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Brief paper

Sampled-data adaptive observer for state-affine systems with uncertain output equation[☆]

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ABSTRACT

The problem of sampled-data observer design is addressed for a class of state- and parameter-affine nonlinear systems. The main novelty in this class lies in the fact that the unknown parameters enter the output equation and the associated regressor is nonlinear in the output. Wiener systems belong to this class. The difficulty with this class of systems comes from the fact that output measurements are only available at sampling times causing the loss of the parameter-affine nature of the model (except at the sampling instants). This makes existing adaptive observers inapplicable to this class of systems. In this paper, a new sampled-data adaptive observer is designed for these systems and shown to be exponentially convergent under specific persistent excitation conditions that ensure system observability and identifiability. The new observer involves an inter-sample output predictor that is different from those in existing observers and features continuous trajectories of the state and parameter estimates.

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1. Introduction

The problem of simultaneous state and parameter estimation, based on sampled measurements, is of great practical interest since most physical systems are continuous-time and subject to parameter uncertainty. It is also of great theoretical interest because the existing observer design and analysis methods are not applicable to specific nonlinear systems and their extension to wider classes constitutes new challenges. The first adaptive observers for nonlinear systems were not sampled-data, see e.g. Bastin and Gevers (1988), Besançon, De León-Morales, and Huerta-Guevara (2006), Marino and Tomei (1996) and Zhang (2002). Their strong nonlinearity makes their direct discretization a highly complex issue. In particular, there is no guarantee that the performances of the original continuous-time adaptive observers are preserved in their approximate discrete-time versions. The first sampled-data adaptive observers, for nonlinear systems, have been developed (Ahmed-Ali, Postoyan, & Lamnabhi-Lagarrigue, 2009; Folin, Ahmed-Ali, Giri, Burlion, & Lamnabhi-Lagarrigue, 2016; Hann & Ahmed-Ali, 2012). In Ahmed-Ali et al. (2009), a class of state affine systems was considered where the unknown parameters come linearly in the state equation and the associated regressor is output-independent. Then, an adaptive observer has been developed using the so-called continuous-discrete design principle. Accordingly, online state estimation is performed using an (open-loop) estimator all the time, except for the sampling instants. At these instants, the state estimate trajectory is corrected using an observer (involving a feedback innovation term). The parameter estimates are only updated at the sampling instants (and kept constant on the rest of the time). It turns out that both the state and the parameter estimate trajectories are discontinuous. Nevertheless, the observer is exponentially convergent, under persistent excitation conditions, if the sampling interval is sufficiently small. A quite different adaptive observer has been proposed in Hann and Ahmed-Ali (2012) for an almost similar class of systems as in Ahmed-Ali et al. (2009). This adaptive observer involves an inter-sample output predictor that is reinitialized at each sampling instant using the output measurements. Its main feature is that the state (resp. the parameters) estimates are generated using all the time the same state estimator (resp. same parameter adaptive law). Therefore, the

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trajectories of both the state and the parameter estimates are continuous. Again, the observer exponential convergence is ensured under PE conditions. A common limitation of the (sampled-output) nonlinear adaptive observers proposed in Ahmed-Ali et al. (2009) and Hann and Ahmed-Ali (2012) is that they are not applicable to systems with output-injection, i.e. those where the regressor (entering the state equation) is output-dependent. This class of systems has been considered in Folin et al. (2016) where an adaptive observer, involving inter-sample output-estimator, has been proposed. Exponential convergence is established, under ad-hoc persistent excitation conditions, provided the sampling period is sufficiently small.

In this paper, the problem of adaptive observer design is considered for a different class of nonlinear systems. Specifically, the unknown parameters enter the output equation, while they entered the state equation in the previous works. Furthermore, the regressor (that is associated with the unknown parameter vector) is output-dependent. Consequently, the system affine nature with respect to the parameters is lost almost all the time, because of the output sampling. Another difficulty with this class of systems is that the output signal enters nonlinearly in the output equation making impossible the construction of dynamic inter-sample output-predictors like those in Folin et al. (2016) or Karafyllis and Kravaris (2009). Therefore, a quite different static (inter-sample output) predictor, reinitialized at sampling times, is designed in this paper. Furthermore, the proposed adaptive observer includes a state estimator and an adaptive parameter law featuring continuous state and parameter estimate trajectories. The observer exponential convergence is established, for small sampling intervals, under persistent excitation (PE) conditions guaranteeing system observability and identifiability. To the authors' knowledge it is the first time that an exponentially convergent adaptive observer is developed for (nonlinear) systems with unknown parameters in the output equation.

The paper is organized as follows: the class of systems under study is described in Section 2 along with the observation objectives; the observer design and analysis are presented in Sections 3 and 4, respectively; simulation results are provided in Section 5; technical proofs are appended.

2. Observation problem statement

2.1. Class of systems

The system under study is described by the following model:

$$\dot{x}(t) = A(u(t))x(t) + b(u(t)), \text{ for all } t > 0 \tag{1}$$

$$y(t) = cx(t) + \psi(u(t), y(t))\theta \tag{2}$$

with,

$$A(u) \in \mathbf{R}^{n\times n};\ c \in \mathbf{R}^{1\times n},\ \theta \in \mathbf{R}^m \tag{3}$$

$$b(u) \in \mathbf{R}^n;\ \psi(u, y) \in \mathbf{R}^{1\times m}, \tag{4}$$

with $x(0)$ arbitrary, where u and y denote the system input and output, respectively; $x \in \mathbf{R}^n$ is the state vector. All quantities in (2)–(4), including the integer n and m, are known except for the parameter vector θ. Furthermore, $A(u)$, $b(u)$, and $\psi(u, y)$ are C^1 functions. The input signal u is bounded and the mapping $u \to x$ (defined by Eq. (1)) is L_∞-stable. Then, it readily follows that, in turn the state x and the output y are bounded. Note that signal boundedness is a usual assumption in the literature of nonlinear observers. The output equation (2) is only accessible to measurements at sampling instants t_k. The latter is any increasing sequence so that $t_k \to \infty$ as $k \to \infty$. The maximum sampling step is $h = \sup_k(t_k - t_{k-1})$.

Remark 1. (1) In case where the output signal $y(t)$ is accessible to measurements all the time (i.e. in the absence of output sampling), the output equation (2) is said to be affine in the unknown parameter vector θ because the quantity $\psi(u, y)$ is then fully known all the time. This is the case in most existing works on sampled-output observers (adaptive or not), e.g. Ahmed-Ali et al. (2009), Hann and Ahmed-Ali (2012) and Karafyllis and Kravaris (2009). The present observer design problem is much harder precisely because the output equation (2) is no longer affine in θ due to the output signal sampling. Indeed, except at sampling times, both $\psi(u(t), y(t))$ and θ are unknown making the quantity $\psi(u(t), y(t))\theta$ subject to a double uncertainty.

(2) Another major difficulty of the present observer design problem, compared to the previous works (Ahmed-Ali et al., 2009; Folin et al., 2016; Hann & Ahmed-Ali, 2012; Karafyllis & Kravaris, 2009), is that the output signal $y(t)$ is *implicitly* defined by the output equation (2). The implicit output definition makes it impossible the design of inter-sample output predictors like those in Ahmed-Ali, Giri, Krstic, and Kahelras (2018), Kahelras, Ahmed-Ali, Giri, and Lamnabhi-Lagarrigue (2018) and Karafyllis and Kravaris (2009).

