Many-body theory of spin-current driven instabilities in magnetic insulators

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We consider a magnetic insulator in contact with a normal metal. We derive a self-consistent Keldysh effective action for the magnon gas that contains the effects of magnon-magnon interactions and contact with the metal to lowest order. Self-consistent expressions for the dispersion relation, temperature, and chemical potential for magnons are derived. Based on this effective action, we study instabilities of the magnon gas that arise due to spin current flowing across the interface between the normal metal and the magnetic insulator. We find that the stability phase diagram is modified by an interference between magnon-magnon interactions and interfacial magnon-electron coupling. These effects persist at low temperatures and for thin magnetic insulators.

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I. INTRODUCTION

Understanding the interplay between magnetization dynamics and spin currents is a fundamental issue that is relevant for spintronic devices [1–4]. In particular, there is a growing interest, both theoretically [5–13] and experimentally [14–20], in magnetic insulators such as yttrium iron garnet (YIG) in contact with heavy metals like Pt (YIG/Pt bilayers). In these hybrid systems, magnons and magnetization dynamics are excited via interfacial spin-transfer torques [21,22]. The realization of a Bose-Einstein condensate (BEC) of magnons through this mechanism has recently been proposed [23]. Auto-oscillations driven by the spin Hall effect [24] and thermal spin current [25,26] have very recently been observed. Earlier, a condensate has been realized at room temperature in YIG by other means, namely via parametric pumping [27]. This is an example of a nonequilibrium condensate of quasiparticles [28–34]. Such nonequilibrium BECs have attracted a great deal of attention and occur in different physical systems such as excitons [35–37], phonons [38], polaritons [39], and photons [40]. Specifically, condensation of magnons has stimulated efforts to control coherent transport of spin waves at room temperature [41].

In this paper, we present a microscopic study of magnon instabilities (such as Bose-Einstein condensation and/or swas- ing [42]) in insulating ferromagnets (F) induced via spin-current injection through the interface with an adjacent normal metal (N; see Fig. 1). The interfacial spin current is generated by the combined effects of a thermal gradient across the interface [7,43] and the spin Hall effect in the normal metal (leading to a spin accumulation in the normal metal at the interface). We derive a Keldysh effective action for the magnons in the F up to second order in the coupling with the normal metal. Through this approach, self-consistent relations are derived for the thermodynamic variables and the dispersion relation of magnons. We use this description to find the stability phase diagram. Besides introducing a different theoretical framework, this approach improves the previous treatment [42] by including interference effects between magnon-magnon interactions and interfacial magnon-electron coupling. These effects are finite at low temperature, and may prevent instabilities if they are very strong.

The remainder of this article is organized as follows. The following Sec. II introduces our model to describe the magnon dynamics and its coupling to electrons. In Sec. III, we proceed to derive the effective action for the magnons within the functional formulation of the Schwinger-Keldysh formalism. In Sec. IV, we construct the stability phase diagram. We end in Sec. V by summarizing our results with a brief discussion and conclusion. In the Appendixes, we detail various technical steps of the calculations.

II. MODEL

The system under consideration is a F in contact with a N as is displayed in Fig. 1. We assume a three-dimensional system of localized spins in quasiequilibrium at temperature \( T_m \) and magnon chemical potential \( \mu_m \) [44,45]. The normal metal is at temperature \( T_e \) and has a spin accumulation \( \Delta \mu \).

The spin Hamiltonian is introduced by labeling the square lattice site by the position \( \mathbf{x} \), with the spin operator \( \hat{S}_x \) at position \( \mathbf{x} \). The nearest-neighbor Hamiltonian is

\[
\hat{H}_S = -J_e \sum_{\langle \mathbf{x}, \mathbf{x} \rangle} \hat{S}_x \cdot \hat{S}_{x'} - B \sum_{\mathbf{x}} \hat{S}_z + K_z \frac{1}{2} \sum_{\mathbf{x}} (\hat{S}_z)^2, \tag{1}
\]

where \( J_e > 0 \) is the exchange coupling between nearest neighbors, \( K_z \) is the easy-plane anisotropy constant, and \( B \) is the external magnetic field in units of energy. We consider linearized spin excitations, magnons, around the \( z \) direction for sufficiently large fields. These are introduced by the
Holstein-Primakoff transformation \cite{46–48} that quantizes the spins in terms of bosons, $\hat{S}_x^{(2)} = \sqrt{2S-b_x \hat{b}_x S_x}$ and $\hat{S}_z^{(2)} = S - \hat{b}_x \hat{b}_x$ with $S$ the spin quantum number. One magnon, created (annihilated) at site $x$ of the lattice by the operator $\hat{b}_x^{\dagger}$, corresponds to changing the total spin with $+h(-h)$. Expanding up to fourth order in the magnon operators, the spin Hamiltonian becomes $\hat{H}_S \simeq \hat{H}_S^{(2)} + \hat{H}_S^{(4)}$, where terms up to order $1/\sqrt{S}$ are kept. In momentum space the quadratic part of the Hamiltonian is given by

$$\hat{H}_S^{(2)} = \sum_{q} \epsilon_q \hat{b}_q^{\dagger} \hat{b}_q.$$  

In the long-wavelength limit, the magnon dispersion is $\epsilon_q = Aq^2 + \epsilon_0$, where $\epsilon_0 = B - SK$ is the magnon gap and $A = 3SL^2a^2$ is the spin stiffness with $a$ the lattice spacing. Furthermore,

$$\hat{H}_S^{(4)} = \sum_{q_1,q_2,q_3,q_4} V^{(2,2)}_{q_1,q_2,q_3,q_4} \hat{b}_{q_1}^{\dagger} \hat{b}_{q_2}^{\dagger} \hat{b}_{q_3} \hat{b}_{q_4},$$  

where momentum conservation is implicit in $V^{(2,2)}$. This part of the Hamiltonian represents magnon-magnon interactions that result from exchange and anisotropy that causes the thermalization of magnons in the $F$ \cite{47}. In the long-wavelength limit the interaction between magnons is dominated by the anisotropy energy. Henceforth we consider magnons with sufficiently long wavelengths since we are interested in long-wavelength instabilities in the $F$. Thus we approximate the scattering amplitude by $V^{(2,2)}_{q_1,q_2,q_3,q_4} = (K_c/2N)\delta_{q_1+q_2-q_3-q_4}$, with $N$ the total number of spins. The exact result containing contributions from the exchange interactions is found in Appendix A.

