Supplementary Information

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1. Parameters for longitudinal waves

In bone, with displacement $\mathbf{u}$ as unknown, the wave equation has the form:

$$\rho \ddot{\mathbf{u}} = \left(\lambda + \mu + (\xi + \eta/3) \frac{\partial}{\partial t}\right) \nabla (\nabla \cdot \mathbf{u}) + \left(\mu + \eta \frac{\partial}{\partial t}\right) \nabla \mathbf{u},$$

(1)

where $\lambda$ and $\mu$ are the Lamé coefficients, $\rho$ is the material density and $\eta$ and $\xi$ are the first and second viscosities. A possible longitudinal plane wave solution for the wave equation (1) is given by:

$$\mathbf{u}(x, y, z, t) = A \hat{\mathbf{k}} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} e^{-\alpha_L \mathbf{k} \cdot \mathbf{r}},$$

(2)

where $\mathbf{k}$ is the wave vector for longitudinal waves, $k = ||\mathbf{k}||$, $\hat{\mathbf{k}} = \mathbf{k}/k$, $\omega$ is the angular frequency, $\mathbf{r} = (x, y, z)$, $A$ is the possible complex amplitude and $\alpha_L$ is the longitudinal waves attenuation coefficient.

Using expression (2), $\nabla \cdot \mathbf{u}$, $\nabla (\nabla \cdot \mathbf{u})$ and $\nabla \mathbf{u}$ can be written as:

$$\nabla \cdot \mathbf{u} = A (ik - \alpha_L) e^{i(k \cdot r - \omega t)} e^{-\alpha_L \mathbf{k} \cdot \mathbf{r}},$$

$$\nabla (\nabla \cdot \mathbf{u}) = (ik - \alpha_L)^2 \mathbf{u},$$

$$\nabla \mathbf{u} = (ik - \alpha_L)^2 \mathbf{u}.$$  

(3)

Substitution of the equations (3) in formula (1) leads to:

$$\rho (-i\omega)^2 = (\lambda + 2\mu)(ik - \alpha_L)^2 + (\xi + 4\eta/3)(-i\omega)(ik - \alpha_L)^2.$$  

(4)

For given material properties $\rho$, $\lambda$, $\mu$, $\xi$, $\eta$ and given angular frequency $\omega$, the attenuation coefficient $\alpha_L$ and the wave number $k$ can be found from equation (4).

In fact, formula (4) can be written as follows:

$$\alpha_L^2 - k^2 - 2ik\alpha_L = \frac{-\omega^2 \rho (\lambda + 2\mu + i\omega (\xi + 4\eta/3))}{(\lambda + 2\mu)^2 + \omega^2 (\xi + 4\eta/3)^2}. $$

(5)

Then, introducing $C$ and $D$ as:

$$C = \sqrt{(\lambda + 2\mu)^2 + \omega^2 (\xi + 4\eta/3)^2},$$

$$D = \sqrt{\lambda + 2\mu + C},$$
the following expressions for $\alpha_L$ and $k$ finally result by solving equation (5):

\[ k = \frac{\omega \sqrt{\rho D}}{\sqrt{2C}}, \]
\[ \alpha_L = \frac{\omega^2 \sqrt{\rho (\zeta + 4\eta/3)}}{\sqrt{2CD}}. \]

(6)

The speed of sound for longitudinal waves can be found from:

\[ c_L = \frac{\omega}{k} = \frac{\sqrt{2C}}{\sqrt{\rho D}}. \]

Hence, longitudinal plane waves of the form (2) exist for each $\omega$ if $k$ and $\alpha_L$ are given by the equation (6).

2. Parameters for shear waves

The shear plane wave solution for equation (1) is given by:

\[ u(x, y, z, t) = B a e^{i(k \cdot r - \omega t)} e^{-\alpha_S k \cdot r}, \]

(7)

where $k$ is the wave vector for shear waves, $k = ||k||$, $\tilde{k} = k/k$, $\alpha_S$ is the shear waves attenuation coefficient, $B$ is the possible complex amplitude and $a$ is the polarization direction vector.

