Optimal control policies for an inventory system with commitment lead time

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Optimal control policies for an inventory system with commitment lead time

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Email: a.g.d.kok@tue.nl

Abstract
We consider a firm which faces a Poisson customer demand and uses a base-stock policy to replenish its inventories from an outside supplier with a fixed lead time. The firm can use a preorder strategy which allows the customers to place their orders before their actual need. The time from a customer’s order until the date a product is actually needed is called commitment lead time. The firm pays a commitment cost which is strictly increasing and convex in the length of the commitment lead time. For such a system, we prove the optimality of bang-bang and all-or-nothing policies for the commitment lead time and the base-stock policy, respectively. We study the case where the commitment cost is linear in the length of the commitment lead time in detail. We show that there exists a unit commitment cost threshold which dictates the optimality of either a buy-to-order (BTO) or a buy-to-stock strategy. The unit commitment cost threshold is increasing in the unit holding and backordering costs and decreasing in the mean lead time demand. We determine the conditions on the unit commitment cost for profitability of the BTO strategy and study the case with a compound Poisson customer demand.

KEYWORDS
advance demand information, commitment cost, inventory management, preorder strategy

1 | INTRODUCTION

The consequences of demand and supply uncertainties and eventual mismatch between demand and supply are well known to many companies. The need for designing company operations such that this mismatch is minimized or avoided has motivated many researchers and resulted in a rich literature on demand and supply management. Among the various methods, information sharing has received a lot of attention. The benefits of acquiring and providing information about future demand are undeniable. Having information on future customer demand helps companies in reducing their inventory levels without sacrificing high service levels. Customers, who provide information on the timing and quantity of their future demand get a high quality service.

One form of advance demand information (ADI) is a preorder strategy in which customers place orders ahead of their actual need. The preorder strategy is characterized by a commitment lead time which is defined as the time that elapses between the moment an order is communicated by the customer and the moment the order must be delivered to the customer. Although in today’s competitive market firms cannot force their customers to place orders before their actual need, they can tempt them to follow the preorder strategy by giving a bonus. In order to make long commitment lead times acceptable and attractive, companies should propose bonuses which increase with the length of commitment lead times. The commitment lead time contracts reduce the companies’ demand uncertainty risk and the customers’ inventory unavailability risk (Lutze & Özer, 2008). Although
the preorder strategy is a form of ADI, it is different than
the form of ADI that is usually used in the existing litera-
ture. In the existing literature, ADI helps to make a better
forecast of the future customer demand. In this paper, the
demand distribution is known. ADI is utilized operationally
to reduce demand-supply mismatching by reducing the lead
time demand uncertainty. Under the optimal solution, lower
lead time demand uncertainty results in lower holding and
backordering costs. This form of ADI works for service and
custom-production companies, where service customers can
make reservations and customers of custom products order in
advance of their needs (Hariharan & Zipkin, 1995).

In the business-to-business (B2B) environment this is typ-
ically the case when the customer is planning production
to efficiently exploit resources, whereby plans are frozen a
few weeks ahead of time. Thus the moment of need for
materials with short shipment lead times is known earlier
and the supplier can be informed about it. By doing so,
the supplier can save on inventory holding cost and exploit
the early demand information to produce more efficiently.
The latter impact is out of scope of this paper. Also in the
business-to-consumer (B2C) environment many online pur-
chases are not time-critical, such as books and electronic
deVICES, and a bonus may seduce the customer to accept a later
moment of delivery. Once identifying the possible mutual
benefit of early order placement, the question arises what
commitment lead time should be chosen. Thus, providing
incentives for customers to inform their supplier earlier than
the point in time determined by the moment of need and the
shipment lead time may substantially reduce cost for the sup-
plier, while hardly having an impact on the customer cost.
This paper discusses both the potential benefits of preordering
and the optimal commitment lead time choice.

We study the preorder strategy of a firm in a single item,
single location setting. The firm faces random customer
demand and uses a continuous-review base-stock policy to
replenish its inventory from an uncapacitated supplier with
a deterministic lead time. Under the aforementioned setting,
the firm offers a preorder strategy to its customers and conse-
quently, they are paid a commitment cost. The commitment
cost function is strictly increasing and convex in the length
of the commitment lead time. Since a commitment lead time
longer than the replenishment lead time does not have any
effect on reducing the lead time demand uncertainty, the
commitment lead time is bounded by zero and the length of
the replenishment lead time. The firm aims to evaluate this
preorder strategy and find the optimal length of the com-
mitment lead time and the optimal base-stock level, which
minimize the total long-run average cost. This cost is the
sum of long-run average holding, backordering, and commit-
ment costs. We formulate the total long-run average cost and
answer the following questions:

1. When and how should the firm use the preorder
strategy?

Based on the structure of the commitment cost, we find the
sufficient conditions under which the firm should offer the
preorder strategy. More specifically, assuming a linear com-
mitment cost per time unit, we find a unit commitment cost
threshold such that for any unit commitment cost below the
threshold it is better for the firm to offer the preorder strategy.
The threshold is a function of the mean lead time demand,
holding and backordering unit costs. By means of this unit
commitment cost threshold the firm can decide whether offer-
ing the preorder strategy is cost effective or not. When the
preorder strategy is beneficial to the firm, the firm should
choose a strategy that is similar in spirit to a make-to-order
production strategy. In our context, we call this a buy-to-order
(BTO) strategy. This strategy works for car dealers, expen-
sive furniture manufacturers, and so on. When preordering is
not beneficial, the firm should use a strategy that is similar in
spirit to a pure make-to-stock production strategy. In our con-
text, we call this a buy-to-stock (BTS) strategy. This strategy
works for grocery products, clothing, and so on.

2. What are the optimal commitment lead time and its
 corresponding optimal base-stock level?

The optimal commitment lead time and the optimal base-
stock level are not independent from each other. We charac-
terize the optimal base-stock level and its corresponding optimal
commitment lead time. We prove the optimality of bang-bang
and all-or-nothing policies for the commitment lead time and
the base-stock policy, respectively. We show that the optimal
commitment lead time is either zero or equal to the replen-
ishment lead time. We show that when the commitment
lead time is zero, the corresponding base-stock level is the
solution of the well-known Newsvendor problem with deter-
ministic lead time and when the commitment lead time equals
the replenishment lead time, the corresponding optimal
base-stock level is zero. Consistent with the literature we call
this policy as an all-or-nothing policy (Lutz & Özer, 2008).

3. Which factors have impact on the benefits of the
preorder strategy?

Through exact sensitivity analysis on the unit commitment
cost threshold, we provide insights on the benefits of the pre-
order strategy. We find that the preorder strategy helps with
high demand uncertainty, even when the unit commitment
cost is high. Similarly, the preorder strategy benefits the firm
when the unit holding and backordering costs increase, even
when the unit commitment cost is high. We also find that
when demand uncertainty is low, the unit commitment cost
threshold is more robust to changes in the unit holding and
backordering costs.

Scholars have studied inventory management with ADI
broadly from different perspectives. They have considered
several bonus conditions for providing ADI. We study the
impact of commitment cost as a function of the commit-
ment lead time in a firm with perfect ADI, continuous-review,
deterministic replenishment lead time, and Poisson demand.
A similar setting has only been studied in Hariharan and Zipkin (1995) but the authors do not assign a cost to the commitment lead time. Assuming a commitment cost strictly increasing and convex in the commitment lead time, we contribute to the literature by characterizing the optimal pre-ordering strategy and the corresponding optimal replenishment strategy. The results of this study can serve as a building block for characterizing the optimal preorder and replenishment strategies for more complicated assemble-to-order systems. In addition, firms can use our results to evaluate the potential of preorder strategies and can make decisions on rejecting or accepting a preorder strategy.

The rest of this paper is organized as follows. In Section 2, we provide a brief review of related literature. In Section 3, we formulate the problem. In Section 4, we find a lower bound for the minimum cost function and characterize the optimal policies in terms of the optimal commitment lead time and the corresponding optimal base-stock level. In Section 5, we study a linear commitment cost and determine the conditions under which the preorder strategy is optimal. In Section 6, we extend our results to the compound Poisson demand case. We provide our concluding remarks in Section 7. We defer the proofs to the Appendix.

2 | LITERATURE REVIEW

The literature on ADI assumes either perfect or imperfect demand information available ahead of the realization of actual demand. This literature can be broadly classified into two categories based on the accuracy of the demand information. These categories are perfect ADI and imperfect ADI.

When a firm has perfect ADI, customers place orders ahead of time in specific quantities to be delivered at specified due dates. Hariharan and Zipkin (1995) are the first to study the perfect ADI situation in a continuous-review setting. After this seminal work, many researchers assume perfect ADI and study different problems. We provide a summary of the most relevant literature on perfect ADI in Table 1.

When a firm has imperfect ADI, customers place their orders in advance but they provide only an estimate of either the actual due dates or order sizes (Gayon, Benjaafar, & De Véricourt, 2009). We provide a summary of the most relevant literature on imperfect ADI in Table 2.

Multiple studies consider both perfect and imperfect ADI. These are listed in Table 3.