The electronic degrees of freedom in the metal are described by the tight-binding Hamiltonian \cite{46–48}

$$\hat{H}_e = -t \sum_{\langle x,x\prime \rangle,\sigma} \hat{\psi}_{x,\sigma}^{\dagger} \hat{\psi}_{x\prime,\sigma} - \mu_{\sigma} \hat{\psi}_{x,\sigma}^{\dagger} \hat{\psi}_{x,\sigma},$$  

in terms of second-quantized operators $\hat{\psi}_{x,\sigma}^{\dagger}$, that create (annihilate) an electron at site $x$ in $M$ with spin $\sigma$. The hopping amplitude is $t$ and the spin-dependent chemical potential is $\mu_{\sigma}$. The latter results from the spin Hall effect in the $M$ and defines a nonzero spin accumulation $\Delta \mu = \mu_+ - \mu_-$. We assume the magnons and electrons predominantly interact via an exchange coupling between the spin density of electrons and localized magnetic moments facing the $F/M$ interface. The Hamiltonian that couples metal and insulator is $\hat{H}_{e-m} = - \sum_{x,x'} J_{xx} \hat{S}_x \cdot \hat{S}_{x'}$, where $J_{xx}$ is the coupling strength that depends on the interface details. The spin density of electrons at site $x$ is $\hat{S}_x = \sum_{\sigma'} \hat{\psi}_{x,\sigma'}^{\dagger} \tau_{\sigma\sigma'} \hat{\psi}_{x,\sigma'}$ with $\tau$ the Pauli matrix vector. After the Holstein-Primakoff transformation on the spins in the insulator, the electron-magnon Hamiltonian is

$$\hat{H}_{e-m} = - \sum_{x,x'} \sqrt{2S} \left( \hat{b}_x^{\dagger} \hat{\psi}_{x,\sigma}^{\dagger} \hat{\psi}_{x',\bar{\sigma}} + H.c. \right)$$

$$+ (S-\hat{b}_x \hat{b}_x) \left( \hat{\psi}_{x,\uparrow}^{\dagger} \hat{\psi}_{x',\downarrow} - \hat{\psi}_{x,\downarrow}^{\dagger} \hat{\psi}_{x',\uparrow} \right),$$

up to quadratic order in the magnon operators. Hence, when an electron flips its spin at the interface it creates (or annihilates) one magnon in the insulating $F$ \cite{5,6}. The electron-magnon interaction has been studied in various related contexts \cite{13,49–54}. Its evaluation through the self-energy has been useful for the study of magnetic damping and noise \cite{55}, magnon-induced superconductivity \cite{56}, and spin transport \cite{57}. In our work we go further by studying, in combination magnon-magnon interactions, its effect on the single-magnon energy. Additionally, thermodynamic properties like temperature and chemical potential of the magnon gas are computed self-consistently with the electron-magnon coupling as the perturbative parameter.

III. NONEQUILIBRIUM THEORY

In this section, we derive an effective action for the magnon gas using a functional Keldysh approach \cite{58}. We also calculate the self-energy that magnons acquire by their interaction with the electrons.

A. Self-energy due to electron-magnon coupling

The starting point is the functional integral

$$\mathcal{Z} = \int D\phi^* D\phi D\psi^* D\psi \exp \left\{ \frac{i}{\hbar} S[\psi, \psi^*, \phi, \phi^*] \right\}. \quad (6)$$

The action is expressed in terms of bosonic fields $\phi_x(t)$, describing magnons, and fermionic fields $\psi_{x,\sigma}(t)$, describing electrons. Thus,

$$S[\psi, \psi^*, \phi, \phi^*] = S_0[\phi, \phi^*] + \int dt dt' \sum_{x,x'} \sum_{\sigma,\sigma'} \psi_{x,\sigma}^{\ast}(t)$$

$$\times \bar{K}_{x,x'}^{\sigma,\sigma'}(t,t') \psi_{x',\sigma}(t'), \quad (7)$$

with $S_0[\phi, \phi^*] = \int dt \int d\phi^* \bar{\phi} \partial_t \phi - H_0[\phi, \phi^*]$, the action for magnons uncoupled with the electrons. The magnon Hamiltonian in the continuum limit given by

$$H_0[\phi, \phi^*] = \int d\phi \left(-\Delta \nabla x^2 + \epsilon_0 + \frac{K_c}{2} |\phi|^2\right),$$

and follows directly from evaluating the Hamiltonian in Eqs. (2) and (3) for the bosonic fields and taking the
continuum limit. The time integration in Eq. (7) is over the Keldysh contour $C^\infty$, whereby the functional integral in Eq. (6) is over all fields $\phi_k(t)$ and $\psi_{x,\sigma}(t)$ that evolve forward in time from $-\infty$ to $t_0$, and backwards from $t_0$ to $-\infty$. The kernel in Eq. (7) is defined as

$$K_{xx'}^{\sigma\sigma'}(t, t') \equiv G_{xx'}^{-1}(t, t')\delta(t, t') + \delta K_{xx'}^{\sigma\sigma'}(t, t'),$$