(3) To illustrate the practical interest of the class of models (1)–(2), consider the following Wiener type system:

$$\dot{x}(t) = Ax(t) + bu(t),\ \ w(t) = cx(t) \tag{5}$$

$$y(t) = f(w(t)) \tag{6}$$

for some triplet (A, b, c) of constant matrices with the dimensions indicated in (3)–(4). That is, the system is constituted of a linear dynamic part, represented by (5), followed in series with a nonlinear static element described by (6). The latter might be the nonlinear characteristic of a sensor. In the system identification literature (e.g. Giri & Bai, 2010; Giri, Radouane, Brouri, & Chaoui, 2014; Radouane, Giri, Ikhouane, Ahmed-Ali, Chaoui, & Brouri, 2017), it is usually supposed that the nonlinear function $f(.)$ is invertible and its inverse is parametrized as follows:

$$w(t) = f^{-1}(y(t)) = y(t) - \sum_{i=1}^{m} \psi_i(y(t))\theta_i \tag{7}$$

with some known function basis $\{\psi_i;\ i = 0, 1, \ldots, m\}$ where $\psi_0(.)$ is the identity function. Combining (7) and (5), one gets a model in the form of (1)–(2). Finally, note that a nonlinear relation like (6) practically stems from the nonlinearity of a sensor or other components. Nonlinear characteristic components are usually met in (renewable) energy systems, see examples in e.g. El Fadil and Giri (2007), El Fadil, Giri, Guerrero, and Tahri (2014), El Fadili, Giri, and El Magri (2014) and Lajouad, El Magri, El Fadili, Chaoui, and Giri (2015). ■

2.2. Observer objectives

The problem at hand consists in designing an observer that provides accurate online estimates of the state $x(t)$ and the parameter vectors θ. State and parameter estimation must only rely on the input signal $u(t)$ and the sampled output measurements $y(t_k)$. A major difficulty in this problem is induced by the term $\psi(u(t), y(t))\theta$ in (2). Indeed, the parameter vector θ is unknown and the output y in $\psi(., .)$ is only accessible to measurement at sampling times. That is, the term $\psi(u(t), y(t))\theta$ is subject to a double uncertainty all time, except at sampling times. This double uncertainty makes currently existing sampled-data adaptive observers inappropriate for the system (1)–(2). Indeed, those proposed in Ahmed-Ali et al. (2009) and Bastin and Gevers (1988) apply to state-affine systems with unknown parameters in the state equation, not in the output equation.

Table 1
Proposed adaptive observer.

$$\dot{\hat{x}}(t) = A(u)\hat{x}(t) + b(u) - S^{-1}(t)c^T\left(\hat{y}(t) - y(t_k)\right) + v(t),\quad \text{for all } t_k \le t < t_{k+1};\ k = 0, 1, 2\ldots \tag{8}$$

$$\hat{y}(t) = c\hat{x}(t_k) + \psi(u(t_k), y(t_k))\hat{\theta}(t),\quad \text{for all } t_k \le t < t_{k+1};\ k = 0, 1, 2\ldots \tag{9}$$

$$v(t) = S^{-1}(t)c^Tc\lambda(t_k)\left(\hat{\theta}(t_k) - \hat{\theta}(t)\right) + \lambda(t)\dot{\hat{\theta}}(t),\quad \text{for all } t_k \le t < t_{k+1};\ k = 0, 1, 2\ldots \tag{10}$$

$$\dot{S}(t) = -\rho S(t) - S(t)A(u) - A^T(u)S(t) + c^Tc \tag{11}$$

with arbitrary $\hat{x}(0)$ and $S(0) = S^T(0) > 0$, and $\rho > 0$ is arbitrary.

$$\dot{\lambda}(t) = A(u)\lambda(t) - S^{-1}(t)c^Tc\lambda(t_k) - S^{-1}(t)c^T\psi(u(t_k), y(t_k))\quad \text{for all } t_k \le t < t_{k+1};\ k = 0, 1, 2\ldots \tag{12}$$

with arbitrary $\lambda(0) \in \mathbf{R}^{n\times m}$.

$$\dot{\hat{\theta}}(t) = -R(t)\Psi^T(t)\tilde{y}(t), \text{ for all } t \ge 0 \tag{13}$$

$$\dot{R}(t) = R(t) - R(t)\Psi^T(t)\Psi(t)R(t), \text{ for all } t \ge 0 \tag{14}$$

$$\Psi(t) = \psi(u(t_k), y(t_k)) + c\lambda(t)\quad \text{for all } t_k \le t < t_{k+1};\ k = 0, 1, 2\ldots \tag{15}$$

with arbitrary $\hat{\theta}(0)$ and $R(0) > 0$.

3. Adaptive observer design

To get online estimates $\hat{x}(t)$ and $\hat{\theta}(t)$, of the state vector x and the unknown parameter vector θ, we propose the adaptive observer of Table 1.

Clearly, the adaptive observer of Table 1 is composed of four main parts:

(i) the state observer (8)–(10) providing the state estimates $\hat{x}(t)$;

(ii) the parameter estimator (13)–(15) providing the parameter estimates $\hat{\theta}(t)$ (which is a least-squares with forgetting factor equal to 1);

(iii) the adaptive law (11) providing the observer matrix gain $S(t)$;

(iv) the filter (12) providing the auxiliary (matrix) signal $\lambda(t) \in \mathbf{R}^{n\times m}$.

Owing to the observer gain $S(t)$, we need it to be positive definite all the time and staying away from the null matrix. This issue has been investigated in Besançon, Bornard, and Hammouri (1996). To state the result established there, the following definitions (also introduced there) are recalled for convenience:

• The transition matrix $\Phi_u(s,t)$ of the system (1) is defined as the solution of the following system:

$$\frac{d}{ds}\Phi_u(s,t) = A(u(s))\Phi_u(s,t), \tag{16}$$

with $\Phi_u(t,t) = I$, where I denotes the identity matrix.

• The observability Gramian associated to (1)–(2) is:

$$\Gamma(t,T,u) = \int_t^{t+T} \Phi_u^T(s,t)c^Tc\Phi_u(s,t)ds \tag{17}$$

Then, the existence result for (11) can be stated as follows Besançon et al. (1996):

Lemma 1. *Suppose the input signal $u(t)$ is such that, there exist $T > 0$, $\alpha_u > 0$ and $t_0 > 0$ so that:*

$$\lambda_{\min}(\Gamma(t,T,u)) > \alpha_u, \text{ for all } t \ge t_0 \tag{18}$$

where $\lambda_{\min}(.)$ refers to the smallest eigenvalue of a matrix. Then, there exists $\rho_0 > 0$ and $\beta_S > 0$ such that, for any $S(0) = S^T(0) > 0$ and $\rho > \rho_0$, there exist $\alpha_S(\rho) > 0$, $\beta_S > 0$, $t_0 > T$ so that:

$$\alpha_S(\rho)I \le S(t) \le \beta_S I, \text{ for all } t \ge t_0 \tag{19}$$

where $\alpha_S(\rho)$ is a decreasing continuous function of ρ ■

Remark 2. (1) Note that property (18) constitutes a persistent excitation condition on the input signal $u(t)$ that guarantees the uniform observability of the system (1)–(2). In the sequel, condition (18) will be assumed to be true so that one can make use of (19). In Besançon et al. (1996), it was shown that $\alpha_S(\rho) = \alpha_u e^{-(\rho+2Ta_{\max})}$ with $a_{\max} = \sup_u \|A(u)\|$.

(2) Another important remark is that output estimation between two sampling instants cannot presently be performed using inter-sample predictors, unlike in e.g. Folin et al. (2016), Karafyllis and Kravaris (2009) and Kahelras et al. (2018). Indeed, those predictors can be obtained when the output derivative $\dot{y}(t)$ can be explicitly expressed in function of the state $x(t)$ only. Presently, the derivative of $\dot{y}(t)$ is a function of both $x(t)$ and $y(t)$. This is a direct result of the fact that $y(t)$ enters the right side of (2) ■

4. Adaptive observer analysis

Let us introduce the following estimation errors:

$$\tilde{x}(t) = \hat{x}(t) - x(t), \qquad \tilde{\theta}(t) = \hat{\theta}(t) - \theta(t), \text{ for all } t > 0 \tag{20}$$

$$\tilde{y}(t) = \hat{y}(t) - y(t_k), \text{ for all } t_k \le t < t_{k+1};\ k = 0, 1, 2\ldots \tag{21}$$

One immediately gets from (9) and (2):

$$\tilde{y}(t) = c\tilde{x}(t_k) + \psi(u(t_k), y(t_k))\tilde{\theta}(t),\quad \text{for all } t_k \le t < t_{k+1};\ k = 0, 1, 2\ldots \tag{22}$$

Subtracting (1) from (8), one gets using (22):

$$\dot{\tilde{x}}(t) = A(u)\tilde{x}(t) - Kc\tilde{x}(t_k) - K\psi(u(t_k), y(t_k))\tilde{\theta}(t) + v(t) \tag{23}$$