Our work belongs to the category of perfect ADI. The closest study to our research is by Hariharan and Zipkin (1995). The authors study a continuous-review, single item, single-location inventory system with perfect ADI. Demand is according to a Poisson process and each customer order has a delivery due date. The firm uses a preorder strategy, which requires that customers place their orders before their actual need. The time between an order placement and its due date is called commitment lead time. Assuming that this commitment lead time is the same for all customers, Hariharan and Zipkin (1995) prove the optimality of a base-stock policy. The authors consider three different settings for both replenishment and commitment lead times; constant, independent stochastic, and sequential stochastic. For each case, they formulate an equivalent inventory model by replacing the actual replenishment lead time with the difference between the replenishment and commitment lead times. We consider the same setting as Hariharan and Zipkin (1995). Different

### Table 1: Literature on perfect ADI

<table>
<thead>
<tr>
<th>Authors</th>
<th>System characteristics</th>
<th>Results</th>
</tr>
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<tbody>
<tr>
<td>Thonemann (2002)</td>
<td>A multiproduct, single-location periodic-review system with two types of ADI</td>
<td>Both the seller and the customers benefit from sharing ADI, but sharing ADI increases the bullwhip effect</td>
</tr>
<tr>
<td>Karaesmen, Liberopoulos, and Dallery (2004)</td>
<td>A capacitated single-location system</td>
<td>Identify the conditions under which ADI may bring significant benefits</td>
</tr>
<tr>
<td>McCardle, Rajaram, and Tang (2004)</td>
<td>Two retailers with an “advance booking discount (ABD)” program</td>
<td>Establish conditions under which the unique equilibrium calls for launching the ABD program at both retailers</td>
</tr>
<tr>
<td>Tang, Rajaram, Alptekinoğlu, and Ou (2004)</td>
<td>A single-location system with two customer demand classes and ABD program</td>
<td>Evaluate the benefits of the ABD program and characterize the optimal discount price</td>
</tr>
<tr>
<td>Wijnegaard and Karaesmen (2007)</td>
<td>A capacitated single-location continuous-review system</td>
<td>Introduce a threshold value that allows order aggregation to find the optimal production policy</td>
</tr>
<tr>
<td>Liberopoulos (2008)</td>
<td>A single-location continuous-review system</td>
<td>Provide a sufficient condition under which the tradeoff between inventory and ADI is linear</td>
</tr>
<tr>
<td>Wang and Toktay (2008)</td>
<td>A single-location periodic-review system with homogeneous and heterogeneous customers</td>
<td>Determine an optimal state-dependent policy and a tractable approximation for homogeneous and exogenous customers, respectively</td>
</tr>
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<td>Papier and Thonemann (2010)</td>
<td>A rental company with two customer demand classes</td>
<td>The optimal admission policy is a threshold policy</td>
</tr>
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<td>Li and Zhang (2013)</td>
<td>A single-location system with two customer demand classes</td>
<td>Accurate demand information may improve product availability</td>
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<tr>
<td>Iida (2015)</td>
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<td>The benefits of demand forecast information may significantly decrease as lead time uncertainty increases</td>
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<tr>
<td>Papier (2016)</td>
<td>A capacitated single-location system with different markets</td>
<td>Determine the optimal solution and an efficient heuristic policies under relaxed conditions and general conditions, respectively</td>
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from them, we have an additional cost component, which is the commitment cost. In addition to the base-stock level, we also optimize the commitment lead time. We prove the optimality of a so-called bang-bang policy for the commitment lead time, that is, it is either 0 or equal to the replenishment lead time, and we show that the corresponding optimal base-stock policy is an all-or-nothing policy.

3 | PROBLEM FORMULATION

We consider a firm managing the inventory of a single item. The firm uses a continuous-review base-stock policy with base-stock level \( s \geq 0 \) to replenish its inventory from an uncapacitated supplier. The replenishment lead time, \( L \), is constant. Customer orders/demands describe a Poisson process with a rate \( \lambda \). Each customer orders a single unit. The firm uses a preorder strategy, which requires that customers place their orders \( \psi \) time units before their actual need. We say that the demand occurs \( \psi \) time units after the corresponding order. We call \( \psi \) the commitment lead time. We assume that \( \psi \) is continuous. However, our results would still hold for discrete \( \psi \). If \( \psi > L \), the firm can meet every demand without holding inventory by placing a replenishment order \( \psi - L \) time units after a customer order occurs. This results in zero holding and backordering costs. In this paper, we analyze the case with \( 0 \leq \psi \leq L \). We define \( \Psi \) as \( \Psi = [0, L] \) and write \( \psi \in \Psi \).

The firm pays a commitment cost \( CC(\psi) \) per customer. We assume that \( CC(0) = 0 \) and \( CC(\psi) \) is strictly increasing and convex in the length of commitment lead time \( \psi \). In addition to the commitment cost, the firm pays an inventory holding cost of \( h \) per unit per time unit. Moreover, there is a commitment to deliver each customer order by the end of the commitment lead time \( \psi \), otherwise, demand is backordered and a backordering cost of \( p \) per unit per time unit is paid to the customer. The customer does not take delivery before the end of commitment lead time since the product is not needed before that. The firm’s objective is to find the commitment lead time, \( \psi \), and the base-stock level, \( s \), which minimize the total average cost.

Consistent with Hariharan and Zipkin (1995), we define \( D^-(t) \) and \( D(t) \) as the cumulative customer orders through

### Table 2: Literature on imperfect ADI

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<th>Authors</th>
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<tbody>
<tr>
<td>Gallego and Özer (2001)</td>
<td>A single item, single-location periodic-review system</td>
<td>State-dependent policies are optimal for systems with and without fixed ordering costs</td>
</tr>
<tr>
<td>Gallego and Özer (2003)</td>
<td>A multistage periodic-review inventory system</td>
<td>State-dependent, echelon base-stock policies for finite and infinite horizons are optimal</td>
</tr>
<tr>
<td>Iyer, Deshpande, and Wu (2003)</td>
<td>A two-stage capacity planning problem with two customer demand classes</td>
<td>Demand postponement enables cost and capacity requirement reductions</td>
</tr>
<tr>
<td>Özer (2003)</td>
<td>A centralized system with one warehouse and multiple retailers</td>
<td>Develop a lower bound and propose a close-to-optimal heuristic</td>
</tr>
<tr>
<td>Zhu and Thomemann (2004)</td>
<td>A single-retailer, multiple demand classes with future demand information (FDI)</td>
<td>Information cost and demand correlation are important factors for determining the optimal extent of FDI sharing</td>
</tr>
<tr>
<td>Ozer and Wei (2004)</td>
<td>A capacitated single-location periodic-review system</td>
<td>Characterize the behavior of optimal policies with respect to capacity and ADI</td>
</tr>
<tr>
<td>Tan, Güllü, and Erkip (2007)</td>
<td>A single-location periodic-review system</td>
<td>The optimal policy is of order-up-to type, where the order level is a function of ADI</td>
</tr>
<tr>
<td>Liberopoulos and Koukoumialos (2008)</td>
<td>A single capacitated/uncapacitated supplier with two demand classes</td>
<td>Investigate the impact of ADI on the optimal decisions</td>
</tr>
<tr>
<td>Gayon et al. (2009)</td>
<td>A capacitated supplier with multiple demand classes</td>
<td>The optimal production and allocation policies are state-dependent base-stock and multilevel rationing policies, respectively</td>
</tr>
<tr>
<td>Wang and Tomlin (2009)</td>
<td>A single-location periodic-review system with forecast updating and lead time uncertainty</td>
<td>The firm becomes less sensitive to lead time variability as the forecast updating process becomes more efficient</td>
</tr>
<tr>
<td>Iida and Zipkin (2010)</td>
<td>A two-echelon serial system in competitive and cooperative settings</td>
<td>The impact of sharing forecasts on profit could be negative in the competitive setting, but it is always positive in the cooperative setting</td>
</tr>
<tr>
<td>Benjaafar, Cooper, and Mardan (2011)</td>
<td>A capacitated supplier with stochastic production times</td>
<td>A state-dependent base-stock policy is optimal</td>
</tr>
</tbody>
</table>

### Table 3: Literature analyzing both imperfect and perfect ADI

<table>
<thead>
<tr>
<th>Authors</th>
<th>Characteristics of the system</th>
<th>Findings and results</th>
</tr>
</thead>
<tbody>
<tr>
<td>Claudio and Krishnamurthy (2009)</td>
<td>A three-echelon capacitated serial system integrating ADI with Kanban-based pull strategy</td>
<td>Integrating ADI and Kanban-based pull strategy corrects inefficiencies</td>
</tr>
<tr>
<td>Bernstein and DeCroix (2014)</td>
<td>A multiproduct, single-location system with capacitated resource selection</td>
<td>Explore the impact of perfect and imperfect ADI on optimal capacities and profit</td>
</tr>
<tr>
<td>Benbitour and Sahin (2015)</td>
<td>A capacitated single-location periodic-review system</td>
<td>The imperfectness of demand information reduces the benefits of ADI</td>
</tr>
</tbody>
</table>
time \( t \) and the cumulative customer demands through time \( t \), respectively. We have \( D(t) = D^*(t - \psi) \). We call the system analyzed in this paper the current system.

Under the assumptions outlined above, the base-stock policy is optimal (Hariharan & Zipkin, 1995). According to the base-stock policy, each customer order triggers a replenishment order. If a customer order occurs at time \( t \), the corresponding item arrives at time \( t + L \). In a conventional base-stock policy a replenishment order is triggered by actual customer demands, which occur \( \psi \) time units later here. If a customer order occurs at time \( t \), the actual demand in the conventional policy occurs at time \( t + \psi \) and the corresponding item arrives at \( t + \psi + (L - \psi) = t + L \). Hence, the supply lead time of the corresponding conventional system is \( L - \psi \).

As a result, the supply and demand processes of the current and conventional systems are identical. This is why the equilibrium net inventories are equivalent and expressed as

\[
s - D^*(L) = s - D(L + \psi),
\]

We refer to Hariharan and Zipkin (1995) for the details.

For practical purposes, from now on we use \( X \) to indicate the demand during \( L - \psi \). \( X \) has a Poisson distribution with a rate \( \lambda(L - \psi) \). We use \( P_X(x, \psi) \) to represent the probability mass function of \( X \). The dependency on \( \psi \) is made explicit since this helps in subsequent analysis. We define \( C(s, \psi) \) as the total average cost as a function of the decision variables \( s \) and \( \psi \). \( C(s, \psi) \) can be written as follows:

\[
C(s, \psi) = h \sum_{x=0}^{\psi} (s-x)P_X(x, \psi) + p \sum_{x=s}^{\infty} (x-s)P_X(x, \psi) + \lambda CC(\psi).
\]

The first term in (1) is the average holding cost, the second term is the average backordering cost, and the final term is the average commitment cost. Defining \( F_X(s, \psi) \) as the cumulative distribution function of \( X \), we obtain the following alternative expression for \( C(s, \psi) \):

\[
C(s, \psi) = (h+p)(sF_X(s, \psi) - \lambda(L - \psi)F_X(s-1, \psi)) + p(\lambda(L - \psi) - s) + \lambda CC(\psi).
\]

Refer to Appendix A.1 for the derivation of this alternative expression. The firm aims to solve the optimization problem \( \min_{s \in \mathbb{N}_0, \psi \in \Psi} C(s, \psi) \) to find the optimal commitment lead time, \( \psi^* \) and the optimal base-stock level, \( s^* \).

