An analytic method for agent-based modeling of spatially inhomogeneous disease dynamics

Citation for published version (APA):
https://doi.org/10.1063/1.4996518

DOI:
10.1063/1.4996518

Document status and date:
Published: 16/08/2017

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.  
• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.  
• You may not further distribute the material or use it for any profit-making activity or commercial gain  
• You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the “Taverne” license above, please follow below link for the End User Agreement:
www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:
openaccess@tue.nl
providing details and we will investigate your claim.

Download date: 24. Apr. 2020
An analytic method for agent-based modeling of spatially inhomogeneous disease dynamics

Wolfgang Bock, Torben Fattler, Isti Rodiah, and Oliver Tse

Citation: AIP Conference Proceedings 1871, 020008 (2017); View online: https://doi.org/10.1063/1.4996518
View Table of Contents: http://aip.scitation.org/toc/apc/1871/1
Published by the American Institute of Physics

Articles you may be interested in

Numerical simulation of agent-based modeling of spatially inhomogeneous disease dynamics
AIP Conference Proceedings 1871, 020009 (2017); 10.1063/1.4996519

Wick type SDEs driven by grey Brownian motion
AIP Conference Proceedings 1871, 020004 (2017); 10.1063/1.4996514

The spin polaron theory as a mechanism for high temperature superconductivity
AIP Conference Proceedings 1871, 020007 (2017); 10.1063/1.4996517

Toward tangible quantum nature
AIP Conference Proceedings 1871, 020001 (2017); 10.1063/1.4996511

On the derivation of the Nakajima-Zwanzig probability density function via white noise analysis
AIP Conference Proceedings 1871, 020006 (2017); 10.1063/1.4996516

On time-dependent diffusion coefficients arising from stochastic processes with memory
AIP Conference Proceedings 1871, 020005 (2017); 10.1063/1.4996515
An Analytic Method for Agent-based Modeling of Spatially Inhomogeneous Disease Dynamics

Wolfgang Bock\textsuperscript{1,b)}, Torben Fattler\textsuperscript{1,a)}, Isti Rodiah\textsuperscript{1,c)} and Oliver Tse\textsuperscript{2,d)}

\textsuperscript{1}Technische Universität Kaiserslautern, P.O. Box 3049, 67653 Kaiserslautern, Germany
\textsuperscript{2}Technische Universiteit Eindhoven, P.O. Box 513, 5600 MB Eindhoven, Netherlands
\textsuperscript{a)}Corresponding author: fattler@mathematik.uni-kl.de
\textsuperscript{b)}bock@mathematik.uni-kl.de
\textsuperscript{c)}rodiah@mathematik.uni-kl.de
\textsuperscript{d)}O.T.C.Tse@tue.nl

Abstract. In this article we set up a microscopic model for the spread of an infectious disease based on configuration space analysis. Using the so-called Vlasov scaling we obtained the corresponding mesoscopic (kinetic) equations, describing the density of susceptible and infected individuals (particles) in space. The resulting system of equations can be seen as a generalization to a 'spatial' SIS-model. The equations showing up in the limiting system are of the type which is know in literature as Fisher–Kolmogorov–Petrovsky–Piscounov type.

INTRODUCTION

In recent years much about the modeling and understanding of various types of disease spreading and epidemic behavior have been studied. In principle one can distinguish two types of models for disease spread. On the one hand there is the classical SIR-model from Kermack and McKendrick [1] which describes the time evolution of the number of susceptible (S), infected (I) and recovered (R) individuals by a system of ordinary differential equations. This model has been developed and extended exhaustively in the last 90 years. Among those extensions are the introduction of new compartments to model vector-bourne diseases, see e.g. [2], delay equations to model incubation time, e.g. [3], models considering the age and wealth structure etc. Recently, models with fractional derivatives have also been considered [4]. Unfortunately, we are unable to provide a detailed account concerning this subject and refer the interested reader to [5]. A main drawback of the models described above is that they do not provide any information about the spatial spread of a disease. Nevertheless, there have been various approaches to link many different SIR-areas to obtain spatial behavior. In the SIR-model case, an advection-diffusion equation has been identified as the limiting equation, see e.g. [6]. Another approach in incorporating spatial information for the SIR-model may also be found in [7].

Although the SIR-model and all its extensions are very flexible in describing the different aspects of disease dynamics, the modeling assumptions of the disease spread is purely on the macroscopic level. However, for many different diseases the infection mechanism is only known on the microscopic, i.e., particle-to-particle or individuum-to-individuum level.

One way to consider both microscopical modeling and spatial resolution is to describe the disease dynamics by means of an interacting particle system with suitable interaction potentials. Fundamental in this area are dynamics in so-called marked configuration spaces [8]. These techniques together with a proper scaling of the microscopic system, the so-called Vlasov scaling, have been recently used to model the dynamics of cancer cells [9]. In our approach the components of particle configurations consist of susceptible and infected/infective particles that interact with one another. One may also easily incorporate other types of particles to model recovery or short time immunity. The microscopic dynamics then results from suitable "spin-flip"-processes (particle changes the type).

The intent of this work is to sketch a strategy to link suitable microscopic disease models with the classical SIS-model via a scaling limit. For the sake of readability we have focused on the analysis part for dynamics without birth,
death and movement of particles. However it is straightforward to include these parts by adding suitable Markov generators in the evolution equations. The methods to construct dynamics on multicomponent configuration spaces have been extensively developed in [8]. A general strategy for the derivation of Vlasov-type equations in the framework of continuous particle systems is described in [10]. Our aim is to apply this approach to the framework of disease dynamics.

This paper is organized as follows: In the first chapter we give a brief introduction to the analysis in configuration spaces needed throughout this work. In the second chapter we model the infection process on the microscopic level using these techniques and show that under a suitable scaling limit we obtain a (spatially inhomogenous) system which resembles the classical SIS-model with constant recovery rate.

