On a generalization of spikes

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ON A GENERALIZATION OF SPIKES

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Abstract. We consider matroids with the property that every subset of the ground set of size $t$ is contained in both an $\ell$-element circuit and an $\ell$-element cocircuit; we say that such a matroid has the $(t, \ell)$-property. We show that for any positive integer $t$, there is a finite number of matroids with the $(t, \ell)$-property for $\ell < 2t$; however, matroids with the $(t, 2t)$-property form an infinite family. We say a matroid is a $t$-spike if there is a partition of the ground set into pairs such that the union of any $t$ pairs is a circuit and a cocircuit. Our main result is that if a sufficiently large matroid has the $(t, 2t)$-property, then it is a $t$-spike. Finally, we present some properties of $t$-spikes.

Key words. matroid, spike, circuit, cocircuit

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1. Introduction. For all $r \geq 3$, a rank-$r$ spike is a matroid on $2r$ elements with a partition $(X_1, X_2, \ldots, X_r)$ into pairs such that $X_i \cup X_j$ is a circuit and a cocircuit for all distinct $i, j \in \{1, 2, \ldots, r\}$. Spikes frequently arise in the matroid theory literature (see, for example, [2, 4, 8, 10]) as a seemingly benign, yet wild, class of matroids. Miller [5] proved that if $M$ is a sufficiently large matroid having the property that every two elements share both a 4-element circuit and a 4-element cocircuit, then $M$ is a spike.

We consider generalizations of this result. We say that a matroid $M$ has the $(t, \ell)$-property if every $t$-element subset of $E(M)$ is contained in both an $\ell$-element circuit and an $\ell$-element cocircuit. It is well known that the only matroids with the $(1, 3)$-property are wheels and whirls, and Miller’s result shows that if $M$ is a sufficiently large matroid with the $(2, 4)$-property, then $M$ is a spike.

We first show that when $\ell < 2t$, there are only finitely many matroids with the $(t, \ell)$-property. However, for any positive integer $t$, the matroids with the $(t, 2t)$-property form an infinite class: when $t = 1$, this is the class of matroids obtained by taking direct sums of copies of $U_{1, 2}$; when $t = 2$, the class contains the infinite family of spikes. Our main result is the following theorem.

Theorem 1.1. There exists a function $f$ such that if $M$ is a matroid with the
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\((t, 2t)\)-property, and \(|E(M)| \geq f(t)\), then \(E(M)\) has a partition into pairs such that the union of any \(t\) pairs is both a circuit and a cocircuit.

We call a matroid with such a partition a \(t\)-spike. (A traditional spike is a 2-spike. Note also that what we call a spike is sometimes referred to as a tipless spike.)

We also prove some properties of \(t\)-spikes, which demonstrate that \(t\)-spikes are highly structured matroids. In particular, a \(t\)-spike has \(2t\) elements for some positive integer \(r\), it has rank \(r\) (and corank \(r\)), any circuit that is not a union of \(t\) pairs avoids at most \(t - 2\) of the pairs, and any sufficiently large \(t\)-spike is \((2t - 1)\)-connected.

We show that a \(t\)-spike’s partition into pairs describes crossing \((2t - 1)\)-separations in the matroid; that is, an appropriate concatenation of this partition is a \((2t - 1)\)-flower (more specifically, a \((2t - 1)\)-anemone), following the terminology of [1]. We also describe a construction of a \((t + 1)\)-spike from a \(t\)-spike, and show that every \((t + 1)\)-spike can be obtained from some \(t\)-spike in this way.

Our methods in this paper are extremal, so the lower bounds on \(|E(M)|\) that we obtain, given by the function \(f\), are extremely large, and we make no attempts to optimize these. For \(t = 2\), Miller [5] showed that \(f(2) = 13\) is best possible, and he described the other matroids with the \((2, 4)\)-property when \(|E(M)| \leq 12\). We see no reason why a similar analysis could not be undertaken for, say, \(t = 3\).

There are a number of interesting variants of the \((t, \ell)\)-property. In particular, we say that a matroid has the \((t_1, \ell_1, t_2, \ell_2)\)-property if every \(t_1\)-element set is contained in an \(\ell_1\)-element circuit, and every \(t_2\)-element set is contained in an \(\ell_2\)-element cocircuit. Although we focus here on the case where \(t_1 = t_2\) and \(\ell_1 = \ell_2\), we show, in section 3, that there are only finitely many matroids with the \((t_1, \ell_1, t_2, \ell_2)\)-property when \(\ell_1 < 2t_1\) or \(\ell_2 < 2t_2\). Oxley et al. [7] recently considered the case where \((t_1, \ell_1, t_2, \ell_2) = (2, 4, 1, k)\) and \(k \in \{3, 4\}\). In particular, they proved, for \(k \in \{3, 4\}\), that a \(k\)-connected matroid \(M\) with \(|E(M)| \geq k^4\) has the \((2, 4, 1, k)\)-property if and only if \(M \cong M(K_{k,n})\) for some \(n \geq k\). This gives credence to the idea that sufficiently large matroids with the \((t_1, \ell_1, t_2, \ell_2)\)-property, for appropriate values of \(t_1, \ell_1, t_2, \ell_2\), may form structured classes. In particular, we conjecture the following generalization of Theorem 1.1.

**Conjecture 1.2.** There exists a function \(f(t_1, t_2)\) such that if \(M\) is a matroid with the \((t_1, 2t_1, t_2, 2t_2)\)-property, for positive integers \(t_1\) and \(t_2\), and \(|E(M)| \geq f(t_1, t_2)\), then \(E(M)\) has a partition into pairs such that the union of any \(t_1\) pairs is a circuit, and the union of any \(t_2\) pairs is a cocircuit.

The study of matroids with the \((t, 2t)\)-property was motivated by problems in matroid connectivity. Tutte proved that wheels and whirls (that is, matroids with the \((1, 3)\)-property) are the only 3-connected matroids with no element whose deletion or contraction preserves 3-connectivity [11]. Moreover, spikes (matroids with the \((2, 4)\)-property) are the only 3-connected matroids with \(|E(M)| \geq 13\) having no triangles or triads, and no pair of elements whose deletion or contraction preserves 3-connectivity [12]. We envision that \(t\)-spikes could also play a role in a connectivity “chain theorem”: they are \((2t - 1)\)-connected matroids, having no circuits or cocircuits of size \(2t - 1\), with the property that for every \(t\)-element subset \(X \subseteq E(M)\), neither \(M/X\) nor \(M\setminus X\) is \((t + 1)\)-connected. We conjecture the following.

**Conjecture 1.3.** There exists a function \(f(t)\) such that if \(M\) is a \((2t - 1)\)-connected matroid with no circuits or cocircuits of size \(2t - 1\), and \(|E(M)| \geq f(t)\), then either

(i) there exists a \(t\)-element set \(X \subseteq E(M)\) such that either \(M/X\) or \(M\setminus X\) is \((t + 1)\)-connected, or
Therefore, each $|D| \leq n$ with $D \subseteq S$ for every subcollection $J \subseteq E(M)$ and every disjoint sets in $S$.

