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Incremental Gain of LTI Systems

P.J.W. Koelewijn and R. Tóth

I. INTRODUCTION

The incremental gain is a notion similar to, but stronger than, the $L_2$-gain to characterize the stability of a dynamical system. In this technical report we prove that for Linear Time Invariant (LTI) systems the $L_2$-gain and incremental gain are equivalent, whereas for nonlinear systems this is generally not the case [1]. Before we will give the proof, we first give the definitions of the $L_2$-gain and incremental gain.

Consider a dynamical system $\Sigma: L_2^{in} \to L_2^{out}$ given by

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t); \\
y(t) &= Cx(t) + Bu(t); \\
x(t_0) &= x_0;
\end{align*}
\]

where $x \in C_1^{in}$, $x_0 \in X \subseteq \mathbb{R}^{n_x}$ is the state variable associated with the considered state-space representation of the system, $u \in L_2^{in}$, taking values in $U \subseteq \mathbb{R}^{n_u}$ is the input, and $y \in L_2^{out}$, taking values in $Y \subseteq \mathbb{R}^{n_y}$ is the output of the system.

Definition I.1 ($L_2$-gain). $\Sigma$, given by (1), is said to be $L_2$-gain stable if for all $u \in L_2^{in}$ and $x_0 \in X$, $\Sigma(u)$ exists and there is a finite $\gamma \geq 0$ and a function $\zeta(x) \geq 0$ with $\zeta(0) = 0$ such that

\[
\|\Sigma(u)\|_2 \leq \gamma \|u\|_2 + \zeta(x_0).
\]

The induced $L_2$-gain of $\Sigma$, denoted by $\|\Sigma\|_2$, is the infimum of $\gamma$ such that (2) still holds.

Definition I.2 (Incremental gain [1], [2]). $\Sigma$, given by (1), is said to be incrementally $L_2$-gain stable, from now on denoted as $L_{i2}$-gain stable, if it is $L_2$-gain stable and, there exist a finite $\eta \geq 0$ and a function $\zeta(x, \tilde{x}) \geq 0$ with $\zeta(0,0) = 0$ such that

\[
\|\Sigma(u) - \Sigma(\tilde{u})\|_2 \leq \eta \|u - \tilde{u}\|_2 + \zeta(x_0, \tilde{x}_0),
\]

for all $u, \tilde{u} \in L_2^{in}$ and $x_0, \tilde{x}_0 \in X$. The induced $L_{i2}$-gain of $\Sigma$, denoted by $\|\Sigma\|_{i2}$, is the infimum of $\eta$ such that (3) holds.

II. MAIN RESULTS

Theorem II.1. For an (LTI) dynamical system given by (1) the $L_2$-gain and $L_{i2}$-gain as defined in Definition I.1 and Definition I.2 are equivalent.

Proof. For the proof we use Theorem 2.7 from [3]. Therefore, formulate the following augmented difference system for the LTI system in (1)

\[
\begin{align*}
y_\Delta &= \Sigma(u) - \Sigma(\tilde{u}) = \Sigma_\Delta(u, \tilde{u}) \\
\dot{y}_\Delta(t) &= (Cx(t) + Du(t)) - (C\tilde{x}(t) + D\tilde{u}(t));
\end{align*}
\]

which has the state-space representation

\[
\begin{bmatrix}
\dot{x}_\Delta(t) \\
y_\Delta(t)
\end{bmatrix} =
\begin{bmatrix}
A_\Delta & B_\Delta \\
C_\Delta & D_\Delta
\end{bmatrix}
\begin{bmatrix}
x_\Delta(t) \\
\dot{u}_\Delta(t)
\end{bmatrix},
\]

where

\[
x_\Delta(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}, \quad u_\Delta(t) = \begin{bmatrix} u(t) \\ \dot{u}(t) \end{bmatrix}, \quad A_\Delta = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad B_\Delta = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}, \quad C_\Delta = \begin{bmatrix} C & -C \end{bmatrix}, \quad D_\Delta = \begin{bmatrix} D & -D \end{bmatrix}.
\]