(9)

where $G^{-1}$ is the inverse of the free Green’s function for the electrons that obeys

$$\sum_x [i\hbar\delta_{xx'} \frac{\partial}{\partial t} + t_{xx',\sigma}^\sigma] G_{xx',\sigma\sigma'}(t, t') = \delta(t, t')\delta_{\sigma\sigma'}\delta_{xx'},$$

(10)

with $t_{xx,\sigma} = \int\! d\sigma \delta_{xx'} \delta_{\sigma\sigma'}$, where the notation $\pm$ denotes all next-nearest neighbors. The interactions between electronic spin and magnetic moments are described by

$$\delta K_{xx'}^{\sigma\sigma'}(t, t') = \sqrt{2S} \delta_{xx'} \sum_x J_{xx'} \left[ \phi_k^\sigma(t) \tau_{\sigma\sigma'}^+ + \phi_k(t) \tau_{\sigma\sigma'}^- \right]$$

$$+ \frac{1}{\sqrt{2S}} \left[ S - \phi_k^\sigma(t) \phi_k(t) \right] \tau_{\sigma\sigma'}^z \delta(t, t'),$$

(11)

where $\tau^\pm = (\tau^+ \pm i\tau^-)/2$.

We now integrate out the electronic degrees of freedom in Eq. (7). The functional integration can be done exactly since the fermionic part of the action is a Gaussian integral. This leads us to an effective theory for magnons in the $F|M$ interface. In order to find concrete results, we give expressions for the retarded/advanced component of the self-energy, $\Sigma_{\alpha\beta}^{\sigma\sigma'}(t, t') = \int\! dt\! J_{xx'} \Pi_{xx'}^{\sigma\sigma'}(t, t') \tau_{\sigma\sigma'}^{\alpha\beta}$.

The self-energy in Eq. (12) is written in momentum space where $\Sigma_{\alpha\beta} = \int\! \frac{d\mathbf{k}}{(2\pi)^3} \Sigma_{\alpha\beta}^{\sigma\sigma'}$, with $\Sigma_{\alpha\beta}^{\sigma\sigma'}$ being the Fourier transform of Eq. (13). The summation on $k_\perp$ represents a sum over all electronic wave vectors perpendicular to the interface. In order to find concrete results, we give expressions for the retarded/advanced component of the self-energy, $\Sigma_{\alpha\beta}^{\sigma\sigma'}(t, t') = \int\! dt\! J_{xx'} \Pi_{xx'}^{\sigma\sigma'}(t, t') \tau_{\sigma\sigma'}^{\alpha\beta}$, which we need later on. Details of its derivation are found in Appendix C. In addition, we assume that the interfacial exchange coupling is nonzero only at the $F|N$ interface, i.e., $J_{xx'} = J_0\delta_{xx'}$, with $x_0$ the position of the interface. Thus the Fourier transform of the electron bubble, in energy and momentum space, is

$$\Pi^{\uparrow,\downarrow}(k, \epsilon) = \frac{1}{2} N(0) a^3 \left[ \frac{1}{3} \left( \frac{k}{2k_F} \right)^2 - 1 \right] - \frac{N(0) a^3 \epsilon}{16\epsilon_F^2}$$

$$\times \left[ \frac{2k_F}{k} \right]^2 + 1 \pm \frac{\pi N(0) a^3 k_F}{8\epsilon_F |k|} (\Delta \mu - \epsilon)$$

$$\times \Theta \left[ 1 - \left( \frac{|k|}{2k_F} \right)^2 \right],$$

(14)

with $N(0)$ the electronic density of states at the Fermi level, and $k_F$ and $\epsilon_F$ the Fermi wave number and energy, respectively. The $\Theta$ function represents the Heaviside step function.

**B. Effective action**

We are interested in real-time dynamics of the magnon gas, thus we expect the effective action to depend only on the retarded part of $\Sigma_{\alpha\beta}^{\sigma\sigma'}(t, t')$. Moreover, since the magnon-magnon interactions resulting from the anisotropy are short ranged, we can use the so-called ladder approximation [58]. We project, in Eq. (12), the fields onto the real-time axis by the substitution $\phi_k = \Psi \pm i\xi/2$, where momentum labels were omitted. $\Psi$ represents the classical field and the symbol $\pm$ refers to the upper and lower branch of the Keldysh contour. The field $\xi$ denotes the quantum fluctuations which by the

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**FIG. 2.** Feynman diagrams contributing to the magnon dynamics due to electron-magnon interactions at the $F|M$ interface. (a) Electron bubble diagram representing the annihilation and creation of magnons via spin-flip processes. (b) Diagram for the enhanced spin polarization. The momentum carried by the magnon is denoted by $p$. Fig. 2(b). This imbalance is defined by $\delta n = \bar{n} - n$, where $\bar{n}_{\sigma}(x_0) = n_{\sigma} - S \sum_{x_0} \int\! dt\! J_{xx'} \Pi_{xx'}^{\sigma\sigma'}(t, t') \tau_{\sigma\sigma'}^{\alpha\beta}$. The self-energy in Eq. (12) is written in momentum space where $\Sigma_{\alpha\beta} = \sum_{k_\perp} \Sigma_{\alpha\beta}^{\sigma\sigma'}$, with $\Sigma_{\alpha\beta}^{\sigma\sigma'}$ being the Fourier transform of Eq. (13). The summation on $k_\perp$ represents a sum over all electronic wave vectors perpendicular to the interface. In order to find concrete results, we give expressions for the retarded/advanced component of the self-energy, $\Sigma_{\alpha\beta}^{\sigma\sigma'}(t, t') = \int\! dt\! J_{xx'} \Pi_{xx'}^{\sigma\sigma'}(t, t') \tau_{\sigma\sigma'}^{\alpha\beta}$, which we need later on. Details of its derivation are found in Appendix C. In addition, we assume that the interfacial exchange coupling is nonzero only at the $F|N$ interface, i.e., $J_{xx'} = J_0\delta_{xx'}$, with $x_0$ the position of the interface. Thus the Fourier transform of the electron bubble, in energy and momentum space, is