2. Preliminaries. Our notation and terminology follow Oxley [6]. We refer to the fact that a circuit and a cocircuit cannot intersect in exactly one element as “orthogonality.” We say that a $k$-element set is a $k$-set. A set $S$ meets a set $S_2$ if $S_1 \cap S_2 \neq \emptyset$. We denote $\{1, 2, \ldots, n\}$ by $[n]$, and, for positive integers $i < j$, we denote $\{i, i + 1, \ldots, j\}$ by $[i, j]$. We denote the set of positive integers by $\mathbb{N}$.

**Lemma 2.1.** There exists a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that, if $S$ is a collection of distinct s-sets and $|S| \geq f(s, n)$, then there is some $S' \subseteq S$ with $|S'| = n$, and a set $J$ with $0 \leq |J| < s$, such that $S_1 \cap S_2 = J$ for all distinct $S_1, S_2 \in S'$.

**Proof.** We define $f(1, n) = n$ and $f(s, n) = (s - 1)f(s - 1, n)$ for $s > 1$. Note that $f$ is increasing. We claim that this function satisfies the lemma. We proceed by induction on $s$. If $s = 1$, then the claim holds with $J = \emptyset$.

Let $T$ be a collection of s-sets with $|T| \geq f(s, n)$. Suppose there are $n$ pairwise disjoint sets in $S$. Then the desired conditions are satisfied if we take $J = \emptyset$. Thus, we may assume that there is some maximal $D \subseteq S$ consisting of pairwise disjoint sets, with $|D| \leq n - 1$. Each $S \in S - D$ meets some $D \in D$. Each such $D$ has $s$ elements. Therefore, each $S \in S$ contains at least one of $(n - 1)s$ elements $e \in \cup D$. By the pigeonhole principle, there is some $e \in \cup D$ such that

$$|\{S \in S : e \in S\}| \geq \frac{f(s, n)}{(n - 1)s} = f(s - 1, n).$$

Let $T = \{S - \{e\} : e \in S \in S\}$. Then, for every $T \in T$, we have $|T| = s - 1$. Moreover, $|T| = |\{S \in S : e \in S\}| \geq f(s - 1, n)$. By the induction assumption, there is a subset $T' \subseteq T$, with $|T'| = n$, and a set $J'$, with $|J'| < s - 1$, such that $T_1 \cap T_2 = J'$ for all distinct $T_1, T_2 \in T'$. Let $S' = T \cup \{e\} : e \in T'$). Then, $S' \subseteq S$ with $|S'| = n$ such that $S_1 \cap S_2 = J' \cup \{e\}$ for all distinct $S_1, S_2 \in S'$ and $|J' \cup \{e\}| < s$. \hfill \Box

3. Matroids with the $(t, \ell)$-property for $\ell < 2t$. Recall that a matroid has the $(t_1, t_2, \ell_2)$-property if every $t_1$-element set is contained in an $t_2$-element circuit, and every $t_2$-element set is contained in an $\ell_2$-element cocircuit. In this section, we prove that there are only finitely many matroids with the $(t_1, \ell_1, t_2, \ell_2)$-property if $\ell_2 < 2t_2$. By duality, the same is true if $\ell_1 < 2t_1$. As a special case, we have that there are only finitely many matroids with the $(t, \ell)$-property for $\ell < 2t$.

**Lemma 3.1.** Let $C$ be a collection of circuits of a matroid $M$ such that, for some $J \subseteq E(M)$ with $|J| \leq k$, we have $C \cap C' = J$ for all distinct $C, C' \in C$. Then, for every subcollection $\{C_1, \ldots, C_{2^k}\} \subseteq C$ of size $2^k$, there is a circuit contained in $\bigcup_{i=1}^{2^k} C_i - J$.

**Proof.** We may assume $|C| \geq 2^k$; otherwise, the result holds vacuously. Also, we may assume $k > 0$ as the result holds for any singleton subcollection of $C$ with $J = \emptyset$. Therefore, $C$ has at least one subcollection $C' = \{C_1, \ldots, C_{2^k}\}$, with $|C'| = 2^k \geq 2$.

Let $x_1, x_2, \ldots, x_{|J|}$ be the elements of $J$. Define $Z_{i,0} = C_i$, for $i \in [2^k]$, and recursively define $Z_{i,j} = Z_{2i-1,j-1} \cup Z_{2i,j-1}$ for $j \in [k]$ and $i \in [2^{k-1}]$. Note that
each \( Z_{i,j} \) is the union of \( 2^j \) members of \( \mathcal{C} \). We will show, by induction on \( j \),
that \( Z_{i,j} \setminus \{x_1, x_2, \ldots, x_j\} \) contains a circuit. This is clear when \( j = 0 \). Now let \( j \geq 1 \).
By the induction hypothesis, \( Z_{2i-1,j-1} \) and \( Z_{2i,j-1} \) each contain a circuit, \( C'_1 \) and \( C'_2 \), respectively, disjoint from \( \{x_1, x_2, \ldots, x_{j-1}\} \), for each \( i \in [2^{k-1}] \). (Moreover, \( C'_1 \neq C'_2 \) since \( C'_1 \cap C'_2 \subseteq Z_{2i-1,j-1} \cap Z_{2i,j-1} \subseteq J \), which is independent since \( J \) is the intersection of at least two circuits.) We may assume that neither \( Z_{2i-1,j-1} \) nor \( Z_{2i,j-1} \) contains a circuit disjoint from \( \{x_1, x_2, \ldots, x_j\} \); otherwise, so does \( Z_{i,j} \). Thus, \( C'_1 \) and \( C'_2 \) both contain \( x_j \). By circuit elimination, there is a circuit \( C' \) contained in \( \langle C'_1 \cup C'_2 \rangle - \{x_j\} \subseteq Z_{i,j} - \{x_1, x_2, \ldots, x_j\} \). This completes the induction argument. In particular, there is a circuit contained in \( Z_{1,k} - \{x_1, x_2, \ldots, x_{|J|}\} = \bigcup_{i=1}^{|X|} C_i - J \), as required.

**Lemma 3.2.** There exists a function \( g : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) such that if \( M \) is a matroid having at least \( g(\ell, d) \)-many \( \ell \)-element circuits, then \( M \) has a collection of \( d \) pairwise disjoint circuits.

**Proof.** Let \( C \) be the collection of \( \ell \)-element circuits of \( M \), let \( f \) be the function of Lemma 2.1, and let \( g(\ell, d) = f(\ell, 2^\ell - d) \). Then, by Lemma 2.1, there is a subset \( C' \subseteq C \), with \( |C'| = 2^\ell - d \), and a set \( J \), with \( 0 \leq |J| \leq \ell - 1 \), such that \( C \cap C' = J \) for every pair \( C, C' \in C' \). Say \( C' = \{C_1, C_2, \ldots, C_{2\ell - d}\} \).

If \( J = \emptyset \), then \( M \) has \( 2^\ell - d \geq d \) pairwise disjoint circuits, as required. Thus, we may assume that \( J \neq \emptyset \). For each \( C_i \in C' \), let \( D_i = C_i - J \), and observe that the \( D_i \)’s are pairwise disjoint. For \( j \in [d] \), let

\[
D'_j = \bigcup_{i=1}^{2^\ell - d} D_{(j-1)(2^\ell - 1)+i}.
\]

By Lemma 3.1, each \( D'_j \) contains a circuit \( C'_j \), and the \( C'_j \)’s are pairwise disjoint. [Q.E.D.]

