W_{B,2}\begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a)\end{bmatrix} = 0 \right\}. \tag{11}$$

Here $H^1((a,b);\mathbb{C}^n)$ are all functions from (a,b) to $\mathbb{C}^n$ which are square integrable and have a derivative which is again square integrable. We remark, that the operator $\mathfrak{A}$ given by (10) and (11) satisfies for $x \in D(\mathfrak{A})$ the following useful condition

$$\operatorname{Re}\langle \mathfrak{A}x, x\rangle_{\mathcal{H}} = \frac{1}{2}\left[(\mathcal{H}x)^*(\zeta)P_1(\mathcal{H}x)(\zeta)\right]_a^b. \tag{12}$$

Finally, we define the operators $\mathfrak{B}$ and $\mathfrak{C}$ by

$$\mathfrak{B} : D(\mathfrak{A}) \to \mathbb{C}^m, \qquad \mathfrak{B}x = W_{B,1}\begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a)\end{bmatrix}, \tag{13}$$

$$\mathfrak{C} : D(\mathfrak{A}) \to \mathbb{C}^k, \qquad \mathfrak{C}x = W_C\begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a)\end{bmatrix}. \tag{14}$$

5 | EXISTENCE OF SOLUTIONS TO THE HOMOGENEOUS EQUATION

A fundamental problem of PDEs is the question of existence and uniqueness of solutions. In this section, we study this question for our port-Hamiltonian system (4) without control function ($u = 0$) and observation, that is, we consider a system of the form

$$\begin{aligned}
\frac{\partial x}{\partial t}(\zeta,t) &= P_1\frac{\partial}{\partial\zeta}(\mathcal{H}(\zeta)x(\zeta,t)) + P_0(\mathcal{H}(\zeta)x(\zeta,t)), \quad \zeta\in(a,b), t>0,\\
x(\zeta,0) &= x_0(\zeta), \qquad \zeta \in (0,1),\\
0 &= W_B\begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t)\end{bmatrix}, \qquad t>0.
\end{aligned} \tag{15}$$

Note that now we have chosen $u \equiv 0$ in Equation (4). In the previous section we rewrote Equation 15 equivalently as (using A instead of $\mathfrak{A}$)

$$\dot{x}(t) = Ax(t), \qquad t \ge 0, \qquad x(0) = x_0, \tag{16}$$

where $A : D(A) \subseteq L^2((a,b);\mathbb{C}^n) \to L^2((a,b);\mathbb{C}^n)$ is given by

$$Ax = P_1\frac{d}{d\zeta}(\mathcal{H}x) + P_0(\mathcal{H}x), \qquad x \in D(A), \tag{17}$$

$$D(A) = \left\{ x\in L^2((a,b);\mathbb{C}^n) \mid \mathcal{H}x \in H^1((a,b);\mathbb{C}^n),\ W_B\begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a)\end{bmatrix} = 0 \right\}. \tag{18}$$

Now the question arises:

Does Equation (16) has a unique solution?

This question is well-studied in operator theory and closely related to the notion of C_0-semigroups. The theory of C_0-semigroup started with the work of Hille, Phillips, and Yosida in the 1950s. By now it is a well-documented theory, for example, Curtain and Zwart,[11] Engel and Nagel,[12] Hille and Phillips,[13] Pazy,[14] and Yosida.[15]

Definition 5.1. Let X be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$. The operator valued function $t \mapsto T(t)$, $t \geq 0$, denoted by $(T(t))_{t\geq 0}$, is a *strongly continuous semigroup* or C_0*-semigroup* if the following holds

1. For all $t \geq 0$, $T(t)$ is a bounded linear operator on X, that is, $T(t) \in \mathcal{L}(X)$;
2. $T(0) = I$;
3. $T(t+\tau) = T(t)T(\tau)$ for all $t, \tau \geq 0$.
4. For all $x_0 \in X$, we have that $\|T(t)x_0 - x_0\|$ converges to zero, when $t \searrow 0$.

The easiest example of a C_0-semigroup is the exponential of a matrix. That is, let A be an $n \times n$ matrix, the matrix-valued function $T(t) = e^{At}$ satisfies the properties of Definition 5.1 on the Hilbert space $\mathbb{C}^n$. Clearly the exponential of a matrix is also defined for $t < 0$. If the semigroup can be extended to all $t \in \mathbb{R}$, then we say that $(T(t))_{t\in\mathbb{R}}$ is a group. We present the formal definition next.

Definition 5.2. Let X be a Hilbert space. $(T(t))_{t\in\mathbb{R}}$ is a *strongly continuous group* or C_0*-group*, if the following holds

1. For all $t \in \mathbb{R}$, $T(t)$ is a bounded linear operator on X;
2. $T(0) = I$;
3. $T(t+\tau) = T(t)T(\tau)$ for all $t, \tau \in \mathbb{R}$.
4. For all $x_0 \in X$, we have that $\|T(t)x_0 - x_0\|$ converges to zero, when $t \to 0$.

It is easy to see that the exponential of a bounded operator is a C_0-group. However, only a few C_0-semigroups are actually a C_0-group. Given the C_0-semigroup $\left(e^{At}\right)_{t\geq 0}$ with A being a square matrix or a linear bounded operator, A can be obtained by differentiating e^{At} and evaluating the derivative at $t = 0$. Next we associate an operator A to a C_0-semigroup $(T(t))_{t\geq 0}$.