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$$\times \left[ \frac{2k_F}{k} \right]^2 + 1 \pm \frac{\pi N(0) a^3 k_F}{8\epsilon_F |k|} (\Delta \mu - \epsilon)$$

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is much smaller than the thermal momenta usual shift of the energy due to the two-particle interaction. Evaluation is carried out using the result given in Eqs. (13).

(a) Feynman diagram for the dressed Green’s function of magnons due to the coupling with the $M$. Bubble diagram represents the self-energy given in Eq. (13). (b) Diagrammatic representation for the interacting self-energy and (c) the $T$-matrix ladder approximation, where the wavy lines denote the contact interaction $K$. Note that the self-energy and $T$ matrix are determined by the dressed and free propagators, respectively.

\[
\text{Re}[\hbar\Sigma^{(+)}_\text{in}(0, \epsilon_0 - \mu_m)] = -a^2 K\int \frac{d\mathbf{q}}{(2\pi)^2} \int \frac{d\epsilon'}{(2\pi)^2} N_B \left( \frac{\epsilon' - \Delta\mu}{k_BT_e} \right) \text{Im}[\hbar\Sigma^{(+)}(\mathbf{q}', \epsilon')]/\epsilon' + (\epsilon' - \mu_m)^2. 
\]

The single-particle dispersion relation of magnons is renormalized by the magnon-magnon interactions and the coupling with electrons, and obeying the self-consistent relation

\[
\epsilon'(\mathbf{q}) = \epsilon(\mathbf{q}) + \text{Re}[\hbar\Sigma^{(+)}(\mathbf{q}; \epsilon'(\mathbf{q}) - \mu_m)].
\]

Here, $\hbar\Sigma = \hbar\Sigma_{\text{el}} + \hbar\Sigma_{\text{magnon}}$ is the sum of contributions due to the coupling with electrons and magnon-magnon interactions, and $\mu_m$ is the chemical potential of magnons. Moreover, we have included in the action Eq. (15) a mean-field interaction with the thermal cloud of magnons, whose density is $n_{\text{th}} = \zeta(3/2)d_F(k_BT_m/4\pi A)^{3/2}$, with $d_F$ the thickness of $F$ and $\zeta$ the Riemann zeta function. The derivation of $n_{\text{th}}$ is straightforward and can be found in Ref. [59]. The self-energy $\hbar\Sigma_{\text{magnon}}$ due to interactions is diagrammatically shown in Fig. 3. The real part of $\hbar\Sigma_{\text{magnon}}$—which we need later on—reads

where $N_B(x) = [e^x - 1]^{-1}$ is the Bose distribution function and $\mathcal{P}$ denotes the principal value of the integral. In Eq. (17) we note that the term on the right-hand side represents the interference between magnons and electrons. This can be seen in Figs. 3(a) and 3(b) where magnons, and thus their interactions, are dressed by their coupling with electrons. Its evaluation is carried out using the result given in Eqs. (13) and (14). Additionally, the second term corresponds to the usual shift of the energy due to the two-particle interaction. To obtain Eq. (17) it has been assumed that the momentum $h\mathbf{q}$ is much smaller than the thermal momenta $\hbar/\Lambda_{\text{th}}$ [58], with $\Lambda_{\text{th}} = \sqrt{3\pi A/k_BT_m}$ the thermal de Broglie wavelength for the magnons.

When the energy of the single-particle state $\epsilon'(\mathbf{q} \to 0)$ becomes less than $\mu_m - a^2 K\cdot n_{\text{th}}$, the magnon system becomes unstable. This signals the formation of a magnon Bose-Einstein condensate, a precessional instability, or magnetization reversal. The criterion for such an instability is thus

\[
\epsilon'(\mathbf{q} \to 0) + a^2 K\cdot n_{\text{th}} - \mu_m < 0. 
\]

Based on this condition, a phase diagram is determined in the next section. It is worthwhile to comment that Eq. (18) involves self-consistent physical quantities such as the magnon energy, Eq. (16), magnon temperature, and chemical potential. Unlike previous works [23,42], all these quantities can be evaluated self-consistently at leading order in the interfacial coupling $J$ with the electrons as we outline below. Before proceeding to evaluate Eq. (18), however, we need to determine the magnon chemical potential and temperature for a given electron spin accumulation and temperature. This is done through the Boltzmann equation for magnons with the metallic coupling acting as an electronic reservoir that transfers spin and energy. Details of this calculation are outlined in Appendix E. A Gilbert damping constant $\alpha$, parametrizing the coupling with phonons, is phenomenologically added. Finally, a steady state is required in the kinetic equations for the total density of magnons and total energy. In this limit, we find relations at thermodynamic equilibrium for the magnon temperature $T_m$ and chemical potential $\mu_m$ in terms of the temperature $T_e$ and spin accumulation $\Delta\mu$ of the electrons. These read

\[
(1 + \chi)|L_{12}(z)| + \frac{e_0(2\chi - \bar{\mu}) - \Delta\mu}{k_BT_m}L_{11}(z) = \frac{\tilde{\epsilon}_0 T_e}{k_BT_m^2}, 
\]