**PRELIMINARIES**

**One-component configuration space**

We consider \( \mathbb{R}^2 \) equipped with the norm \(| \cdot |\) given by the euclidean scalar product \((\cdot, \cdot)_{\mathbb{R}^2}\). The configuration space \( \Gamma \) over \( \mathbb{R}^2 \) is defined by

\[
\Gamma := \Gamma_{\mathbb{R}^2} := \{ \gamma \subset \mathbb{R}^2 \mid \#(\gamma \cap K) < \infty \text{ for all } K \subset \mathbb{R}^2 \text{ compact} \},
\]

where \( \#S \) denotes the cardinality of a set \( S \). One can identify each \( \gamma \in \Gamma \) with the positive, integer-valued Radon measure

\[
\sum_{x \in \gamma} \delta_x \in \mathcal{M}(\mathbb{R}^2) \quad \text{with} \quad \delta_x(A) := \begin{cases} 1 & \text{for } x \in A, \\ 0 & \text{else} \end{cases}, \quad x \in \mathbb{R}^2, \ A \in \mathcal{B}(\mathbb{R}^2),
\]

where \( \sum_{x \in \gamma} \delta_x := \text{zero measure} \) and \( \mathcal{M}(\mathbb{R}^2) \) denotes the set of positive Radon measures on the Borel-\( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}^2) \) of \( \mathbb{R}^2 \). This identification allows \( \Gamma \) to be endowed with the topology induced by the vague topology on \( \mathcal{M}(\mathbb{R}^2) \), i.e., the coarsest topology on \( \Gamma \) with respect to which all the mappings

\[
\Gamma \ni \gamma \mapsto \langle f, \gamma \rangle := \sum_{x \in \gamma} f(x), \quad f \in C_c(\mathbb{R}^2),
\]

are continuous. Here \( C_c(\mathbb{R}^2) \) denotes the set of all continuous functions on \( \mathbb{R}^2 \) having compact support. We denote by \( \mathcal{B}(\Gamma) \) the corresponding Borel-\( \sigma \)-algebra on \( \Gamma \).

Let us now consider the space of finite configurations

\[
\Gamma_0 := \bigsqcup_{n=0}^{\infty} \Gamma^{(n)}, \text{ where } \Gamma^{(n)} := \{ \gamma \in \Gamma \mid \#\gamma = n \} \text{ for } n \in \mathbb{N} \text{ and } \Gamma^{(0)} := \{ \varnothing \}.
\]

For \( n \in \mathbb{N} \) there is a natural bijection between the spaces \( \Gamma^{(n)} \) and the symmetrization \( (\mathbb{R}^2)^n / S_n \), of \( (\mathbb{R}^2)^n := \{ (x_1, \ldots, x_n) \in (\mathbb{R}^2)^n \mid x_i \neq x_j \text{ if } i \neq j \} \) under the permutation group \( S_n \) over \( \{1, \ldots, n\} \) acting on \( (\mathbb{R}^2)^n \) by permuting the coordinate indexes. This bijection induces a metrizable topology on \( \Gamma^{(n)} \) and we endow \( \Gamma_0 \) with the metrizable topology of disjoint union of topological spaces. We denote the corresponding Borel-\( \sigma \)-algebra on \( \Gamma^{(n)} \) and \( \Gamma_0 \) by \( \mathcal{B}(\Gamma^{(n)}) \) and \( \mathcal{B}(\Gamma_0) \), respectively. Let \( \mathcal{B}(\mathbb{R}^2) \) denote the family of all Borel sets of \( \mathbb{R}^2 \) that have compact closure and for \( \Lambda \in \mathcal{B}(\mathbb{R}^2) \) let \( \Gamma_\Lambda := \{ \gamma \in \Gamma \mid \gamma \subset \Lambda \} \). Evidently,

\[
\Gamma_\Lambda = \bigsqcup_{n=0}^{\infty} \Gamma^{(n)}_\Lambda, \quad \text{where } \Gamma^{(n)}_\Lambda := \Gamma_\Lambda \cap \Gamma^{(n)}, \quad n \in \mathbb{N}_0 := \mathbb{N} \cap \{0\},
\]

leading to a situation similar to the one of \( \Gamma_0 \), described above. We endow \( \Gamma_\Lambda \) with the topology of disjoint union of topological spaces and with the corresponding Borel-\( \sigma \)-algebra \( \mathcal{B}(\Gamma_\Lambda) \).
Two-component configuration space

Given two copies of the space $\Gamma$, denoted by $\Gamma^+$ and $\Gamma^-$, let

$$\Gamma^2 := \{(y^+, y^-) \in \Gamma^+ \times \Gamma^- \mid y^+ \cap y^- = \emptyset\}.$$ 

Similarly, given two copies of the space $\Gamma_0$, denoted by $\Gamma_0^+$ and $\Gamma_0^-$, we consider the space

$$\Gamma_0^2 := \{(\eta^+, \eta^-) \in \Gamma_0^+ \times \Gamma_0^- \mid \eta^+ \cap \eta^- = \emptyset\}.$$ 

We endow $\Gamma^2$ and $\Gamma_0^2$ with the topology induced by the product of the topological spaces $\Gamma^+ \times \Gamma^-$ and $\Gamma_0^+ \times \Gamma_0^-$, respectively, and with the corresponding Borel-$\sigma$-algebras, denoted by $\mathcal{B}(\Gamma^2)$ and $\mathcal{B}(\Gamma_0^2)$, respectively.

**Remark 1**

(i) A $\mathcal{B}(\Gamma_0^2)$-measurable function $G: \Gamma_0^2 \to \mathbb{R}$ is called quasi observable.

(ii) A bounded quasi observable $G$ has bounded support ($G \in B_{bs}(\Gamma_0^2)$, for short), whenever

$$G|_{\Gamma_0^2} \left(\prod_{\mathbb{N}_0} \Gamma_0^+ \times \prod_{\mathbb{N}_0} \Gamma_0^-\right) = 0 \quad \text{for some } N^+, N^- \in \mathbb{N}_0, \quad \Lambda^+, \Lambda^- \in \mathcal{B}(\mathbb{R}^2).$$

In this way, we can define the (two-component) K-transform $\mathcal{K}$.

**Definition 2 (Two-component K-transform)**

Given a function $G \in B_{bs}(\Gamma_0^2)$ the mapping

$$\Gamma_0^2 \ni (\eta^+, \eta^-) \mapsto (\mathcal{K}G)(\eta^+, \eta^-) := \sum_{\eta^+ \subset \eta^-} \sum_{\eta^+ \subset \eta^-} G(\eta^+, \eta^-) \in \mathbb{R}$$

is well-defined and is called the (two-component) K-transform $\mathcal{K}$ of $G$.