**Theorem 3.3.** Let \( t_1, \ell_1, t_2, \) and \( \ell_2 \) be positive integers. If \( \ell_1 < 2t_1 \) or \( \ell_2 < 2t_2 \),
then there is a finite number of matroids with the \((t_1, \ell_1, t_2, \ell_2)\)-property.

**Proof.** By duality, it suffices to prove the result when \( \ell_2 < 2t_2 \). So let \( \ell_2 < 2t_2 \), and let \( g \) be the function given in Lemma 3.2.

Suppose \( M \) has at least \( g(\ell_1, t_2) \)-many \( \ell_1 \)-element circuits. By Lemma 3.2, \( M \) has a collection of \( t_2 \) pairwise disjoint circuits. Call this collection \( \mathcal{C} = \{C_1, \ldots, C_{t_2}\} \). Let \( b_i \) be an element of \( C_i \), for each \( i \in [t_2] \). By the \((t_1, \ell_1, t_2, \ell_2)\)-property, there is an \( \ell_2 \)-element cocircuit \( C^* \) containing \( \{b_1, \ldots, b_{t_2}\} \). By orthogonality, for each \( i \in [t_2] \) there is an element \( b'_i \neq b_i \) such that \( b'_i \in C_i \cap C^* \). This implies that \( \ell_2 = |C^*| \geq 2t_2 \); a contradiction. Thus, \( M \) has fewer than \( g(\ell_1, t_2) \)-many \( \ell_1 \)-element circuits.

Suppose \( |E(M)| \geq \ell_1 \cdot g(\ell_1, t_2) \). Partition a subset of \( E(M) \) into \([\ell_1/t_1] \cdot g(\ell_1, t_2) \) pairwise disjoint \( t_1 \)-sets. By the \((t_1, \ell_1, t_2, \ell_2)\)-property, each of these \( t_1 \)-sets is contained in an \( \ell_1 \)-element circuit. The collection consisting of these \( \ell_1 \)-element circuits contains at least \( g(\ell_1, t_2) \) distinct circuits. This contradicts the fact that \( M \) has fewer than \( g(\ell_1, t_2) \)-many \( \ell_1 \)-element circuits. Therefore, \( |E(M)| < \ell_1 \cdot g(\ell_1, t_2) \). The result follows. [Q.E.D.]

Note that there may still be infinitely many matroids where every \( \ell_1 \)-element set is in an \( \ell_1 \)-element circuit for fixed \( \ell_1 < 2t_1 \); it is necessary that the matroids in Theorem 3.3 have the property that every \( t_2 \)-element set is in an \( \ell_2 \)-element cocircuit, for fixed \( t_2 \) and \( \ell_2 \). To see this, observe that projective geometries on at least three elements form an infinite family of matroids with the property that every pair of elements is in a 3-element circuit.
Corollary 3.4. Let $t$ and $\ell$ be positive integers. When $\ell < 2t$, there is a finite number of matroids with the $(t, \ell)$-property.

4. Echidnas and $t$-spikes. We now focus on matroids with the $(t, 2t)$-property. In section 5, we will show that every sufficiently large matroid with the $(t, 2t)$-property has a partition into pairs such that the union of any $t$ of these pairs is both a circuit and a cocircuit. We call such a matroid a $t$-spike. We first define a related structure: a $t$-echidna.

Definition 4.1. Let $M$ be a matroid. A $t$-echidna of order $n$ is a partition $(S_1, \ldots, S_n)$ of $E(M)$ such that

1. $|S_i| = 2$ for all $i \in [n]$ and
2. $\bigcup_{i \in I} S_i$ is a circuit for all $I \subseteq [n]$ with $|I| = t$.

For $i \in [n]$, we say $S_i$ is a spine. We say $(S_1, \ldots, S_n)$ is a $t$-coechidna of $M$ if

$(S_1, \ldots, S_n)$ is a $t$-echidna of $M^*$.

Definition 4.2. A matroid $M$ is a $t$-spike of order $r$ if there exists a partition $\pi = (A_1, \ldots, A_\ell)$ of $E(M)$ such that $\pi$ is a $t$-echidna and a $t$-coechidna, for some $\ell \geq t$. We say $\pi$ is the associated partition of the $t$-spike $M$, and $A_i$ is an arm of the $t$-spike for each $i \in [\ell]$.

Note that if $M$ is a $t$-spike, then $M^*$ is a $t$-spike.

In this section, we prove, as Lemma 4.5, that if $M$ is a matroid with the $(t, 2t)$-property, and $M$ has a $t$-echidna of order $4t - 3$, then $M$ is a $t$-spike.

Lemma 4.3. Let $M$ be a matroid with the $(t, 2t)$-property. If $M$ has a $t$-echidna $(S_1, \ldots, S_n)$, where $n \geq 3t - 1$, then $(S_1, \ldots, S_n)$ is also a $t$-coechidna of $M$.

Proof. Let $S_i = \{x_i, y_i\}$ for each $i \in [n]$. By definition, if $J$ is a $t$-element subset of $[n]$, then $\bigcup_{i \in J} S_i$ is a circuit. Consider such a circuit $C$; without loss of generality, we let $C = \{x_1, y_1, \ldots, x_t, y_t\}$. By the $(t, 2t)$-property, there is a $2t$-element cocircuit $C^*$ that contains $\{x_1, \ldots, x_t\}$.

Suppose that $C^* \neq C$. Then there is some $i \in [t]$ such that $y_i \notin C^*$. Without loss of generality, say $y_t \notin C^*$. Let $I$ be a $(t-1)$-element subset of $[t+1, n]$. For any such $I$, the set $S_I \cup \bigcup_{i \in I} S_i$ is a circuit that meets $C^*$. By orthogonality, $\bigcup_{i \in I} S_i$ meets $C^*$ for every $(t-1)$-element subset $I$ of $[t+1, n]$. Thus, $C^*$ avoids at most $t-2$ of the $S_i$’s for $i \in [t+1, n]$. In fact, as $C^*$ meets each $S_i$ with $i \in [t]$, the cocircuit $C^*$ avoids at most $t-2$ of the $S_i$’s with $i \in [n]$. Thus $|C^*| \geq n - (t-2) \geq (3t-1) - (t-2) = 2t + 1 > 2t$; a contradiction. Therefore, we conclude that $C^* = C$, and the result follows.

Lemma 4.4. Let $M$ be a matroid with the $(t, 2t)$-property, and let $(S_1, \ldots, S_n)$ be a $t$-echidna of $M$ with $n \geq 3t - 1$. Let $I$ be a $(t-1)$-element subset of $[n]$. For $z \in E(M) - \bigcup_{i \in I} S_i$, there is a $2t$-element circuit and a $2t$-element cocircuit each containing $\{z\} \cup (\bigcup_{i \in I} S_i)$.