Again, $\nabla \cdot u$, $\nabla (\nabla \cdot u)$ and $\nabla^2 u$ can be written as:

\[ \nabla \cdot u = 0, \]
\[ \nabla (\nabla \cdot u) = 0, \]
\[ \nabla^2 u = (ik - \alpha_S)^2 u, \]

(8)

due to the fact that $a$ and $k$ are perpendicular. Substitution of equations (8) in formula (1) leads to:

\[ \rho (-i\omega)^2 = (\mu - i\omega \eta)(ik - \alpha_S)^2, \]

which can be written as:

\[ k^2 - \alpha_S^2 + 2ik\alpha_S = \frac{\rho \omega^2 \mu + i\rho \omega^3 \eta}{M^2}, \]

(9)

where $M = \sqrt{\mu^2 + \omega^2 \eta^2}$. The imaginary part of equation (9) gives:

\[ \alpha_S = \frac{\rho \omega^3 \eta}{2kM^2}. \]

(10)

Substitution of formula (10) in the real part of equation (9) leads to:

\[ k^4 - \frac{\rho \omega^2 \mu}{M^2} k^2 - \frac{\rho^2 \omega^6 \eta^2}{4M^4} = 0. \]

Since $M > \mu$, the only positive solution allowed is:

\[ k^2 = \frac{\rho \omega^2}{2M^2} (\mu + M). \]
Defining \( N = \sqrt{\mu + M} \), the resultant expressions for \( k \) and \( \alpha_S \):
\[
\begin{align*}
    k &= \frac{\rho \omega N}{\sqrt{2M}}, \\
    \alpha_S &= \frac{\rho \omega^2 \eta}{\sqrt{2MN}}.
\end{align*}
\] (11)

Finally, the speed of sound for longitudinal waves can be found from:
\[
    c_S = \frac{\omega}{k} = \frac{\sqrt{2M}}{\sqrt{\rho N}}.
\]

Hence, shear plane waves of the form (2) exist for each \( \omega \) if \( k \) and \( \alpha_S \) are given by the equation (11).

3. Derivation of the total power loss

The total acoustic energy \( E(t) \), in a volume \( V \), due to the ultrasound waves propagation in a solid lossy material, is given by:
\[
    E(t) = \int_V \mathcal{E}(x,y,z,t) \, dV,
\] (12)

where \( \mathcal{E}(x,y,z,t) \) is the energy density on position \((x,y,z)\) at time \( t \). The energy density \( \mathcal{E} \) is defined by:
\[
    \mathcal{E} = \frac{1}{2} \rho \dot{u}^2 + \mathcal{W}_p(u),
\] (13)

where \( u(x,y,z,t) \) is the displacement field, with components \( u_1, u_2, u_3 \) and \( \mathcal{W}_p \) is the potential energy density. The potential energy density in equation (13) can be written as:
\[
    \mathcal{W}_p(u) = \frac{1}{2} \sum_{l,j,m,n} c_{l j m n} u_l u_j u_m u_n,
\] (14)

where the summation is from 1 to 3 for all the indices and the term \( c_{l j m n} \) represents the components of the elasticity tensor. Substituting in equation (14) the values of \( c_{l j m n} \) for an homogenous, isotropic material (Beltzer 1988) leads to:
\[
    \mathcal{W}_p(u) = \frac{1}{2} (\lambda + 2\mu)(u_{1,1}^2 + u_{2,2}^2 + u_{3,3}^2) + \lambda (u_{1,1} u_{2,2} + u_{1,1} u_{3,3} + u_{2,2} u_{3,3}) + \frac{1}{2} \mu (u_{1,1}^2 + u_{3,3}^2 + u_{1,3}^2 + u_{2,3}^2 + u_{3,1}^2 + u_{2,1}^2).
\] (15)