## 4 | ANALYSIS

In this section, we initially analyze the properties of \( C(s, \psi) \) and prove its convexity with respect to the decision variables and construct a lower bound on it (Section 4.1). Then, we prove the structure of the optimal policy (Section 4.2).

### 4.1 | Analysis of the cost function

\( C(s, \psi) \) is a continuous function with respect to \( \psi \) and a discrete function with respect to \( s \). In this section, we investigate the properties of \( C(s, \psi) \), which help in proving the structure of the optimal policy. The proofs of the results can be found in Appendix A.

In Lemmas 1 and 2 we prove the convexity of \( C(s, \psi) \) with respect to the commitment lead time and the base-stock level, respectively.

**Lemma 1** For each \( s \in \mathbb{N}_0 \), the cost function \( C(s, \psi) \) is convex in \( \psi \).

**Lemma 2** For each \( \psi \in \Psi \), the cost function \( C(s, \psi) \) is convex in \( s \).

These results imply that we can find the optimal value of each decision variable by fixing the other one. Initially, for a given value of \( \psi \), we minimize \( C(s, \psi) \) with respect to \( s \). Using the first order conditions, we find that the optimal base-stock level for a given value of \( \psi \in \Psi \) is the base-stock level \( s \) that satisfies the following inequalities:

\[
F_X(s-1, \psi) < \frac{p}{h + p} \leq F_X(s, \psi).
\]

With the following theorem, we prove that the optimal base-stock level is nonincreasing in the length of the commitment lead time. Hence, the maximum value of the optimal base-stock level can be found by setting the commitment lead time to zero. Similarly, the minimum value of the optimal base-stock level, which is \( 0 \), occurs when the commitment lead time is at its maximum value, that is, \( L \).

**Theorem 3** The optimal base-stock level is nonincreasing in \( \psi \in \Psi \). The set of optimal base-stock levels can be written as \( \tilde{S} = \{S, S-1, \ldots, 2, 1, 0\} \), where \( S \) is the optimal base-stock level corresponding to \( \psi = 0 \) and 0 is the optimal base-stock level corresponding to \( \psi = L \).

We refer to Appendix A.4 for the proof. This result implies that the whole interval \( \Psi \) can be partitioned into \( S + 1 \) subintervals \( \Psi_i \), where subinterval \( \Psi_i \), covers all the commitment lead times for which the corresponding optimal base-stock level is \( s \).

Define \( C(\psi) \) as \( C(\psi) = \min_{s \in \mathbb{N}_0} C(s, \psi) \). According to Theorem 3, there is a finite sequence of continuous convex functions \( C(s, \psi) \) each defined on \( \psi \in \Psi \) for which

\[
C(\psi) = \min_{s=0,1,2,\ldots,S} C(s, \psi), \text{ for all } \psi \in \Psi.
\]

Hence, the real valued function \( C(\psi) \) is continuous piecewise-convex in \( \Psi \). In fact, \( C(\psi) \) constitutes a tight lower bound for \( C(s, \psi) \) (refer to Figure 1 for an example).

### 4.2 | The structure of the optimal policy

In this section, we prove the main result of this paper. We assume a general structure for the commitment cost, \( CC(\psi) \),
The policy implies that the optimal commitment lead time is either zero or the maximum possible value, which is $L$. Hence, it is one of the endpoints of the interval $\Psi$. In addition, we show that the corresponding optimal base-stock level is zero or nothing.

**Theorem 4** Let $CC(\psi) = \int_0^\psi \phi(x) dx$ be the commitment cost function, where $\phi(x)$ is a positive and differentiable function such that $\Phi(x) = \phi(x) - \frac{C(0)}{\mu}$ is either a constant function or a nonconstant function without a root in the interval $(0, L)$. Then the optimal commitment lead time policy on $\Psi$ is a “bang-bang” policy and its corresponding optimal base-stock policy is of “all-or-nothing” type.

We refer to Appendix A.5 for the proof. $\Phi(x)$ and $\phi(x)$ are two auxiliary functions. $\Phi(x)$ is used for presenting the sufficient condition for the validity of Theorem 4 and all the other relevant proofs. It does not have any specific meaning. $\phi(x)$ is used to build the amount of commitment cost paid to each customer as $CC(\psi) = \int_0^\psi \phi(x) dx$. Generally speaking $\phi(x)$ has no meaning; however, when $\phi(x)$ is a constant function $\phi(x) = c$, it can be interpreted as the unit commitment cost per time unit per customer. For a linear commitment cost, that is, $CC(\psi) = c\psi$, we have $\phi(x) = c$ and $\Phi(x) = c - \frac{C(0)}{\mu}$. Given that $\Phi(x)$ is a constant function, the condition in Theorem 4 is satisfied. Hence, the bang-bang policy is optimal for linear commitment costs.

For a nonlinear commitment cost if the condition in Theorem 4 does not hold, the result may or may not hold. Theorem 4 provides the sufficient condition but not a necessary condition. We provide two numerical examples to clarify this further. We use the same parameters in both examples with different commitment costs (please refer to Figure 2).

In both examples, we have $C(0) = 12.69$, $h = 4$, $p = 20$, and $L = 10$. Since $\Phi(x) = c - \frac{C(0)}{\mu}$ is a constant function, the condition in Theorem 4 is satisfied. Therefore, for both cases, the optimal commitment lead time is zero, the optimal base-stock level is at its maximum value $S$ (refer to Theorem 3). Otherwise, the optimal lead time is equal to the replenishment lead time $L$ and the corresponding optimal base-stock level is zero or nothing.

**FIGURE 1** Representation of $C(\psi) = \min_x C(x, \psi)$ [Color figure can be viewed at wileyonlinelibrary.com]

**FIGURE 2** $C(\phi)$ when the sufficient condition does not hold [Color figure can be viewed at wileyonlinelibrary.com]
sufficient condition does not hold. However, as it can be seen in Figure 2, the bang-bang policy may or may not hold. It does not hold for the first example, while it holds for the second one.

The proof of Theorem 4 relies on constructing a monotone lower bound for $C(\psi)$ on $\Psi$. We need the conditions in Theorem 4 to hold for constructing this lower bound. Let $C_{LB}(\psi)$ be the lower bound. We construct it so that the endpoints of $C(\psi)$ and $C_{LB}(\psi)$ coincide, that is, $C(0) = C_{LB}(0)$ and $C(L) = C_{LB}(L)$. Therefore, the minimum of $C(\psi)$ on $\Psi$ is equal to the minimum of $C_{LB}(\psi)$ on $\Psi$. Given that $C_{LB}(\psi)$ is a monotone function on a closed interval $\Psi$, the minimum of $C(\psi)$ always happens at the endpoints of $\Psi$. Hence, $\psi^* \in \{0, L\}$, that is, the optimal commitment lead time policy is a bang-bang policy (refer to Appendix A.5 for the details). In Figure 3, we provide two examples for the monotonic lower bounds; one where $\psi^* = 0$ and another where $\psi^* = L$.

With Corollary 5, we provide sufficient conditions for the optimality of BTS and BTO strategies.

**Corollary 5** When $\forall x \in (0, L)$, $\Phi(x) > 0$ the optimal strategy is the buy-to-stock strategy and when $\forall x \in (0, L)$, $\Phi(x) < 0$ it is the buy-to-order strategy.

This corollary implies that the firm either uses the standard base-stock policy without any commitment lead time (BTS strategy) or holds no inventory and uses a commitment lead time of $L$ time units (BTO strategy).

## 5 Linear Commitment Cost

In this section, we consider a linear commitment cost function. We define $c$ as the commitment cost per time unit and write $CC(\psi) = c\psi$. With the following corollary, we prove that the sufficient condition in Theorem 4 is satisfied and, therefore, the bang-bang policy is optimal under the linear commitment cost. In addition, we show that there is a threshold value $c_0$ such that if $c > c_0$, the optimal strategy is the BTS strategy and if $c \leq c_0$, it is the BTO strategy.

**Corollary 6** Let the commitment cost function be $CC(\psi) = c\psi$, then

1. The optimal commitment lead time policy on $\Psi$ is a “bang-bang policy” and its corresponding optimal base-stock policy is of “all-or-nothing” type.
2. There is a threshold $c_0$ such that if $c > c_0$, then $\psi^* = 0$, otherwise, $\psi^* = L$. The expression for $c_0$ is

$$c_0 = \frac{(h + p)(S\Phi_x(S, 0) - \mu\Phi_x(S - 1, 0) + p(\mu - S))}{\mu}$$

where $S = \inf \left\{ s \in \mathbb{N}_0 : \Phi_x(s, 0) \geq \frac{p}{p + h} \right\}$ and $\mu = \lambda L$.

The intuition behind this result is that, under the linear commitment cost structure, when a firm has two decisions to make, one on the base-stock level and another on the commitment lead time, it has to optimize the total cost by considering the trade-off of having more/less inventory or longer/shorter commitment lead time. If $c \geq c_0$, that is, the unit commitment cost is higher than the threshold, it is cheaper to hold inventory. This is why a solution with no investment on commitment cost but highest investment on inventory is chosen. On the other hand, if $c \leq c_0$, that is, the unit commitment cost is lower than the threshold, it is cheaper to have a long commitment lead time. This is why a solution with maximum investment on commitment cost but no investment on inventory is chosen.