Proof. By duality, it suffices to show that there is a $2t$-element circuit containing $\{z\} \cup (\bigcup_{i \in I} S_i)$. For $i \in [n]$, let $S_i = \{x_i, y_i\}$. By the $(t, 2t)$-property, there is a $2t$-element circuit $C$ containing $\{z\} \cup \{x_i : i \in I\}$. Let $J$ be a $(t-1)$-element subset of $[n]$ such that $C$ and $\bigcup_{i \in J} S_i$ are disjoint (such a set exists since $|C| = 2t$ and $n \geq 3t - 1$). For $i \in I$, let $C'_i = S_i \cup (\bigcup_{j \in J} S_j)$, and observe that $x_i \in C'_i \cap C$, and $C'_i \cap C \subseteq S_i$. By Lemma 4.3, $(S_1, \ldots, S_n)$ is a $t$-coechidna as well as a $t$-echidna; therefore, $C'_i$ is a cocircuit. Now, for each $i \in I$, orthogonality implies that $|C'_i \cap C| \geq 2$, and hence $y_i \in C$. So $C$ contains $\{z\} \cup (\bigcup_{i \in I} S_i)$, as required.

Let $(S_1, \ldots, S_n)$ be a $t$-echidna of a matroid $M$. If $(S_1, \ldots, S_m)$ is a $t$-echidna of
Thus, \((S_1, \ldots, S_n)\) is a \(t\)-echidna since \((t, m)\) is a \(t\)-spike. Our primary goal is to show that a sufficiently large matroid with the \((t, m)\)-property is a \(t\)-spike. We claim that \((S_1, \ldots, S_m)\) is also a \(t\)-spike. Let \(X = \bigcup_{i=1}^m S_i\). By Lemma 4.3, \(X\) is a \(t\)-spike as well as a \(t\)-echidna.

**Proof.** Suppose that \((S_1, \ldots, S_n)\) extends to \(\pi = (S_1, \ldots, S_m)\), where \(\pi\) is maximal. Let \(X = \bigcup_{i=1}^m S_i\). By Lemma 4.3, \(\pi\) is a \(t\)-echidna as well as a \(t\)-spike. The result holds if \(X = E(M)\). Therefore, towards a contradiction, we suppose that \(E(M) - X \neq \emptyset\). Let \(z \in E(M) - X\). By Lemma 4.4, there is a \(2t\)-element circuit \(C = \{z, z'\} \cup (\bigcup_{i\in[t-1]} S_i)\), for some \(z' \in E(M) - (\{z\} \cup (\bigcup_{i\in[t-1]} S_i))\).

We claim that \(z' \notin X\). Towards a contradiction, suppose that \(z' \in S_k\) for some \(k \in [t, m]\). Let \(J\) be a \(t\)-element subset of \([t, m]\) containing \(k\). Then, since \((S_1, \ldots, S_{m})\) is a \(t\)-echidna, \(\bigcup_{j \in J} S_j\) is a cocircuit that contains \(z\). Now, by orthogonality, \(z \in X\); a contradiction. Thus, \(z' \notin X\), as claimed.

We next show that \((\{z, z'\}, S_1, S_{t+1}, \ldots, S_{m})\) is a \(t\)-echidna. It suffices to show that \(\{z, z'\} \cup (\bigcup_{i \in I} S_i)\) is a cocircuit for each \((t - 1)\)-element subset \(I\) of \([t, m]\). Let \(I\) be such a set. Lemma 4.4 implies that there is a \(2t\)-element cocircuit \(C^*\) of \(M\) containing \(\{z\} \cup (\bigcup_{i \in I} S_i)\). By orthogonality, \(|C \cap C^*| > 1\). Therefore, \(z' \in C^*\). Thus, \((\{z, z'\}, S_1, S_{t+1}, \ldots, S_{m})\) is a \(t\)-echidna. Since this \(t\)-echidna has order \(1 + m - (t - 1) \geq 3t - 1\), the dual of Lemma 4.3 implies that \((\{z, z'\}, S_1, S_{t+1}, \ldots, S_{m})\) is also a \(t\)-echidna.

Now, we claim that \((\{z, z'\}, S_1, S_{2}, \ldots, S_{m})\) is a \(t\)-echidna. It suffices to show that \(\{z, z'\} \cup (\bigcup_{i \in I} S_i)\) is a cocircuit for any \((t - 1)\)-element subset \(I\) of \([m]\). Let \(I\) be such a set, and let \(J\) be a \((t - 1)\)-element subset of \([t, m]\) - \(I\). By Lemma 4.4, there is a \(2t\)-element cocircuit \(C^*\) containing \(\{z\} \cup (\bigcup_{i \in I} S_i)\). Moreover, \(C = \{z, z'\} \cup (\bigcup_{i \in J} S_i)\) is a circuit since \((\{z, z'\}, S_1, S_{t+1}, \ldots, S_{m})\) is a \(t\)-echidna. By orthogonality, \(z' \in C^*\). Therefore, \((\{z, z'\}, S_1, S_{2}, \ldots, S_{m})\) is a \(t\)-echidna. By the dual of Lemma 4.3, it is also a \(t\)-echidna, contradicting the maximality of \((S_1, \ldots, S_m)\). \(\square\)

**5. Matroids with the \((t, 2t)\)-property.** In this section, we prove that every sufficiently large matroid with the \((t, 2t)\)-property is a \(t\)-spike. Our primary goal is to show that a sufficiently large matroid with the \((t, 2t)\)-property has a large \(t\)-echidna or \(t\)-echidna; it then follows, by Lemma 4.5, that the matroid is a \(t\)-spike.

**Lemma 5.1.** Let \(M\) be a matroid with the \((t, 2t)\)-property, and let \(X \subseteq E(M)\).

(i) If \(r(X) < t\), then \(X\) is independent.

(ii) If \(r(X) = t\), then \(M|X \cong U_{t,|X|}\) and \(|X| < 3t\).

**Proof.** Clearly, as \(M\) has the \((t, 2t)\)-property, \(M\) has no circuits of size at most \(t\). Thus, if \(r(X) < t\), then \(X\) contains no circuits and is therefore independent. If \(r(X) = t\), then a subset of \(X\) is a circuit if and only if it has size \(t + 1\). Therefore, \(M|X \cong U_{t,|X|}\).

Suppose towards a contradiction that \(M|X \cong U_{t,3t}\). Let \(x \in X\), and let \(C^*\) be a cocircuit of \(M\) containing \(x\). Then \(E(M) - C^*\) is closed, so \(\text{cl}(X - C^*) \subseteq \text{cl}(E(M) - C^*) = E(M) - C^*\). Therefore, \(r(X - C^*) < r(X) = t\), implying that \(|C^*| > 2t\). But then every cocircuit containing \(x\) has size greater than \(2t\), contradicting the \((t, 2t)\)-property. \(\square\)

**Lemma 5.2.** Let \(M\) be a matroid with the \((t, 2t)\)-property. Let \(C_1^*, C_2^*, \ldots, C_{t-1}^*\) be a collection of \(t-1\) pairwise disjoint cocircuits of \(M\), and let \(Y = E(M) - \bigcup_{i \in [t-1]} C_i^*\). For all \(y \in Y\), there is a \(2t\)-element circuit \(C_y\) containing \(y\) such that either
(i) \(|C_y \cap C_j^*| = 2| for all \(i \in [t-1]\) or
(ii) \(|C_y \cap C_j^*| = 3| for some \(j \in [t-1]\), and \(|C_y \cap C_i^*| = 2| for all \(i \in [t-1] - \{j\}\).
Moreover, if \(C_y = S \cup \{y\}\) satisfies (ii), then there are at most \(3t-1| elements \(w \in Y\) such that \(S \cup \{w\}\) is a circuit.

