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Citation for published version (APA):

Document status and date:
Published: 27/02/2019

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

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Download date: 25. Dec. 2019
Incremental Gain of LTI Systems

P.J.W. Koelewijn and R. Tóth

I. INTRODUCTION

The incremental gain is a notion similar to, but stronger than, the $L_2$-gain to characterize the stability of a dynamical system. In this technical report we prove that for Linear Time Invariant (LTI) systems the $L_2$-gain and incremental gain are equivalent, whereas for nonlinear systems this is generally not the case [1]. Before we will give the proof, we first give the definitions of the $L_2$-gain and incremental gain.

Consider a dynamical system $\Sigma: \mathcal{L}_2^{nu} \to \mathcal{L}_2^{ny}$ given by

$$
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t); \\
y(t) &= Cx(t) + Bu(t); \\
x(t_0) &= x_0;
\end{align*}
$$

where $x \in \mathcal{C}_1^{nu}$ with $x_0 \in X \subseteq \mathbb{R}^{nu}$ is the state variable associated with the considered state-space representation of the system, $u \in \mathcal{L}_2^{nu}$ taking values in $U \subseteq \mathbb{R}^{nu}$ is the input, and $y \in \mathcal{L}_2^{ny}$ taking values in $Y \subseteq \mathbb{R}^{ny}$ is the output of the system.

**Definition I.1** ($L_2$-gain). $\Sigma$, given by (1), is said to be $L_2$-gain stable if for all $u \in \mathcal{L}_2^{nu}$ and $x_0 \in X$, $\Sigma(u)$ exists and there is a finite $\gamma \geq 0$ and a function $\zeta(x) \geq 0$ with $\zeta(0) = 0$ such that

$$
\|\Sigma(u)\|_2 \leq \gamma \|u\|_2 + \zeta(x_0).$$

The induced $L_2$-gain of $\Sigma$, denoted by $\|\Sigma\|_2$, is the infimum of $\gamma$ such that (2) still holds.

**Definition I.2** (Incremental gain [1], [2]). $\Sigma$, given by (1), is said to be incrementally $L_2$-gain stable, from now on denoted as $L_{i2}$-gain stable, if it is $L_2$-gain stable and, there exist a finite $\eta \geq 0$ and a function $\zeta(x, \tilde{x}) \geq 0$ with $\zeta(0,0) = 0$ such that

$$
\|\Sigma(u) - \Sigma(\tilde{u})\|_2 \leq \eta \|u - \tilde{u}\|_2 + \zeta(x_0, \tilde{x}_0),$$

for all $u, \tilde{u} \in \mathcal{L}_2^{nu}$ and $x_0, \tilde{x}_0 \in X$. The induced $L_{i2}$-gain of $\Sigma$, denoted by $\|\Sigma\|_{i2}$, is the infimum of $\eta$ such that (3) holds.

II. MAIN RESULTS

**Theorem II.1.** For an (LTI) dynamical system given by (1) the $L_2$-gain and $L_{i2}$-gain as defined in Definition I.1 and Definition I.2 are equivalent.

**Proof.** For the proof we use Theorem 2.7 from [3]. Therefore, formulate the following augmented difference system for the LTI system in (1)

$$
\begin{align*}
y_\Delta &= \Sigma(u) - \Sigma(\tilde{u}) = \Sigma_\Delta(u, \tilde{u}) \\
x_\Delta(t) &= x(t) - \tilde{x}(t), \\
y_\Delta(t) &= y(t) - \tilde{y}(t);
\end{align*}
$$

which has the state-space representation

$$
\begin{bmatrix}
\dot{x}_\Delta(t) \\
y_\Delta(t)
\end{bmatrix} =
\begin{bmatrix}
A_\Delta & B_\Delta \\
C_\Delta & D_\Delta
\end{bmatrix}
\begin{bmatrix}
x_\Delta(t) \\
u_\Delta(t)
\end{bmatrix},
$$

where

$$
x_\Delta(t) = \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix}, \quad u_\Delta(t) = \begin{bmatrix} u(t) \\ \tilde{u}(t) \end{bmatrix}, \quad A_\Delta = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad B_\Delta = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}, \quad C_\Delta = \begin{bmatrix} C & -C \end{bmatrix}, \quad D_\Delta = \begin{bmatrix} D & -D \end{bmatrix}.
$$