In the rest of this section, we perform sensitivity analysis on the unit commitment cost threshold, and we illustrate the behavior of $C(\psi)$ through numerical examples. In addition, we consider the profit maximization version of the problem and determine the conditions on the unit commitment cost for profitability of the BTO strategy.
The unit commitment cost threshold plays a critical role in determination of the optimal strategy. According to Corollary 6, $c_0$ is a function of the unit holding cost $h$, unit backordering cost $p$, the mean lead time demand $\mu$ and distribution function of the lead time demand $F_X(x, \psi$) for the following pairs of $x$ and $\psi$: $(S, 0)$ and $(S - 1, 0)$. Since the values of $F_X(S, 0)$ and $F_X(S - 1, 0)$ also depend on $h$, $p$, and $\mu$, the effect of a change in one of these parameters on $c_0$ is not straightforward. In Lemma 7 we characterize the effect of each parameter on $c_0$.

**Lemma 7** The unit commitment cost threshold $c_0$ is

1. increasing in the unit holding cost $h$,
2. increasing in the unit backordering cost $p$
3. decreasing in the mean lead time demand $\mu$.

We refer to Appendix A.8 for the proof. According to Lemma 7, the constraint $c < c_0$ is likely to hold more often when $h$ and $p$ increase and $\mu$ decreases. Hence, the BTO strategy becomes preferable (refer to Corollary 6). Inventory management is difficult under high demand uncertainty. Firms may keep excess inventory to protect against a stock-out situation or they might keep low inventory to prevent a surplus situation. Under Poisson demand a low mean lead time demand $\mu$ implies a high coefficient of variation ($\frac{1}{\sqrt{\mu}}$) and, therefore, high demand uncertainty. The BTO, that is, preorder, strategy becomes more beneficial as the demand uncertainty increases. This holds even for high values of the commitment cost. When the unit holding and backordering costs increase the surplus and stock-out situations become more expensive compared to paying a commitment cost. This is why the preorder strategy becomes more beneficial. Figure 4 confirms this conclusion.

Figure 4 also illustrates that when the firm has less demand uncertainty, the unit commitment cost threshold is more sensitive to changes in unit holding and backordering costs. In addition, the unit commitment cost threshold is more sensitive to changes in the unit holding cost compared to changes in the unit backordering cost.

Consistent with Lemma 7, Figure 5 shows that the firm prefers the BTO strategy as demand uncertainty increases; inventory related costs outweigh the commitment cost, and the preorder strategy becomes preferable even for high values of the unit commitment cost.

**5.2 Numerical analysis on $C(\psi)$**

Next, we conduct multiple numerical analysis analyses to illustrate the behavior of the function $C(\psi)$ and the optimality of the bang-bang policy when the commitment cost is linear in $\psi$. As a base case, we consider the following parameter values: $L = 10$, $\lambda = 1$, $h = 4$, $p = 20$, and $c = 2$. In Figure 6, we keep the values of other parameters at their base case values and vary the values of $h$, $p$ and $\mu$ one-by-one and show the impact of these parameters on $C(\psi)$. For each parameter, we draw $C(\psi)$ for three values. The values are chosen such that we observe the change in the policy structure. More specifically, for each parameter, the values.
are chosen such that we observe the following three optimal structures:

1. a single optimal solution \((s^*, \psi^*) = (S, 0)\) indicated by a dotted line,
2. two alternative optimal solutions \((s^*_1, \psi^*_1) = (S, 0)\) and \((s^*_2, \psi^*_2) = (0, L)\) indicated by a continuous line and
3. a single optimal solution \((s^*, \psi^*) = (0, L)\) indicated by a dashed plot.

In Figure 6, the behavior of \(C(\psi)\) with respect to its argument \(\psi\) for different parameter values is depicted. For each parameter combination \(C(\psi)\) is a continuous piecewise function of \(\psi\). Each “piece” is for a specific base-stock level and each piece is convex (Lemma 1). From Theorem 3, we know that the optimal base-stock level is nonincreasing in \(\psi\). Since the base-stock levels take integer values, by increasing \(\psi\) the corresponding optimal base-stock level may decrease by one unit. This is why the optimal base-stock level remains the same for an interval of \(\psi\) values and in each convex piece the optimal base-stock level is the same. As soon as the optimal base-stock level decreases (as a response to increasing \(\psi\)), another convex piece emerges.

Figure 6 suggests that for low values of \(h\), \(\psi^*\) is 0 and, hence, the firm follows a BTS strategy. For sufficiently large \(h\) values \(\psi^*\) becomes \(L\) and the firm follows a BTO strategy. This result does not only follow intuitively but also directly from Corollary 6. Note that as \(h \to 0\), \(S \to \infty\), and \(c_0 \to 0\). Hence, \(c > c_0\) holds and Corollary 6 implies the optimality of BTS strategy.

As the unit backordering cost \(p\) increases we observe a similar behavior, that is, the strategy changes from BTS to BTO. In the classical inventory theory, the effects of \(h\) and \(p\) are usually opposite. However, in our problem, their effects are similar. The reason is that the optimality of a policy depends on the value of \(c_0\). As \(p \to \infty\), \(S \to \infty\), and \(c_0 \to \infty\). Hence, \(c_0 > c\) holds and Corollary 6 implies the optimality of the BTO strategy. Therefore, a BTS strategy is optimal for low values of \(p\) and a BTO strategy becomes optimal as \(p\) gets sufficiently large. This implies the same trend in the optimal policy structure as \(h\) changes. Figure 6 confirms this conclusion.

As the mean lead time demand \(\mu\) increases the strategy changes from BTO to BTS. This is in line with what we observe in the classical inventory theory. We also observe that as \(\mu\) increases \(C(\psi)\) becomes a smooth concave function. As explained above, in each convex piece in \(C(\psi)\) the optimal base-stock level is the same. When \(\mu\) increases, the optimal base-stock level becomes more sensitive to \(\psi\). Hence, the interval of \(\psi\) for which a base-stock level remains optimal gets shorter, that is, convex pieces become smaller. As observed in Figure 7, when mean lead time demand increases the number of convex pieces increases and for high
In the previous sections, we study the cost minimization following inequality

\[ C = \text{mean lead time demand} \]

**Optimal strategies**

**FIGURE 7** Behavior of \( C(\psi) \) for different values of \( \mu \)

mean lead time demand \( C(\psi) \) becomes a smooth concave function.

5.3 **Profitability of the optimal policy**

In the previous sections, we study the cost minimization problem \( \min_{s \in \mathbb{N}, \psi \in \Psi} C(s, \psi) \). We prove that the optimal strategy is either BTO or BTS. In this section, we concentrate on the BTO strategy and analyze the average total profit. If the firm follows the BTO strategy, the only positive cost component is the commitment cost \( cL \). Define \( v \) as the unit selling price and \( \Theta \) as the average total profit of the firm per time unit. Then we formulate the long-run average profit of the firm under the BTO strategy as follows:

\[ \Theta_{\text{BTO}} = \lambda v - \lambda cL = \lambda(v - cL). \]

The long-run average profit of the firm is nonnegative when \( c < \frac{v}{L} \). In addition, the optimality of the BTO strategy requires \( c \leq c_0 \). Hence, the unit commitment cost should satisfy the following inequality

\[ c < \min \left\{ c_0, \frac{v}{L} \right\}. \]

In Figure 8, we consider the unit commitment cost and the mean lead time demand and determine the region where the BTO strategy is profitable.

**FIGURE 8** Optimal strategies

**FIGURE 9** Behavior of \( C(\psi) \) for different values of \( \mu \)

Figure 8 helps managers in making strategic decisions when, in addition to the inventory-related and commitment costs, the revenue from selling the product is considered. This figure suggests that profitability of the BTO strategy depends on the value of \( c \) and, although it is cost optimal, the firm should not follow this strategy for high values of \( c \) as it is non-profitable. When the optimal strategy is the BTS strategy, the unit commitment cost does not play a role in the total profit of the firm.

6 **COMPOUND POISSON DEMAND**

In this section, we assume that the customer demand follows a compound Poisson process. By definition, compound Poisson demand means that the size of customer demand is a stochastic variable. The demand size is independent of other customer demands and the distribution of the customer arrival process. Similar to our previous assumption, we assume that the customer arrival process is a Poisson process with parameter \( \lambda \). We define \( D \) as a random variable representing the total demand in the time interval \( L - \psi \). \( P_D(x, \psi) \) is the probability that \( D \) takes the value \( x \) when the commitment lead time is \( \psi \).

We define \( F_D(s, \psi) = \sum_{x=0}^{s} P_D(x, \psi) \) as the cumulative distribution function of \( D \) and write the long-run average total cost \( C(s, \psi) \) consisting of holding, backordering and commitment costs as follows:

\[
C(s, \psi) = h\mathbb{E}[(s - D)^+] + p\mathbb{E}[(D - s)^+] + \mu\lambda CC(\psi)
= (h + p) \left( sF_D(s, \psi) - \sum_{x=0}^{s} xP_D(x, \psi) \right) + p(\mu_D - s) + \mu\lambda CC(\psi).
\]

Here \( \mu_D \) equals \( \mu(\lambda - \psi) \). Refer to Appendix B.1 for expressions and detailed derivations. Similar to the pure Poisson demand case, we investigate the behavior of \( C(s, \psi) \) with respect to the decisions variables. In Lemma 8 we prove the convexity of \( C(s, \psi) \) with respect to the base-stock level.

**Lemma 8** For each \( \psi \in \Psi \), the cost function \( C(s, \psi) \) is convex in \( s \).

Refer to Appendix B.2 for the proof. Lemma 8 implies that for a given value of \( \psi \in \Psi \), \( C(s, \psi) \) can be minimized with respect to \( s \). For a given \( \psi \in \Psi \) value, the optimal base-stock level satisfies the following inequalities:

\[
F_D(s - 1, \psi) \leq \frac{p}{p + h} \leq F_D(s, \psi).
\] (4)

Similar to the pure Poisson customer demand case, the state space \( \Psi \) is divided into multiple subintervals with the optimal base-stock level being different in each subinterval.
Next with Conjecture 9, we claim that the convexity result also holds with respect to the commitment lead time.