Also, it readily follows from (13) and (22) that:

$$\dot{\tilde{\theta}}(t) = \dot{\hat{\theta}}(t) = -R(t)\Psi^T(t)(c\tilde{x}(t_k) + \psi(u(t_k), y(t_k))\tilde{\theta}(t)),\quad \text{for all } t_k \le t < t_{k+1};\ k = 0, 1, 2\ldots \tag{24}$$

Now, introduce the coordinate change:

$$z(t) = \tilde{x}(t) - \lambda(t)\tilde{\theta}(t) \tag{25}$$

Differentiating (25) with respect to time yields, using (23) and (24):

$$\dot{z}(t) = A(u)z(t) - Kcz(t_k), \text{ for all } t_k \le t < t_{k+1};\ k = 0, 1, 2\ldots \tag{26}$$

Also, using (25) it follows from (22) that:

$$\tilde{y}(t) = \Psi(t)\tilde{\theta}(t) + cz(t) - c\left(\tilde{x}(t) - \tilde{x}(t_k)\right),\quad \text{for all } t_k \le t < t_{k+1};\ k = 0, 1, 2\ldots \tag{27}$$

where $\Psi(t)$ is as in (51). In turn, the quantity $\tilde{x}(t) - \tilde{x}(t_k)$ in (27) develops as follows, using (25):

$$\begin{aligned}\tilde{x}(t) - \tilde{x}(t_k) &= z(t) - z(t_k) + \lambda(t)\tilde{\theta}(t) - \lambda(t_k)\tilde{\theta}(t_k)\\ &= z(t) - z(t_k) + (\lambda(t) - \lambda(t_k))\tilde{\theta}(t) + \lambda(t_k)\left(\tilde{\theta}(t) - \tilde{\theta}(t_k)\right)\\ &= z(t) - z(t_k) + (\lambda(t) - \lambda(t_k))\tilde{\theta}(t) + \lambda(t_k)\int_{t_k}^t \dot{\tilde{\theta}}(s)ds\end{aligned} \tag{28}$$

which together with (27) yields:

$$\begin{aligned}\tilde{y}(t) &= \Psi(t)\tilde{\theta}(t) + cz(t) - c(z(t) - z(t_k))\\ &\quad - c\left((\lambda(t) - \lambda(t_k))\tilde{\theta}(t) + \lambda(t_k)\int_{t_k}^t \dot{\tilde{\theta}}(s)ds\right),\\ &\quad \text{for all } t_k \le t < t_{k+1};\ k = 0, 1, 2\ldots\end{aligned} \tag{29}$$

Using (29), Eq. (24) becomes:

$$\begin{aligned}\dot{\tilde{\theta}}(t) &= -R(t)\Psi(t)\Psi^T(t)\tilde{\theta}(t) - R(t)\Psi(t)cz(t)\\ &\quad + R(t)\Psi(t)c\left(z(t) - z(t_k)\right) + R(t)\Psi(t)c\lambda(t_k)\int_{t_k}^t \dot{\tilde{\theta}}(s)ds\end{aligned}$$

$$+ R(t)\Psi(t)c\,(\lambda(t) - \lambda(t_k))\,\widetilde{\theta}(t),$$
$$\text{for all } t_k \le t < t_{k+1};\; k = 0, 1, 2 \ldots \quad (30)$$

This equation is to be completed with the adaptation gain equation (14). The complete error system, including (26) and (30), is rewritten for future referencing:

$$\dot{z}(t) = A(u)z(t) - Kcz(t_k),\ \text{for all } t_k \le t < t_{k+1};\; k = 0, 1, 2 \ldots \quad (31)$$

$$\dot{\widetilde{\theta}}(t) = -R(t)\Psi(t)\Psi^T(t)\widetilde{\theta}(t) + R(t)\Psi(t)c\,(\lambda(t) - \lambda(t_k))\,\widetilde{\theta}(t)$$
$$+ R(t)\Psi(t)c\lambda(t_k)\int_{t_k}^{t}\dot{\widetilde{\theta}}(s)ds - \mathrm{R\,(t)\,\Psi\,(t)\,cz\,(t_k)}$$
$$- R(t)\Psi(t)cz(t_k),$$
$$\text{for all } t_k \le t < t_{k+1};\; k = 0, 1, 2 \ldots \quad (32)$$

$$\frac{d}{dt}R^{-1}(t) = -R^{-1}(t) + \Psi^T(t)\Psi(t) \quad (33)$$

$$\dot{\lambda}(t) = A(u)\lambda(t) - S^{-1}(t)c^Tc\lambda(t_k) - S^{-1}(t)c^T\psi(u(t_k), y(t_k)),$$
$$\text{for all } t_k \le t < t_{k+1};\; k = 0, 1, 2 \ldots \quad (34)$$

The complementary equation (34) is identical to (12), this is rewritten here for convenience. Eq. (32) is equivalent to (14), it is more convenient because it is re-expressed in terms of the inverse gain $R^{-1}(t)$. The exponential stability of the error system (31)–(34) is stated in the following theorem:

Theorem 1 (*Main Result*)**.** *Consider the system (1)–(2) and let it be excited by an input signal $u(t)$ satisfying (18) so that (19) holds. Consider the adaptive observer described by Table 1 and the corresponding estimation error system described by Eqs. (30) through (33). Then, there exists a $h_M > 0$ such that if $0 < h < h_M$ then the following properties hold:*

(1) The transformed estimation error $z(t)$ converges exponentially to the origin.

(2) The auxiliary matrix signal $\lambda(t)$ is bounded.

(3) If the input signal $u(t)$ is such that $\lambda(t)$ is persistently exciting in the sense that,
$\int_t^{t+\delta}\Psi^T(s)\Psi(s)ds > \alpha_\Psi I$, for all $t \ge 0$ for some real constants $\delta > 0$, $\alpha_\Psi > 0$, then the parameter estimation error $\widetilde{\theta}(t)$ exponentially converges to the origin.

(4) Under the above conditions, the state estimation error $\widetilde{x}(t)$ exponentially converges to the origin.

Proof. See the Appendix.

5. Simulation

The proposed approach is illustrated on a Wiener system (see Remark 1). A noise-free case is considered first. Afterwards, a case with output measurement noise is considered.

The considered Wiener system is inspired on Example 1 in Wigren (1993), which considers a Wiener model that describes a valve for control of fluid flow. The linear dynamic part is a continuous-time equivalent (using zero-order-hold) of the discrete-time model presented in Wigren (1993). The continuous-time model has state-space matrices $A = \begin{bmatrix} -4 & -\frac{25}{4} \\ 4 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, and $c = \begin{bmatrix} 0 & \frac{25}{8} \end{bmatrix}$. The nonlinear function in Wigren (1993) is $f(w) = \frac{w}{\sqrt{0.1+0.9w^2}}$ and its inverse is $f^{-1}(y) = \frac{\sqrt{0.1}y}{\sqrt{1-0.9y^2}}$. The results here are obtained for a system where $f^{-1}(y)$ is replaced by its least-squares fifth-order polynomial approximation and where the data are simulated

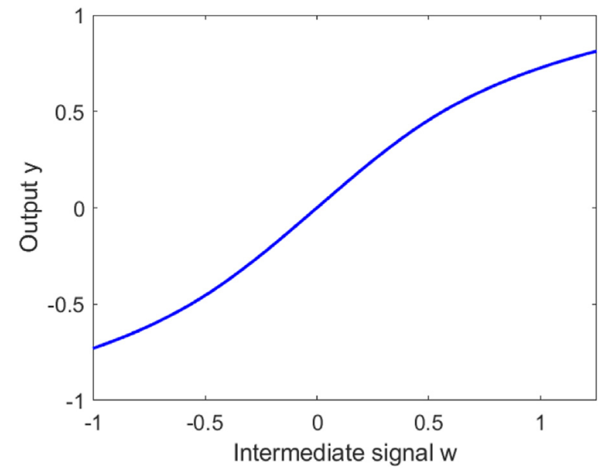

Fig. 1. The nonlinear block of the considered Wiener system.