The time derivative of the total acoustic energy in a volume \( V \), due to the ultrasound waves propagation in a lossy material, is related to the total power loss according to:
\[
    E = - \int_V \dot{Q} \, dV - \int_{\partial V} \mathbf{F} \cdot \mathbf{n} \, dS,
\] (16)

where the first term represents the total power loss \( Q \) integrated over the volume \( V \) and the second term represents the energy flux outside the volume, with \( \mathbf{F} \) the acoustic Poynting vector, \( \mathbf{n} \) the outward pointing normal on the surface \( S \) and \( dS \) the surface element on \( \partial V \).
Substituting equation (13) in formula (12), the time derivative of the total energy in the
volume $V$ becomes:

$$
\dot{E} = \int_V \left( \rho \dot{\mathbf{u}} \cdot \mathbf{u} + \sum_{l,j=1}^{3} \frac{\partial W_p}{\partial u_{l,j}} \dot{u}_{l,j} \right) dV ,
$$  \hspace{1cm} (17)

Substituting equation (1) in equation (17) leads to:

$$
\dot{E} = \int_V \left( \nabla \cdot (\nabla \cdot \mathbf{u}) + \mu \triangle \mathbf{u} \right) + \sum_{l,j=1}^{3} \frac{\partial W_p}{\partial u_{l,j}} \dot{u}_{l,j} dV + \int_V \left( (\xi + \eta/3) \nabla \cdot (\nabla \cdot \dot{\mathbf{u}}) + \eta \triangle \dot{\mathbf{u}} \right) dV ,
$$  \hspace{1cm} (18)

The terms with factor $\lambda$ in the first integral of equation (18) can be written as:

$$
\lambda \dot{\mathbf{u}} \cdot (\nabla \cdot \mathbf{u}) = \lambda \nabla \cdot ((\nabla \cdot \mathbf{u}) \dot{\mathbf{u}}) - \lambda (\nabla \cdot \mathbf{u})(\nabla \cdot \dot{\mathbf{u}}) ,
$$  \hspace{1cm} (19)

while the ones with factor $\mu$ becomes after some rewriting:

$$
\mu \sum_{j} \left( \sum_{l} (\dot{u}_{l,j} u_{j,l} + \dot{u}_{j,l} u_{l,j}) \right) .
$$  \hspace{1cm} (20)

Then, from equations (19) and (20), and using the divergence theorem, the first term in
formula (18) becomes:

$$
\int_{\partial V} \left( \lambda (\nabla \cdot \mathbf{u}) \dot{\mathbf{u}} + \mu \mathbf{G} \right) \cdot \mathbf{n} dS ,
$$  \hspace{1cm} (21)

where $\mathbf{G}$ is a vector with components

$$
G_j = \sum_{l} (\dot{u}_{l,j} u_{j,l} + \dot{u}_{j,l} u_{l,j}) .
$$

Formula (21) describes an energy flux, so the first integral does not contribute to the total
power loss.

Consider now the second integral in equation (18), which contains the attenuation terms, and
is responsible for the heat deposition. The term with coefficient $\xi + \eta/3$ can be written as:

$$
(\xi + \eta/3) \dot{\mathbf{u}} \cdot (\nabla \cdot \dot{\mathbf{u}}) = (\xi + \eta/3) \nabla \cdot ((\nabla \cdot \dot{\mathbf{u}}) \dot{\mathbf{u}}) - (\xi + \eta/3)(\nabla \cdot \dot{\mathbf{u}})^2 .
$$  \hspace{1cm} (22)

The first term in the right-hand side of equation (22) is a divergence, and contributes to the
energy flux, while the second term contributes to the total power loss.

Then, to rewrite the second integral in equation (18) with coefficient $\eta$, the vector field $\mathbf{D}$ is
introduced, with components:

$$
D_l = \eta \sum_{j} (\dot{u}_{l,j} \dot{u}_{j,l} - \dot{u}_{j,l} \dot{u}_{l,j}) .
$$

Then

$$
\eta \dot{\mathbf{u}} \cdot (\triangle \dot{\mathbf{u}}) = \eta \sum_{l,j} (\dot{u}_{j,l} \dot{u}_{l,j}) - \nabla \cdot \left( \frac{1}{2} \eta \nabla (\dot{\mathbf{u}}^2) + \mathbf{D} \right) - \eta \sum_{l,j} (\dot{u}_{l,j}^2 + \dot{u}_{j,l} \dot{u}_{l,j} - \dot{u}_{l,j} \dot{u}_{l,j}) .
$$  \hspace{1cm} (23)
The first part of the last formula is a divergence, and it becomes part of the flux $F$, while the second term contributes to the power loss $Q$.