**Conjecture 9** For each \( s \in \mathbb{N}_0 \), the cost function \( C(s, \psi) \) is convex in \( \psi \).

Our proof on optimality of the bang-bang policy relies on constructing a monotone lower bound with end points of the lower bound coinciding with the end points of the cost function. We aim to do the same for the compound Poisson demand case. We define \( \hat{C}(s, \psi) = \hat{C}(s, \psi) - \mu \lambda CC(\psi) \). We have the following conjecture:

**Conjecture 10** Let \( \hat{C}(\psi) \) be a tight lower bound of \( C(s, \psi) \) on \( \Psi \), where for all \( \psi \in \Psi \),

\[
\hat{C}(\psi) = \min_{s=0,1,2,...,S} C(s, \psi). \text{ Then } \frac{d (\hat{C}(\psi))}{d \psi} \geq 0.
\]

If Conjecture 10 holds, the optimality of the “bang-bang policy” follows. We are unable to provide a formal proof. We summarize our attempt and findings in Appendix B.3. Our extensive numerical analysis confirms the correctness of Conjectures 8 and 10. In Appendix B.4 we represent some of our numerical results.

7 | CONCLUDING REMARKS

In this paper, we investigated the impact of the commitment cost on the replenishment strategy of a firm under continuous-review setting. The firm faces a Poisson customer demand and uses a preorder strategy which requires that customers place their orders ahead of the actual need based on a predetermined time window called commitment lead time. The firm pays a commitment cost which is strictly increasing and convex in the length of the commitment lead time. The firm uses a base-stock replenishment policy with a deterministic lead time. The firm’s objective is to find the commitment lead time and the base-stock level which minimize the total average cost consisting of inventory holding, backordering and commitment costs. This is a nonlinear mixed-integer optimization problem. We have a continuous commitment lead time and a discrete base-stock level as our decision variables and they are dependent on each other.

We proved the convexity of the average cost function in each decision variable. We showed that the optimal base-stock level is nonincreasing in the length of the commitment lead time. The average cost as a function of a commitment lead time and the corresponding optimal base-stock level served as a continuous piecewise-convex lower bound on the original average cost function. We constructed another monotone lower bound on this piecewise-convex lower bound. This monotone lower bound implies the optimality of the bang-bang policy for the commitment lead time. Under this policy the commitment lead time is either zero or the maximum possible value, which is the replenishment lead time. In addition, we showed that the corresponding optimal base-stock policy is of all-or-nothing type. Hence, the optimal base-stock level is either zero or the solution of the well-known Newsvendor problem with complete replenishment lead time.

The optimality of the bang-bang policy holds for general commitment cost structures under a very mild condition. As a specific case, we studied a linear commitment cost. For this case, we found a unit commitment cost threshold which dictates the optimality of either a BTO or a BTS strategy. More specifically, we showed that when the unit commitment cost is less than the unit commitment cost threshold, the optimal ordering strategy is a BTO strategy (the optimal commitment lead time is equal to replenishment lead time and the optimal base-stock level is zero) and when the unit commitment cost is more than the unit commitment cost threshold, the optimal strategy is a pure BTS strategy (the optimal commitment lead time is zero and optimal base-stock level is nonzero). Moreover, we showed that the unit commitment cost threshold is increasing in the unit holding and backordering costs and decreasing in the mean lead time demand. For a given base-stock level, we developed a simple and accurate approximation for the corresponding optimal commitment lead time. We determined the conditions on the unit commitment cost for profitability of the BTO strategy and study the case with a compound Poisson customer demand.

In this paper, we studied a system with a single location and a single item with a cost minimization objective. A natural extension is to replace the backordering cost with a service level and/or waiting time constraints and check whether the optimality of a bang-bang policy remains. Another extension is to consider a system where each customer places an order for a product which needs to be assembled from multiple components with different replenishment lead times. Then, the firm needs to find the optimal replenishment policy for each component and also the optimal commitment lead time (Ahmadi, 2019; Ahmadi, Atan, de Kok, & Adan, 2019; Atan, Ahmadi, Stegehuis, de Kok, & Adan, 2017). More general assembly structures can also be analyzed. The combination of planned lead time and commitment lead time decisions can also be an interesting and challenging problem (Atan, de Kok, Dellaert, Janssen, & van Boxel, 2016; Janssen, Atan, Adan, & de Kok, 2018, 2019).

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**APPENDIX: PURE POISSON DEMAND**

A.1 | Alternative cost expression

\[ C(s, \psi) = h \sum_{i=0}^{s} (s - i)P_X(x, \psi) \]
\[
+ p \sum_{x=\infty}^{\infty} (x-s)P_X(x, \psi) + \lambda CC(\psi)
\]
\[
= h \sum_{x=0}^{\infty} (s-x)P_X(x, \psi) + p \sum_{x=\infty}^{\infty} (x-s)P_X(x, \psi) + \lambda CC(\psi)
\]
\[
= h \sum_{x=0}^{\infty} sP_X(x, \psi) - h \sum_{x=0}^{\infty} xP_X(x, \psi) + p \sum_{x=\infty}^{\infty} xP_X(x, \psi) - p \sum_{x=\infty}^{\infty} sP_X(x, \psi) + \lambda CC(\psi)
\]
\[
= h \sum_{x=0}^{\infty} sP_X(x, \psi) - h \sum_{x=0}^{\infty} xP_X(x, \psi) + p \left( \lambda(L - \psi) - \sum_{x=0}^{s} xP_X(x, \psi) \right) - p \left( \sum_{x=0}^{s} xP_X(x, \psi) \right) + \lambda CC(\psi)
\]
\[
= (h + p) \left( \sum_{x=0}^{s} sP_X(x, \psi) - \sum_{x=0}^{s} xP_X(x, \psi) \right) + p(\lambda(L - \psi) - s) + \lambda CC(\psi)
\]
\[
= (h + p) \left( sF_X(s, \psi) - \sum_{x=0}^{s} x^2(L - \psi)^x e^{-\lambda(L - \psi)} x! \right) + p(\lambda(L - \psi) - s) + \lambda CC(\psi)
\]

\[
\frac{d}{d\psi} \frac{dP_X(x, \psi)}{d\psi} \text{ is as follows:}
\]
\[
= \frac{d}{d\psi} \left( \sum_{x=0}^{n} P_X(x, \psi) \right)
\]
\[
= \sum_{x=0}^{n} \frac{d}{d\psi} P_X(x, \psi) = \sum_{x=0}^{n} \left( \lambda - \frac{x}{L - \psi} \right) P_X(x, \psi)
\]
\[
= \lambda P_X(x, \psi) - \frac{1}{L - \psi} \sum_{x=0}^{n} xP_X(x, \psi)
\]
\[
= \lambda F_X(n, \psi) - \frac{1}{L - \psi} \sum_{x=0}^{n} \frac{x^2(L - \psi)^x}{x!} e^{-\lambda(L - \psi)} x!
\]
\[
= \lambda F_X(n, \psi) - \frac{\lambda(L - \psi)}{L - \psi}
\]
\[
= \lambda F_X(n, \psi) - \lambda \sum_{x=0}^{n} P_X(x, \psi)
\]
\[
= \lambda \left( F_X(n, \psi) - \sum_{x=0}^{n} P_X(x, \psi) \right)
\]
\[
= \lambda (F_X(n, \psi) - F_X(n-1, \psi))
\]
\[
= \lambda F_X(n, \psi).
\]

A.2 | Proof of Lemma 1

We show that \( \forall s \in \mathbb{N}_0 \), the second derivative of \( C(s, \psi) \) with respect to \( \psi \in \Psi \) is nonnegative. For this purpose, we initially calculate the derivatives of \( P_X(x, \psi) \) and \( F_X(n, \psi) = \sum_{x=0}^{n} P_X(x, \psi) \) with respect to \( \psi \). We define \( \frac{dG(z)}{dz} \) as the derivative of function \( G(z) \) with respect to \( z \). We have \( P_X(x, \psi) = \frac{\lambda^x(L - \psi)^x}{x!} e^{-\lambda(L - \psi)} x \in \{0, 1, \ldots\} \). \( \frac{dP_X(x, \psi)}{d\psi} \) is as
Using these results, we obtain the first and second derivatives of \( C(s, \psi) \) with respect to \( \psi \):

\[
\frac{dC(s, \psi)}{d\psi} = (h + p) \left( s \frac{dF_X(s, \psi)}{d\psi} - \lambda \frac{d}{d\psi}((L - \psi)F_X(s - 1, \psi)) \right) + p \frac{d}{d\psi}(\lambda s - F_X(s, \psi) - \lambda s - F_X(s, \psi)) = (h + p)(s \lambda P_X(s, \psi) - \lambda F_X(s, \psi)) + \lambda dC(s, \psi) = (h + p)(s \lambda P_X(s, \psi) - \lambda F_X(s, \psi)) + \lambda dC(s, \psi).
\]

Using the fact that \( \lambda(L - \psi)P_X(s - 1, \psi) = s P_X(s, \psi) \), we write

\[
\frac{dC(s, \psi)}{d\psi} = (h + p)(s \lambda P_X(s, \psi) - \lambda F_X(s, \psi)) + \lambda dC(s, \psi) = (h + p)(s \lambda P_X(s, \psi) - \lambda F_X(s, \psi)) + \lambda dC(s, \psi).
\]

Using the last expression, we write \( C(s, \psi) \) as

\[
C(s, \psi) = (h + p)(s \lambda P_X(s, \psi) - \lambda F_X(s, \psi)) + \lambda dC(s, \psi).
\]

Since \( \forall s \in \mathbb{N}_0 \) and \( \psi \in \Psi \), we have \( \frac{dC(s, \psi)}{d\psi} \geq 0 \) and \( \frac{d^2C(s, \psi)}{d\psi^2} \geq 0 \). It means that for all \( s \in \mathbb{N}_0 \), then \( C(s, \psi) \) is convex with respect to \( \psi \).