Definition 5.3. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on the Hilbert space X. If the limit $\lim_{t\searrow 0} \frac{T(t)x_0 - x_0}{t}$ exists, then we say that $x_0 \in X$ is an element of the *domain* of A, shortly $x_0 \in D(A)$, and we define

$$Ax_0 = \lim_{t\searrow 0} \frac{T(t)x_0 - x_0}{t}.$$

We call A *infinitesimal generator* of the C_0-semigroup $(T(t))_{t\geq 0}$.

The relevance of C_0-semigroup is given by the next theorem.

Theorem 5.4 ([12, Theorem II.6.7])**.** *Let A be a linear operator with non-empty resolvent set, that is, there exists an $s \in \mathbb{C}$ such that the inverse of $sI - A$ exists as a bounded operator on X. Then the following statements are equivalent.*

1. *A is the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$.*
2. *For every $x_0 \in D(A)$ Equation (16) has a unique classical solution.*

Moreover, for $x_0 \in D(A)$, the function $x(t) := T(t)x_0$ equals the unique classical solution of (16).

Although $T(t)x_0$ only satisfies (16) if $x_0 \in D(A)$, $x(t) = T(t)x_0$ is still called the *solution* of (16) if $x_0 \in X$. To distinguish it from the classical solution, it is also called *mild* solution. So far we associated an infinitesimal generator to a C_0-semigroup, but usually, we have a differential operator A and we need to decide whether A generates a C_0-semigroup. The Hille-Yosida theorem (see [12, Theorem II.3.8]) provides an equivalent characterization. Here we formulate the Lumer-Phillips theorem which characterizes generators of contraction semigroups.

Definition 5.5. *A C_0-semigroup $(T(t))_{t\geq 0}$ is called a contraction semigroup if $\|T(t)x_0\| \leq \|x_0\|$ for all $x_0 \in X$ and all $t \geq 0$.*
A C_0-group is called a unitary group if $\|T(t)x_0\| = \|x_0\|$ for all $x_0 \in X$ and all $t \in \mathbb{R}$.

Theorem 5.6 (Lumer-Phillips theorem [12, Theorem II.3.15])**.** *An operator A defined on the Hilbert space X is the infinitesimal generator of a contraction semigroup on X if and only if the following conditions hold*

1. *For all $x_0 \in D(A)$ we have that* $\operatorname{Re}\langle Ax_0, x_0\rangle \leq 0$*;*
2. *For all $z \in X$ there exists a $x_0 \in D(A)$ such that $(I - A)x_0 = z$.*

For unitary groups there is a similar characterization.

Corollary 5.7. *An operator A defined on the Hilbert space X is the infinitesimal generator of a unitary group on X if and only if*

1. *For all $x_0 \in D(A)$ we have that* $\operatorname{Re}\langle Ax_0, x_0\rangle = 0$;
2. *For all $z \in X$ there exists a $x_0 \in D(A)$ such that $(I - A)x_0 = z$.*
3. *For all $z \in X$ there exists a $x_0 \in D(A)$ such that $(I + A)x_0 = z$.*

We are now in the position to characterize the existence of a unique solution to (15).

Theorem 5.8 ([16], [17, Theorem 7.2.4], [18]). *Consider the port-Hamiltonian operator A defined by (17) and (18), where we assume that P_0, P_1, W_B and $\mathcal{H}(\zeta)$ are as in Section 3. $\tilde{W}_B := W_B \left[\begin{smallmatrix} P_1 & -P_1 \\ I & I \end{smallmatrix}\right]^{-1}$ and $\Sigma := \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$. Here I denotes the $n \times n$-identity matrix. Then the following statements are equivalent:*

1. *A generates a contraction semigroup on $L^2((a,b);\mathbb{C}^n)$ with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$;*
2. $\operatorname{Re}\langle Ax, x\rangle_{\mathcal{H}} \le 0$ *for every $x \in D(A)$;*
3. *For $x \in D(A)$ there holds $(\mathcal{H}x)^*(b)P_1(\mathcal{H}x)(b) - (\mathcal{H}x)^*(a)P_1(\mathcal{H}x)(a) \le 0$;*
4. *The matrix W_B has rank n and $\tilde{W}_B\Sigma\tilde{W}_B^* \ge 0$.*

Moreover, the following statements are equivalent:

1. *A generates a unitary group on $L^2((a,b);\mathbb{C}^n)$ with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$;*
2. $\operatorname{Re}\langle Ax, x\rangle_{\mathcal{H}} = 0$ *for every $x \in D(A)$;*
3. *For $x \in D(A)$ there holds $(\mathcal{H}x)^*(b)P_1(\mathcal{H}x)(b) - (\mathcal{H}x)^*(a)P_1(\mathcal{H}x)(a) = 0$;*
4. *The matrix W_B has rank n and $\tilde{W}_B\Sigma\tilde{W}_B^* = 0$.*

We note, that whether A generates a contraction semigroup or unitary group is independent of the matrix-function $\mathcal{H}$. In Jacob et al.,[18] equivalent conditions for C_0-semigroup and C_0-group generation of port-Hamiltonian operator A defined by (17) and (18) are given. Again these are easy verifiable matrix conditions. The results are further generalized for infinitely many coupled port-Hamiltonian systems in Jacob and Kaiser.[19] Next we apply this theorem to our Example 2.1.