Hence, from equations (22) and (23), the total power loss becomes:

$$Q = (\xi + \eta/3)(\sum_l \ddot{u}_{l,l})^2 + \eta \sum_{l,j} \dddot{u}_{l,j} \dddot{u}_{j,l}.$$  \hfill (24)

Finally, from equations (21), (22) and (23), the total energy flux (Poynting vector) is defined as:

$$F = -\lambda(\nabla \cdot \mathbf{u})\hat{k} - \mu \mathbf{G} - (\xi + \eta/3)(\nabla \cdot \mathbf{u})\hat{k} + \eta \nabla \cdot (\frac{1}{2} \ddot{u}^2) - \mathbf{D}.$$ \hfill (25)

The formulas (24) and (25) are consistent with the expression for $Q$ and $F$ given by Auld & Auld (1973) and Beltzer (1988) for the general case of a general elasticity tensor.

4. Time averaged flux for a longitudinal wave in solids

Equation (2) can be written as:

$$\mathbf{u} = \text{Re}(U)\hat{k},$$  \hfill (26)

where now $\mathbf{u}$ is the real-valued displacement and $U = A e^{i(k \cdot r - \omega t)} e^{-\alpha_L \hat{k} \cdot r}$. The spatial and temporal derivatives of the components of $\mathbf{u}$ in equation (26) are:

$$\ddot{u}_{l,j} = \ddot{k}_l \ddot{k}_j \text{Re}(-i\omega(ik - \alpha_L)U),$$

which yields:

$$\nabla \cdot \mathbf{u} = \omega \sqrt{k^2 + \alpha_L^2} \text{Re}(U_2),$$  \hfill (27)

where $U_2 = e^{i\beta} U$ with $\cos(\beta) = k/\sqrt{k^2 + \alpha_L^2}$. For a longitudinal wave, taking expression (27) into account, the first term in equation (25) can be written as:

$$-\lambda(\nabla \cdot \mathbf{u})\hat{k} = -\lambda \text{Re}((ik - \alpha_L)U) \omega \text{Re}(-iU)\hat{k}$$

$$= \lambda \omega \text{Re}((ik - \alpha_L)U) \text{Re}(iU)\hat{k}$$

$$= \lambda \omega \sqrt{k^2 + \alpha_L^2} \text{Re}(e^{i\phi} U) \text{Re}(e^{i\pi/2} U)\hat{k},$$

where $ik - \alpha_L = \sqrt{k^2 + \alpha_L^2} e^{i\phi}$, so $\phi = \pi/2 + \beta$ with $\cos(\beta) = k/\sqrt{k^2 + \alpha_L^2}$. To compute the time average of this part flux, note that:

$$\text{Re}(e^{i\phi} U) \text{Re}(e^{i\pi/2} U) = e^{-2\alpha_L \hat{k} \cdot r} \|A\|^2 \cos(\omega t + \phi_0 + \beta) \cos(\omega t + \phi_0),$$

where $\phi_0 = k \cdot r + \phi_A$. The common phase $\phi_0$ can be omitted for the integral over a time period. Since

$$\cos(\omega t + \beta) \cos(\omega t) = (\cos(\omega t) \cos(\beta) - \sin(\omega t) \sin(\beta)) \cos(\omega t)$$

and the average values of $\cos^2(\omega t)$ and $\sin(\omega t) \cos(\omega t)$ over a period are $\frac{1}{2}$ and 0 respectively, the time average becomes:

$$\langle -\lambda(\nabla \cdot \mathbf{u})\hat{k} \rangle = \frac{1}{2} \lambda \omega k e^{-2\alpha_L \hat{k} \cdot r} \|A\|^2 \hat{k}.$$
The second term of (25) is computed as:
\[ G_j = \sum_i (\dot{u}_i u_{j,i} + \ddot{u}_i u_{j,i}) \]
\[ = \omega \text{Re}(-iU) \text{Re}((i k - \alpha_L)U) \sum_l \tilde{k}_l (2\tilde{k}_j \tilde{k}_l) \]
\[ = -\omega \text{Re}(iU) \text{Re}((i k - \alpha_L)U) 2\tilde{k}_j \]
\[ = -2\omega \sqrt{k^2 + \alpha_L^2} \text{Re}(e^{i\phi}) \text{Re}(e^{i\pi/2}U) \tilde{k}_j , \]
where \( \phi \) is the same as above. For the time average this yields again
\[ \langle G_j \rangle = -2\omega \sqrt{k^2 + \alpha_L^2} e^{-2\alpha_L \cdot r} \|A\|^2 \frac{1}{2} \cos(\beta) \]
\[ = -\omega \text{Re}(-iU) \text{Re}((i k - \alpha_L)U) \tilde{k}_j . \]

The third term of (25) is similar to the first term, only now the divergence is from the time
derivative (velocity) \( \dot{u} \).
\[ - (\xi + \eta/3)(\nabla \cdot \dot{u}) \dot{u} = -(\xi + \eta/3) \omega^2 \text{Re}((-i\omega i k - \alpha_L)U) \text{Re}(-i\omega U) \tilde{k} \]
\[ = (\xi + \eta/3) \omega^2 \text{Re}((k + i\alpha_L)U) \text{Re}(iU) \tilde{k} \]
\[ = (\xi + \eta/3) \omega^2 \sqrt{k^2 + \alpha_L^2} \text{Re}(e^{i\beta} U) \text{Re}(e^{i\pi/2}U) \tilde{k} , \]
where, as before, \( \cos(\beta) = k/\sqrt{k^2 + \alpha_L^2} \). The time average of the product of the two real parts
depends again on their phase difference \( \gamma = \pi/2 - \beta \), so the time average of the third term is then:
\[ \langle -(\xi + \eta/3)(\nabla \cdot \dot{u}) \dot{u} \rangle = (\xi + \eta/3) \omega^2 \sqrt{k^2 + \alpha_L^2} e^{-2\alpha_L \cdot r} \|A\|^2 \frac{1}{2} \cos(\gamma) \tilde{k} \]
\[ = (\xi + \eta/3) \omega^2 \sqrt{k^2 + \alpha_L^2} e^{-2\alpha_L \cdot r} \|A\|^2 \frac{1}{2} \sin(\beta) \tilde{k} \]
\[ = \frac{1}{2} (\xi + \eta/3) \omega^2 \alpha_L e^{-2\alpha_L \cdot r} \|A\|^2 \tilde{k} . \]

The \( l \)-th component of the fourth term of formula (25) can be written as:
\[ - \eta (\frac{1}{2} \ddot{u}_l^2) = -\eta \sum_j \ddot{u}_j \dot{u}_{j,l} \]
\[ = -\eta \text{Re}(-i\omega U) \text{Re}(-i\omega (i k - \alpha_L)U) \sum_j \tilde{k}_j \tilde{k}_l \]
\[ = -\eta \text{Re}(-i\omega U) \text{Re}(-i\omega (i k - \alpha_L)U) \tilde{k}_j . \]
\[ = \eta \omega^2 \text{Re}(iU) \text{Re}((k + i\alpha_L)U) \tilde{k}_j . \]

Then, the time average becomes:
\[ \langle -\eta (\frac{1}{2} \ddot{u}_l^2) \rangle = \frac{1}{2} \eta \omega^2 \alpha_L e^{-2\alpha_L \cdot r} \|A\|^2 \tilde{k}_j . \]

Finally, the fifth term of (25) has components:
\[ D_l = \eta \sum_j (\dot{u}_i \dot{u}_{j,j} - \ddot{u}_j \dot{u}_{j,j}) \]
\[ = \sum_l \text{Re}((-i\omega i k - \alpha_L)U) \text{Re}(-i\omega U) (\tilde{k}_l \tilde{k}_j^2 - \tilde{k}_l \tilde{k}_j^2) \]
\[ = 0 . \]