A.3 Proof of Lemma 2

The concept of convexity for a real valued function on a discrete domain is not common. According to Van Houtum and Kranenburg (2015) this concept can be defined as follows.

**Definition 11** Let \( f(x) \) be a function on \( \mathbb{Z} \) and \( x_0 \in \mathbb{Z} \), then \( f(x) \) is convex for \( x \geq x_0 \) if

\[
\Delta^2 f(x) = \Delta f(x + 1) - \Delta f(x) \geq 0, \quad x \geq x_0,
\]

where \( \Delta f(x) = f(x + 1) - f(x) \).

The function \( C(s, \psi) \) is convex in \( s \in \mathbb{N}_0 \) if for all \( \psi \in \Psi \) we have \( \Delta^2 C(s, \psi) \geq 0 \), where \( \Delta^2 C(s, \psi) \) is defined as follows:

\[
\Delta^2 C(s, \psi) = \Delta \Delta C(s + 1, \psi) - \Delta C(s, \psi) = (C(s + 1, \psi) - C(s + 1, \psi)) - (C(s + 1, \psi) - C(s, \psi)) = C(s + 1, \psi) - 2C(s + 1, \psi) + C(s, \psi).
\]

Next, we determine the expressions for \( C(s + 1, \psi) \) and \( C(s + 2, \psi) \).

\[
C(s + 1, \psi) = (h + p)(s + 1)F_X(s + 1, \psi) - \lambda s F_X(s, \psi) + p(\lambda s - F_X(s, \psi)) + \lambda CC(s, \psi) = (h + p)(s F_X(s + 1, \psi) + F_X(s + 1, \psi)) - \lambda s F_X(s, \psi) + p(\lambda s - F_X(s, \psi) + \lambda CC(s, \psi) = (h + p)(s F_X(s, \psi) + s P_X(s + 1, \psi) - \lambda s F_X(s, \psi) + p(\lambda s - F_X(s, \psi) + \lambda CC(s, \psi).
\]

From the Poisson probability mass function we know \( \lambda(L - \psi)P_X(s, \psi) = (s + 1)F_X(s + 1, \psi) \).

\[
C(s + 1, \psi) = (h + p)(s F_X(s, \psi) + s P_X(s + 1, \psi) + F_X(s + 1, \psi) - \lambda s F_X(s, \psi) + p(\lambda s - F_X(s, \psi) + \lambda CC(s, \psi).
\]

Using the last expression, we write \( C(s + 2, \psi) \) as

\[
C(s + 2, \psi) = (h + p)F_X(s + 1, \psi) - p + C(s + 1, \psi) = (h + p)F_X(s + 1, \psi) - p + C(s, \psi) + F_X(s, \psi) - 2p + C(s, \psi).
\]

Using these expressions, we write \( \Delta^2 C(s, \psi) \) as

\[
\Delta^2 C(s, \psi) = (h + p)(F_X(s + 1, \psi) + F_X(s, \psi)) - 2p + C(s, \psi) = (h + p)(F_X(s + 1, \psi) - F_X(s, \psi)) = (h + p)F_X(s + 1, \psi).
\]

Since \( \forall s \in \mathbb{N}_0 \) and \( \psi \in \Psi \), \( F_X(s + 1, \psi) \geq 0 \), then \( \Delta^2 C(s, \psi) \geq 0 \). Hence, \( \forall \psi \in \Psi \), \( C(s, \psi) \) is convex with respect to \( s \).

A.4 Proof of Theorem 3

We use the following lemma in proving the result.

**Lemma 12** Let \( F_X(s, \psi) \) be the Poisson cumulative distribution function of \( X \) with mean \( \lambda(L - \psi) \), then,

1. for all \( s \in \mathbb{N}_0 \) and \( \lambda \geq 0 \), \( F_X(s, \psi) \) is increasing in \( \psi \).
2. for all \( \psi \in \Psi \), \( F_X(s, \psi) \) is increasing in \( s \).

**Proof** For all \( s \in \mathbb{N}_0 \), \( F_X(s, \psi) \) is a continuous and differentiable function with respect to \( \psi \), then from Appendix A, we have \( \frac{dF_X(s, \psi)}{d\psi} = \lambda P_X(s, \psi) \). For each \( s \in \mathbb{N}_0 \) and \( \psi \in \Psi \), \( P_X(s, \psi) \geq 0 \). Then, for all \( s \geq 0 \), \( F_X(s, \psi) \) is increasing in \( \psi \).

For all \( \psi \in \Psi \), \( F_X(s, \psi) \) is a discrete function with respect to \( s \), then by calculating the first order forward difference, we have

\[
\Delta F_X(s, \psi) = F_X(s, \psi) - F_X(s - 1, \psi) = P_X(s, \psi) \geq 0
\]

then, \( F_X(s, \psi) \) is increasing in \( s \). Define \( S \) as the set all optimal base-stock levels \( \forall \psi \in \Psi \). We
Proof of Theorem 4

We need an additional result to prove this theorem. We define $\tilde{C}(s, \psi)$ on $\Psi$ as follows;

$$\tilde{C}(s, \psi) = C(s, \psi) - \lambda CC(\psi).$$  \hfill (A4)

**Lemma 13** Let $\tilde{C}(\psi)$ be the lower bound of $C(s, \psi)$ on $\Psi$, where for all $\psi \in \Psi$, $\tilde{C}(\psi) = \min_{s \in S} \tilde{C}(s, \psi)$, then the fraction of $\frac{\tilde{C}(\psi)}{L-\psi}$ is non-decreasing in $\psi$.

**Proof** We need to show that $\forall s \in \tilde{S}$, $\frac{d\tilde{C}(\psi)}{d\psi} \geq 0$. Remember that $\tilde{S}$ is the set of optimal base-stock levels (refer to Theorem 3). $\tilde{C}(s, \psi)$ is defined as $\tilde{C}(s, \psi) = C(s, \psi) - \lambda CC(\psi)$. In Appendix A, we provide an alternative expression for $C(s, \psi)$. Using this alternative, we obtain the following expression for $\tilde{C}(s, \psi)$:

$$\tilde{C}(s, \psi) = (h + p)(sF_X(s, \psi) - \lambda (L - \psi)FX(s - 1, \psi)) + p(\lambda (L - \psi) - s).$$

Then, the derivative of $\tilde{C}(s, \psi)$ with respect to $\psi$ is

$$\frac{d\tilde{C}(s, \psi)}{d\psi} = (h + p) \left( \frac{dF_X(s, \psi)}{d\psi} - \lambda \frac{d}{d\psi} ((L - \psi)FX(s - 1, \psi)) \right)$$

$$+ p \frac{d}{d\psi} (\lambda (L - \psi) - s)$$

$$= (h + p)(sPX(s, \psi) - \lambda (FX(s - 1, \psi) + \lambda (L - \psi)P_X(s - 1, \psi)) - p\lambda.$$ 

Using the fact that for Poisson distribution we have $\lambda (L - \psi)P_X(s - 1, \psi) = sP_X(s, \psi)$, we write $\frac{d\tilde{C}(s, \psi)}{d\psi}$ as

$$\frac{d\tilde{C}(s, \psi)}{d\psi} = \lambda(h + p)FX(s - 1, \psi) - p\lambda.$$

Then, we have

$$\int_{L-\psi}^{s} \frac{d\tilde{C}(s, \psi)}{d\psi} = \frac{1}{(L-\psi)^2} \left\{ (L-\psi) \left( \frac{d}{d\psi} \tilde{C}(s, \psi) \right) \right\}$$

$$- \int_{(L-\psi)^2}^{(s+(L-\psi)^2)} \frac{1}{(L-\psi)^2} \left\{ (L-\psi)(\lambda(h + p)FX(s - 1, \psi) - p\lambda) \right\}$$

$$= \frac{1}{(L-\psi)^2} \left\{ (L-\psi)(\lambda(h + p)FX(s - 1, \psi) - p\lambda) \right\}$$

$$+ \frac{1}{(L-\psi)^2} \left\{ sFX(s, \psi) - \lambda (L - \psi)FX(s - 1, \psi) - p\lambda \right\}$$

$$= \frac{1}{(L-\psi)^2} \left\{ sFX(s, \psi) - \lambda (L - \psi)FX(s - 1, \psi) - ps \right\}$$

$$= \frac{1}{(L-\psi)^2} \left\{ sFX(s, \psi) - \frac{p}{h + p} \right\}.$$ 

Since we subtract $CC(\psi)$ from $C(s, \psi)$ and $CC(\psi)$ does not depend on $s$, we have the same set $\tilde{S}$ as we had before for the lower bound of $C(s, \psi)$. Also, from inequality (3) we know $FX(s, \psi) \geq \frac{p}{h + p}$. Then for all $s \in \tilde{S}$, $\frac{d}{d\psi} \frac{\tilde{C}(s, \psi)}{L-\psi} \geq 0$.