(18)

\[
1 + \frac{2}{\chi} L_{11}(z) + \frac{e_0(2\chi - \bar{\mu}) - \Delta\mu}{k_BT_m}L_{12}(z) 
\]

\[
+ \frac{e_0(2\chi - \bar{\mu}) - \Delta\mu}{k_BT_m^2}L_{11}(z) = \frac{\tilde{\epsilon}_0 T_e}{6k_BT_m^3}, 
\]

(19)

(20)

where we used the PolyLogarithm function $L_i(z) = \Gamma^{-1}(s)\int_0^\infty dy y^{i-1}/(e^y z^{i-1} - 1)$, with $\Gamma(s)$ the gamma function.
expression for the instability criterion is obtained via $F[\delta \mu, \eta] = e_0 + e_1 \eta^{3/2} - e_2 \delta \mu N_B(-\delta \mu) - \delta \mu$. The coefficients $e_0$, $e_1$, and $e_2$ are dimensionless parameters that obey

$$e_0 = \frac{\epsilon_0 - \mu_m}{k_B T_e} \left( \frac{1}{JN(0)\alpha^3} - 1 \right) + \frac{2JS}{k_B T_e} - \frac{K_z^2}{4aK_BT_eJN(0)} \times \int \frac{d\mathbf{q}'}{(2\pi)^2} \int \frac{d\mathbf{q}''}{(2\pi)^2} N_B \left( \frac{\epsilon(\mathbf{q}'') - \mu_m}{k_BT_m} \right) \times N_B \left( \frac{\epsilon(\mathbf{q}'') - \mu_m}{k_BT_m} \right) \mathcal{P} \frac{\mathbf{q}' \cdot \mathbf{q}''}{\mathbf{q}' \cdot \mathbf{q}''},$$

$$e_1 = \left[ \frac{\pi}{2} \right] \left( \frac{k_BT_e}{A} \right)^{1/2} \frac{K_d F}{\pi J N(0) a},$$

$$e_2 = \frac{a^3 k_F J S}{2\epsilon_0 F} \int \frac{d\mathbf{q}'}{(2\pi)^2} \mathcal{P} \ln \frac{1 + \sqrt{1 - (q/2k_BT_e)^2}}{1 - \epsilon(\mathbf{q}')/\epsilon_0^2}.$$ 

These parameters show the effect from interfacial coupling, magnon-magnon interactions, and thermal cloud on the magnon gap. Note that $e_2$ is proportional to the magnon-magnon interaction $K_z$ and the magnon-electron coupling $J$, thus representing the interference between magnons and electrons. In fact, the two-particle interaction occurs between magnons that are dressed by their coupling with electrons. Taking $\epsilon_0 \sim K_z$ and the critical temperature of the $F_T c \sim A/a^2 k_B \ll T_c$ we estimate the dimensionless parameters as

$$e_0 \sim \left[ 1 - \left( \frac{K_z}{A/aS} \right)^2 \frac{J \epsilon_0}{k_BT_e} \right], \quad e_1 \sim \left( \frac{T_c}{T_e} \right)^{1/2} \left( \frac{K_z}{k_BT_e} \right) \left( \frac{\epsilon_0}{\epsilon_0^2} \right), \quad e_2 \sim \left( \frac{\epsilon_0}{\epsilon_0^2} \right).$$

For typical values we expect $e_2 \ll 1$, $e_0 = O(1)$, but that $e_1$ can be rather large.

The phase diagram is shown in Fig. 5 as a function of the effective chemical potential $\delta \mu$ and thermal imbalance $\eta$, respectively. It consists of a stable and unstable magnon phase, separated by the line $F[\delta \mu, \eta] = 0$, for different values of $e_2$ [panel (a)] and $e_1$ [panel (b)]. It is worthwhile to note that in the limit $e_2 \ll 1$, i.e., for $J \ll \epsilon_F$, the criterion for instability reduces to that for a Bose-Einstein condensation in a Bose gas in the Popov approximation [59], for which the critical temperature is $\propto (\delta \mu - \epsilon_0)^2/3$. On the other hand, the unstable region diminishes as we increase both parameters $e_1$ and $e_2$. In particular, when $e_2 > 1$ the unstable region is suppressed. This can be further analyzed by taking the limit $\eta \rightarrow 0$ in Eq. (21), where we see that $F \rightarrow e_0 + (e_2 - 1)\delta \mu$, for large $\delta \mu$. Clearly, when $e_2 > 1$ the magnon gas never shows an instability if $e_0 > 0$. The term proportional to $e_2$ stems from combined effects of magnon-electron coupling and magnon-magnon interactions. We thus find that if these combined effects are sufficiently strong, an instability does not occur. While our perturbative approach is not valid in this regime, it still hints at interesting strong-coupling effects. Finally, we remark that at zero spin accumulation, $e_0$ and $e_2$ reflect the shift in the energy of single-particle ground state due to the electron-magnon coupling.

V. DISCUSSION AND CONCLUSIONS

In this paper, we have presented a formalism to microscopically investigate how spin currents across a $F'M$ interface lead to instabilities in the magnetic insulator. Our study has
of relevance once interfacial electron-magnon interactions are being experimentally explored beyond the YIG-Pt paradigm.

Here, we have investigated electron-magnon and magnon-magnon interactions perturbatively. Future work should address the effects of strong interactions also by other means, i.e., by using the renormalization group. Another interesting direction is extending our theory beyond the linear stability analysis performed here to include the description of the dynamics in the unstable region of the phase diagram.