using parameterization (1)–(2) and (7) instead of (5)–(6), since, although the fifth-order polynomial $f^{-1}(y)$ is invertible (see Fig. 1), it is in general impossible to find an analytical expression for its inverse (as stated by the Abel–Ruffini theorem). To obtain a perfect fit with the parameterization (7), this polynomial approximation's linear coefficient, say γ, should be one. This is achieved by dividing the linear block with γ and replacing the nonlinear block with $f(\gamma w)$. Note that a Wiener system with a linear block with transfer function $\frac{1}{\gamma}c\,(sI - A)^{-1}b$ and with a nonlinear block $f(\gamma w)$ has the same input/output behavior as the original system. Like this, there is a perfect fit with the parameterization (7) with $\psi_1 = -1$, $\psi_2 = -y^2$, $\psi_3 = -y^3$, $\psi_4 = -y^4$, $\psi_5 = -y^5$, and with true parameters $\theta_1 = 1.4094 \cdot 10^{-4}$, $\theta_2 = -6.0928 \cdot 10^{-3}$, $\theta_3 = 3.3077 \cdot 10^{-1}$, $\theta_4 = 2.4600 \cdot 10^{-2}$, $\theta_5 = 7.1310 \cdot 10^{-1}$, and we can show convergence towards these true parameters. The input consists of 1000 samples of a white Gaussian noise with standard deviation 0.202, which is selected such that the standard deviation of the non-scaled intermediate signal w is 0.1 on average. Note that this input is bounded and persistently exciting for the considered Wiener system.

Input and output signal measurements are collected over the time interval 0 s through 100 s, equidistantly sampled with time step $h = 0.1$ s. The system and the observer are simulated using Matlab/Simulink with a variable time step, and with a maximum step of 0.0002 s. The simulation protocol is such that the system initial state is zero, $x(0) = 0$. The initial state estimates are $\hat{x}_1(0) = \hat{x}_2(0) = 1$ and the initial parameter estimates are $\hat{\theta}_1(0) = \hat{\theta}_2(0) = \hat{\theta}_3(0) = 0$. The observer gains are fixed or initialized as follows: $\rho = 1$, $S(0) = I_n$, $R(0) = 10^4 I_m$, and $\lambda(0)$ is a zero $n \times m$ matrix, where I_n and I_m are $n \times n$ and $m \times m$ identity matrices, respectively.

From Figs. 2 to 4, it is seen that the predicted output, generated by (9), coincides with the true output at the sampling times. Note that, after convergence is achieved, the output-predictor has a zero-order-hold inter-sample behavior (Fig. 4), while before convergence, it shows an integrating behavior (Fig. 3). Furthermore, the state estimation error (Fig. 5) and the parameter estimation error (Fig. 6) converge to zero. The trajectories of the state and parameter estimates are continuous.

Next, a zero mean white Gaussian noise $e(t)$ with a signal-to-noise ratio (SNR) of circa 20 dB (there are slight variations depending on the noise realization) is added as follows in (2): $y(t) = cx(t) + \psi(y(t))\,\theta + e(t)$. Note that the parameter estimation errors do not converge to a fixed value (Fig. 7). In fortunate cases, the final parameter error is close to zero, resulting in good estimates of the nonlinear block, while in unfortunate cases, the error is large. This

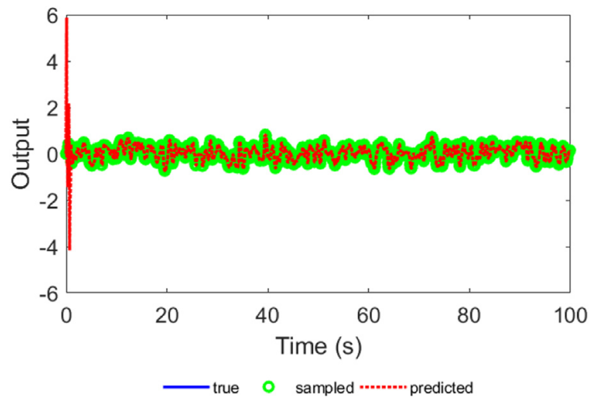

Fig. 2. True output and predicted output provided by the observer which only has access to output samples. It is seen that the predicted output converges close to the true output after a few seconds.

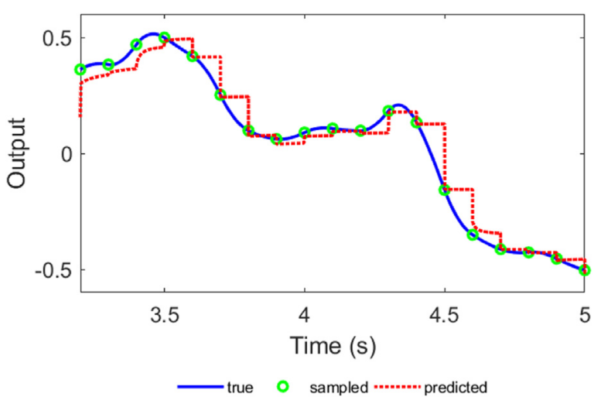

Fig. 3. Zoom on the true and predicted outputs around $t = 4$ s.

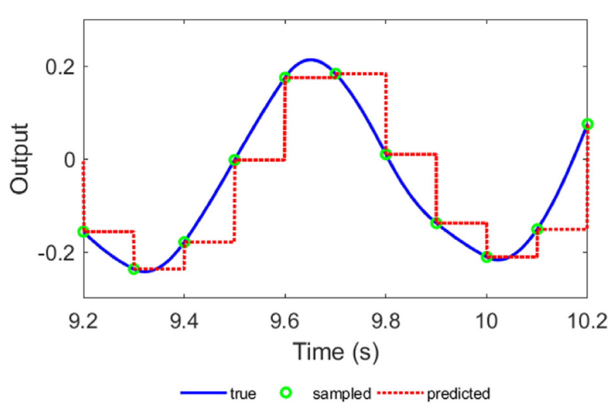

Fig. 4. The predicted output has perfectly converged, at sampling instants, to the true output around $t = 10$ s.

jumping up and down of the parameter estimation errors can be counteracted by decreasing the exponential forgetting. Removing the exponential forgetting completely (by putting the forgetting factor equal to zero, which corresponds to removing the first term in the right side of (14)) results in parameter estimation errors that tend to converge to a fixed value (Fig. 8). With the considered noise setting and without exponential forgetting, no bias is expected as the number of samples goes to infinity. When choosing the exponential forgetting factor, a trade-off should be made between parameter convergence and adaptability to system changes.

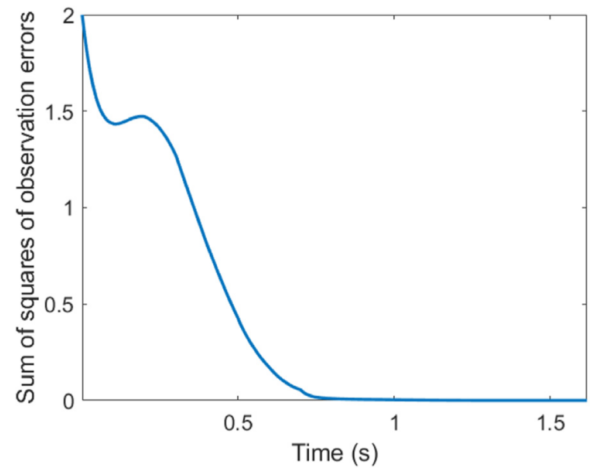

Fig. 5. The observed states perfectly converge to the true states of the considered Wiener system.

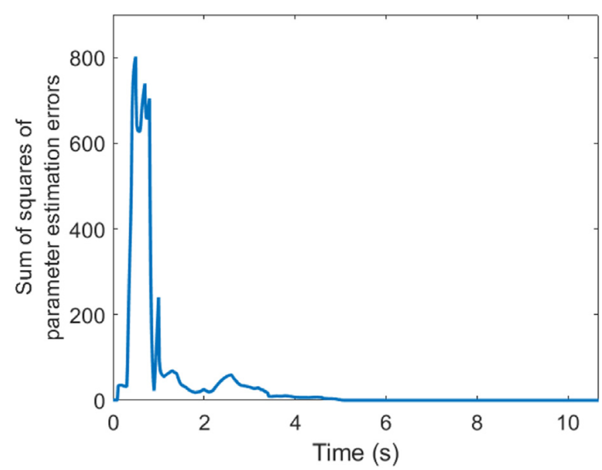

Fig. 6. The estimated parameters converge to their true counterparts.