Proof. Choose an element \(c_i \in C_i^*\) for each \(i \in [t-1]\). By the \((t,2t)-property, there is a \(2t\)-element circuit \(C_y\) containing \(\{c_1,c_2,\ldots,c_{t-1},y\}\), for each \(y \in Y\). By orthogonality, \(C_y\) satisfies (i) or (ii).

Suppose \(C_y\) satisfies (ii), and let \(S = C_y - Y = C_y - \{y\}\). Let \(W = \{w \in Y : S \cup \{w\}\ is a circuit\}.\) It remains to prove that \(|W| < 3t\). Observe that \(W \subseteq \text{cl}(S) \cap Y\), and, since \(S\) contains \(t-1\) elements in pairwise disjoint cocircuits that avoid \(Y\), we have \(r(\text{cl}(S) \cup Y) \geq r(Y) + (t-1)\). Thus,

\[
\begin{align*}
    r(W) & \leq r(\text{cl}(S) \cap Y) \\
    & \leq r(\text{cl}(S)) + r(Y) - r(\text{cl}(S) \cup Y) \\
    & \leq (2t-1) + r(Y) - (r(Y) + (t-1)) \\
    & = t,
\end{align*}
\]

using submodularity of the rank function at the second line.

Now, by Lemma 5.1(i), if \(r(W) < t\), then \(W\) is independent, so \(|W| = r(W) < t\). On the other hand, by Lemma 5.1(ii), if \(r(W) = t\), then \(M[W] \cong U_{t,|W|}\) and \(|W| < 3t\), as required.

Lemma 5.3. There exists a function \(h\) such that if \(M\) is a matroid with the \((t,2t)-property and having at least \(h(t,d)\) \(\ell\)-element circuits, then \(M\) has a collection of \(d\) pairwise disjoint \(2\ell\)-element cocircuits.

Proof. By Lemma 3.2, there is a function \(g\) such that if \(M\) has at least \(g(t,d)\) \(\ell\)-element circuits, then \(M\) has a collection of \(d\) pairwise disjoint circuits. We define \(h(t,d) = g(t,td)\), and claim that a matroid with the \((t,2t)-property and having at least \(h(t,d)\) \(\ell\)-element circuits has a collection of \(d\) pairwise disjoint \(2t\)-element cocircuits.

Let \(M\) be such a matroid. By Lemma 3.2, \(M\) has a collection of \(td\) pairwise disjoint circuits. We partition these into \(d\) groups of size \(t\): call this partition \((C_1,\ldots,C_d)\). Since the \(t\) circuits in any cell of this partition are pairwise disjoint, it now suffices to show that, for each \(i \in [d]\), there is a \(2t\)-element cocircuit contained in the union of the members of \(C_i\). Let \(C_i = \{C_{i,1},\ldots,C_{i,t}\}\) for some \(i \in [d]\). Pick some \(c_j \in C_j\) for each \(j \in [t]\). Then, by the \((t,2t)-property, \(\{c_1,c_2,\ldots,c_t\}\) is contained in a \(2t\)-element cocircuit, which, by orthogonality, is contained in \(\bigcup_{j \in [t]} C_j\).

Lemma 5.4. There exists a function \(g\) such that if \(M\) is a matroid with the \((t,2t)-property and \(|E(M)| \geq g(t,q)\), then, for some \(M' \in \{M,M^*\}\), the matroid \(M'\) has \(t-1\) pairwise disjoint cocircuits \(C_1^*,C_2^*,\ldots,C_{t-1}^*\), and there is some \(Z \subseteq E(M') - \bigcup_{i \in [t-1]} C_i^*\) such that

(i) \(r_{M'}(Z) \geq q\) and
(ii) for each \(z \in Z\), there exists an element \(z' \in Z - \{z\}\) such that \(\{z,z'\}\) is contained in a \(2t\)-element circuit \(C\) of \(M'\) with \(|C \cap C_i^*| = 2\) for each \(i \in [t-1]\).

Proof. By Lemma 5.3, there is a function \(h\) such that if \(M'\) has at least \(h(t,d)\) \(\ell\)-element circuits, for \(M' \in \{M,M^*\}\), then \(M'\) has a collection of \(d\) pairwise disjoint \(2t\)-element cocircuits.

Suppose \(|E(M)| \geq 2t \cdot h(2t,t-1,t)\). Then, by the \((t,2t)-property, \(M'\) has at least \(h(2t,t-1,t)\) distinct \(2t\)-element circuits. Hence, by Lemma 5.3, \(M'\) has a collection
of \( t-1 \) pairwise disjoint \( 2t \)-element cocircuits \( C^*_1, C^*_2, \ldots, C^*_t \).

Let \( X = \bigcup_{i \in [t-1]} C^*_i \) and \( Y = E(M) - X \). By Lemma 5.2, for each \( y \in Y \) there is a \( 2t \)-element circuit \( C^*_y \) containing \( y \) such that \( |C^*_y \cap C^*_j| = 3 \) for at most one \( j \in [t-1] \) and \( |C^*_y \cap C^*_i| = 2 \) otherwise. Let \( W \) be the set of all \( w \in Y \) such that \( w \) is in a \( 2t \)-element circuit \( C \) with \( |C \cap C^*_j| = 3 \) for some \( j \in [t-1] \), and \( |C \cap C^*_i| = 2 \) for all \( i \in [t-1] - \{j\} \). Now, letting \( Z = Y - W \), we see that (ii) is satisfied for both \( M' = M \) and \( M' = M^* \).

Since the \( C^*_i \)'s have size \( 2t \), there are \((t-1)\binom{2t}{3}\binom{t-2}{2}^{t-2}\) sets \( X' \subseteq X \) with \( |X' \cap C^*_j| = 3 \) for some \( j \in [t-1] \) and \( |X' \cap C^*_i| = 2 \) for all \( i \in [t-1] - \{j\} \). It follows, by Lemma 5.2, that \(|W| \leq s(t)\) where

\[
s(t) = (3t-1) \left[ (t-1) \binom{2t}{3} \binom{2t}{2}^{t-2} \right].
\]

We define

\[
g(t,q) = \max \left\{ 2t \cdot h(2t, t-1, t), 2(q + s(t) + 2t(t-1)) \right\}.
\]

Suppose that \(|E(M)| \geq g(t,q)\). Recall that (ii) holds for both \( M' = M \) and \( M' = M^* \). Moreover, we can choose \( M' \in \{M, M^*\} \) such that \( r(M') \geq q+s(t)+2t(t-1) \). Then,

\[
r_{M'}(Z) \geq r_{M'}(Y) - |W|
\]

\[
\geq (r(M') - 2t(t-1)) - s(t)
\]

\[
\geq q,
\]

so (i) holds as well, as required.