**Remark 3**

(i) It is clear by (1) that for a given $G \in B_{bs}(\Gamma_0^2)$, its K-transform $\mathcal{K}G$ is a polynomially bounded cylinder function such that $\mathcal{K}G(y^+, y^-) = \mathcal{K}G(y^+ \cap \Lambda^+, y^- \cap \Lambda^-)$ for all $(y^+, y^-) \in \Gamma^2$ and, for each constant $C \geq |G|$, 

$$|\mathcal{K}G(y^+, y^-)| \leq C(1 + \#(y^+ \cap \Lambda^+))^{N^+} (1 + \#(y^- \cap \Lambda^-))^{N^-} \quad \text{for all } (y^+, y^-) \in \Gamma^2.$$

(ii) Moreover, $\mathcal{K}: B_{bs}(\Gamma_0^2) \to \mathcal{F}(\mathcal{P}(\Gamma^2)) := \mathcal{K}(B_{bs}(\Gamma_0^2))$ is a linear and positivity preserving isomorphism whose inverse mapping is defined by

$$\Gamma_0^2 \ni (\eta^+, \eta^-) \mapsto (\mathcal{K}^{-1}F)(\eta^+, \eta^-) := \sum_{\xi^+, \xi^- \subset \eta^+, \eta^-} (-1)^{\#(\xi^+ \cap \eta^+)} F(\xi^+, \xi^-) \in \mathbb{R}.$$  

**Remark 4**

Given any $\mathcal{B}(\Gamma^2)$-measurable function $F$, also called observable, note that the right-hand side of (2) is also well defined for $F|_{\Gamma_0^2}$. In this case since there will be no risk of confusion, we will denote the right-hand side of (2) by $\mathcal{K}^{-1}F$.

Let $\mathcal{M}^1_{\text{fin}}(\Gamma^2)$ denote the set of probability measures $\mu$ on $(\Gamma^2, \mathcal{B}(\Gamma^2))$ with finite local moments of all orders,

$$\int_{\Gamma^2} \left(\#(y^+ \cap \Lambda)^n \#(y^- \cap \Lambda)^n\right) d\mu(y^+, y^-) < \infty \quad \text{for all } n \in \mathbb{N} \text{ and all } \Lambda \in \mathcal{B}(\mathbb{R}^2).$$

Given $\mu \in \mathcal{M}^1_{\text{fin}}(\Gamma^2)$, the so-called correlation measure $\rho_\mu$ corresponding to $\mu$ is a measure on $(\Gamma_0^2, \mathcal{B}(\Gamma_0^2))$ defined for all $G \in B_{bs}(\Gamma_0^2)$ by

$$\int_{\Gamma_0^2} G(\eta^+, \eta^-) d\rho_\mu(\eta^+, \eta^-) = \int_{\Gamma^2} (\mathcal{K}G)(\eta^+, \eta^-) d\mu(\eta^+, \eta^-).$$

(4)

Note that under these assumptions $\mathcal{K}\mu$ is $\mu$-integrable and thus (4) is well-defined. In terms of correlation measures this means that $B_{bs}(\Gamma_0^2) \subset L^1(\Gamma_0^2, \rho_\mu)$. In fact, $B_{bs}(\Gamma_0^2)$ is dense in $L^2(\Gamma_0^2, \rho_\mu)$. Moreover, by (4) the inequality $\|\mathcal{K}\mu\|_{L^1(\Gamma^2, \rho_\mu)} \leq \|G\|_{L^1(\Gamma_0^2, \rho_\mu)}$ holds on $B_{bs}(\Gamma_0^2)$, allowing an extension of the K-transform to a bounded linear operator.
\( \mathcal{K} : L^1(\Gamma^0_\mu) \to L^1(\Gamma^2; \mu) \) in such a way that (4) still holds for \( G \in L^1(\Gamma^2; \mu) \). For the extended operator the explicit form (1) still holds now \( \mu \)-a.e. In terms of correlation measures, property (3) means that \( \rho_\mu \) is locally finite, i.e., 
\[ \rho_\mu((\Gamma^0_\lambda \times \Gamma^0_\lambda) \cap \Gamma^0_\mu) < \infty \]
for all \( n, m \in \mathbb{N}_0 \) and \( \Lambda \in \mathcal{B}_r(\mathbb{R}^2) \).

Let \( \sigma \) be a non-atomic Radon measure on \((\mathbb{R}^2, \mathcal{B}(\mathbb{R}^d))\) with \( \sigma(\mathbb{R}^2) = +\infty \). Consider e.g., the Lebesgue measure on \((\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))\). We equip \((\mathbb{R}^2)^n, \mathcal{B}(\mathbb{R}^2)^n)\) with the \( n \)-dimensional product measure \( \sigma^{(n)} := \sigma_1 \cdots \sigma_n \). For \( z \in (0, \infty) \) the Lebesgue–Poisson measure on \((\Gamma_0, \mathcal{B}(\Gamma_0))\) with intensity measure \( \sigma_z := z \sigma \) is given by
\[ \lambda_{\sigma_z} := \sum_{n=0}^{\infty} \frac{z^n}{n!} \sigma^{(n)}, \]
where \( \sigma^{(n)}, n \in \mathbb{N} \), is the image measure on \((\Gamma^n, \mathcal{B}(\Gamma^n))\) of the product measure \( \sigma^{(n)} \) under the mapping
\[ \text{sym}^{(n)}: (\mathbb{R}^2)^n / \mathcal{S}_n \ni (x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\} \in \Gamma^n. \]