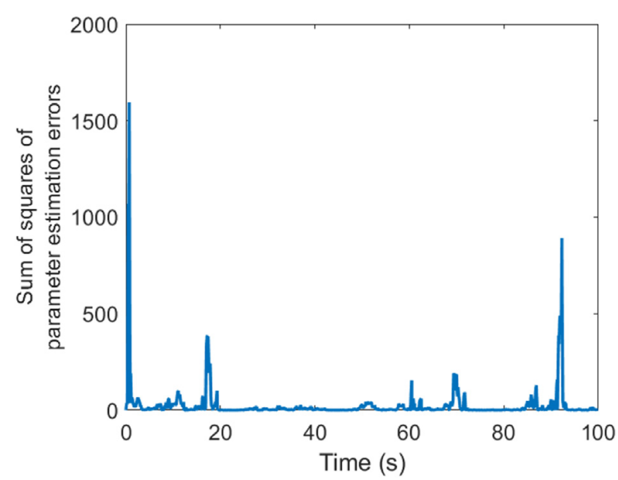

Fig. 7. The parameter estimation errors do not converge to a fixed value in the presence of measurement noise with forgetting factor one.

6. Concluding remarks

The problem of sampled-data adaptive observer design has been addressed for the class of state- and parameter-affine systems

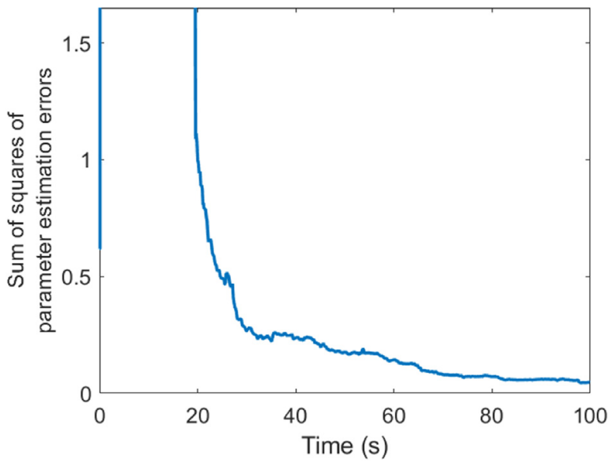

Fig. 8. The parameter estimation errors tend to converge to a fixed value in the presence of measurement noise if the exponential forgetting is removed.

described by (1)–(2). This class of systems is quite different from those in the existing works (on adaptive observers) because the unknown parameter vector θ comes presently into the output equation. Furthermore, the corresponding regressor $\psi(u(t), y(t))$ depends on the output $y(t)$ which is not accessible to measurements, except at the sampling times. Then, the affine character of the system is lost all the time. The proposed adaptive observer is also quite different from the existing one. Among the numerous differences one has: (i) the varying gain in the state estimator (8), the inter-sample output-observer (9), the filter generated by (12), the transformed regressor defined by (15) used in the adaptive parameter law (13)–(14). The present work offers several research perspectives e.g. accounting in the model (5)–(6) for (output) delays and output-dependent state matrix (Ahmed-Ali et al., 2018; Cuny, Lajouad, Giri, Ahmed-Ali, & Van Assche, 2019), the investigation of the problem where a nonlinearity is present both in the state and the output equation, and the case where the nonlinearity is memory (Giri, Rochdi, Chaoui, & Brouri, 2008),

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Appendix. Proof of Theorem 1

Proof of Part 1. Exponential stability of (27). Consider the Lyapunov function candidate:

$$V_z = z^T S z + \int_{t-h}^{t}\int_{s}^{t} \|\dot z(\xi)\|^2 \, d\xi \, ds, \ \text{for } t_k \le t < t_{k+1};\ k \in \mathbf{N} \quad (35)$$

This can be rewritten as follows (Fridman, Dambrine, & Yeganifar, 2008):

$$V_z(t) = z^T(t)S(t)z(t) + \int_{t-h}^{t} (s - t + h)\,\|\dot z(s)\|^2 \, ds,$$
$$\text{for all } t_k \le t < t_{k+1};\ k \in \mathbf{N} \quad (36)$$

Using (25) and (11), one gets by differentiating (35):

$$\begin{aligned}\dot V_z &= \dot z^T S z + z^T S \dot z + z^T \dot S z + h\,\|\dot z(t)\|^2 - \int_{t-h}^{t} \|\dot z(s)\|^2\, ds \\ &= (A(u)z(t) - S^{-1}c^T c z(t_k))^T S z + z^T S (A(u) z(t) - S^{-1} c^T c z(t_k)) \\ &\quad + z^T \dot S z + h\,\|\dot z(t)\|^2 - \int_{t-h}^{t} \|\dot z(s)\|^2\, ds \\ &= z^T(t)(A(u)^T S + S A(u)) z - 2 z^T c^T c z(t_k) \\ &\quad + z^T(-\rho S - S A(u) - A^T(u) S + c^T c) z \\ &\quad + h\,\|\dot z(t)\|^2 - \int_{t-h}^{t} \|\dot z(s)\|^2\, ds \\ &= -\rho z^T S z - z^T c^T c z - 2 z^T c^T c \int_{t_k}^{t} \dot z(s) ds \\ &\quad + h\,\|\dot z(t)\|^2 - \int_{t-h}^{t} \|\dot z(s)\|^2\, ds \end{aligned} \quad (37)$$

Let

$$\tau(t) = t - t_k \quad \text{for all } t_k \le t < t_{k+1};\ k = 0, 1, 2\ldots \quad (38)$$

Then, one has:

$$\begin{aligned} -z^T c^T c z - 2 z^T c^T c \int_{t_k}^{t} \dot z(s) ds &\le \left(c \int_{t_k}^{t} \dot z(s) ds \right)^T c \int_{t_k}^{t} \dot z(s) ds \\ &\le \left(\int_{t-\tau(t)}^{t} \|\dot z(s)\|\, ds \right)^T c^T c \int_{t-\tau(t)}^{t} \|\dot z(s)\|\, ds \\ &\le \left(\int_{t-h}^{t} \|\dot z(s)\|\, ds \right)^T c^T c \int_{t-h}^{t} \|\dot z(s)\|\, ds, \\ &\quad \text{for all } t_k \le t < t_{k+1};\ k = 0, 1, 2\ldots \end{aligned} \quad (39)$$

using the fact that $t - \tau(t) = t_k$ and $\tau(t) \le h$ (due to (38)).

Combining (37) and (39), one obtains:

$$\dot V_z \le -\rho z^T S z + I^T(t) c^T c I(t) + h\,\|\dot z(t)\|^2 - \int_{t-h}^{t} \|\dot z(s)\|^2\, ds \quad (40)$$

with:

$$I(t) = \int_{t-h}^{t} \|\dot z(s)\|\, ds \quad \text{for all } t_k \le t < t_{k+1};\ k = 0, 1, 2\ldots \quad (41)$$

By Jensen's inequality, one has:

$$\|I(t)\|^2 \le h \int_{t-h}^{t} \|\dot z(s)\|^2\, ds, \quad \text{for all } t_k \le t < t_{k+1};\ k = 0, 1, 2\ldots \quad (42)$$