The differential dissipation inequality (DDI) is given by

\[
\partial_x S(x(t)) f(x(t), u(t)) \leq w(u(t), y(t)),
\]

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where $S(x)$ is a storage function, $w(u, y)$ a supply function and $f(x, u)$ the state equation. In our case, per Theorem 2.7 from [3], as storage function we take (omitting time dependence for brevity)
\[
S(x, \dot{x}) = S(x_\Delta) = (x - \dot{x})^TP(x - \dot{x}) = x_\Delta^T \begin{bmatrix} P & -P \\ -P & P \end{bmatrix} x_\Delta,
\]

and as supply function we take
\[
w_\Delta(u, \bar{u}, y_\Delta) = \eta^2 \|u - \bar{u}\|^2 - \|y_\Delta\|^2.
\]

The state equation, based on (5), is given by
\[
f(x_\Delta, u_\Delta) = A_\Delta x_\Delta + B_\Delta u_\Delta.
\]

Combining (6)-(9) results in
\[
2x_\Delta^T \bar{P} (A_\Delta x_\Delta + B_\Delta u_\Delta) \leq \eta^2 \|u - \bar{u}\|^2 - \|y_\Delta\|^2,
\]

which can be rewritten as
\[
\begin{bmatrix} x_\Delta \\ u_\Delta \end{bmatrix}^T \begin{bmatrix} I & 0 \\ A_\Delta & B_\Delta \end{bmatrix} \begin{bmatrix} 0 & P \\ \bar{P} & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A_\Delta & B_\Delta \end{bmatrix} \begin{bmatrix} x_\Delta \\ u_\Delta \end{bmatrix} \leq \begin{bmatrix} x_\Delta \\ u_\Delta \end{bmatrix}^T \begin{bmatrix} 0 & I \\ C_\Delta & D_\Delta \end{bmatrix} \begin{bmatrix} 0 & -I \\ C_\Delta & D_\Delta \end{bmatrix} \begin{bmatrix} x_\Delta \\ u_\Delta \end{bmatrix},
\]

which needs to hold for all $x_\Delta$ and $u_\Delta$ values over all $t$, with
\[
H = \begin{bmatrix} \eta^2 I & -\eta^2 I \\ -\eta^2 I & \eta^2 I \end{bmatrix}.
\]

Next, (11) holds if and only if
\[
\begin{bmatrix} I & 0 \\ A_\Delta & B_\Delta \end{bmatrix}^T \begin{bmatrix} 0 & P \\ \bar{P} & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A_\Delta & B_\Delta \end{bmatrix} \begin{bmatrix} I & 0 \\ C_\Delta & D_\Delta \end{bmatrix} \leq 0.
\]

Collapsing (12) gives
\[
\begin{bmatrix} M_{11} & -M_{11} & M_{12} & -M_{12} \\ -M_{11} & M_{11} & -M_{12} & M_{12} \\ M_{12} & -M_{12} & M_{22} & -M_{22} \\ -M_{12} & M_{12} & -M_{22} & M_{22} \end{bmatrix} \leq 0,
\]

where
\[
M_{11} = A^T P + PA + C^T C,
M_{12} = PB + C^T D,
M_{22} = D^T D - \eta^2 I.
\]

Introduce the non-singular
\[
\mathcal{I} = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix},
\]

By using $\mathcal{I}$ as a congruence transformation, (13) can equivalently be written as
\[
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathcal{I} \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix} \mathcal{I}^T \leq 0.
\]

We can reduce (16) to
\[
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \leq 0,
\]

and to
\[
\begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - \eta^2 I \end{bmatrix} \leq 0,
\]

which is equivalent with the bounded real lemma [4]. This shows that the $\mathcal{L}_2$-gain and $\mathcal{L}_{12}$-gain are equivalent for LTI systems. \qed
REFERENCES