For \( n = 0 \) one sets \( \sigma^{(0)}(\{\emptyset\}) := 1 \). Taking into account that
\[ \lambda_{\sigma_z}(\Gamma_\lambda) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sigma^{(n)}(\Gamma_\lambda) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sigma(\Lambda)^n = \exp \left( \sigma_z(\Lambda) \right), \]
we define a probability measure \( \pi_{\sigma_z}^\lambda := \exp(-\sigma_z(\Lambda)) \lambda_{\sigma_z} \) on \((\Gamma_\lambda, \mathcal{B}(\Gamma_\lambda))\). Using for \( \Lambda \in \mathcal{B}_r(\mathbb{R}^d) \) the projection
\[ \Gamma \ni \gamma \mapsto p_\lambda(\gamma) := \gamma \cap \Lambda \in \Gamma_\lambda, \]
and applying a version of Kolmogorov's theorem for projective limit spaces we obtain a unique measure \( \pi_{\sigma_z} \), as the limit of the family \((\Gamma_\lambda, \mathcal{B}(\Gamma_\lambda)) | \Lambda \in \mathcal{B}_r(\mathbb{R}^d)\), \( \pi_{\sigma_z} \), is called Poisson measure on \((\Gamma, \mathcal{B}(\Gamma))\) with respect to the intensity measure \( \sigma_z \). The product measure \( \pi_{\sigma_z}^\lambda := \pi_{\sigma_z} \otimes \lambda_{\sigma_z} \) defines a probability measure on \((\Gamma^2, \mathcal{B}(\Gamma^2))\).

If for \( \mu \in M^{1}_m(\Gamma^2) \) the corresponding correlation measure \( \rho_\mu \) is absolutely continuous with respect to the Lebesgue–Poisson measure \( \lambda^\mu := \lambda_{\sigma_z}^\mu := \lambda_{\sigma_z} \otimes \lambda_{\sigma_z} \) on \((\Gamma^2_\mu, \mathcal{B}(\Gamma^2_\mu))\). Its Radon–Nikodym derivative \( k_\mu := dp_\mu / d\lambda^\mu \) is called the correlation functional corresponding to \( \mu \).

### Markov generators and related evolution equations

Heuristically, the stochastic evolution of an infinite two-component particle system is described by a Markov process on \( \Gamma^2 \), which is determined by a Markov generator \( L \) defined on a proper space of functions on \( \Gamma^2 \). In particular, we consider the following space of functions on \( \Gamma^2_\mu \): For \( \kappa, \kappa_- \in (0, \infty) \) we define \( \kappa := (\kappa, \kappa_-) \) and
\[ \Gamma^2_\mu \ni (\eta^+, \eta^-) \mapsto \varphi_\kappa(\eta^+, \eta^-) := \exp \left( \kappa \cdot \eta^+ + \kappa_- \cdot \eta^- \right) \in (0, \infty). \]

Moreover, we consider the measure \( \nu_\kappa := \varphi_\kappa \lambda^2 \) on \((\Gamma^2_\mu, \mathcal{B}(\Gamma^2_\mu))\) and the corresponding Banach space of integrable functions \( \mathbb{B}_\kappa := L^1(\Gamma^2_\mu; \nu_\kappa) \), where the canonical norm is denoted by \( \|\cdot\|_\kappa \). The dual space of correlation functions \( k \in \mathcal{B}_{-\kappa} = L^\infty(\Gamma^2_\mu; \nu_{-\kappa}) \), where \( \nu_{-\kappa} := \varphi_\kappa^{-1} \lambda^2 \) and \( \|k\|_{-\kappa} := \lambda^2 \text{-ess sup}_{(\eta^+, \eta^-) \in \Gamma^2_\mu} [k(\eta^+, \eta^-)] \varphi_\kappa^{-1} \). The dual pairing of the spaces \( \mathbb{B}_\kappa \) and \( \mathbb{B}_{-\kappa} \) is given by
\[ \langle G, k \rangle := \int_{\Gamma^2_\mu} G(\eta^+, \eta^-) k(\eta^+, \eta^-) \, d\lambda^2(\eta^+, \eta^-), \] (5)
and we have \( \|\langle G, k \rangle\| \leq \|G\|_\kappa \|k\|_{-\kappa} \). For pairs \( \kappa' := (\kappa'_+, \kappa'_-) \) and \( \kappa = (\kappa_+, \kappa_-) \) we write \( \kappa' < \kappa \) if \( \kappa'_+ < \kappa_+ \) and \( \kappa'_- < \kappa_- \) holds. In such a case for an operator \( \hat{\mathcal{L}} \in \mathcal{L}(\mathbb{B}_{\kappa_-}, \mathbb{B}_{\kappa'_-}) \) for \( \kappa' < \kappa \) and its dual operator \( \mathcal{L}^* \in \mathcal{L}(\mathbb{B}_{\kappa'_+}, \mathbb{B}_{\kappa_-}) \) we have
\[ \epsilon \|\hat{\mathcal{L}}\|\kappa_- = \epsilon' \|\mathcal{L}^*\|_{\kappa'_-}, \]
where \( \epsilon \|\cdot\|_{\kappa'}, \epsilon' \|\cdot\|_{\kappa'_-} \) denote the corresponding operator norms. Assume there exists a measurable function \( M_\kappa : \Gamma^2_\mu \to (0, \infty) \) such that
\[ \|\hat{\mathcal{L}}G\|_\kappa = \int_{\Gamma^2_\mu} |\hat{\mathcal{L}}G(\eta^+, \eta^-)| \varphi_\kappa(\eta^+, \eta^-) \, d\lambda^2(\eta^+, \eta^-) \leq \int_{\Gamma^2_\mu} M_\kappa(\eta^+, \eta^-) |G(\eta^+, \eta^-)| \varphi_\kappa(\eta^+, \eta^-) \, d\lambda^2(\eta^+, \eta^-) = \|M_\kappa G\|_\kappa. \]
Thus the operator \((\hat{L}, D(M_0))\) is well-defined on \(D(M_0) := \{ G \in \mathcal{B}_K | M_\varepsilon \cdot G \in \mathcal{B}_K \} \).