The differential dissipation inequality (DDI) is given by

$$
\partial_x S(x(t), x(u)) \leq w(u(t), y(t)),
$$

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where $S(x)$ is a storage function, $w(u, y)$ a supply function and $f(x, u)$ the state equation. In our case, per Theorem 2.7 from [3], as storage function we take (omitting time dependence for brevity)

$$S(x, \dot{x}) = S(x_\Delta) = (x - \dot{x})^T P(x - \dot{x}) = x_\Delta^T \begin{bmatrix} P & -P \\ -P & P \end{bmatrix} x_\Delta,$$

and as supply function we take

$$w_\Delta(u, \ddot{u}, y_\Delta) = \eta^2 \|u - \ddot{u}\|^2 - \|y_\Delta\|^2.$$  

The state equation, based on (5), is given by

$$f(x_\Delta, u_\Delta) = A_\Delta x_\Delta + B_\Delta u_\Delta.$$  

Combining (6)-(9) results in

$$2 x_\Delta^T \bar{P} (A_\Delta x_\Delta + B_\Delta u_\Delta) \leq \eta^2 \|u - \ddot{u}\|^2 - \|y_\Delta\|^2,$$

which can be rewritten as

$$\begin{bmatrix} x_\Delta^T \\ u_\Delta^T \end{bmatrix} \begin{bmatrix} I & 0 \\ A_\Delta & B_\Delta \end{bmatrix} \begin{bmatrix} I & 0 \\ \bar{P} & 0 \end{bmatrix} \begin{bmatrix} x_\Delta \\ u_\Delta \end{bmatrix} \leq \begin{bmatrix} x_\Delta^T \\ u_\Delta^T \end{bmatrix} \begin{bmatrix} 0 & I \\ 0 & -I \end{bmatrix} \begin{bmatrix} C_\Delta & D_\Delta \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_\Delta \\ u_\Delta \end{bmatrix},$$

which needs to hold for all $x_\Delta$ and $u_\Delta$ values over all $t$, with

$$H = \begin{bmatrix} \eta^2 I & -\eta^2 I \\ -\eta^2 I & \eta^2 I \end{bmatrix}.$$

Next, (11) holds if and only if

$$\begin{bmatrix} I & 0 \\ A_\Delta & B_\Delta \end{bmatrix} \begin{bmatrix} 0 & \bar{P} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \leq \begin{bmatrix} I & 0 \\ A_\Delta & B_\Delta \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = 0.$$  

Collapsing (12) gives

$$\begin{bmatrix} M_{11} & -M_{11} & M_{12} & -M_{12} \\ -M_{11} & M_{11} & -M_{12} & M_{12} \\ M_{12}^T & -M_{12}^T & M_{22} & -M_{22} \\ -M_{12}^T & M_{12} & -M_{22} & M_{22} \end{bmatrix} \leq 0,$$

where

$$\begin{align*}
M_{11} &= A^T P + PA + C^T C, \\
M_{12} &= PB + C^T D, \\
M_{22} &= D^T D - \eta^2 I.
\end{align*}$$

Introduce the non-singular

$$\mathcal{I} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & -I_n & 0 \\ 0 & 0 & I_{n_n} \\ 0 & 0 & -I_{n_n} \end{bmatrix}.$$  

By using $\mathcal{I}$ as a congruence transformation, (13) can equivalently be written as

$$\mathcal{I} \begin{bmatrix} 0 & 0 & 0 \\ 0 & M_{11} & M_{12} \\ 0 & M_{12} & M_{22} \\ 0 & 0 & 0 \end{bmatrix} \mathcal{I}^T \leq 0.$$  

We can reduce (16) to

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & M_{11} & M_{12} \\ 0 & M_{12}^T & M_{22} \end{bmatrix} \leq 0,$$

and to

$$\begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - \eta^2 I \end{bmatrix} \leq 0,$$

which is equivalent with the bounded real lemma [4]. This shows that the $\mathcal{L}_2$-gain and $\mathcal{L}_{12}$-gain are equivalent for LTI systems. \qed
REFERENCES