Example 5.9. We consider the lossless transmission line the spatial interval $[a,b]$ as discussed in Examples 2.1 and 3.1 with $u = 0$. For the lossless transmission line (1) we have $n = 2$,

$$P_1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \; P_0 = 0, \; \mathcal{H}(\zeta) = \begin{bmatrix} \frac{1}{C(\zeta)} & 0 \\ 0 & \frac{1}{L(\zeta)} \end{bmatrix}$$

and

$$W_B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & -R & 0 & 0 \end{bmatrix},$$

see Example 3.1 and note, that we have no control function, that is, $u = 0$. Clearly, the matrix W_B has rank 2. An easy calculation shows

$$\tilde{W}_B\Sigma\tilde{W}_B^* = \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} \ge 0.$$

Thus, by Theorem 5.8 the differential operator associated to the PDE (1) with boundary condition $V(a,t) = 0$, $V(b,t) = RI(b,t)$ generates a contraction semigroup on the energy space $L^2((a,b);\mathbb{C}^2)$. Furthermore, if $R = 0$, then this operator generates a unitary group on $L^2((a,b);\mathbb{C}^2)$.

6 | PORT-HAMILTONIAN SYSTEMS WITH DISTRIBUTED CONTROL AND OBSERVATION

So far, we have only considered systems without control, that is, $u(t) = 0$. In this section, we study port-Hamiltonian systems with distributed control and observation, that is, we consider systems of the form

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}(\zeta)x(\zeta,t)) + P_0(\mathcal{H}(\zeta)x(\zeta,t)) + b(\zeta)u(\zeta,t), \\ x(\zeta,0) &= x_0(\zeta), \qquad \zeta \in (0,1), \\ 0 &= W_B \left[\begin{smallmatrix} (\mathcal{H}(b)x(b,t) \\ (\mathcal{H}(a)x(a,t) \end{smallmatrix}\right], \qquad t > 0, \\ y(t) &= \int_a^b c(\zeta)x(\zeta,t)d\zeta, \qquad t > 0. \end{aligned} \tag{19}$$

Here we assume that $b \in L^\infty((a,b);\mathbb{C}^{n\times m})$, $c \in L^\infty((a,b);\mathbb{C}^{k\times n})$ and $u \in L^2((a,b)\times[0,\infty);\mathbb{C}^m)$. As the choice $u(t) = 0$ is possible, we further assume that the operator A given by

$$Ax = P_1 \frac{d}{d\zeta}(\mathcal{H}x) + P_0(\mathcal{H}x), \qquad x \in D(A),$$
$$D(A) = \left\{ x \in L^2((a,b);\mathbb{C}^n) \mid \mathcal{H}x \in H^1((a,b);\mathbb{C}^n),\ W_B \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix} = 0 \right\}$$

generates a C_0-semigroup $(T(t))_{t\geq 0}$ on $L^2((a,b);\mathbb{C}^n)$. With a slight abuse of notation, we write $u(\cdot,t) = (u(t))(\cdot)$ and we define the bounded operator $B : L^2((a,b);\mathbb{C}^m) \to L^2((a,b);\mathbb{C}^n)$ by $(Bu)(\zeta) := b(\zeta)u(\zeta)$. Furthermore, let $C : L^2((a,b);\mathbb{C}^n) \to \mathbb{C}^k$ be defined by $Cx := \int_a^b c(\zeta)x(\zeta)d\zeta$. Thus we "forget" the spatial dependence, and we write the port-Hamiltonian system in the form

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad t \geq 0, \qquad x(0) = x_0, \qquad y(t) = Cx(t).$$

The following theorem provides a formula for the solution of (19).

Theorem 6.1. *If the operator A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on $L^2((a,b);\mathbb{C}^n)$, then for $x_0 \in D(A)$ and $u \in C^1([0,\tau], L^2((a,b);\mathbb{C}^m))$*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)ds, \quad t \in [0,\tau], \tag{20}$$
$$y(t) = CT(t)x_0 + \int_0^t CT(t-s)Bu(s)ds, \quad t \in [0,\tau],$$

is the unique classical solution of (19) on $[0,\tau]$.

If $x_0 \in L^2((a,b);\mathbb{C}^n)$ and $u \in L^2((a,b)\times[0,\infty);\mathbb{C}^m)$, then we call (20) the *mild solution of* (19).

7 | PORT-HAMILTONIAN SYSTEMS WITH BOUNDARY CONTROL AND OBSERVATION

In this section we study the existence of solution of the port-Hamiltonian system (4).

Using the operator theoretic approach of Section 4, we can rewrite the port-Hamiltonian system (4) equivalently as

$$\dot{x}(t) = \mathfrak{A}x(t), \qquad u(t) = \mathfrak{B}x(t), \qquad y(t) = \mathfrak{C}x(t) \qquad t \geq 0, \qquad x(0) = x_0, \tag{21}$$

where $\mathfrak{A}$ is given by (10) and (11), $\mathfrak{B}$ by (13) and $\mathfrak{C}$ by (14). Now the question arises:

Does there exist for every initial condition $x_0 \in X$ and every control function $u \in L^2$ a solution of (21)*?*

This question is closely related to the notion of boundary control systems, see Curtain and Zwart,[11] Staffans,[20] and Tuscnak and Weiss,[21] which we define next.

Definition 7.1. Let X, U and Y be Hilbert spaces. Furthermore, assume that $\mathfrak{A} : D(\mathfrak{A}) \subset X \to X$, $\mathfrak{B} : D(\mathfrak{A}) \to U$ and $\mathfrak{C} : D(\mathfrak{A}) \to Y$ are linear operators. Then $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is a *boundary control and observation system* if the following hold:

1. The operator $A : D(A) \to X$ with $D(A) = D(\mathfrak{A}) \cap \ker(\mathfrak{B})$ and
$$Ax = \mathfrak{A}x \quad \text{for} \quad x \in D(A) \tag{22}$$
is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X;
2. There exists a right inverse $\tilde{B} \in \mathcal{L}(U,X)$ of $\mathfrak{B}$ in the sense that for all $u \in U$ we have $\tilde{B}u \in D(\mathfrak{A})$, $\mathfrak{A}\tilde{B} \in \mathcal{L}(U,X)$ and
$$\mathfrak{B}\tilde{B}u = u, \qquad u \in U. \tag{23}$$
3. The operator $\mathfrak{C}$ is bounded from $D(A)$ to Y, where $D(A)$ is equipped with the graph norm of A.