In applications there is a need of knowledge on certain characteristics of the stochastic evolution in terms of mean values rather than pointwise. These characteristics concern e.g. observables, i.e., functions defined on \(\Gamma^2\) for which expected values are given by

\[
\langle F, \mu \rangle := \int_{\Gamma^2} F(\gamma^+, \gamma^-) d\mu(\gamma^+, \gamma^-),
\]

where \(\mu\) is a probability measure on \(\mathcal{B}(\Gamma^2)\), i.e., a state of the system. This leads to the following time evolution problem on states,

\[
\frac{d}{dt} \langle F, \mu_t \rangle = \langle LF, \mu_t \rangle, \quad \mu_t|_{t=0} = \mu_0.
\]

(6)

For \(F\) being of type \(F = \mathcal{K}G, G \in B_{\mathcal{B}_K}(\Gamma^2_0)\), (6) may be rewritten in terms of correlation functionals \(k_t := k_{\mu_t}\) corresponding to the measures \(\mu_t\) provided these functionals exist (or, more generally, in terms of correlation measures \(\rho_t := \rho_{\mu_t}\)), yielding

\[
\frac{d}{dt} \langle G, k_t \rangle = \langle \hat{L}^* G, k_t \rangle, \quad k_t|_{t=0} = k_0,
\]

(7)

where \(\hat{L} := \mathcal{K}^{-1}L\mathcal{K}\) and \(\langle \cdot, \cdot \rangle\) is the usual pairing provided in (5). A strong version of Equation (7) reads

\[
\frac{d}{dt} k_t = \hat{L}^* k_t, \quad k_t|_{t=0} = k_0,
\]

(8)

with \(\hat{L}^*\) being the dual operator of \(\hat{L}\) in the sense defined in (5).

One may associate to any function \(k\) on \(\Gamma^2_0\) a double sequence \(\{k^{(n,m)}\}_{n,m \in \mathbb{N}_0}\), where

\[
k^{(n,m)} := k|_{((q^+, q^-) \in \Gamma^2_0 | \#q^+ = n, \#q^- = m)},
\]

(9)

are symmetric functions on \((\mathbb{R}^2)^n \times (\mathbb{R}^2)^m, n, m \in \mathbb{N}_0\).

Remark 5 For \(n, m \in \mathbb{N}_0\), \(k^{(n,m)}\) describes the moments of the state \(\mu\) and in the special case \(n = m = 1\) the function \(k^{(1,1)}\) is the density of the system, whereas \(k^{(1,0)}\) and \(k^{(0,1)}\) correspond to the density of (+) and (-) particles, respectively.

Related to (8) one has a countably infinite number of equations having a hierarchical structure,

\[
\frac{d}{dt} k^{(n,m)}_t = \hat{L}^* k^{(n,m)}_t, \quad k^{(n,m)}_t|_{t=0} = k^{(n,m)}_0, \quad n, m \in \mathbb{N}_0,
\]

(10)

where each equation only depends on a finite number of coordinates. As a result we have reduced the infinite dimensional problem (6) to the infinite system of equations (10). However, recall that due to (7) we are only interested in weak solutions to (10). To derive solutions to (6) from solutions to (7) an additional analysis is needed, namely, to distinguish the correlation functionals from the set of solutions to (7).

**General strategy of Vlasov scaling**

The so-called Vlasov scaling or kinetic scaling starts with scaling the Markov pre-generator of the underlying dynamics with respect to parameter \(\varepsilon > 0\) in a proper way. Therefore, we get a scaled version of equations (8) and (10), i.e., \(\hat{L}_\varepsilon^*\) instead of \(\hat{L}^*\). The next important step to construct the Vlasov scaling concerns the proper rescaling of the initial state of the system. Or, equivalently, in the language of correlation functions it means the proper rescaling of the initial conditions of the evolution of correlation functions. More precisely, at the beginning we scale \(k_0\) with parameter \(\varepsilon > 0\) in such a way that the resulting functions \(k_{0,\varepsilon}\) as \(\varepsilon \to 0\) behave as follows:

\[
k^{(n,m)}_{0,\varepsilon}(\eta) := \varepsilon^{n+m} k^{(n,m)}_{0,\varepsilon}(\eta) \to k^{(n,m)}_0(\eta), \quad \varepsilon \to 0, \quad \eta \in \Gamma_0,
\]

(11)

where for \(n, m \in \mathbb{N}_0\) the symmetric functions \(k^{(n,m)}_0\) are a subject of choice for our concrete example.
An important case is to take
\[
 r_0^{(n,m)}(x_1, \ldots, x_n, y_1, \ldots, y_m) = \rho_0^+(x_1) \cdots \cdot \rho_0^+(x_1) \cdot \rho_0^+(y_1) \cdots \cdot \rho_0^+(y_m), \quad \rho_0^+, \rho_0^- : \mathbb{R}^d \to (0, +\infty),
\]
corresponding to an independent initial distribution of (+) and (-) particles.

It is clear that such a rescaling of the initial condition leads to a singular function with respect to \( \varepsilon > 0 \). In applications, this fact can be interpreted as the growth of density of the system with \( \varepsilon \to 0 \). For \( n, m \in \mathbb{N}_0 \) we denote by \( k_{t, \varepsilon}^{(n,m)} \) the solution of the functional evolution
\[
 \frac{d}{dt} k_{t, \varepsilon}^{(n,m)} = L^* k_{t, \varepsilon}^{(n,m)} , \quad k_{t, \varepsilon}^{(n,m)}|_{t=0} = k_{0, \varepsilon}^{(n,m)}.
\]
On can expect that this solution will be also singular with respect to \( \varepsilon > 0 \). Moreover, we should choose a type of scaling of the generators which preserves the order of this singularity. Namely, for \( n, m \in \mathbb{N}_0 \) and \( \varepsilon > 0 \) we consider
\[
 r_{t, \varepsilon}^{(n,m)} := e^{n+m t} r_{t, \varepsilon}^{(n,m)},
\]
and assume that
\[
 r_{t, \varepsilon}^{(n,m)} \to r_{t, 0}^{(n,m)}, \quad \varepsilon \to 0.
\]