Hence, also the time average over a period also vanishes.
Combining the results of the five parts, the time averaged energy flux for a pressure wave is
obtained as:
\[ \langle F_L \rangle = \frac{1}{2} e^{-2\alpha_L \cdot r} \|A\|^2 \omega ((\lambda + 2\mu)k + (\xi + 4\eta/3)\omega \alpha_L) \tilde{k} . \]  
(28)
Finally, the intensity for a longitudinal wave is defined as the length of the Poynting vector expressed in formula (28). Hence, the time averaged intensity of a longitudinal wave in a solid is:

\[ I_L = \frac{1}{2} \omega A^2 ((\lambda + 2\mu)k + (\xi + 4\eta/3)\omega\alpha_L). \]

### 5. Time averaged flux for a shear wave in solids

Equation (7) can be written as:

\[ \mathbf{u} = \text{Re}(V)\mathbf{a}, \]

where now \( u \) is the real-valued displacement and \( V = B e^{i(kr - \omega t)} e^{-\alpha_k r} \).

The terms \( \nabla \cdot \mathbf{u}, \nabla \cdot \dot{\mathbf{u}} \) can be written as:

\[ \nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \dot{\mathbf{u}} = 0, \]

since \( \mathbf{a} \) and \( \tilde{k} \) are perpendicular.

For a shear wave, taking expressions (30) into account, the first and the third term of equation (25) vanish.

The second term of formula (25) can be written as:

\[ G_j = \omega \text{Re}(-iV)\text{Re}((ik - \alpha_S) V) \sum_l a_l (a_l \tilde{k}_l + a_l \tilde{k}_j) \]

\[ = -\omega \text{Re}(iV)\text{Re}((ik - \alpha_S)V) \tilde{k}_j \]

\[ = -\omega \sqrt{k^2 + \alpha_S^2} \text{Re}(e^{i\pi/2}V)\text{Re}(e^{i(\pi/2 + \gamma)}V) \tilde{k}_j, \]

where again \( \cos(\gamma) = k/\sqrt{k^2 + \alpha_S^2} \). For the time average the phase difference between the two real parts leads again to a factor \( \frac{1}{2} \cos \gamma \):

\[ \langle G_j \rangle = -\omega \sqrt{k^2 + \alpha_S^2} e^{-2\alpha_S k} ||B||^2 \frac{1}{2} \cos(\gamma) \tilde{k}_j \]

\[ = -\frac{1}{2} \omega ke^{-2\alpha_S k} ||B||^2 \tilde{k}_j. \]

The \( l \)-th component of the fourth term of formula (25) can be written as:

\[-\eta(\frac{1}{2} \dot{u}_l^2)_l = -\eta \sum_j \dot{u}_j \dot{u}_{j,l} \]

\[ = -\eta \text{Re}(-i\omega V)\text{Re}(-i\omega (ik - \alpha_S)V) \sum_j a_j^2 \tilde{k}_l \]

\[ = -\eta \text{Re}(-i\omega V)\text{Re}(-i\omega (ik - \alpha_S)V) \tilde{k}_l \]

\[ = \eta \omega^2 \text{Re}(iV)\text{Re}((k + i\alpha_S)V) \tilde{k}_l \]

\[ = \eta \omega^2 \sqrt{k^2 + \alpha_S^2} \text{Re}(e^{i\pi/2}V)\text{Re}(e^{i\gamma V}) \tilde{k}_l, \]

where \( \cos(\gamma) = k/\sqrt{k^2 + \alpha_S^2} \). The phase difference between the two real parts is \( \pi/2 - \gamma \), and \( \cos(\pi/2 - \gamma) = \sin(\gamma) = \alpha_S/\sqrt{k^2 + \alpha_S^2} \). Hence, the time average leads to a factor \( \frac{1}{2} \alpha_S / \sqrt{k^2 + \alpha_S^2} \) and it can be written as:

\[ \langle -\eta(\frac{1}{2} \dot{u}_l^2)_l \rangle = \frac{1}{2} \eta \omega^2 \alpha_S e^{-2\alpha_S k} ||B||^2 \tilde{k}_l. \]
Finally, the fifth term of (25) becomes:
\[
D_l = \eta \sum_j (\dot{u}_{l,j} \dot{u}_j - \ddot{u}_{l,j}) \\
= \sum_j \text{Re}(-i \omega (ik - \alpha_S) V) \text{Re}(-i \omega V) (\alpha_l \tilde{k}_j \alpha_j - \alpha_j \tilde{k}_j) \\
= 0.
\]
Hence, also the time average over a period also vanishes.