Theorem 7.2 ([17, Lemma 13.1.5])**.** *Assume that $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is a boundary control and observation system. Let $x_0 \in D(\mathfrak{A})$ and $u \in C^2([0,\tau];\mathbb{C}^m)$ satisfying $\mathfrak{B}x_0 = u(0)$. Then system (21) has a unique (classical) solution given by*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)\mathfrak{A}\tilde{B}u(s)ds - A\int_0^t T(t-s)\tilde{B}u(s)ds, \tag{24}$$
$$y(t) = \mathfrak{C}T(t)x_0 + \mathfrak{C}\int_0^t T(t-s)\mathfrak{A}\tilde{B}u(s)ds - \mathfrak{C}A\int_0^t T(t-s)\tilde{B}u(s)ds.$$

For port-Hamiltonian systems (4) it is easy to verify whether they are boundary control and observation systems.

Theorem 7.3. ([17, Theorem 11.3.2 and Theorem 11.3.5]) *Let $U = \mathbb{C}^m$, $Y = \mathbb{C}^k$, $X = L^2((a,b);\mathbb{C}^n)$, $\mathfrak{A}$ be given by (10) and (11), $\mathfrak{B}$ by (13) and $\mathfrak{C}$ by (14). If the operator*

$$Ax = P_1 \frac{d}{d\zeta}(\mathcal{H}x) + P_0(\mathcal{H}x), \qquad x \in D(A), \tag{25}$$

$$D(A) = \{x \in L^2((a,b);\mathbb{C}^n) \mid \mathcal{H}x \in H^1((a,b);\mathbb{C}^n), \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix} = 0\}.$$

generates a C_0-semigroup $(T(t))_{t\geq 0}$ on $L^2((a,b);\mathbb{C}^n)$, then $(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is a boundary control and observation system, that is, the port-Hamiltonian system (4) is a boundary control and observation system.

Furthermore, the operator $\tilde{B} : \mathbb{C}^m \to L^2((a,b);\mathbb{C}^n)$ can be defined as follows

$$(\tilde{B}u)(\zeta) := (\mathcal{H}(\zeta))^{-1}\left(S_{11}\frac{\zeta - a}{b-a} + S_{21}\frac{b-\zeta}{b-a}\right)u,$$

where S_{11} and S_{21} are $n \times m$-matrices given by

$$S := \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} := \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}^{-1} W_B^*(W_B W_B^*)^{-1} \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}$$

Note that $W_B = \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix}$ has full rank.

So far, we have only considered classical solutions of boundary control and observation system. Existence of (mild) solutions for any initial condition $x_0 \in X$ and any input $u \in L^2([0,\tau];U)$, such that x is a continuous X-valued function and $y \in L^2([0,\tau];Y)$ is called *well-posedness*.

Definition 7.4. We call a boundary control and observation system $(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ *well-posed* if there exist a $\tau > 0$ and $m_\tau \geq 0$ such that for all $x_0 \in D(\mathfrak{A})$ and $u \in C^2([0,\tau];U)$ with $u(0) = \mathfrak{B}x_0$ we have

$$\|x(\tau)\|_X^2 + \int_0^\tau \|y(t)\|^2 dt \leq m_\tau \left(\|x_0\|_X^2 + \int_0^\tau \|u(t)\|^2 dt\right). \tag{26}$$

There exists a rich literature on well-posed systems, see, for example, the monographs Staffans[20] and Tuscnak and Weiss.[21] In general it is not easy to show that a boundary control system is well-posed. However, there is a special class of systems for which well-posedness can be proved easily.

Proposition 7.5 ([17, Proposition 13.1.4])**.** *If every classical solution of a boundary control and observation system satisfies*

$$\frac{d}{dt}\|x(t)\|^2 \leq \|u(t)\|^2 - \|y(t)\|^2, \tag{27}$$

then the system is well-posed.

Boundary control and observation systems satisfying (27) are called *scattering passive*.

We now return to the port-Hamiltonian system (4). For every $\zeta \in (a,b)$, the matrices P_1 and $\mathcal{H}(\zeta)$ are self-adjoint and thus there exists a diagonal matrix $\Delta(\zeta)$ and an invertible matrix $S(\zeta)$ such that

$$P_1\mathcal{H}(\zeta) = S^{-1}(\zeta)\Delta(\zeta)S(\zeta).$$

Definition 7.6. We say that the port-Hamiltonian system (4) satisfies the *Condition (C)* if the matrix-valued functions S and Δ can be chosen such that both are continuous differentiable on (a,b).

Theorem 7.7 ([22], [17, Theorem 13.2.2])**.** *Assume that the port-Hamiltonian system (4) satisfies Condition (C). Then the following statements are equivalent.*

1. *The operator A, as defined in (25), generates a C_0-semigroup on $L^2((a,b);\mathbb{C}^n)$,*
2. *The port-Hamiltonian system (4) is a well-posed boundary control and observation system.*

Theorem 5.8 implies the following two corollaries.