This is equivalent to investigating the Cauchy problem for the operators \( \hat{L}_{\varepsilon,x} = R^{*} \hat{L}_{\varepsilon, x}^* R^{-1} \), \( \varepsilon > 0 \), where for \( \delta > 0 \) and a correlation function \( k \) we have \( (R_{\delta} k)(\eta^+, \eta^-) := \delta^{|\eta^+|+|\eta^-|} k(\eta^+, \eta^-) \). Hence the associated Cauchy problem reads
\[
 \frac{d}{dt} r_{t, x} = \hat{L}_{\varepsilon, x} r_{t, x}, \quad r_{t, x}|_{t=0} = r_{0, x},
\]
where we use the identification via (9). We seek for the limit \( \hat{L}_{\varepsilon, x} \to V \) as \( \varepsilon \to 0 \). Using the initial condition \( r_{t, x}|_{t=0} = r_0 \), where \( r_0 \) is associated to the sequence \( r_{0, m}^{(n,m)} \) in (11), the solution \( r_t \) of Vlasov equation
\[
 \frac{d}{dt} r_t = V r_t, \quad r_t|_{t=0} = r_0,
\]
clearly implies that the associated sequence \( r_{t, 0}^{(n,m)} \) is again of the form
\[
 r_{t, 0}^{(n,m)}(x_1, \ldots, x_n, y_1, \ldots, y_m) = \rho_t^+(x_1) \cdots \cdot \rho_t^+(x_1) \cdot \rho_t^+(y_1) \cdots \cdot \rho_t^+(y_m), \quad \rho_t^+, \rho_t^- : \mathbb{R}^d \to (0, +\infty),
\]
where \( \rho_t^+, \rho_t^- \) are determined by the kinetic equations
\[
 \frac{\partial}{\partial t} \rho_t^+ = v^+(\rho_t^+, \rho_t^-), \quad \frac{\partial}{\partial t} \rho_t^- = v^-(\rho_t^+, \rho_t^-),
\]
(13)
where \( (v^+, v^-) \) are derived from the (nonlinear) limiting operator \( V \).

In other words, considering the Vlasov equation (12) with initial condition \( r_0 \) of the form
\[
 r_0(\eta^+, \eta^-) = e_d(\rho_0^+, \rho_0^-, \eta^+, \eta^-) := \prod_{x \in \eta^+} \rho_0^+(x) \prod_{y \in \eta^-} \rho_0^-(y), \quad \eta^+ \in \Gamma^+, \eta^- \in \Gamma^-,
\]
the solution \( r_t \) at time \( t \in [0, \infty) \) is of the same type, i.e.
\[
 r_t(\eta^+, \eta^-) = e_d(\rho_t^+, \rho_t^-, \eta^+, \eta^-) := \prod_{x \in \eta^+} \rho_t^+(x) \prod_{y \in \eta^-} \rho_t^-(y), \quad \eta^+ \in \Gamma^+, \eta^- \in \Gamma^-,
\]
and \( \rho_t^+ \) and \( \rho_t^- \) for \( t \in [0, \infty) \) are determined by the kinetic equations (13).
MARKOV EVOLUTION

Evolution of observables

Modeling infection of particles

We consider the evolution of a two-component system in the state space $\Gamma^2$ such that, at each random moment of time, a mark $(\pm)$ may flip to $(-)$, while keeping the site:

$$(\gamma^+, \gamma^-) \mapsto (\gamma^+ \setminus \{x\}, \gamma^- \cup \{x\}), \quad x \in \gamma^+.$$  

The Markov pre-generator $L^\text{flip}_{\pm}$ for the evolution of observables $F \in \mathcal{FP}(\Gamma^2) := K(B_0(\Gamma^2))$ is described by

$$(L^\text{flip}_{\pm}F)(\gamma^+, \gamma^-) := \sum_{x \in \gamma^+} c^+(x, \gamma^-)(F(\gamma^+ \setminus \{x\}, \gamma^- \cup \{x\}) - F(\gamma^+, \gamma^-)),  \quad (14)$$

where $c^+(x, \gamma^-) \geq 0$ is the rate at which a $(\pm)$ particle at $x \in \gamma^+$ flips to a $(-)$ particle in dependence of the surrounding $(-)$ particles.

Specification of the flip rate. In our model the prescribed flip from a $(\pm)$ particle to a $(-)$ particle shall be interpreted as a healthy particle getting infected by the surrounding infected particles with a certain rate of infection $c^+$. We describe the particular form of the flip rate $c^+$ next. With $R \in (0, \infty)$ we denote the maximal distance of possible infection and set $i_0 \in (0, 1]$ to be the risk of infection for a single healthy individual at direct contact with an infected one. Via the function

$$(0, \infty) \ni r \mapsto \phi_R(r) := \phi_0(0, r) \equiv \phi_R \in (0, \infty),$$

we describe the risk of infection for a healthy individual depending on distance to a single infected individual, where $\phi_R$ is e.g. of the form given in Figure 1. For fixed $x \in \mathbb{R}^2$ the rate of infection for a single healthy individual at location $x$ in the surrounding $\gamma^- \in \Gamma^-$ of infected individuals is given by

$$c^+(x, \gamma^-) := \sum_{y \in \gamma^-} \phi_R(|x - y|), \quad x \in \mathbb{R}^2, \quad \gamma^- \in \Gamma^-.$$  \quad (15)$$

We want (15) to serve as a flip rate in the Markov pre-generator (14).

![Figure 1](https://example.com/figure1.png)

**FIGURE 1.** Distance dependent risk of infection for a susceptible individual in range of an infected one ($R = 0.05$)

Modeling recovery of particles

We consider a second type of evolution in our two-component system in the state space $\Gamma^2$. Namely, at each random moment of time, a $(-)$ particle may flip to $(\pm)$, while keeping the site:

$$(\gamma^+, \gamma^-) \mapsto (\gamma^+ \cup \{y\}, \gamma^- \setminus \{y\}), \quad y \in \gamma^-.$$
The corresponding Markov pre-generator \( L_{\text{flip}}^{+} \) is described by
\[
(L_{\text{flip}}^{+} F)(\gamma^{+}, \gamma^{−}) := \alpha \sum_{y \in \Gamma} \left(F(\gamma^{+} \cup \{ y \}, \gamma^{−} \setminus \{ y \}) - F(\gamma^{+}, \gamma^{−})\right),
\]
where \( \alpha \in [0, 1] \) is the constant rate at which a (−) particle at \( y \in \gamma^{−} \) flips to a (+) particle.