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APPENDIX A: MAGNON-MAGNON INTERACTION

Here we provide the exact expression for the interacting Hamiltonian of magnons. This is obtained by introducing the Holstein-Primakoff bosons approximated up to order $1/\sqrt{Z}$, $\hat{S}_x^z \approx \sqrt{2SZ\hat{b}_x} - (2\sqrt{Z})^{-1}\hat{b}_x^+\hat{b}_x$, $\hat{S}_z \approx \sqrt{2SZ}\hat{b}_x - (2\sqrt{Z})^{-1}\hat{b}_x^+\hat{b}_x$, and $\hat{S}_x = (S - \hat{b}_x^+\hat{b}_x)$. Then we expand up to fourth order in magnon operators on the spin Hamiltonian Eq. (1). Thus in momentum space the strength of interaction obeys

$$V^{(2,2)}_{\mathbf{q},\mathbf{q},\mathbf{q},\mathbf{q}} = \frac{J_z N}{4N}\left[\gamma_{\mathbf{q}} + \gamma_{\mathbf{-q}} + \gamma_{\mathbf{-q}} + \gamma_{\mathbf{-q}} - 4\gamma_{\mathbf{q}\mathbf{-q}}\right] + \frac{K_z}{2N} \times \delta_{\mathbf{q}+\mathbf{q},\mathbf{-q}+\mathbf{-q}},$$

(A1)

where $N$ is the number of spins and $\gamma_{\mathbf{q}} = z^{-1}\sum_{\alpha} e^{\mathbf{q}\alpha}$, with $z$ the coordination number and the sum runs over next-nearest neighbours. In the long-wavelength limit ($\mathbf{q} \rightarrow 0$) the strength of interactions becomes independent on the exchange coupling, $V^{(2,2)}_{\mathbf{q},\mathbf{q},\mathbf{q},\mathbf{q}} = (K_z/2N)\delta_{\mathbf{q}+\mathbf{q},\mathbf{-q}+\mathbf{-q}}$.

APPENDIX B: DERIVATION OF EFFECTIVE ACTION

$S_\infty[\phi, \phi^*]$ (Eq. (12))

In this section we derive the effective action given in Eq. (12) from Eq. (7). To start with, we consider explicitly the generating functional [Eq. (6)] as

$$Z = \int D\phi^* D\phi D\psi^* D\psi \exp\left\{ \frac{i}{\hbar} S_0[\phi, \phi^*] \right\} + \frac{i}{\hbar} \int dt dt' \sum_{xx} \sum_{\sigma\sigma'} \psi_x^{\sigma*}(t) \mathcal{K}_{\sigma\sigma'}^{xx}(t, t') \psi_{x'}^{\sigma'}(t') \right\}$$

$$= \int D\phi^* D\phi D\psi^* D\psi \exp\left\{ \frac{i}{\hbar} S_0[\phi, \phi^*] \right\} \det\left[ -\frac{i}{\hbar} \mathcal{K} \right],$$

(B1)

where $\mathcal{K} = G^{-1} - \delta \mathcal{K}$ is the Kernel defined in Eq. (9). Next, we use the identity $\det[A] = \exp[\text{Tr}[\text{Log}(A)]]$ to expand Eq. (B1) in powers of $\delta \mathcal{K}$. Approximating up to second order
in the electron-magnon coupling we obtain
\[
\det[K] = \exp[\text{Tr} \ln(G^{-1} - \delta K)]
\approx \exp \left\{ \text{Tr} \ln G^{-1} - \frac{1}{\hbar} \text{Tr} [G \delta K] - \frac{1}{2 \hbar^2} \text{Tr} [G \delta KK \delta K] \right\},
\]
where in the second line we have used Eq. (10). The two last terms on the right-hand side of Eq. (B2) can be explicitly written as
\[
\text{Tr}[G \delta KK \delta K] = \int_{-\infty}^{\infty} dt dt' \sum_{xx} \sum_{\sigma \sigma'} G_{xx,\sigma \sigma}(t, t') \delta K_{\sigma \sigma}(t, t')
\]
and
\[
\text{Tr}[G \delta K \delta K] = \frac{1}{\hbar} \int_{-\infty}^{\infty} dt dt' \sum_{xx} \sum_{\sigma \sigma'} G_{xx,\sigma \sigma}(t, t') \times \delta K_{\sigma \sigma}(t, t_2) \delta K_{\sigma \sigma}(t_2, t_3) \delta K_{\sigma \sigma}(t_3, t).
\]
Using the definition for \( \delta K \) given by Eq. (11) we rewrite the sum of Eqs. (B3) and (B4) as
\[
\frac{1}{\hbar} \text{Tr}[G \delta K] + \frac{1}{2 \hbar^2} \text{Tr}[G \delta K G \delta K]
= i \hbar \int_{-\infty}^{\infty} dt dt' \sum_{xx} \phi_x(t) \Sigma_{xx}(t, t') \phi_x(t'),
\]
with the self-energy defined by Eq. (13). Note that linear, cubic contributions and quartic in \( \phi \) do not appear in Eq. (B5). The first two terms involve non-spin-conserving processes, e.g., like those induced by spin-orbit coupling, which are not considered in the model. Quartic terms in \( \phi \) are nonzero but their contribution at low temperatures becomes negligible with corrections being of the order of \( \mathcal{O} [\Delta \mu / (k_B T)^2] \).

**APPENDIX C: MAGNON SELF-ENERGY DUE TO COUPLING WITH ELECTRONS**

In this Appendix, we evaluate the self-energy of the magnons due to their coupling with electrons. We start out with some general remarks for functions on the Keldysh contour.