Corollary 7.8. *Assume that the port-Hamiltonian system (4) satisfies Condition (C). If the matrix W_B has rank n, and $\tilde{W}_B\Sigma\tilde{W}_B^* \geq 0$, where $\tilde{W}_B$ and Σ are defined as in Theorem 5.8, then the port-Hamiltonian system (4) is a well-posed boundary control and observation system.*

Definition 7.9. We call the port-Hamiltonian system (4) *impedance passive*, if $m = k = n$ and for every $x \in D(\mathfrak{A})$ we have

$$\mathrm{Re}\,\langle \mathfrak{A}x, x\rangle_{\mathcal{H}} \leq \mathrm{Re}\,\langle \mathfrak{B}x, \mathfrak{C}x\rangle_{\mathbb{C}^m}. \tag{28}$$

Furthermore, the port-Hamiltonian system (4) is *impedance energy preserving*, if $m = k = n$ and for every $x \in D(\mathfrak{A})$ we have

$$\mathrm{Re}\,\langle \mathfrak{A}x, x\rangle_{\mathcal{H}} = \mathrm{Re}\,\langle \mathfrak{B}x, \mathfrak{C}x\rangle_{\mathbb{C}^m}, \tag{29}$$

Theorem 5.8 implies that the operator A of an impedance passive port -Hamiltonian system (4) generates a contraction semigroup, which is even a unitary group if the system (4) is impedance energy preserving.

Corollary 7.10. *Assume that the port-Hamiltonian system (4) satisfies Condition (C) and is impedance passive. Then the port-Hamiltonian system (4) is a well-posed boundary control and observation system.*

Theorem 7.11 ([17, Lemma 13.1.5 and Theorem 13.1.7])**.** *Let* $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *be a well-posed boundary control and observation system. Then for every* $t > 0$ *there exists linear and bounded operators*

$$\begin{aligned} \Phi_t &: L^2((0,t); U) \to X, \\ \Psi_t &: X \to L^2((0;t), Y), \\ F_t &: L^2((0,t); U) \to L^2((0,t); Y). \end{aligned}$$

such that for every $x_0 \in D(\mathfrak{A})$ *and* $u \in C^2([0,t]; \mathbb{C}^m)$ *satisfying* $\mathfrak{B}x_0 = u(0)$ *we have*

$$\begin{aligned} \Phi_t u &= \int_0^t T(t-s)\mathfrak{A}\tilde{B}u(s)ds - A\int_0^t T(t-s)\tilde{B}u(s)ds, \\ (\Psi_t x_0)(\tau) &= \mathfrak{C}T(\tau)x_0, \qquad \tau \in [0,t], \\ (F_t u)(\tau) &= \mathfrak{C}\left(\int_0^\tau T(\tau-s)\mathfrak{A}\tilde{B}u(s)ds - A\int_0^\tau T(\tau-s)\tilde{B}u(s)ds\right), \ \tau \in [0,t], \end{aligned}$$

and therefore

$$x(t) = T(t)x_0 + \Phi_t u, \qquad y = \Psi_t x_0 + F_t u. \tag{30}$$

Definition 7.12. Let $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a well-posed boundary control and observation system. Then for every $x_0 \in X$ and $u \in L^2((0,\infty), U)$ the *mild solutions* of (21) are given by (30).

Summarizing, if the port-Hamiltonian system (4) satisfies Condition (C) and the operator A, as defined in (25), generates a C_0-semigroup on $L^2((a,b), \mathbb{C}^n)$, then the port-Hamiltonian system (4) is a well-posed boundary control and observation system and it possesses for every initial condition $x_0 \in X$ and every $u \in L^2((0,\infty), \mathbb{C}^m)$ a mild solution.

8 | STANDARD CONTROL OPERATOR FORMULATION

In this section we aim to rewrite the control part of our port-Hamiltonian system (4) in the standard control operator formulation

$$\dot{x}(t) = A_{-1}x(t) + Bu(t), \quad x(0) = x_0, \qquad t \geq 0, \tag{31}$$

for some operators A_{-1} and B. Due to the fact that we have a system with boundary control, we cannot expect B to be a bounded operator from the input space $\mathbb{C}^m$ to the state space $X = L^2((a,b), \mathbb{C}^n)$. Indeed B will map into a larger space. Throughout this section we assume that the port-Hamiltonian system (4) is a well-posed boundary control and observation system.

Let the operator A be given by (25). Using the operator theoretic approach to port-Hamiltonian system, properties of the operators A play an important role. Here we list some of these. The operator A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on $X = L^2((a,b), \mathbb{C}^n)$, and therefore A is a densely defined, closed operator. Moreover, A has a compact resolvent.[23] Thus the spectrum of A is contained in some closed left half plane of $\mathbb{C}$, it consists of eigenvalues with finite multiplicity only and the spectrum of A is a discrete set without any finite accumulation point.