**Lemma 5.5.** Let \( M \) be a matroid with the \((t,2t)\)-property. Suppose \( M \) has \( t-1 \) pairwise disjoint cocircuits \( C^*_1, C^*_2, \ldots, C^*_t \), and, for some positive integer \( p \), there is some \( Z \subseteq E(M) - \bigcup_{i \in [t-1]} C^*_i \) such that

(a) \( r_M(Z) \geq (2^{t-1})^{t-1} (p + 2(t-1)) \) and

(b) for each \( z \in Z \), there exists an element \( z' \in Z - \{z\} \) such that \( \{z, z'\} \) is contained in a \( 2t \)-element circuit \( C \) of \( M \) with \( |C \cap C^*_j| = 2 \) for each \( i \in [t-1] \).

Then there exist a subset \( Z' \subseteq Z \) and a partition \( Z' = (Z^*_1, \ldots, Z^*_p) \) of \( Z' \) into pairs such that

(i) each circuit of \( M[Z'] \) is a union of pairs in \( Z' \) and

(ii) the union of any \( t \) pairs of \( Z' \) contains a circuit.

**Proof.** We first prove the following claim.

**Claim 5.5.1.** There exist a \((2t - 2)\)-element set \( X \), with \( |X \cap C^*_j| = 2 \) for each \( i \in [t-1] \), and a set \( Z' \subseteq Z \), with a partition \( Z' = (Z'_1, \ldots, Z'_p) \) into \( p \) pairs, such that

(I) \( X \cup Z'_i \) is a circuit for each \( i \in [p] \) and

(II) \( Z' \) partitions the ground set of \((M/X)[Z']\) into parallel classes, and we have that \( r_{M/X}(\bigcup_{i \in [p]} Z'_i) = p \).

**Proof.** For each \( z \in Z \), there exist an element \( z' \in Z - \{z\} \) and a set \( X' \) such that \( \{z, z'\} \cup X' \) is a circuit of \( M \), and \( X' \) is the union of pairs \( Y_i \) for \( i \in [t-1] \), with \( Y_i \subseteq C^*_i \). There are \((2^t)^{t-1}\) choices of such pairs \( Y_i \subseteq C^*_i \) for \( i \in [t-1] \). Thus, for some \( m \leq (2^t)^{t-1} \), there are \((2t - 2)\)-element sets \( X_1, \ldots, X_m \), each of which intersects \( C^*_i \) in two elements for each \( i \in [t-1] \), and sets \( Z_1, \ldots, Z_m \) whose union is \( Z \), such that
for each $j \in [m]$ and each $z_j \in Z_j$, there is an element $z'_j \in Z_j$ such that $X_j \cup \{z_j, z'_j\}$ is a circuit. Moreover, $r(Z_1) + \cdots + r(Z_m) \geq r(Z)$. Thus, by the pigeonhole principle, there exists some $j \in [m]$ with

$$r(Z_j) \geq \frac{r(Z)}{\left(\binom{2}{2}\right)^{t-1}} \geq p + 2(t - 1).$$

Let $Z' = Z_j$ and $X = X_j$. Now, observe that $X \cup \{z, z'\}$ is a circuit, for some pair $\{z, z'\} \subseteq Z'$, if and only if $\{z, z'\}$ is a parallel pair in $M/X$. So the ground set of $(M/X)|Z'$ has a partition into parallel classes, where each parallel class has size at least two. Let $Z' = \{\{z_1, z'_1\}, \ldots, \{z_n, z'_n\}\}$ be a collection of pairs from each parallel class such that $\{z_1, z_2, \ldots, z_n\}$ is independent in $(M/X)|Z'$. Since $r_{M/X}(Z') = r(Z' \cup X) - r(X) \geq r(Z') - 2(t - 1) \geq p$, there exists such a collection $Z'$ of size $p$, and this collection satisfies Claim 5.5.1.

Let $X$ and $Z' = \{Z'_1, \ldots, Z'_p\}$ be as described in Claim 5.5.1, let $Z' = \bigcup_{i \in [p]} Z'_i$, and let $X = \{X_1, \ldots, X_{t-1}\}$, where $X_i = \{x_i, x'_i\} = X \cap C_i$.

Claim 5.5.2. Each circuit of $M|(X \cup Z')$ is a union of pairs in $X \cup Z'$.

Proof. Let $C$ be a circuit of $M|(X \cup Z')$. If $x_i \in C$, for some $\{x_i, x'_i\} \in X$, then, by orthogonality with $C_i$, we have $x'_i \in C$. Towards a contradiction, say $\{z, z'\} \in Z'$, and $C \cap \{z, z'\} = \{z\}$. Choose $W$ to be the union of the pairs of $Z'$ that contain elements of $(C \setminus \{z\}) \cap Z'$. Then $z \in \text{cl}(X \cup W)$. Hence $z \in \text{cl}_{M/X}(W)$, contradicting Claim 5.5.1(I).

Claim 5.5.3. The union of any $t$ pairs of $X \cup Z'$ contains a circuit.

Proof. Let $W$ be a subcollection of $X \cup Z'$ of size $t$. We prove by induction on the number of pairs in $W \cap Z'$. If there is only one pair in $W \cap Z'$, then the union of the pairs in $W$ contains a circuit (indeed, is a circuit) by Claim 5.5.1(I). Suppose the result holds for any subcollection containing $k$ pairs in $Z'$, and let $W$ be a subcollection containing $k + 1$ pairs in $Z'$. Let $\{x, x'\}$ be a pair in $X \setminus W$, and let $W = \bigcup_{W' \subseteq W} W'$. By induction hypothesis, $W \cup \{x, x'\}$ contains a circuit $C_1$. If $\{x, x'\} \subseteq \bar{E}(M) - C_1$, then $C_1 \subseteq W$, in which case the union of the pairs in $W$ contains a circuit, as desired. Therefore, we may assume, by Claim 5.5.2, that $\{x, x'\} \subseteq C_1$. Since $X$ is independent, there is a pair $\{z, z'\} \subseteq Z' \cap C_1$. By induction hypothesis, there is a circuit $C_2$ contained in $(W - \{z, z'\}) \cup \{x, x'\}$. Observe that $C_1$ and $C_2$ are distinct, and $\{x, x'\} \subseteq C_1 \cap C_2$. By circuit elimination on $C_1$ and $C_2$, and Claim 5.5.2, there is a circuit $C_3 \subseteq (C_1 \cup C_2) - \{x, x'\} \subseteq W$, as desired. The result now follows by induction.

Now, Claim 5.5.3 implies that the union of any $t$ pairs of $Z'$ contains a circuit, and the result follows.

In order to prove Theorem 1.1, we use some hypergraph Ramsey theory [9].

Theorem 5.6 (Ramsey’s theorem for $k$-uniform hypergraphs). For positive integers $k$ and $n$, there exists an integer $r_k(n)$ such that if $H$ is a $k$-uniform hypergraph on $r_k(n)$ vertices, then $H$ has either a clique on $n$ vertices, or a stable set on $n$ vertices.

We now prove Theorem 1.1, restated below as Theorem 5.7.