A function \( F(t, t') \), whose arguments are defined on the Keldysh contour, can be decomposed into analytic parts by means of
\[
F(t, t') = F^\delta(t, t') + \Theta(t, t') F^> (t, t') + \Theta(t', t) F^< (t, t'),
\]
with \( \Theta(t, t') \) the Heaviside step function on the Keldysh contour and \( F^\delta(t) \) represents a possible \( \delta \) singularity. The retarded and advanced components of \( F(t, t') \) are related to the analytic parts by
\[
F^{(\pm)}(t, t') = \pm \Theta[\pm (t, t')] [F^>(t, t') - F^<(t, t')],
\]
where \( \Theta[\pm (t, t')] \equiv \Theta[\pm (t - t')] \). We also have the Keldysh component
\[
F^K(t, t') = F^> (t, t') + F^< (t, t')
\]
that typically is associated to the strength of fluctuations.

Applying the above definitions to Eq. (13), we see that the Fourier transform of the retarded (advanced) electron bubble is
\[
\Pi^{\sigma \sigma, (\pm)}(k, \epsilon) = \int \frac{d\epsilon'}{(2\pi \hbar)} \int \frac{d\epsilon''}{(2\pi \hbar)} \sum_{k'} A_{k+k'}(\epsilon') A_{k'}(\epsilon'') \times N_F(\epsilon'' - \epsilon_\sigma) - N_F(\epsilon'' - \epsilon_\sigma)
\]
\[
\times \frac{\epsilon''-\epsilon''-\epsilon_{\pm}}{\epsilon''-\epsilon''-\epsilon_{\pm}}
\]
where \( A_k(\epsilon) \) denotes the spectral function (\( k \) being a three-dimensional wave vector), \( N_F(\epsilon) = \left[ e^{\epsilon/k_B T} + 1 \right]^{-1} \) the Fermi distribution function, and \( \mu_\sigma \) the chemical potential of electrons with spin projection \( \sigma \). Ignoring electronic lifetime effects, we use \( A_k(\epsilon) = 2\pi \hbar \delta(\epsilon - \epsilon_k) \) and approximate up to first order in spin accumulation. We then find that in the long-wavelength and small frequency limit the retarded (advanced) and Keldysh components of electron bubble at low temperatures read
\[
\Pi^{\uparrow \downarrow, (\pm)}(k, \epsilon) = \frac{1}{2} N(0) a^2 \left[ \frac{1}{3} \left( \frac{k}{2k_F} \right)^2 - 1 \right] - \frac{N(0) a^2 \epsilon_\Delta}{16 \epsilon^2}
\]
\[
\times \left[ \frac{2k_F}{k} \right]^2 + 1 \pm i \pi N(0) a^2 k_F |(\Delta \mu - | \epsilon )| \times \Theta \left[ 1 - \left( \frac{|k|}{2k_F} \right)^2 \right]
\]
\[
\times \Theta \left[ 1 - \left( \frac{|k|}{2k_F} \right)^2 \right]
\]
and
\[
\Pi^{\uparrow \downarrow, K}(k, \epsilon) = -\frac{i \pi N(0) a^2 m k_F T_0}{\hbar^2 k_F |k|} \Theta \left[ 1 - \left( \frac{|k|}{2k_F} \right)^2 \right]
\]
with \( a \) the lattice constant, \( m \) the mass of electrons, and \( N(0) = m k_F / \pi^2 \hbar^2 \) the electronic density of states at the Fermi level. The imaginary part of the self-energy Eq. (13) represents the rate of change of the number of magnons. From Eq. (14), we see that the evolution of the number of magnons corresponds to the competition between the spin transfer \( (\propto \Delta \mu \equiv \mu_\uparrow - \mu_\downarrow) \) and spin pumping mechanisms \( (\propto \epsilon \epsilon) \). In principle, the magnon-electron coupling matrix element can be determined in terms of the mixing conductance, but this identification will not be pursued here [5,6].

**APPENDIX D: LADDER APPROXIMATION**

In Sec. III A, we derived an action for the gas of magnons, which are excited by the combined effects of a thermal gradient and the spin-transfer torque across the \( FM \) interface. Here, we discuss more details of this derivation.

We introduce the order parameter \( \langle \phi_q(t) \rangle \), that characterizes the instability. This is accomplished by performing a Legendre transformation [61] on the magnon field variables that ultimately leads to an effective action for \( \Psi_q(t) \equiv \langle \phi_q(t) \rangle \). With this aim we start out by introducing the
generating functional for Keldysh Green’s functions, following Ref. [58], as
\[ Z[J, J^*] = \int \mathcal{D}[\phi^*] \mathcal{D}[\phi] \times \exp \left\{ \frac{i}{\hbar} \mathcal{S}_m[\phi, \phi^*] + i(J^a \phi_a + \text{c.c.}) \right\}, \] (D1)
where \( J_a \) and \( J^a \) are sources that are defined on the Keldysh contour and summation over repeated indices means an integration over space and time coordinates. Then the Legendre transformation [61]
\[ \Gamma[\Psi, \Psi^*] \equiv \Psi^a J_a + J^a \Psi_a - W[J, J^*], \] (D2)
where \( W[J, J^*] = -i \hbar \ln Z[J, J^*] \) is the generating functional for connected Green’s functions, which can be evaluated in terms of a perturbation series. The order parameter is then \( \Psi_a = \hbar W/\partial J^a = (\phi_a) \). The functional \(-\hbar \Gamma[\phi, \phi^*] \), that generates all one-particle irreducible diagrams, corresponds to the effective action. We now do perturbation theory in the interaction to evaluate the effective action in terms of
\[ \hbar \Sigma_m(q; t, t') = i \int \frac{dq}{(2\pi)^2} \hbar \Gamma^{(4)}(q - q', q - q', q + q'; t, t') \delta(q', t'). \] (D5)
In Fourier space the expression for the retarded component of the interacting self-energy Eq. (D5) take the form
\[ \hbar \Sigma_m^{(+)}(q; \epsilon) = i \int \frac{dq'}{(2\pi)^2} \int \frac{d\epsilon'}{(2\pi)^2} \Gamma^{(+)}(q - q', q - q', q + q'; \epsilon + \epsilon') \hbar \Sigma_m^{(+)}(q; \epsilon') \mathcal{G}^{(-)}(q'; \epsilon') \] \[ + i \int \frac{dq'}{(2\pi)^2} \int \frac{d\epsilon'}{(2\pi)^2} \Gamma^{(+)}(q - q', q - q', q + q'; \epsilon + \epsilon') \mathcal{G}^{(+)}(q'; \epsilon'). \] (D6)
The evaluation of Eq. (D6) is carried out by expanding up to second order in the coupling with the leads. In this approach the various components of the Green’s function for magnons are approximated by
\[ \mathcal{G}^{(+)} = \mathcal{G}_0^{(+)} + \mathcal{G}_0^{(+)} \hbar \Sigma_m^{(+)} + \cdots, \] (D7)
\[ \mathcal{G}^{(-)} = \mathcal{G}_0^{(-)} \hbar \Sigma_m^{(-)} + \cdots, \] (D8)
as can be seen in Fig. 3(b). On the other hand, the magnon-magnon interactions will be approximated by a contact interaction, and therefore \( T^{(2)}(q, q', Q; \epsilon) \approx \frac{K}{2} \). After some manipulations [58], we arrive at the semiclassical effective action Eq. (15), describing the low-energy dynamics of the interacting magnons.