In order to define the operator B we first need to derive the Hilbert space adjoint of A. In the remaining of this section we assume that A generates a contraction semigroup $(T(t))_{t\geq 0}$ on $X = L^2((a,b), \mathbb{C}^n)$. Then there exist an invertible $n \times n$-matrix S and a $n \times n$-matrix V with $VV^* \leq I$ such that[19]

$$\begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} = S\,[I + V \;\; I - V] \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}. \tag{32}$$

Theorem 8.1. ([23,24]) *The Hilbert space adjoint* $A^* : D(A^*) \subset X \to X$ *of* A *is given by*

$$\begin{aligned} A^*x &= -P_1 \frac{d}{d\zeta}(\mathcal{H}x) - P_0(\mathcal{H}x), \qquad x \in D(A^*), \\ D(A^*) &= \left\{ x \in X \mid \mathcal{H}x \in H^1((a,b); \mathbb{C}^n), \;\; [I + V^* \;\; I - V^*] \begin{bmatrix} -P_1 & P_1 \\ I & I \end{bmatrix} \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix} = 0 \right\}. \end{aligned}$$

Let X_{-1} is the completion of X with respect to the norm $\|x\|_{X_{-1}} = \|(\beta - A)^{-1}x\|_X$ for some β in the resolvent set $\rho(A)$ of A, that is,

$$X \subset X_{-1}$$

and X is continuously embedded and dense in X_{-1}. The semigroup $(T(t))_{t\geq 0}$ extends uniquely to a C_0-semigroup $(T_{-1}(t))_{t\geq 0}$ on X_{-1} whose generator A_{-1}, with domain equal to X, is an extension of A, see, for example, Engel and Nagel.[12] Moreover, we can identify X_{-1} with the dual space of $D(A^*)$ with respect to the pivot space X, see Tucsnak and Weiss,[21] that is $X_{-1} = D(A^*)'$.

For $x_0 \in D(\mathfrak{A})$ and $u \in C^2([0,\tau];\mathbb{C}^m)$ satisfying $\mathfrak{B}x_0 = u(0)$, we can rewrite the classical solution (24) as

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)\left(\mathfrak{A}\tilde{B} - A_{-1}\tilde{B}\right)u(s)ds. \tag{33}$$

This equation is well-defined for every $x_0 \in X$, every $u \in L^2((0,\tau);\mathbb{C}^m)$ and $0 < t < \tau$, and satisfies $x \in C([0,\tau];X)$ for every $\tau > 0$. Thus the operator Φ_t of Theorem 7.11 is given by

$$\Phi_t u = \int_0^t T_{-1}(t-s)\left(\mathfrak{A}\tilde{B} - A_{-1}\tilde{B}\right)u(s)ds.$$

We recall, that if $x_0 \in X$ and $u \in L^2([0,\tau];\mathbb{C}^m)$ the function x given by (33) is called the *mild solution* of our port-Hamiltonian system (4). This function can also be interpreted as the mild solution of the abstract differential equation

$$\dot{x}(t) = A_{-1}x(t) + \left(\mathfrak{A}\tilde{B} - A_{-1}\tilde{B}\right)u(t), \quad x(0) = x_0, \qquad t \geq 0,$$

that is, the operator $B : \mathbb{C}^m \to X_{-1}$ in (31) is given by

$$B = \left(\mathfrak{A}\tilde{B} - A_{-1}\tilde{B}\right).$$

Furthermore, the Hilbert space adjoints of B are often needed. As an application we mention the duality of the notions controllability and observability.[21] Therefore, we conclude the section with the calculation of the Hilbert space adjoint of B.

Theorem 8.2. ([23]) *The Hilbert space adjoint $B^* : D(A^*) \to \mathbb{C}^m$ of $B : \mathbb{C}^m \to X_{-1}$ is given by*

$$B^*x = \frac{1}{4}S^{-*}(I+VV^*)^{-1}\left[I - V \;\; I + V\right]\begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}\begin{bmatrix} \mathcal{H}(b)x(b) \\ \mathcal{H}(a)x(a) \end{bmatrix}, \quad x \in D(A^*),$$

where V is given by (32) and $D(A^*)$ by Theorem 8.1.

9 | STABILITY

This chapter is devoted to stability of linear (homogeneous) port-Hamiltonian systems (15) ($u = 0$), that is, we study the question whether the solution of (15) tends to zero as time tends to infinity. For infinite-dimensional systems there are different notions of stability such as strong stability, polynomial stability, and exponential stability. In this chapter, we restrict ourselves to exponential stability and again we use an operator theoretic approach. Therefore, we start with the definition of exponential stability for abstract differential equations of the form $\dot{x}(t) = Ax(t)$, where A generates a C_0-semigroup $(T(t))_{t\geq 0}$.

Definition 9.1. A C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X is *exponentially stable* if there exist positive constants M and ω such that

$$\|T(t)\| \leq Me^{-\omega t} \qquad \text{for } t \geq 0.$$

The constant ω is called the *decay rate*.

If $(T(t))_{t\geq 0}$ is exponentially stable, then the solution of the abstract Cauchy problem

$$\dot{x}(t) = Ax(t),\ t \geq 0, \qquad x(0) = x_0,$$

tends to zero exponentially fast as $t \to \infty$.

Theorem 9.2 ([17, Theorem 8.1.3])**.** *Suppose that A is the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X. Then the following are equivalent*

1. *$(T(t))_{t\geq 0}$ is exponentially stable;*
2. *There exists a positive operator $P \in \mathcal{L}(X)$ such that*

$$\langle Ax, Px\rangle + \langle Px, Ax\rangle \leq -\langle x, x\rangle \qquad \textit{for all } x \in D(A). \tag{34}$$

Equation (34) is called a *Lyapunov inequality*. In Zwart,[25] it is shown that the condition

$$\operatorname{Re}\langle Ax,x\rangle < 0, \quad x\in D(A), x\neq 0, \tag{35}$$

is in general not sufficient for exponential stability of the semigroup generated by A. However, for port-Hamiltonian systems a weaker but more structured condition than (35) implies even exponential stability.