From Lemma (13) we know that $\frac{\tilde{C}(s, \psi)}{L-\psi}$ is non-decreasing in $\psi$ on closed interval $\Psi$. Then, the fraction $\frac{\tilde{C}(s, \psi)}{L-\psi}$ has its minimum value at the beginning of the interval $\Psi = [0, L]$. Hence, for all $\psi \in \Psi$ we have $\frac{\tilde{C}(s, \psi)}{L-\psi} \geq \frac{C(s, \psi)}{L-\psi}$. Since for all...
\( \psi \in \Psi, \) we have \( L - \psi \geq 0, \) then,

\[
\tilde{C}(\psi) \geq \frac{C(0)}{L} (L - \psi). \quad \text{(A7)}
\]

By adding \( \lambda CC(\psi) \) to both sides of (A7), we have

\[
\tilde{C}(\psi) + \lambda CC(\psi) \geq \frac{C(0)}{L} (L - \psi) + \lambda CC(\psi). \quad \text{(A8)}
\]

We know that \( CC(0) = 0. \) Also, from expression (A6), we have \( \tilde{C}(\psi) = \psi - \lambda CC(\psi). \) Then, \( \tilde{C}(0) = C(0) \) and we can rewrite expression (A8) as follows.

\[
C(\psi) \geq \frac{C(0)}{L} (L - \psi) + \lambda CC(\psi). \quad \text{(A9)}
\]

Let \( C_{LB}(\psi) = \frac{C(0)}{L} (L - \psi) + \lambda CC(\psi). \) Then, \( C_{LB}(\psi) \) is a lower bound for \( C(\psi) \) on \( \Psi. \) Since \( C(0) = C_{LB}(0) \) and \( C(L) = C_{LB}(L), \) then \( C(\psi) \) and \( C_{LB}(\psi) \) coincide at the endpoints of the interval \( \Psi. \) In addition, \( C_{LB}(\psi) \) is a continuous function by construction. Now, we want to show that \( C_{LB}(\psi) \) is a monotone function on \( \Psi. \) To do this, we show that for all \( \psi \in \Psi, \) \( \frac{dC_{LB}(\psi)}{d\psi} \neq 0. \) By taking the first derivative of \( C_{LB}(\psi) \) we have

\[
\frac{dC_{LB}(\psi)}{d\psi} = \frac{C(0) d(L - \psi)}{L} + \lambda \frac{dCC(\psi)}{d\psi} = -\frac{C(0)}{L} + \lambda \phi(\psi) = \lambda \left( \phi(\psi) - \frac{C(0)}{\mu} \right). \quad \text{(A10)}
\]

For all \( \psi \in (0, L), \) we have \( \|\phi(\psi)\| > 0, \) then \( \frac{dC_{LB}(\psi)}{d\psi} \neq 0. \) It means that \( C_{LB}(\psi) \) is a monotone lower bound for \( C(\psi) \) on \( \Psi. \) Then, the minimum of \( C(\psi) \) on \( \Psi \) is equal to the minimum of \( C_{LB}(\psi) \) on \( \Psi. \) Since \( C_{LB}(\psi) \) is a monotone function on \( \Psi \) and \( \Psi \) is a closed interval, then the minimum of \( C(\psi) \) always happens at the endpoints of \( \Psi. \) Then, \( \psi^* \in \{0, L\}. \) The optimality of “all-or-nothing” base-stock policy follows directly from Theorem 3.

### A.6 Proof of Corollary 5

From expression (A10), it is obvious that when \( \phi(\psi) > \frac{C(0)}{\mu}, \) then \( C_{LB}(\psi) \) is a strictly increasing function in \( \psi \) and its minimum occurs at zero, that is, \( \psi^* = 0. \) And when \( \phi(\psi) < \frac{C(0)}{\mu}, \) then \( C_{LB}(\psi) \) is a strictly decreasing function in \( \psi \) and its minimum occurs at \( L, \) that is, \( \psi^* = L. \)

### A.7 Proof of Corollary 6

1. This part follows directly from Theorem 4. When \( CC(\psi) = c\psi, \) then \( C_{LB}(\psi) = C(0) + \left( c\lambda - \frac{C(0)}{L} \right) \psi \) which is a straight line. Then, \( \forall \psi \in \Psi, \) \( C_{LB}(\psi) \) is a monotone function. Also, \( C_{LB}(0) = C(0) \) and \( C_{LB}(L) = C(L). \) Then the optimal \( \psi \) on \( \Psi \) is either zero or \( L. \)

2. From \( C_{LB}(\psi) = C(0) + \left( c\lambda - \frac{C(0)}{L} \right) \psi, \) when \( c\lambda - \frac{C(0)}{L} > 0, \) then \( C_{LB}(\psi) \) is increasing in \( \psi \) and \( C_{LB}(0) \) is its minimum value on \( \Psi. \) Otherwise, \( C_{LB}(\psi) \) is decreasing in \( \psi \) and \( C_{LB}(L) \) is its minimum value on \( \Psi. \) Also, when \( c\lambda - \frac{C(0)}{L} = 0, \) then \( C_{LB}(\psi) = C(0). \) So, \( C_{LB}(\psi) \) is a constant function on \( \Psi \) and the optimal commitment lead time is \( 0 \) and \( L. \)

### A.8 Proof of Lemma 7

For each \( h, p, \) and \( \mu \neq 0, c_0 \) is continuous and differentiable.

1. The first derivative of \( c_0 \) with respect to \( h \) is

\[
\frac{dc_0}{dh} = \frac{S}{\mu} \left( F_X(S, 0) - F_X(S - 1, 0) \right) = \frac{S}{\mu} \left( F_X(S, 0) - \frac{S}{\mu} F_X(S - 1, 0) \right) . \quad \text{(A11)}
\]

Now we need to show that \( F_X(S, 0) \geq \frac{S}{\mu} F_X(S - 1, 0). \) For \( x = 0, 1, 2, \ldots, S - 1 \) and \( \mu \geq 0, \) we can write \( S \geq x + 1. \) Then, \( \frac{1}{\mu} \leq \frac{1}{x + 1}. \) We can multiply both sides of the last inequality by \( e^{\mu x + 1} \geq 0, \) then

\[
\frac{1}{S} \left( e^{-\mu x + 1} x! \right) \leq \frac{1}{x + 1} \left( e^{-\mu x + 1} x! \right) = \frac{e^{-\mu x + 1}}{(x + 1)!}.
\]

Using this inequality, we write

\[
\frac{\mu}{S} F_X(S - 1, 0) = \frac{\mu}{S} \sum_{x=0}^{S-1} \frac{e^{-\mu x + 1}}{x!} = \frac{\mu}{S} \sum_{x=0}^{S-1} \frac{e^{-\mu x + 1}}{(x + 1)!} \leq \frac{\mu}{S} \sum_{x=0}^{S-1} \frac{e^{-\mu x + 1}}{x!} = \frac{e^{-\mu x + 1}}{x!} - P_X(0, 0) \leq \frac{F_X(S, 0) - P_X(0, 0)}{F_X(S, 0)}. \]

Hence, \( \frac{S}{\mu} F_X(S - 1, 0) < F_X(S, 0). \) Then, from expression (A11), we have \( \frac{S}{\mu} > 0. \) As a result, \( c_0 \) is increasing in \( h. \)

2. The first derivative of \( c_0 \) with respect to \( p \) is

\[
\frac{dc_0}{dp} = \frac{S}{\mu} \left( F_X(S, 0) - F_X(S - 1, 0) + 1 - \frac{S}{\mu} \right) = \frac{S}{\mu} F_X(S - 1, 0) + \frac{S}{\mu} P_X(S, 0) - F_X(S - 1, 0) + 1 - \frac{S}{\mu} \leq \frac{S}{\mu} (1 - F_X(S - 1, 0)) + (1 - F_X(S - 1, 0)) + \frac{S}{\mu} P_X(S, 0) \leq \frac{S}{\mu} \sum_{x=S}^{\infty} \frac{e^{-\mu x + 1}}{x!} + \frac{S}{\mu} P_X(S, 0) = \frac{S}{\mu} \left( \sum_{x=S}^{\infty} \frac{e^{-\mu x + 1}}{x!} - P_X(S, 0) \right) + \frac{S}{\mu} \sum_{x=S}^{\infty} \frac{e^{-\mu x + 1}}{x!}.
\]
\[ \frac{\partial c_0}{\partial \mu} = S(h+p) \frac{d}{d \mu} \left( \frac{F_X(S,0)}{\mu} \right) \]

\[ - (h+p) \left( \frac{d F_X(S-1,0)}{d \mu} \right) - \frac{d (S \frac{\mu}{e^{\mu}}} {d \mu} \]

\[ = S(h+p) \left( \frac{\mu \left( \frac{d F_X(S,0)}{d \mu} \right) - F_X(S,0)}{\mu^2} \right) \]

\[ - (h+p) \frac{d F_X(S-1,0)}{d \mu} - \left( -pS \frac{\mu}{\mu^2} \right). \]

Using the equalities \( \frac{d F_X(S,0)}{d \mu} = -P_X(S,0) \) and \( \frac{d F_X(S-1,0)}{d \mu} = -P_X(S-1,0) \), we write:

\[ = S(h+p) \left( -\mu P_X(S,0) - F_X(S,0) \right) \]

\[ + (h+p) P_X(S-1,0) + \frac{pS}{\mu^2} \]

\[ = \frac{S}{\mu^2} (-\mu(h+p)P_X(S,0) - (h+p)F_X(S,0)) \]

\[ + \frac{\mu^2}{S} (h+p)P_X(S-1,0) + p \]

\[ = \frac{S}{\mu^2} (-\mu(h+p)P_X(S,0) - (h+p)F_X(S,0)) \]

\[ + \mu(h+p) \left( \frac{\mu}{S} P_X(S-1,0) \right) + p. \]

For Poisson probability distribution we know that \( \mu P_X(S-1,0) = \mu P_X(S,0) \). Then, the last expression reduces to:

\[ \frac{\partial c_0}{\partial \mu} = - \frac{S(h+p)}{\mu^2} \left( F_X(S,0) - \frac{p}{p+h} \right). \]

Since \( S \leq \), then \( F_X(S,0) \geq \frac{p}{p+h} \) and \( \frac{\partial c_0}{\partial \mu} \leq 0 \). As a result, \( c_0 \) is decreasing in \( \mu \).

**APPENDIX: COMPOUND POISSON DEMAND**

### B.1 Derivation of the cost function

By definition, compound Poisson demand means that the size of customer demand \( X \) is a stochastic variable. It is independent of other customer demands and the distribution of the customer arrival process. Similar to our previous assumption, we assume that the customer arrival process is a Poisson process with parameter \( \lambda \). Then the number of customers \( N \) in a time interval of length \( L - \psi \) (ie, lead time demand) has a Poisson distribution with mean \( \lambda (L - \psi) \). We define \( P_N(n, \psi) \) as the probability that the random variable \( N \) takes the value \( n \) when the commitment lead time is \( \psi \). We have

\[ P_N(n, \psi) = \frac{(\lambda (L - \psi))^n}{n!} e^{-\lambda (L - \psi)}, \quad n = 0, 1, 2, \ldots \quad (B1) \]

Let \( f_x \) be the probability that a customer demands \( x \) units, that is, the probability that \( X \) takes the value \( x = 1, 2, \ldots \) units. Assume that the mean demand size is \( \mu \). When \( f_1 = 1 \), the demand process is a pure Poisson process. Then total demand in a given interval is equal to the number of customer arrivals.