Theorem 5.7. There exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that if $M$ is a matroid with the $(t, 2t)$-property, and $|E(M)| \geq f(t)$, then $M$ is a $t$-spike.
Proof. We first consider the case where \( t = 1 \). Let \( M \) be a nonempty matroid with the \((1,2)\)-property. Then, for every \( e \in E(M) \), the element \( e \) is in a parallel pair \( P \) and a series pair \( S \). By orthogonality, \( P = S \), and \( P \) is a connected component of \( M \). Then \( M \cong U_{1,2} \oplus M\setminus P \), and the result easily follows.

We may now assume that \( t \geq 2 \). We define the function \( h_k : \mathbb{N} \to \mathbb{N} \), for each \( k \in [t] \), as follows:

\[
h_k(t) = \begin{cases} 
4t - 3 & \text{if } k = t, \\
r_k(h_{k+1}(t)) & \text{if } k \in [t-1], 
\end{cases}
\]

where \( r_k(n) \) is the Ramsey number described in Theorem 5.6. Note that \( h_k(t) \geq h_{k+1}(t) \geq 4t - 3 \), for each \( k \in [t-1] \). Let \( p(t) = h_1(t) \), and let \( q(t) = \left( \frac{3t}{2} \right)^{t-1} (p(t) + 2(t - 1)) \).

By Lemma 5.4, there exists a function \( g \) such that if \( |E(M)| \geq q(t,q(t)) \), then, for some \( M' \in \{M,M^*\} \), the matroid \( M' \) has \( t - 1 \) pairwise disjoint cocircuits \( C_1^*,C_2^*,\ldots,C_{t-1}^* \), and there is some \( Z' \subseteq E(M') - \bigcup_{i \in [t-1]} C_i^* \) such that \( r_{M'}(Z') \geq q(t) \), and, for each \( z \in Z' \), there exists an element \( z' \in Z' - \{z\} \) such that \( \{z,z'\} \cup (\bigcup_{i \in [t-1]} (x_i,x_i')) \) is a circuit of \( M' \), where \( \{x_i,x_i'\} \subseteq C_i^* \).

Let \( f(t) = q(g(t,q(t))) \), and suppose that \( |E(M)| < f(t) \). For ease of notation, we assume that \( M' = M \). Then, by Lemma 5.5, there exist a subset \( Z \subseteq Z' \) and a partition \( Z = (Z_1,\ldots,Z_{p(t)}) \) of \( Z \) into \( p(t) \) pairs such that

(1) each circuit of \( M|Z \) is a union of pairs in \( Z \) and

(2) the union of any \( t \) pairs of \( Z \) contains a circuit.

By Lemma 4.5, and since \( t \geq 2 \), it suffices to show that \( M \) has a \( t \)-echidna or a \( t \)-coechidna of order \( 4t - 3 \). If the smallest circuit in \( M\setminus Z \) has size \( 2t \), then, by (II), \( Z \) is a \( t \)-echidna of order \( p(t) \geq 4t - 3 \). So we may assume that the smallest circuit in \( M\setminus Z \) has size \( 2j \) for some \( j \in [t-1] \).

Claim 5.7.1. If the smallest circuit in \( M\setminus Z \) has size \( 2j \), for \( j \in [t-1] \), and \( |Z| \geq h_j(t) \), then either

(i) \( M \) has a \( t \)-echidna of order \( 4t - 3 \) or

(ii) there exists some \( Z'' \subseteq Z \) that is the union of \( h_{j+1}(t) \) pairs of \( Z \) for which the smallest circuit in \( M\setminus Z'' \) has size at least \( 2(j+1) \).

Proof. Let \( 2j \) be the size of the smallest circuit in \( M\setminus Z \). We define \( H \) to be the \( j \)-uniform hypergraph with vertex set \( Z \) whose hyperedges are the \( j \)-subsets of \( Z \) that are partitions of circuits in \( M\setminus Z \). By Theorem 5.6 and the definition of \( h_k \), as \( H \) has at least \( h_j(t) \) vertices, it has either a clique or a stable set, on \( h_{j+1}(t) \) vertices. If \( H \) has a stable set \( Z'' \) on \( h_{j+1}(t) \) vertices, then clearly (ii) holds, with \( Z'' = \bigcup_{P \in Z''} P \).

So we may assume that there are \( h_{j+1}(t) \) pairs in \( Z \) such that the union of any \( j \) of these pairs is a circuit. Let \( Z'' \) be the union of these \( h_{j+1}(t) \) pairs. We claim that the union of any set of \( t \) pairs contained in \( Z'' \) is a cocircuit. Let \( T \) be a transversal of \( t \) pairs of \( Z \) contained in \( Z'' \), and let \( C^* \) be the \( 2t \)-element cocircuit containing \( T \). Towards a contradiction, suppose that there exists some pair \( P \in Z \) with \( P \subseteq Z'' \) such that \( |C^* \cap P| = 1 \). Select \( j-1 \) pairs \( Z''_1,\ldots,Z''_{j-1} \) of \( Z \) that are each contained in \( Z'' - C^* \) (these exist since \( h_{j+1}(t) \geq 3t - 1 \geq 2t + j - 1 \)). Then \( P \cup (\bigcup_{i \in [j-1]} Z''_i) \) is a circuit that intersects the cocircuit \( C^* \) in a single element, contradicting orthogonality. We deduce that the union of any \( t \) pairs of \( Z \) that are contained in \( Z'' \) is a cocircuit. So \( M \) has a \( t \)-coechidna of order \( h_{j+1}(t) \geq 4t - 3 \), satisfying (i).

We now apply Claim 5.7.1 iteratively, for a maximum of \( t - j \) iterations. If (i) holds, at any iteration, then \( M \) has a \( t \)-coechidna of order \( 4t - 3 \), as required.

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Otherwise, we let \( Z' \) be the partition of \( Z' \) induced by \( Z \); then, at the next iteration, we relabel \( Z = Z' \) and \( Z = Z' \). If (ii) holds for each of \( t-j \) iterations, then we obtain a subset \( Z' \) of \( Z \) such that the smallest circuit in \( M|Z' \) has size \( 2t \). Then, by (II), \( M \) has a \( t \)-echidna of order \( h_{t}(t) = 4t-3 \). This completes the proof.

6. Properties of \( t \)-spikes. In this section, we prove some properties of \( t \)-spikes, which demonstrate that \( t \)-spikes form a class of highly structured matroids. In particular, we show that a \( t \)-spike has order at least \( 2t - 1 \); a \( t \)-spike of order \( r \) has \( 2r \) elements and rank \( r \); the circuits of a \( t \)-spike that are not a union of \( t \) arms meet all but at most \( t-2 \) of the arms; and a \( t \)-spike of order at least \( 4t-4 \) is \((2t-1)\)-connected. We also show that an appropriate concatenation of the associated partition of a \( t \)-spike is a \((2t-1)\)-anemone, following the terminology of [1].

It is straightforward to see that the family of 1-spikes consists of matroids obtained by taking direct sums of copies of \( U_{1,2} \). We also describe a construction that can be used to obtain a \((t+1)\)-spike from a \( t \)-spike, and show that every \((t+1)\)-spike can be constructed from some \( t \)-spike in this way.