Theorem 9.3 ([26], [17, Theorem 9.1.3])**.** *Let A be defined by (17) and (18). If for some positive constant k one of the following conditions is satisfied for all $x\in D(A)$*

$$\operatorname{Re}\langle Ax,x\rangle_{\mathcal{H}} \le -k\|\mathcal{H}(b)x(b)\|^2,$$
$$\operatorname{Re}\langle Ax,x\rangle_{\mathcal{H}} \le -k\|\mathcal{H}(a)x(a)\|^2,$$

then A generates an exponentially stable C_0-semigroup.

Example 9.4. We consider the lossless transmission line on the spatial interval $[a,b]$ as discussed in Examples 2.1, 3.1 and 5.9 with $u(t)=0$ and $R>0$. Let $x\in D(A)$. Using (12) and (7) we get

$$\begin{aligned}\operatorname{Re}\langle Ax,x\rangle_{\mathcal{H}} &= \frac{1}{2}\left[(\mathcal{H}x)^*(\zeta)P_1(\mathcal{H}x)(\zeta)\right]_a^b\\ &= \frac{x_2}{L}(a)\frac{x_1}{C}(a) - \frac{x_2}{L}(b)\frac{x_1}{C}(b).\end{aligned}$$

As $x\in D(A)$, which in particular implies $\frac{x_1}{C}(a)=0$ and $\frac{x_1}{C}(b)=R\frac{x_2}{L}(b)$, we obtain

$$\operatorname{Re}\langle Ax,x\rangle_{\mathcal{H}} = -R\left(\frac{x_2}{L}(b)\right)^2 = -\frac{R}{1+R}\|\mathcal{H}(b)x(b)\|^2.$$

Thus, if $R>0$, then by Theorem 9.3 the differential operator associated to the PDE (1) with boundary condition $\frac{Q}{C}(a,t)=0$, $\frac{Q}{C}(b,t)=RI(b,t)$ generates an exponentially stable semigroup on the energy space $L^2((a,b),\mathbb{C}^2)$.

10 | SYSTEM THEORETIC PROPERTIES

We have introduced an operator theoretic approach to linear port-Hamiltonian systems. This approach has been successfully used to prove system theoretic properties, to develop controllers and to couple port-Hamiltonian systems with finite-dimensional linear and nonlinear systems. In this section we list some of these results.

10.1 | Stabilizability by output feedback

There are several different possibilities to stabilize a port-Hamiltonian system by static linear output feedback, dynamic linear output feedback or nonlinear static or dynamic feedback. Linear static and dynamic output feedback has been studied by Villegas.[24] In particular, he investigated the well-posedness and stability of feedback connections. Augner[23] extended these results for nonlinear static and dynamic output feedback. Related results for nonlinear dynamic output feedback were obtained in Ramirez et al.[27] Here we only mention the following results for impedance passive port-Hamiltonian systems.

If the port-Hamiltonian system (4) is impedance passive, see (28), then the corresponding operator A generates a contraction semigroup. Thus, if additionally Condition (C) is satisfied, it is a well-posed boundary control and observation system. Moreover, the port-Hamiltonian system (4) is impedance passive if and only if

$$\begin{bmatrix}\tilde W_B\Sigma\tilde W_B^* & \tilde W_B\Sigma\tilde W_C^*\\ \tilde W_C\Sigma\tilde W_B^* & \tilde W_C\Sigma\tilde W_C^*\end{bmatrix} \le \Sigma,$$

see [23, Proposition 3.2.16]. Here $\tilde W_B := W_B\begin{bmatrix}P_1 & -P_1\\ I & I\end{bmatrix}^{-1}$, $\tilde W_C := W_C\begin{bmatrix}P_1 & -P_1\\ I & I\end{bmatrix}^{-1}$ and $\Sigma := \begin{bmatrix}0 & I\\ I & 0\end{bmatrix}$.

Theorem 10.1. ([28]) *Any impedance passive port-Hamiltonian system (4) can be exponentially stabilized by negative output feedback $u(t)=-ky(t)$, for any $k>0$, that is, the operator $A : D(A)\subseteq L^2((a,b);\mathbb{C}^n)\to L^2((a,b);\mathbb{C}^n)$ given by*

$$Ax = P_1\frac{d}{d\zeta}(\mathcal{H}x) + P_0(\mathcal{H}x), \qquad x\in D(A),$$
$$D(A) = \left\{x\in L^2((a,b);\mathbb{C}^n) \mid \mathcal{H}x\in H^1((a,b);\mathbb{C}^n),\ [W_B+kW_C]\begin{bmatrix}(\mathcal{H}x)(b)\\(\mathcal{H}x)(a)\end{bmatrix}=0\right\}$$

generates an exponentially stable C_0-semigroup.

In Humaloja and Paunonen,[28] robust regulation of port-Hamiltonian systems is studied.

10.2 | Observability

Observability is a measure of how well internal states of a system can be inferred from knowledge of its external outputs. Throughout this subsection we assume that the port-Hamiltonian system (4) is a well-posed boundary control and observation system.

Definition 10.2. The system (4) is called *exactly observable*, if there exists a time $\tau > 0$ and a constant $c \geq 0$ such that for every $x_0 \in X$ the corresponding mild solution x (with $u = 0$) satisfies

$$\|x_0\|^2 \leq c \int_0^\tau \left\| W_C \begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{bmatrix} \right\|^2 dt.$$

Moreover, the port-Hamiltonian system (4) is called *finally observable*, if there exists a time $\tau > 0$ and a constant $c \geq 0$ such that for every $x_0 \in X$ the corresponding mild solution x (with $u = 0$) satisfies

$$\|x(\tau)\|^2 \leq c \int_0^\tau \left\| W_C \begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{bmatrix} \right\|^2 dt.$$

Lemma 9.1.2 in Jacob and Zwart[17] implies a sufficient condition for final observability.