Combining the results of the five parts, the time averaged energy flux for a shear wave is obtained as:
\[
\langle \tilde{P}_S \rangle = \frac{1}{2} e^{-2\alpha_S \tilde{k} \cdot r} \| B \|^2 (\mu \omega k + \eta \omega^2 \alpha_S) \tilde{k}.
\]
Finally, the intensity for a shear wave is defined as the length of the Poynting vector expressed in formula (31). Hence, the time averaged intensity of a longitudinal wave in a solid is:
\[
I_L = \frac{1}{2} B^2 (\mu \omega k + \eta \omega^2 \alpha_S).
\]

6. Time and spatial differentiation of the displacement for longitudinal waves

For longitudinal plane waves, the (complex) displacement in a point \( \mathbf{r}_0 \) is given by:
\[
u(\mathbf{r}_0) = \hat{k} A e^{i \phi_L}.
\]
The complex displacement around the point \( \mathbf{r}_0 \) is:
\[
u(\mathbf{r}) = \hat{k} A e^{i \phi_L + (ik - \alpha_L) \tilde{k} \cdot (r - r_0)}.
\]
Since the time behavior of \( \mathbf{u} \) is given by \( e^{-i \omega t} \), the complex velocity around the point \( \mathbf{r}_0 \) is:
\[
v(\mathbf{r}) = (-i \omega) \hat{k} A e^{i \phi_L + (ik - \alpha_L) \tilde{k} \cdot (r - r_0)}.
\]
The \( m \)-th component of the velocity is:
\[
v_m(\mathbf{r}) = (-i \omega) \tilde{k}_m A e^{i \phi_L + (ik - \alpha_L) \tilde{k} \cdot (r - r_0)}.
\]
Differentiation with respect to coordinate \( n \) in \( \mathbf{r}_0 \) leads to:
\[
v_{m,n}(\mathbf{r}_0) = (-i \omega) (ik - \alpha) \tilde{k}_m \tilde{k}_n A e^{i \phi_L}.
\]

7. Time and spatial differentiation of the displacement for shear waves

For plane shear waves, the (complex) displacement in a point \( \mathbf{r}_0 \) is given by:
\[
u(\mathbf{r}_0) = \mathbf{a} B e^{i \phi_S}.
\]
The complex displacement around the point \( \mathbf{r}_0 \) is:
\[
u(\mathbf{r}) = B e^{i \phi_S + (ik - \alpha_S) \tilde{k} \cdot (r - r_0)} \mathbf{a}.
\]
Since the time behaviour of \( \mathbf{u} \) is given by \( e^{-i \omega t} \), the complex velocity around the point \( \mathbf{r}_0 \) is:
\[
v(\mathbf{r}) = (-i \omega) \mathbf{a} B e^{i \phi_S + (ik - \alpha_S) \tilde{k} \cdot (r - r_0)}.
\]
The \( m \)-th component of the velocity is:

\[
v_m(r) = (-i\omega)a_m Be^{i\phi_S + (ik - \alpha_S)k \cdot (r - r_0)}.
\]

Differentiation with respect to coordinate \( n \) in \( r_0 \) leads to:

\[
v_{m,n}(r_0) = (-i\omega)(ik - \alpha_S)a_m \tilde{k}_n Be^{i\phi_S}.
\]

### 8. Loss factors and attenuation coefficients

The relation between the loss factor for shear modulus \( \eta_g \) and the shear wave attenuation \( \alpha_S \) is straightforward (Dyson et al. 1995):

\[
\eta_g = \frac{2\alpha_S c_S}{\omega},
\]

(32)

The loss factor \( \eta_g \) is defined as (Dyson et al. 1995):

\[
\eta_g = \frac{\mu''}{\mu'},
\]

(33)

where and \( \mu' \) and \( \mu'' \) are the real and the imaginary part of the shear modulus \( \mu \).