**APPENDIX E: SELF-CONSISTENT RELATIONS: CHEMICAL POTENTIALS AND TEMPERATURES**

In this section we compute the chemical potential and temperature of magnons assuming that magnons are sufficiently close to equilibrium.

The total spin-current flowing across the interface is quantified by the rate of change of magnons in the \( F \), that may be obtained following standard methods described in Ref. [58]. It consists of analyzing the stochastic dynamics of magnons due to the coupling with the \( M \), that ultimately turns out in a Boltzmann equation. For this purpose we split the magnon field, in Eq. (12), into semiclassical and fluctuating parts according to \( \phi_0(t) = \phi(t) + \xi(t)/2 \), where \( t_0 \) refers to the forward and backward branches of the Keldysh contour, respectively, and \( \xi(t) \) the fluctuations. After integrating out the fluctuations \( \xi(t) \) in the action, Eq. (12), we find that the field \( \phi_0(t) \) obeys the Langevin equations
\[ \frac{i\hbar}{\partial t} \mathcal{G}_0^{(+)}(t) = \epsilon(t) - \mu_m \] \[ + \int dt' \hbar \Sigma_m^{(+)}(t, t') \mathcal{G}_0^{(+)}(t') + \eta_0(t) \] (E1)
and
\[ -i\hbar \frac{\partial}{\partial t} \mathcal{G}_0^{(-)}(t) = \epsilon(t) - \mu_m \mathcal{G}_0^{(-)}(t) \] \[ + \int dt' \mathcal{G}_0^{(+)}(t') \hbar \Sigma_m^{(-)}(t, t') + \eta_0^{*}(t) \] (E2)
with the Gaussian stochastic noise \( \eta_0(t) \) and \( \eta_0^{*}(t) \) is zero on average and has the correlations
\[ \langle \eta_0(t) \eta_0^{*}(t') \rangle = \frac{i\hbar}{2} \Sigma_m^{(2)}(t, t'). \] (E3)
In the low-energy approximation we see that the strength of the noise is evaluated directly from the combination of
Eqs. (13) and (C6). This relation between noise and damping stems from the fluctuation-dissipation theorem [62] and ensures us that the magnon gas relaxes to thermal equilibrium. Note that to obtain Eqs. (E1) and (E2), as a first step, we have not taken into account the interaction between magnons. However, the collision terms will be included next in the Boltzmann equation. We take into account the leading low-energy contribution of the self-energy, i.e., \( \int dt' \bar{h} \Sigma_{st}^{\lambda}(q; t, t') \). Finally, the rate equation for the magnons due to the coupling at the interface with the electron reservoir is written explicitly as

\[
i \hbar \left( \frac{\partial n(q, t)}{\partial t} \right)_{st} = 2i \text{Im}[\bar{h} \Sigma_{st}^{\lambda}(q, 0)](\epsilon_q - \mu_m)n(q, t) + 2i \text{Im}[\bar{h} \Sigma_{st}^{\lambda}(q, 0)]n(q, t) - \frac{1}{2} \bar{h} \Sigma_{st}^{\mu}(q, 0)
\]

(4)

with \( n(q, t) = \langle \phi_q^* \phi_q \rangle \) and \( \langle \ldots \rangle \) stand for averaging over noise realization. Therefore, the full dynamics for the distribution of magnons in the \( F \) is determined by the Boltzmann equation

\[
\frac{\partial n(q, t)}{\partial t} = -2\alpha \omega_q n(q, t) + \left( \frac{\partial n(q, t)}{\partial t} \right)_{st}
\]

(E5)

with \( \alpha \) the Gilbert damping constant and where the collisions term has been considered. Taking moments of Eq. (E5) we obtain a closed set of equations for the total number of magnons and energy. In equilibrium, there will be neither spin flow nor energy transfer through the interface. This is implemented by requiring \( \partial n(t)/\partial t \equiv \frac{\partial}{\partial t} \sum_q n(q, t) = 0 \) and \( \partial \epsilon(t)/\partial t \equiv \frac{\partial}{\partial t} \sum_q \epsilon_q n(q, t) = 0 \), that turn out in the pair of Eqs. (19) and (20).