Theorem 10.3. *Assume that the operator A corresponding to the port-Hamiltonian system (4) generates a contraction semigroup, $\mathcal{H}$ is Lipschitz continuous and that $n = k$. If $W_C = [W\, 0]$ or $W_C = [0\, W]$ with W invertible, then the port-Hamiltonian system (4) is finally observable.*

If additionally, A generates a C_0-group, then the port-Hamiltonian system (4) is exactly observable.

10.3 | Controllability

Exact controllability denotes the ability to move a system from any initial state to every given state in some time τ. Furthermore, a system is null controllable if the system can be steered from every initial condition to the zero state. Throughout this subsection we assume that the port-Hamiltonian system (4) is a well-posed boundary control and observation system.

Definition 10.4. The system (4) is called *exactly controllable*, if there exists a time $\tau > 0$ such that for every $x_1 \in X$ there is a control function $u \in L^2((0,\tau);\mathbb{C}^m)$ such that the corresponding mild solution x (with $x_0 = 0$) satisfies

$$x(0) = 0, \quad x(\tau) = x_1.$$

Moreover, the port-Hamiltonian system (4) is called *null controllable*, if there exists a time $\tau > 0$ such that for every $x_0 \in X$ there is a control function $u \in L^2((0,\tau);\mathbb{C}^m)$ such that the corresponding mild solution x satisfies

$$x(0) = x_0, \quad x(\tau) = 0.$$

For linear systems controllability and observability are dual notion.[21] Therefore, Theorems 10.3 and 8.1 imply the following theorem.

Theorem 10.5. *Assume that the operator A corresponding to the port-Hamiltonian system (4) generates a contraction semigroup, $\mathcal{H}$ is Lipschitz continuous and that $n = m$. If $W_B = [W\ 0]$ or $W_B = [0\ W]$ with W invertible, then the port-Hamiltonian system (4) is null controllable.*

If additionally, A generates a C_0-group, then the port-Hamiltonian system (4) is exactly controllable.

Exact controllability is closely related to the notion of stabilizability. We remark, that the following results even holds for more general control systems.

Theorem 10.6 ([29, Corollary 2.2])**.** *Assume that the operator A corresponding to the port-Hamiltonian system (4) generates a C_0-group $(T(t))_{t\in\mathbb{R}}$ satisfying $\|T(t)x\| \geq c\|x\|$ for some $c > 0$ and every $x \in X$ and $t \geq 0$. Then the following statements are equivalent*

1. *The port-Hamiltonian system (4) is exactly controllable,*
2. *For every $x_0 \in X$ there exists a control function $u \in L^2([0,\infty);\mathbb{C}^m)$ such that the corresponding mild solution x satisfies $x \in L^2([0,\infty), L^2((a,b);\mathbb{C}^n))$.*

Note that by Theorem 5.8 the operator A of an impedance energy preserving port-Hamiltonian system (4) generates a unitary group. Thus Theorem 10.1 implies the following corollary.

Corollary 10.7. *Assume that the port-Hamiltonian system (4) is impedance energy preserving, then the port-Hamiltonian system (4) is exactly controllable.*

10.4 | Input-to-state stability

The concept of *input-to-state stability*, introduced by E. Sontag in 1989,[30] is a well-studied stability notion of control systems with respect to external inputs. For well-posed boundary control and observation systems input-to-state stability is equivalent to exponential stability of the corresponding semigroup.[31] Here we give an input-to state stability estimate for port-Hamiltonian systems.

Theorem 10.8. ([31]) *Assume that the port-Hamiltonian system (4) is a well-posed boundary control and observation system and that the corresponding operator A is exponentially stable. Then there exists constants $M \geq 1$, $\omega > 0$ and $c > 0$ such that for every initial condition $x_0 \in X$ and every input function $u \in L^\infty((0,\infty);\mathbb{C}^m)$ the corresponding mild solution x satisfies for every $t \geq 0$*

$$\|x(t)\|_X \leq Me^{-\omega t}\|x_0\|_X + c\left(\int_0^t \|u(s)\|^2_{\mathbb{C}^m} ds\right)^2,$$

$$\|x(t)\|_X \leq Me^{-\omega t}\|x_0\|_X + c \operatorname*{ess\,sup}_{s\in[0,t]} \|u(s)\|_{\mathbb{C}^m}.$$

11 | CONCLUSIONS

In this article we presented an operator theoretic approach to infinite-dimensional systems theory. However, we only consider port-Hamiltonian systems, where the partial differential operator $\mathfrak{A}$ is given by $\mathfrak{A}x = P_1 \frac{d}{d\zeta}[\mathcal{H}x] + P_0[\mathcal{H}x]$. In order to model examples like the Euler-Bernoulli beam, Schrödinger equation or Airy's equation, more general port-Hamiltonian systems of the form

$$\frac{\partial x}{\partial t}(\zeta,t) = \sum_{j=0}^{N} P_j \frac{\partial^j}{\partial \zeta^j}[\mathcal{H}(\zeta)x(\zeta,t)]$$

need to be investigated. Several results extend to this more general class of port-Hamiltonian systems. The contraction semigroup generation results have been shown by Le Gorrec, Zwart and Maschke,[16] see also Jacob and Kaiser.[19] Furthermore, stability has been investigated in Augner[23] and Augner and Jacob.[32] It is known that a general result like presented in Theorem 7.7 does not hold for port-Hamiltonian systems of higher order. However, which of these systems are well-posed is still an open question.

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