**Modeling the disease**

Combining to above defined Markov pre-generators we obtain a Markov pre-generator
\[
(L_{\text{flip}}^{+} F)(\gamma^{+}, \gamma^{−}) := (L_{\text{flip}}^{-} F)(\gamma^{+}, \gamma^{−}) + (L_{\text{flip}}^{−} F)(\gamma^{+}, \gamma^{−}) = \alpha \sum_{x \in \Gamma} c^{+}(x, \gamma^{−}) (F(\gamma^{+} \setminus \{ x \}, \gamma^{−} \cup \{ x \}) - F(\gamma^{+}, \gamma^{−}))
\]

which describes the evolution of a two-component system in the state space \( \Gamma^{2} \) that models the spread of an infectious disease.

**Evolution of quasi observables**

At this point we are able to transform the generators \( L_{\text{flip}}^{+} \) and \( L_{\text{flip}}^{-} \) to act on quasi observables. Therefore, we consider \( \hat{L}_{\text{flip}}^{+} := \mathcal{K} L_{\text{flip}}^{+} \mathcal{K} \) and \( \hat{L}_{\text{flip}}^{-} := \mathcal{K} L_{\text{flip}}^{-} \mathcal{K} \), respectively. In particular, we have the following result.

**Proposition 6** For \( G \in B^{0}_{\text{dis}}(\Gamma^{2}) \) the generators \( \hat{L}_{\text{flip}}^{+} := \mathcal{K} L_{\text{flip}}^{+} \mathcal{K} \) corresponding to \( L_{\text{flip}}^{+} \) (see (14) and (15)) and \( \hat{L}_{\text{flip}}^{-} := \mathcal{K} L_{\text{flip}}^{-} \mathcal{K} \) corresponding to \( L_{\text{flip}}^{-} \) (see (16)) are given by
\[
(\hat{L}_{\text{flip}}^{+} G)(\gamma^{+}, \gamma^{−}) = \sum_{x \in \Gamma} \sum_{y \in \Gamma} \phi(x) G(\gamma^{+} \setminus \{ y \}, \gamma^{−} \cup \{ x \}) - G(\gamma^{+}, \gamma^{−})
\]

In particular, we have
\[
(L_{\text{flip}}^{-} G)(\gamma^{+}, \gamma^{−}) = \sum_{x \in \Gamma} \sum_{y \in \Gamma} \phi(x) G(\gamma^{+} \setminus \{ y \}, \gamma^{−} \cup \{ x \}) - G(\gamma^{+}, \gamma^{−})
\]

**Evolution of correlation functions**

To describe the evolution of correlation functions we calculate the corresponding dual operator of \( L_{\text{dis}} \), see Corollary 6, with respect to the dual pairing (5).

**Corollary 7** The dual operator \( \hat{L}_{\text{dis}}^{+} \) corresponding to \( L_{\text{dis}} \) is given by
\[
(\hat{L}_{\text{dis}}^{+} k)(\gamma^{+}, \gamma^{−}) = \sum_{x \in \Gamma} \sum_{y \in \Gamma} \phi(x) k(\gamma^{+} \setminus \{ y \}, \gamma^{−} \cup \{ x \}) - \sum_{x \in \Gamma} \sum_{y \in \Gamma} \phi(x) k(\gamma^{+}, \gamma^{−}) + \alpha \sum_{y \in \Gamma} \int_{\mathbb{R}^{2}} \phi(x) k(\gamma^{+} \cup \{ y \}, \gamma^{−} \setminus \{ y \} \cup \{ x \}) dx - \alpha \sum_{y \in \Gamma} \int_{\mathbb{R}^{2}} \phi(x) k(\gamma^{+}, \gamma^{−} \cup \{ y \}) dy
\]

where \( \alpha \in [0, 1] \) is the constant rate at which a (−) particle at \( y \in \gamma^{−} \) flips to a (+) particle.
In order to examine the density of particles of a particular type, we consider the evolution of correlation functions of the following type,
\[
\mathbb{R}^2 \ni x \mapsto k^+(x) := k\big((x), \emptyset\big) = k^{(1,0)}(\eta^+, \eta^-) \in \mathbb{R},
\]
\[
\mathbb{R}^2 \ni y \mapsto k^-(y) := k\big(\emptyset, (y)\big) = k^{(0,1)}(\eta^+, \eta^-) \in \mathbb{R},
\]
and \( \mathbb{R}^2 \times \mathbb{R}^2 \ni (x, y) \mapsto k^{\pm}(x, y) := k\big((x), (y)\big) = k^{(1,1)}(\eta^+, \eta^-) \in \mathbb{R}, \)

(cf. Remark 5). Therefore, the evolution of these functions at \( x \in \mathbb{R}^2 \) for \( t \geq 0 \), conform to (8), is given by
\[
\frac{d}{dt}k^+_t(x) = (L_{\text{dis}}^* k^+_t)(x) = (L_{\text{dis}}^* k_t)(x) = -\int_{\mathbb{R}^2} \phi(x-y)k_t(x, \{y\}) dy + \alpha k(\emptyset, \{x\})
\]
\[
= -\int_{\mathbb{R}^2} \phi(x-y)k^+_t(x, y) dy + \alpha k^-(x),
\]
and
\[
\frac{d}{dt}k^-_t(x) = (L_{\text{dis}}^* k^-_t)(x) = (L_{\text{dis}}^* k_t)(\emptyset, \{x\}) = \int_{\mathbb{R}^2} \phi(x-y)k_t(x, \{y\}) dy - \alpha k(\emptyset, \{x\})
\]
\[
= \int_{\mathbb{R}^2} \phi(x-y)k^-_t(x, y) dy - \alpha k^-(x).
\]

The resulting system of equations reads
\[
\begin{cases}
\frac{d}{dt}k^+_t(x) = -\int_{\mathbb{R}^2} \phi(x-y)k^+_t(x, y) dy + \alpha k^-(x), \\
\frac{d}{dt}k^-_t(x) = \int_{\mathbb{R}^2} \phi(x-y)k^-_t(x, y) dy - \alpha k^-(x).
\end{cases}
\]

Using the interpretation given in Remark 5 we note that the time evolution of the density of particles of (+) and (-), \( k^+_t \) and \( k^-_t \), respectively, depends on the time evolution of the particle density \( k^{\pm} \) of the whole two-component system. Thus we are dealing with a non-closed system.