Now, let us focalize on the penultimate quantity on the right side of (40). It follows from (31) that:

$$\begin{aligned} \|\dot z(t)\| &= \left\| \left(A(u) - S^{-1} c^T c\right) z(t) + S^{-1} c^T c \,(z(t) - z(t_k)) \right\|, \\ &\le \left\| \left(A(u) - S^{-1} c^T c\right) z(t) \right\| + \left\| S^{-1} c^T c \int_{t_k}^{t} \dot z(s) \right\| \\ &\le \kappa_1(\rho)\,\|z(t)\| + \kappa_2(\rho) \left\| \int_{t_k}^{t} \dot z(s) \right\| \\ &\quad \text{for all } t_k \le t < t_{k+1};\ k = 0, 1, 2\ldots \end{aligned} \quad (43)$$

with

$$\kappa_1(\rho) = \sup_t \left\| A(u(t)) - S^{-1}(t) c^T c \right\|, \qquad \kappa_2(\rho) = \alpha_S^{-1}(\rho) \left\| c^T c \right\| \quad (44)$$

using (19). By Lemma 1, the real quantities $\kappa_1(\rho)$ and $\kappa_2(\rho)$ are continuous increasing functions of ρ. Using (41) and (36), inequality (43) implies:

$$\begin{aligned} \|\dot z(t)\|^2 &\le 2\kappa_1^2(\rho)\,\|z(t)\|^2 + 2\kappa_2^2(\rho) \left\| \int_{t-h}^{t} \dot z(s) \right\|^2 \\ &\le \kappa_3(\rho) \left(V_z(t) + I^2(t) \right), \quad \text{for all } t \end{aligned} \quad (45)$$

with:

$$\kappa_3(\rho) = \max\left(\frac{2\kappa_1^2(\rho)}{\alpha_S(\rho)}, 2\kappa_2^2(\rho)\right) \tag{46}$$

where we have used (19). Using (45) and (42), it follows from (40) that:

$$\begin{aligned}\dot{V}_z &\le -\rho z^T S z + \kappa_4 h \int_{t-h}^{t} \|\dot{z}(s)\|^2 \, ds \\ &\quad + h\left(\kappa_3(\rho)\left(V_z(t) + I^2(t)\right)\right) - \int_{t-h}^{t} \|\dot{z}(s)\|^2 \, ds \\ &\le -\rho V_z(t) + \rho \int_{t-h}^{t} (s-t+h)\|\dot{z}(s)\|^2 \, ds + \kappa_4 h \int_{t-h}^{t} \|\dot{z}(s)\|^2 \, ds \\ &\quad + h\left(\kappa_3(\rho)\left(V_z(t) + I^2(t)\right)\right) - \int_{t-h}^{t} \|\dot{z}(s)\|^2 \, ds \\ &\le -\sigma_z(h,\rho)V_z(t) - (1 - h(\kappa_4 + h\kappa_3(\rho) + \rho)) \int_{t-h}^{t} \|\dot{z}(s)\|^2 \, ds \\ &\quad \text{for all } t_k \le t < t_{k+1};\ k = 0, 1, 2 \ldots\end{aligned} \tag{47}$$

with $\sigma_z(h, \rho) = \rho - h\kappa_3(\rho) > 0$ and $\kappa_4 = \lambda_{\max}(c^T c)$, where we have used the inequality $0 \le s - t + h \le h$ (which holds for all $t_k \le t < t_{k+1};\ k = 0, 1, 2 \ldots$). Given any real $0 < \varepsilon_1 < 1$, there exists a $h_z > 0$ such that for any $0 < h < h_z$ one has

$$\sigma_z(h,\rho) > 0 \text{ and } 1 - h(\kappa_4 + h\kappa_3(\rho) + \rho) > 0 \tag{48}$$

Then, it follows from (47) that $\dot{V}_z \le -\sigma_z(\rho)V_z(t)$ implying that the subsystem (31) is exponentially stable and so $z(t)$ tends exponentially to the origin as $t \to \infty$.

Proof of Part 2. Introduce the notation $\lambda(t) = [\lambda_1(t) \ldots \lambda_m(t)]$ where the λ_i's denote the column vectors of $\lambda(t) \in \mathbf{R}^{n\times m}$. Similarly, we let:

$$\psi(u(t_k), y(t_k)) = [\psi_1(u(t_k), y(t_k)) \cdots \psi_m(u(t_k), y(t_k))].$$

Then, the subsystem (34) can be rewritten in the following disassembled form:

$$\begin{aligned}\dot{\lambda}_i(t) &= A(u)\lambda_i(t) - S^{-1}(t)c^T c\lambda_i(t_k) - S^{-1}(t)c^T \psi_i(u(t_k), y(t_k)), \\ &\quad \text{for all } t_k \le t < t_{k+1};\ k = 0, 1, 2 \ldots\end{aligned} \tag{49}$$

Interestingly, the nominal system defined by (49) without the disturbance input $K\psi_i(u(t_k), y(t_k))$ is identical to (31). Therefore, we consider the following Lyapunov function, similar to (36), to analyze the properties of (49):

$$\begin{aligned}V_\lambda &= \lambda_i^T S \lambda_i + \int_{t-h}^{t} (s-t+h)\left\|\dot{\lambda}_i(s)\right\|^2 ds, \\ &\quad \text{for all } t_k \le t < t_{k+1};\ k = 0, 1, 2 \ldots\end{aligned} \tag{50}$$

where the argument t has been (and will continued to be) omitted to alleviate expressions. Just as for (37), we get the following derivative:

$$\begin{aligned}\dot{V}_\lambda &= -\rho\lambda_i^T S\lambda_i - \lambda_i^T c^T c\lambda_i - 2\lambda_i^T c^T c \int_{t_k}^{t} \dot{\lambda}_i(s) ds \\ &\quad + h\left\|\dot{\lambda}_i(t)\right\|^2 - \int_{t-h}^{t} \left\|\dot{\lambda}_i(s)\right\|^2 ds - 2\lambda_i^T(t)c^T \psi_i^T(u(t_k), y(t_k))\end{aligned} \tag{51}$$

Following similar steps as those from (38) to (40), it follows from (51) that:

$$\begin{aligned}\dot{V}_\lambda &\le -\rho\lambda_i^T S\lambda_i + \kappa_4 h \int_{t-h}^{t} \left\|\dot{\lambda}_i(s)\right\|^2 ds \\ &\quad + h\left\|\dot{\lambda}_i(t)\right\|^2 - \int_{t-h}^{t} \left\|\dot{\lambda}_i(s)\right\|^2 ds \\ &\quad + \xi \left\|\psi_i^T(u(t_k), y(t_k))c\right\|^2 + \frac{1}{\xi}\|\lambda_i(t)\|^2, \\ &\quad \text{for all } t_k \le t < t_{k+1};\ k = 0, 1, 2 \ldots\end{aligned} \tag{52}$$

with $\kappa_4 = \lambda_{\max}(c^T c)$, where ξ is any positive real number. Also, following the steps (43) through (47), it follows from (52):

$$\begin{aligned}\left\|\dot{\lambda}_i(t)\right\| &\le \kappa_1(\rho)\|\lambda_i(t)\| + \kappa_2(\rho)\left\|\int_{t_k}^{t} \dot{\lambda}_i(s) ds\right\| \\ &\quad + \left\|S^{-1}(t)c^T\right\| \|\psi(u(t_k), y(t_k))\|\end{aligned} \tag{53}$$

where $\kappa_1(\rho)$ and $\kappa_2(\rho)$ are as in (44). Just as we did with (45), we obtain by taking the square of both sides of (53) and using Young's inequality:

$$\begin{aligned}\left\|\dot{\lambda}_i(t)\right\|^2 &\le 3\kappa_1(\rho)\|\lambda_i(t)\|^2 + 3\kappa_2(\rho)\left\|\int_{t_k}^{t} \dot{\lambda}_i(s) ds\right\|^2 \\ &\quad + 3\beta_S^2 \|c\|^2 \|\psi(u(t_k), y(t_k))\|^2 \\ &\le \kappa_5(\rho)\left(V_\lambda(t) + h\int_{t-h}^{t} \left\|\dot{\lambda}_i(s)\right\|^2 ds\right) \\ &\quad + 3\beta_S^2 \|c\|^2 \|\psi(u(t_k), y(t_k))\|^2, \\ &\quad \text{for all } t_k \le t < t_{k+1};\ k = 0, 1, 2 \ldots\end{aligned} \tag{54}$$

with $\kappa_5(\rho) = \max\left(\frac{3\kappa_1^2(\rho)}{\alpha_S(\rho)}, 3\kappa_2^2(\rho)\right)$, where the last inequality has been obtained using Jenssen's inequality (as for (42)). Combining (54) and (52) implies:

$$\begin{aligned}\dot{V}_\lambda &\le -\rho V_\lambda(t) + \rho\int_{t-h}^{t} (s-t+h)\left\|\dot{\lambda}_i(s)\right\|^2 ds \\ &\quad + \kappa_4 h\int_{t-h}^{t} \left\|\dot{\lambda}_i(s)\right\|^2 ds \\ &\quad + h\kappa_5(\rho)\left(V_\lambda(t) + h\int_{t-h}^{t} \left\|\dot{\lambda}_i(s)\right\|^2 ds\right) \\ &\quad + 3h\beta_S^2\|c\|^2 \|\psi(u(t_k), y(t_k))\|^2 - \int_{t-h}^{t} \left\|\dot{\lambda}_i(s)\right\|^2 ds \\ &\quad + \xi\left\|\psi_i^T(u(t_k), y(t_k))c\right\|^2 + \frac{\alpha_S^{-1}(\rho)}{\xi}V_\lambda(t) \\ &\le -\left(\rho - \frac{\alpha_S^{-1}(\rho)}{\xi} - h\kappa_5(\rho)\right)V_\lambda(t) \\ &\quad - (1 - h(\rho + \kappa_4 + h\kappa_5(\rho)))\int_{t-h}^{t} \left\|\dot{\lambda}_i(s)\right\|^2 ds \\ &\quad + (3h\beta_S^2\|c\|^2 + \xi)\|\psi(u(t_k), y(t_k))\|^2, \\ &\le -\sigma_\lambda(h,\rho)V_\lambda(t) + (3h\beta_S^2\|c\|^2 + \frac{2\alpha_S^{-1}(\rho)}{\rho})\|\psi(u(t_k), y(t_k))\|^2 \\ &\quad - (1 - h(\rho + \kappa_4 + h\kappa_5(\rho)))\int_{t-h}^{t} \left\|\dot{\lambda}_i(s)\right\|^2 ds \\ &\quad \text{for all } t_k \le t < t_{k+1};\ k = 0, 1, 2 \ldots\end{aligned} \tag{55}$$

with $\sigma_\lambda(h,\rho) = \frac{\rho}{2} - h\kappa_5(\rho)$. Let the free parameter ξ be set such that $\frac{\alpha_S^{-1}(\rho)}{\xi} = \frac{\rho}{2}$ and consider any $0 < \varepsilon_2 < 1$. There exists a $h_\lambda > 0$ such that for any $0 < h < h_\lambda$ one has:

$$\sigma_\lambda(h,\rho) > 0 \text{ and } 1 - h(\rho + \kappa_4 + h\kappa_5(\rho)) > 0 \tag{56}$$

Then, inequality (55) yields:

$$\begin{aligned}\dot{V}_\lambda(t) &\le -\sigma_\lambda(h,\rho)V_\lambda(t) + (3h\beta_S^2\|c\|^2 \\ &\quad + \frac{2\alpha_S^{-1}(\rho)}{\rho})\|\psi(u(t_k), y(t_k))\|^2, \\ &\quad \text{for all } t_k \le t < t_{k+1};\ k = 0, 1, 2 \ldots\end{aligned} \tag{57}$$

which implies that $V_\lambda(t)$ is bounded and

$$\limsup_{t\to\infty} V_\lambda(t) \leq \frac{1}{\sigma_\lambda(h,\rho)}\left(3h\beta_S^2\left\|c\right\|^2 + \frac{2\alpha_S^{-1}(\rho)}{\rho}\right)\psi_{\max} \tag{58}$$

with $\psi_{\max} = \sup_t \left\|\psi(u(t), y(t))\right\|^2$. Using (50) and (19), it follows from (58):

$$\limsup_{t\to\infty} \left\|\lambda(t)\right\|^2 \leq \lambda_M^2(h,\rho) \tag{59}$$

with,

$$\lambda_M(h,\rho) = \left[\frac{\psi_{\max}}{\alpha_S(\rho)\sigma_\lambda(h,\rho)}\left(3h\beta_S^2\left\|c\right\|^2 + \frac{2}{\alpha_S(\rho)\rho}\right)\right]^{\frac{1}{2}} \tag{60}$$

Replacing the various quantities in (60) by their explicit expressions, the above bound rewrites as follows:

$$\lambda_M(h,\rho) = \left[\frac{\psi_{\max}e^{(\rho+2Ta_{\max})}}{\alpha_u(\frac{\rho}{2} - h\kappa_5(\rho))}\left(3h\beta_S^2\left\|c\right\|^2 + \frac{2e^{(\rho+2Ta_{\max})}}{\rho\alpha_u}\right)\right]^{\frac{1}{2}} \tag{61}$$

We now establish an additional useful property that concerns the bounding of the quantity $\lambda(t) - \lambda(t_k)$. To this end, one gets integrating (49):

$$\lambda(t) - \lambda(t_k) = \int_{t_k}^t A(u(s))\lambda_i(s)ds - \left(\int_{t_k}^t S^{-1}(s)ds\right)c^Tc\lambda_i(t_k)$$
$$- \left(\int_{t_k}^t S^{-1}(s)ds\right)c^T\psi_i(u(t_k), y(t_k)),$$
$$\text{for all } t_k \leq t < t_{k+1};\ k = 0,1,2\dots \tag{62}$$

Using (59) and (19), it readily follows from (62) that:

$$\left\|\lambda(t) - \lambda(t_k)\right\| \leq \widetilde{\lambda}(h,\rho),\ \text{for all } t_k \leq t < t_{k+1};\ k = 0,1,2\dots \tag{63}$$

with

$$\widetilde{\lambda}(h,\rho) = ha_{\max}\lambda_M(h,\rho) + h\alpha_S(\rho)\left\|c^Tc\right\|\lambda_M(h,\rho)$$
$$+ h\alpha_S(\rho)\left\|c\right\|\lambda_M(h,\rho)$$

which together with (63) gives:

$$\limsup_{h\to 0} \widetilde{\lambda}(h,\rho) = 0 \tag{64}$$

Proof of Part 3. It is shown in many places that, under the PE assumption of Part 3 (of Theorem 1), the solution of (33) satisfies the following property (e.g. Ioannou & Sun, 1996):

$$R^{-1}(t) > \alpha_R I,\ \text{for all } t \geq 0 \tag{65}$$

for some real constant $\alpha_R > 0$. Consider the following Lyapunov function candidate:

$$V_\theta = \widetilde{\theta}^T R^{-1}\widetilde{\theta} + \int_{t-h}^t (s-t+h)\left\|\dot{\widetilde{\theta}}(s)\right\|^2 ds \tag{66}$$

Deriving this along the trajectory of (32)–(33) gives:

$$\dot{V}_\theta = 2\widetilde{\theta}^T R^{-1}\dot{\widetilde{\theta}} + \widetilde{\theta}^T\frac{dR^{-1}}{dt}\widetilde{\theta} + h\left\|\dot{\widetilde{\theta}}(t)\right\|^2 - \int_{t-h}^t\left\|\dot{\widetilde{\theta}}(s)\right\|^2 ds$$
$$= -2\widetilde{\theta}^T(t)\Psi(t)\Psi^T(t)\widetilde{\theta}(t) + 2\widetilde{\theta}^T\Psi(t)c\left(\lambda(t) - \lambda(t_k)\right)\widetilde{\theta}(t)$$
$$+ 2\widetilde{\theta}^T\Psi(t)\Psi(t)c\lambda(t_k)\int_{t_k}^t\dot{\widetilde{\theta}}(s)ds$$
$$- 2\widetilde{\theta}^T\Psi(t)\Psi(t)cz(t_k) - \widetilde{\theta}^TR^{-1}(t)\widetilde{\theta} + \widetilde{\theta}^T\Psi(t)\Psi^T(t)\widetilde{\theta}$$
$$+ h\left\|\dot{\widetilde{\theta}}(t)\right\|^2 - \int_{t-h}^t\left\|\dot{\widetilde{\theta}}(s)\right\|^2 ds$$
$$= -\widetilde{\theta}^T(t)R^{-1}(t)\widetilde{\theta}(t) - \widetilde{\theta}^T(t)\Psi(t)\Psi^T(t)\widetilde{\theta}(t)$$
$$+ 2\widetilde{\theta}^T(t)\Psi(t)c\left(\lambda(t) - \lambda(t_k)\right)\widetilde{\theta}(t)$$
$$+ 2\widetilde{\theta}^T\Psi(t)\Psi(t)c\lambda(t_k)\int_{t_k}^t\dot{\widetilde{\theta}}(s)ds$$
$$- 2\widetilde{\theta}^T\Psi(t)\Psi(t)cz(t_k) + h\left\|\dot{\widetilde{\theta}}(t)\right\|^2 - \int_{t-h}^t\left\|\dot{\widetilde{\theta}}(s)\right\|^2 ds,$$
$$\text{for all } t_k \leq t < t_{k+1};\ k = 0,1,2\dots \tag{67}$$