For handling the general case with varying demand sizes, we define \( f^n \) as the probability that \( n = 1, 2, \ldots \) customers demand \( x \) units. Since customer demand with the size of zero is not rational, then without loss of generality we assume that \( f_0 = 0 \) (Feeney and Sherbrooke, 1966). Knowing that \( f_0 = 0 \) and \( f_1 = f_x \), \( f^n \) can be calculated recursively as follows (see Axsäter, 2015, p. 79):

\[ f^n = \sum_{i=1}^{n-1} f^{n-1} f_{x-i}, \quad n = 2, 3, \ldots \quad (B2) \]

We define \( D \) as a random variable representing the total demand in the time interval \( L - \psi \). In addition, we define \( P_D(x, \psi) \) as the probability that \( D \) takes the value \( x \) when the commitment lead time is \( \psi \). Using expressions (B1) and (B2), we write \( P_D(x, \psi) \) as follows:

\[ P_N(x, \psi) = \begin{cases} \sum_{n=0}^{x} f^n P_N(n, \psi) & x = 0 \\ f^n P_N(x, \psi) & x = 1, 2, \ldots. \end{cases} \quad (B3) \]

Then, the cumulative distribution function of \( D \) is calculated as \( F_D(s, \psi) = \sum_{x=0}^{s} P_D(x, \psi) \).

Based on these definitions, the long-run average total cost \( C(s, \psi) \) consisting of holding, backordering and commitment costs can be calculated as follows:

\[ C(s, \psi) = h \mathbb{E} \{ [s - D]_+ \} + p \mathbb{E} \{ [D - s]_+ \} + \mu \lambda CC(\psi) \]

\[ = h \sum_{x=0}^{s} (s-x) P_D(x, \psi) \]
Then we need to derive the expression for\( \frac{dP_D(x, \psi)}{d\psi} \). First, we calculate \( \frac{dP_D(x, \psi)}{d\psi} \), \( \frac{dF_D(s, \psi)}{d\psi} \), and \( \frac{d\tilde{C}(s, \psi)}{d\psi} \) as follows:

\[
\frac{dP_D(x, \psi)}{d\psi} = \sum_{n=0}^{\infty} f^n_{x} \left( \frac{dP_{\lambda}(n, \psi)}{d\psi} \right)
\]

Hence, the expression for \( \frac{d\tilde{C}(s, \psi)}{d\psi} \) becomes

\[
\frac{d\tilde{C}(s, \psi)}{d\psi} = (h+p) \left( s \left( \frac{dF_D(s, \psi)}{d\psi} \right) - \sum_{x=0}^{s} x \left( \frac{dP_D(x, \psi)}{d\psi} \right) \right) + p \frac{d\mu_D}{d\psi}
\]
\[
\frac{d}{d\psi} \left( \frac{\mathcal{C}(s,\psi)}{L}\right) = \frac{1}{(L-\psi)^2} \left( (L-\psi) \frac{d\mathcal{C}(s,\psi)}{d\psi} + \mathcal{C}(s,\psi) \right)
\]

\[
= \frac{(L-\psi)\lambda(\mathcal{C}(s,\psi) - p(\mu_D-s) - (h+p) \sum_{s=0}^{\infty} \sum_{n=0}^{\infty} (s-x)f_n^{\mu}(n-1,\psi) - p\mu) + \mathcal{C}(s,\psi)}{(L-\psi)^2}
\]

\[
= \frac{\lambda(L-\psi) + 1)\mathcal{C}(s,\psi) - \lambda(L-\psi) (p(\mu_D-s) - (h+p) \sum_{s=0}^{\infty} \sum_{n=0}^{\infty} (s-x)f_n^{\mu}(n-1,\psi) + p\mu)}{(L-\psi)^2}
\]

For all \(\psi \in \Psi\) and corresponding \(s\) which satisfies (4), we need to have \(\frac{d}{d\psi} \left( \frac{\mathcal{C}(s,\psi)}{L}\right) \geq 0\). However, we could not prove this analytically.

### B.4 Numerical results for the compound demand case

We confirm the correctness of Conjectures (9) and (10) through numerical analysis. Without loss of generality, we assume that the demand size \(X\) has a shifted Poisson distribution by one unit, that is, \(X = Y + 1\), where \(Y \sim \text{Poisson}(\mu_Y)\). Then, \(P_D(x,\psi)\) can be written as follows:

\[
P_D(0,\psi) = P_{X'}(0,\psi)
\]

\[
P_D(x,\psi) = \sum_{n=1}^{\infty} P(X_1 + X_2 + \cdots + X_n = x)P_{X'}(n,\psi)
\]

\[
= \sum_{n=1}^{\infty} P(Y_1 + Y_2 + \cdots + Y_n = x-n)P_{X'}(n,\psi)
\]

\[
= \sum_{n=1}^{\infty} \left( \frac{\mu_Y - n}{(x-n)!} \right) \frac{(\lambda(L-\psi))^n}{n!} e^{-\lambda(L-\psi)}, \quad x = 1, 2, \ldots
\]

Note that we shifted Poisson distribution by one unit to avoid generating zero demand size. Then the mean demand size per customer is \(\mu = \mu_Y + 1\). Hence, we have \(\mu_D = \mu(\lambda(L-\psi)) = (\mu_Y + 1)\lambda(L-\psi)\). By considering linear commitment cost, we can write

\[
C(s,\psi) = (h+p) \left( sF_D(s,\psi) - \sum_{x=0}^{s} xP_D(x,\psi) \right)
\]

\[
+ p(\mu_D-s) + c\mu \lambda \psi
\]

\[
= (h+p) \left( s \sum_{n=0}^{s} F_Z(s-n)P_{X'}(n,\psi) - \sum_{n=0}^{s} nF_Z(s-n) + \mu_T F_Z(s-n-1)P_{X'}(n,\psi) \right)
\]

\[
+ p((\mu_Y + 1)\lambda(L-\psi) + s) + c(\mu_Y + 1)\lambda \psi
\]

\[
= (h+p) \sum_{n=0}^{s} ((s-n)F_Z(s-n) - n\mu_T F_Z(s-n-1)P_{X'}(n,\psi) + p((\mu_Y + 1)\lambda(L-\psi) - s) + c(\mu_Y + 1)\lambda \psi.
\]

The expressions for \(\sum_{x=0}^{s} xP_D(x,\psi)\) and \(F_D(s,\psi)\) rely on the following derivations:

\[
\sum_{x=0}^{s} xP_D(x,\psi) = \sum_{x=0}^{s} \sum_{n=0}^{x} \left( \frac{(\mu_Y - n)\lambda(x-n)}{(x-n)!} e^{-\mu_Y \lambda} \right) P_{X'}(n,\psi)
\]

since we know that \(n \leq x\), then by changing the order of sums, we have

\[
= \sum_{n=0}^{s} \sum_{x=n+1}^{s} \left( \frac{(\mu_Y - n)\lambda(x-n)}{(x-n)!} e^{-\mu_Y \lambda} \right) P_{X'}(n,\psi)
\]

\[
= \sum_{n=0}^{s} P_{X'}(n,\psi) \sum_{x=n+1}^{s} \left( \frac{(\mu_Y - n)\lambda(x-n)}{(x-n)!} e^{-\mu_Y \lambda} \right)
\]
changing the order of sums, we have
\[ \sum_{n=0}^{s} P_{X}(n, \psi) \sum_{x=n}^{s} (n \mu Y)^{s-n} (x-n)! e^{-n Y} \]
\[ = \sum_{n=0}^{s} P_{X}(n, \psi) \sum_{x=0}^{s-n} (n \mu Y)^{s-n} x! e^{-n Y} \]
\[ = \sum_{n=0}^{s} P_{X}(n, \psi) \sum_{x=n}^{s} Z(n, \psi) \]
\[ = \sum_{n=0}^{s} F_{Z}(s-n) P_{X}(n, \psi) \]

For proving the optimality of the bang-bang policy we need to show that \( \frac{d(\hat{C}(s, \psi))}{d\psi} \geq 0 \). First we need to calculate \( \frac{d\hat{C}(s, \psi)}{d\psi} \).

We define \( Q(s, n) \) as \( (s-n)F_{Z}(s-n) - n \mu Y F_{Z}(s-n-1) \), then \( \hat{C}(s, \psi) = (h + p) \sum_{n=0}^{s} Q(s, n) P_{X}(n, \psi) + p(\mu Y + 1) \lambda (L - \psi - s) \).

changing the order of sums, we have
\[ = \sum_{n=0}^{s} P_{X}(n, \psi) \sum_{x=n}^{s} (n \mu Y)^{s-n} (x-n)! e^{-n Y} \]
\[ = \sum_{n=0}^{s} P_{X}(n, \psi) \sum_{x=0}^{s-n} (n \mu Y)^{s-n} x! e^{-n Y} \]
\[ = \sum_{n=0}^{s} P_{X}(n, \psi) \sum_{x=n}^{s} Z(n, \psi) \]
\[ = \sum_{n=0}^{s} F_{Z}(s-n) P_{X}(n, \psi) \]

For a parameter setting shown in Figure A1, we can see that \( C(s, \psi) \) is convex in \( \psi \). We also observe that consistent with our previous analysis for pure Poisson demand, the optimal base-stock is nonincreasing in \( \psi \).

For the same parameter setting, in Figure A2 we observe that \( \frac{d(\hat{C}(s, \psi))}{d\psi} \geq 0 \). We run similar numerous experiments with different parameter settings and observe the same results. Therefore, we conjecture the optimality of the bang-bang commitment lead time policy.