**Remark 8** Note that the operator \( L_{\text{dis}} \) is just an auxiliary object in order to obtain the evolution of the correlation functions. It has no interpretation in terms of the underlying model.

In order to close the system of equations (17) we apply the method of Vlasov scaling. For \( \varepsilon > 0 \) we first obtain the renormalized pre-generator for the dynamics of quasi-observables, which reads
\[
\big(L_{\text{ren}}^* G\big)(\eta^+, \eta^-) = \sum_{\eta^{-}\in \mathcal{I}^-} \sum_{\eta^+\in \mathcal{I}^+} \phi(\eta^- \setminus \{y\}) G(\eta^+ \setminus \{x\}, \eta^- \setminus \{y\} \cup \{x\})- \sum_{\eta^{-}\in \mathcal{I}^-} \sum_{\eta^+\in \mathcal{I}^+} \phi(\eta^- \setminus \{y\}) G(\eta^+ \setminus \{x\}, \eta^- \setminus \{y\} \cup \{x\})
\]
\[
+ \alpha \sum_{\eta^{-}\in \mathcal{I}^-} \left(G(\eta^+ \cup \{y\}, \eta^- \setminus \{y\}) - G(\eta^+, \eta^-)\right).
\]

Its corresponding dual operator with respect to \( \langle \cdot, \cdot \rangle \) is given by
\[
(Vk)(\eta^+, \eta^-) := (L_{\text{ren}}^* k)(\eta^+, \eta^-) = \sum_{\eta^{-}\in \mathcal{I}^-} \int_{\mathbb{R}^2} \phi(\eta^- \setminus \{y\}) k^+ \big(\eta^+ \cup \{y\}, \eta^- \setminus \{y\} \cup \{x\}\big) dx - \sum_{\eta^{-}\in \mathcal{I}^-} \sum_{\eta^+\in \mathcal{I}^+} \int_{\mathbb{R}^2} \phi(\eta^- \setminus \{y\}) k^+ \big(\eta^+ \cup \{y\}, \eta^- \setminus \{y\} \cup \{x\}\big) dy
\]
\[
+ \alpha \sum_{\eta^{-}\in \mathcal{I}^-} k^+ \big(\eta^+ \setminus \{x\}, \eta^- \setminus \{x\}\big) - \alpha \#\eta^- \cdot k(\eta^+, \eta^-).
\]

Hence we analyze the Vlasov equation
\[
\frac{d}{dt} r_t(\eta^+, \eta^-) = (Vr_t)(\eta^+, \eta^-),
\]
subject to the initial condition
\[
r_0(\eta^+, \eta^-) = e_t(\rho^+_0, \rho^-_0, \eta^+, \eta^-) = \prod_{x \in \mathcal{I}^+} \rho^+_0(x) \prod_{y \in \mathcal{I}^-} \rho^-_0(y), \quad \eta^+ \in \Gamma^+, \ \eta^- \in \Gamma^-.
\]

020008-9
For $t \geq 0$ in the particular situation

$$
\mathbb{R}^2 \ni x \mapsto \rho_t^+(x) := r_t([x], \emptyset) = r_t^{(1,0)}(\eta^+, \eta^-) \in \mathbb{R},
$$

$$
\mathbb{R}^2 \ni y \mapsto \rho_t^-(y) := r_t(\emptyset, [y]) = r_t^{(0,1)}(\eta^+, \eta^-) \in \mathbb{R},
$$

this analysis results in

$$
\begin{cases}
\frac{d}{dt} \rho_t^+(x) = -\int_{\mathbb{R}^2} \phi(|x-y|) \rho_t^+(y) dy \rho_t^+(x) + \alpha \rho_t^-(x), \\
\frac{d}{dt} \rho_t^-(x) = \int_{\mathbb{R}^2} \phi(|x-y|) \rho_t^+(y) dy \rho_t^+(x) - \alpha \rho_t^-(x), \\
\rho_0^+(x) = \rho^+(x), \quad \rho_0^-(x) = \rho^-(x), \quad x \in \mathbb{R}^2, \quad t \geq 0,
\end{cases}
$$

or in other words

$$
\begin{cases}
\frac{d}{dt} \rho_t^*(x) = -\left(\phi \ast \rho_t^*(x)\right) \rho_t^+(x) + \alpha \rho_t^-(x), \\
\frac{d}{dt} \rho_t^-(x) = \left(\phi \ast \rho_t^*(x)\right) \rho_t^+(x) - \alpha \rho_t^-(x), \\
\rho_0^+(x) = \rho^+(x), \quad \rho_0^-(x) = \rho^-(x), \quad x \in \mathbb{R}^2, \quad t \geq 0,
\end{cases}
$$

(18)

where $\ast$ denotes the convolution operator. (18) is the system of kinetic equations to be studied.

**Remark 9** One easily observes from (18) that $\rho_t^+(x) + \rho_t^-(x) = \rho_0^+(x) + \rho_0^-(x)$ for all $t \geq 0$ and $x \in \mathbb{R}^2$. Denoting $\rho(x) := \rho^+(x) + \rho^-(x)$, we find that $\rho_t^+(x) = \rho(x) - \rho_t^-(x)$. Inserting this into the second equation in (18) yields a closed equation for $\rho_t^-(x)$:

$$
\frac{d}{dt} \rho_t^-(x) = \left(\phi \ast \rho_t^-(x)\right) (\rho(x) - \rho_t^-(x)) - \alpha \rho_t^-(x), \\
\rho_0^-(x) = \rho^-(x),
$$

which resembles the nonlocal Fisher–KPP equation known in literature (cf. [11]).

**CONCLUSION**

In this article we set up a microscopic model for the spread of an infectious disease based on configuration space analysis. Using the so-called Vlasov-scaling we obtained the corresponding mesoscopic equations, describing the density of susceptible and infected individuals (particles) in space. The resulting system of equations can be seen as a generalization to a ‘spatial’ SIS-model. The equations showing up in the limiting system are of type which is known in literature as Fisher–Kolmogorov–Petrovsky–Piscounov type, see e.g. [11] and the references therein.

A numerical evaluation of our considerations can be found in [12].

**ACKNOWLEDGMENTS**


We wish Chris Bernido all the best for his 60th birthday.

**REFERENCES**