The relation between \( \eta_l \) and the longitudinal attenuation coefficient \( \alpha_L \) is given by (Dyson et al. 1995):

\[
\eta_l = \frac{2\alpha_L c_L}{\omega},
\]

(34)

where \( c_L \) is the speed of sound for longitudinal waves. Equation (34) holds only if \( \frac{\alpha_L c_L}{\omega} < 0.1 \).

According to the parameters in Table 1, \( \frac{\alpha_L c_L}{\omega} = 0.094 \), so the condition is satisfied. Moreover, the longitudinal loss factor \( \eta_l \) is defined as (Dyson et al. 1995):

\[
\eta_l = \frac{L''}{L'},
\]

(35)

where and \( L' \) and \( L'' \) are the real and the imaginary part of the constrained tensile modulus \( L \).

From formula (32), \( \eta_g = 0.148 \) and from formula (34), \( \eta_l = 0.188 \). Therefore, using expressions (33) and (35), the complex quantities \( \mu = \mu' + i\mu'' \) and \( L = L' + iL'' \) can be found. Then, the complex Young modulus \( E = E' + iE'' \) can be derived from:

\[
E = \frac{3\mu L - 4\mu^2}{L - \mu}.
\]

Finally, \( \eta_E \) is found using the definition:

\[
\eta_E = \frac{E''}{E'}.
\]

With the used parameters, \( \eta_E = 0.149 \).

### 9. Comparison with a finite element approach (3D cylindrical bone)

In this section, a comparison between a finite element approach and the raytracers has been done in terms of the displacement field in a cylindrical bone. The frequency used is 1.2 MHz and the initial power is 1 W, and the soft tissue and bone parameters are shown in Table 1 in the manuscript. The equations to solve with the finite element approach can be found in section 2.3 in the manuscript, and the software used is COMSOL.
9.1. Set-up

The simulation set-up for the raytracer model is shown in figure 1. In COMSOL, due to the symmetrical nature of the configuration, the wave equation has been solved only for 1/4 of the geometry (see figure 2). The boundary conditions used are impedance boundary conditions for the outer boundaries of the soft tissue, and low-reflection boundary conditions for the outer boundaries of the bone. Moreover, a perfect matching layer (PML) has been used to reduce the reflection in the bone.

Figure 1. Simulations set-up for the comparison between the raytracer and the solutions of the Helmholtz equation and of the frequency-domain wave equation with a cylindrical bone. The transducer element and the bone are immersed in soft tissue. The cylinder has a radius of 2 cm with a centre located at (3.6, 0, 0) cm.

Figure 2. Simulations set-up in COMSOL for the comparison between the raytracer and the frequency-domain wave equation with a cylindrical bone.
9.2. Limitations

The current set-up in COMSOL presents many limitations to reduce the memory usage. In fact a small geometry has been used, as well as a bigger maximum mesh element size (compared to the one in the raytracer). The small geometry could lead to unwanted reflection at the boundaries, despite the use of impedance boundary conditions, low-reflection boundary conditions and a PML layer. Using a bigger maximum mesh element size (in the COMSOL model this value is 0.03 cm, while in the raytracer model this value is 0.02 cm) may reduce the accuracy.

9.3. Results

A comparison of the x- and y- displacements in the lateral focal plane has been done. The results are shown in figure 3.

![Figure 3](image)

**Figure 3.** Component $d_1$ computed by the raytracer (a) and by solving the frequency-domain wave equation (c), and $d_2$ for the the raytracer (b) and by solving the frequency-domain wave equations (d) when the bone is modeled as a cylinder.

The two models presents two displacements with comparable shape and values, however with more differences compared to the case with only flat interfaces. This could be due
to the unwanted reflection which occurs in the COMSOL model, as well as the choice of the maximum element size of the mesh. The oscillating effect of the reflection waves are particularly evident in the region with $x > 2.1$ cm for the $d_2$ (see figure 3(d)).

References