Using (66) and Young's and Jenssen's inequalities, it follows from (67) that:

$$\dot{V}_\theta(t) \leq -V_\theta(t) + h\int_{t-h}^t\left\|\dot{\widetilde{\theta}}(s)\right\|^2 ds - (\widetilde{\theta}^T(t)\Psi(t))^2$$
$$+ \zeta(\widetilde{\theta}^T(t)\Psi(t))^2 + \frac{\left\|c\right\|^2}{\zeta}\left\|\lambda(t) - \lambda(t_k)\right\|^2\left\|\widetilde{\theta}(t)\right\|^2$$
$$+ \varsigma(\widetilde{\theta}^T\Psi(t))^2 + \frac{\left\|\Psi(t)c\lambda(t_k)\right\|^2}{\varsigma}\left(\int_{t_k}^t\dot{\widetilde{\theta}}(s)ds\right)^2$$
$$+ \varsigma(\widetilde{\theta}^T\Psi(t))^2 + \frac{1}{\varsigma}\left\|\Psi(t)c\right\|^2\left\|z(t_k)\right\|^2$$
$$+ h\left\|\dot{\widetilde{\theta}}(t)\right\|^2 - \int_{t-h}^t\left\|\dot{\widetilde{\theta}}(s)\right\|^2 ds$$
$$\leq -(1 - \frac{\left\|c\right\|^2\widetilde{\lambda}^2(h,\rho)}{\varsigma\alpha_R})V_\theta(t)$$
$$- \left(1 - h - h\frac{\left\|\Psi(t)c\right\|^2\lambda_M^2(h,\rho)}{\varsigma}\right)\int_{t-h}^t\left\|\dot{\widetilde{\theta}}(s)\right\|^2 ds$$
$$- (1-3\varsigma)(\widetilde{\theta}^T(t)\Psi(t))^2$$
$$+ \frac{1}{\varsigma}\left\|\Psi(t)c\right\|^2\left\|z(t_k)\right\|^2 + h\left\|\dot{\widetilde{\theta}}(t)\right\|^2,$$
$$\text{for all } t_k \leq t < t_{k+1};\ k = 0,1,2\dots \tag{68}$$

with ς any real such that $1 - 3\varsigma < 0$, where we have used (65) and (63). To find an upper bound of the last term on the right side of (68), we take the square of both sides of (32) and apply Young's inequality:

$$\left\|\dot{\widetilde{\theta}}(t)\right\|^2 \leq 4\left\|R(t)\Psi(t)\Psi^T(t)\right\|^2\left\|\widetilde{\theta}(t)\right\|^2$$
$$+ 4\left\|R(t)\Psi(t)c\right\|^2\left\|\lambda(t) - \lambda(t_k)\right\|^2\left\|\widetilde{\theta}(t)\right\|^2$$
$$+ \left\|R(t)\Psi(t)c\right\|^2\left\|\lambda(t_k)\right\|^2\left\|\int_{t_k}^t\dot{\widetilde{\theta}}(s)ds\right\|^2$$
$$+ 4\left\|R(t)\Psi(t)c\right\|^2\left\|z(t_k)\right\|^2$$
$$\leq (4\alpha_R^{-1}\psi_{\max}^4 + 4\alpha_R^{-2}\psi_{\max}^2\left\|c\right\|^2\lambda_M^2(h,\rho))\left\|\widetilde{\theta}(t)\right\|^2$$
$$+ 4h\alpha_R^{-2}\psi_{\max}^2\left\|c\right\|^2\lambda_M^2(h,\rho)\int_{t_k}^t\left\|\dot{\widetilde{\theta}}(s)\right\|^2 ds$$
$$+ 4\alpha_R^{-2}\psi_{\max}^2\left\|c\right\|^2\lambda_M^2(h,\rho)\left\|z(t_k)\right\|^2$$
$$\text{for all } t_k \leq t < t_{k+1};\ k = 0,1,2\dots \tag{69}$$

where the last inequality is obtained using Jenssen's inequality. Combining (69) and (68) gives, using (66) and the fact that $1-3\varsigma < 0$:

$$\dot{V}_\theta(t) \leq -\sigma_\theta(h,\rho)V_\theta(t) - \kappa_5(h,\rho,t)\int_{t-h}^t\left\|\dot{\widetilde{\theta}}(s)\right\|^2 ds$$
$$+ \kappa_6(h,\rho,t)\left\|z(t_k)\right\|^2,$$
$$\text{for all } t_k \leq t < t_{k+1};\ k = 0,1,2\dots \tag{70}$$

with

$$\sigma_\theta(h,\rho) = -\left[1 - \frac{\|c\|^2\,\widetilde{\lambda}^2(h,\rho)}{\varsigma\alpha_R} + h\alpha_R^{-1}\left(4\alpha_R^{-1}\psi_{\max}^4 + 4\alpha_R^{-2}\psi_{\max}^2\,\|c\|^2\,\lambda_M^2(h,\rho)\right)\right] \quad (71)$$

$$\kappa_5(h,\rho,t) = 1 - h - h\frac{\|\Psi(t)c\|^2\,\lambda_M^2(h,\rho)}{\varsigma} - 4h^2\alpha_R^{-2}\psi_{\max}^2\,\|c\|^2\,\lambda_M^2(h,\rho) \quad (72)$$

$$\kappa_6(h,\rho,t) = \frac{1}{\varsigma}\,\|\Psi(t)c\|^2 + 4h\alpha_R^{-2}\psi_{\max}^2\,\|c\|^2\,\lambda_M^2(h,\rho) \quad (73)$$

Using (63), it readily follows from (72)–(73) that:

$$\limsup_{h\to 0}\sigma_\theta(h,\rho) = \limsup_{h\to 0}\kappa_5(h,\rho,t) = 1 \quad (74)$$

$$0 < \kappa_6(h,\rho,t) \le \frac{1}{\varsigma}\Psi_{\max}^2\,\|c\|^2 + 4h\alpha_R^{-2}\psi_{\max}^2\,\|c\|^2\,\lambda_M^2(h,\rho) < \infty \quad (75)$$

with $\Psi_{\max} = \sup_t \|\Psi(t)\|$ is finite due to (59) and (15). Given any real number $0 < \varepsilon_3 < 1$, it follows from (71), (74) that there exists a real number $0 < h_\theta < \min(h_z, h_\lambda)$(where h_z and h_λ are as in (56) and (48)) such that, for any $0 < h < h_\theta$ one has:

$$0 < \sigma_\theta(h,\rho) < 1 - \varepsilon_2 \quad \text{and} \quad \kappa_5(h,\rho,t) > 0 \quad (76)$$

which together with (69) implies that:

$$\dot{V}_\theta(t) \le -(1-\varepsilon_\theta)V_\theta(t) + \kappa_6(h,\rho,t)\,\|z(t_k)\|^2, \quad \text{for all } t_k \le t < t_{k+1};\ k = 0,1,2\dots \quad (77)$$

It has already been proved that $\|z(t)\|$ is exponentially vanishing. Then, it follows from (70) that so is $V_\theta(t)$. The same result applies to $\widetilde{\theta}(t)$, due to (65)–(66).

Proof of Part 4. From (25) one gets $\|\widetilde{x}(t)\| \le \|z(t)\| + \|\lambda(t)\|\,\|\widetilde{\theta}(t)\|$ which implies that $\widetilde{x}(t)$ is exponentially vanishing because $z(t)$ and $\widetilde{\theta}(t)$ are so and $\|\lambda(t)\|$ is bounded.

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