Basic properties.

**Lemma 6.1.** Let \( M \) be a \( t \)-spike of order \( r \). Then \( r \geq 2t-1 \).

**Proof.** Let \((A_{1}, \ldots, A_{r})\) be the associated partition of \( M \). By definition, \( r \geq t \). Let \( J \) be a \( t \)-element subset of \([r]\), and let \( Y = \bigcup_{j \in J} A_{j} \). Pick some \( y \in Y \). Since \( Y \) is a cocircuit and a circuit, \( Z = (E(M) - Y) \cup \{y\} \) spans and cospans \( M \). Since \( |Z| = 2(r-t) + 1 \),

\[
2r = |E(M)| = r(M) + r^{*}(M) \leq (2(r-t)+1) + (2(r-t)+1).
\]

It follows that \( r \geq 2t-1 \).

**Lemma 6.2.** Let \( M \) be a \( t \)-spike of order \( r \). Then \( r(M) = r^{*}(M) = r \).

**Proof.** Let \((A_{1}, \ldots, A_{r})\) be the associated partition of \( M \), and label \( A_{i} = \{x_{i}, y_{i}\} \) for each \( i \in [r] \). Pick \( I \subseteq J \subseteq [r] \) such that \(|I| = t-1 \) and \(|J| = r-t \). Let \( X = \bigcup_{i \in I} A_{i} \cup \{x_{j} : j \in J \} \), and observe that \(|X| = |I| + |J| = r-1 \). Now, since \((A_{1}, \ldots, A_{r})\) is a \( t \)-echidna, \( \bigcup_{j \in J} A_{j} \subseteq cl(X) \). As \( E(M) - \bigcup_{j \in J} A_{j} \) is a cocircuit, we deduce that \( r(M) - 1 \leq r(X) \leq |X| = r-1 \), so \( r(M) \leq r \). Similarly, as \((A_{1}, \ldots, A_{r})\) is a \( t \)-coechidna, we deduce that \( r^{*}(M) \leq r \). Since \( r(M) + r^{*}(M) = |E(M)| = 2r \), the lemma follows.

The next lemma shows that a circuit \( C \) of a \( t \)-spike is either a union of \( t \) arms, or else \( C \) meets all but at most \( t-2 \) of the arms.

**Lemma 6.3.** Let \( M \) be a \( t \)-spike of order \( r \) with associated partition \((A_{1}, \ldots, A_{r})\), and let \( C \) be a circuit of \( M \). Then either

(i) \( C = \bigcup_{j \in J} A_{j} \) for some \( t \)-element set \( J \subseteq [r] \)

(ii) \( |\{i \in [r] : A_{i} \cap C \neq \emptyset\}| \geq r - (t-2) \) and \(|\{i \in [r] : A_{i} \subseteq C\}| < t \).

**Proof.** Let \( S = \{i \in [r] : A_{i} \cap C \neq \emptyset\} \), so \( S \) is the minimal subset of \([r]\) such that \( C \subseteq \bigcup_{i \in S} A_{i} \). If \( C \) is properly contained in \( \bigcup_{j \in J} A_{j} \) for some \( t \)-element set \( J \subseteq [r] \), then \( C \) is independent: a contradiction. So \( |S| \geq t \). If \(|S| = t \), then \( C = \bigcup_{i \in S} A_{i} \), implying \( C \) is a circuit, which satisfies (i). So we may assume that \(|S| > t \). Now \(|\{i \in [r] : A_{i} \subseteq C\}| < t \); otherwise \( C \) properly contains a circuit. Thus, there exists some \( j \in S \) such that \( A_{j} \cap C \neq \emptyset \). If \( |S| \geq r - (t-2) \), then (ii) holds; thus we assume that \(|S| \leq r -(t-1) \). Let \( T = ([r] - S) \cup \{j\} \). Then \(|T| \geq t \), so \( \bigcup_{i \in T} A_{i} \) contains a cocircuit that intersects \( C \) in one element, contradicting orthogonality.\[\square\]
Connectivity. Let $M$ be a matroid with ground set $E$. Recall that the connectivity function of $M$, denoted by $\lambda$, is defined as

$$\lambda(X) = r(X) + r(E - X) - r(M)$$

for all subsets $X$ of $E$. It is easily verified that

$$\lambda(X) = r(X) + r^*(X) - |X|. \quad (6.1)$$

A subset $X$ or a partition $(X, E - X)$ of $E$ is $k$-separating if $\lambda(X) < k$. A $k$-separating partition $(X, E - X)$ is a $k$-separation if $|X| \geq k$ and $|E - X| \geq k$. The matroid $M$ is $n$-connected if, for all $k < n$, it has no $k$-separations.

**Lemma 6.4.** Suppose $M$ is a $t$-spike with associated partition $(A_1, \ldots, A_r)$. Then, for all partitions $(J, K)$ of $[r]$ with $|J| \leq |K|$,

$$\lambda\left(\bigcup_{j \in J} A_j\right) = \begin{cases} 2|J| & \text{if } |J| < t, \\ 2t - 2 & \text{if } |J| \geq t. \end{cases} \quad (6.1)$$

**Proof.** Let $(J, K)$ be a partition of $[r]$ with $|J| \leq |K|$.

**Claim 6.4.1.** The lemma holds when $|J| \leq t$.

**Proof.** Suppose $|J| < t$. Since $(A_1, \ldots, A_r)$ is a $t$-spike (respectively, $t$-coincidence), $\bigcup_{j \in J} A_j$ is independent (respectively, coindependent). So, by (6.1), $\lambda\left(\bigcup_{j \in J} A_j\right) = 2|J| + 2|J| - |J| - 2|J| = 2t - 2$. Now suppose $|J| = t$. Then, by definition, $\bigcup_{j \in J} A_j$ is a circuit and a cocircuit. So $\lambda\left(\bigcup_{j \in J} A_j\right) = (2t - 1) + (2t - 1) - 2t = 2t - 2$, by (6.1).

**Claim 6.4.2.** Let $X \subseteq Y \subseteq [r]$ such that $|X| \geq t - 1$. Then

$$\lambda\left(\bigcup_{x \in X} A_x\right) \geq \lambda\left(\bigcup_{y \in Y} A_y\right).$$

**Proof.** Let $X'$ be a $(t - 1)$-element subset of $X$, and let $y \in Y - X$. Then $\lambda\left(\bigcup_{x \in X}, A_x\right) = 2(t - 1)$, and $\lambda(A_y \cup \left(\bigcup_{x \in X'}, A_x\right)) = 2t - 2$, by Claim 6.4.1. By submodularity of the connectivity function,

$$\lambda\left(A_y \cup \bigcup_{x \in X} A_x\right) \leq \lambda\left(A_y \cup \bigcup_{x \in X'} A_x\right) + \lambda\left(\bigcup_{x \in X} A_x\right) - \lambda\left(\bigcup_{x \in X'} A_x\right)$$

$$= \lambda\left(A_y \cup \bigcup_{x \in X'} A_x\right) + (2t - 2)$$

$$= \lambda\left(\bigcup_{x \in X} A_x\right).$$

Claim 6.4.2 now follows by induction. \qed

Now suppose $|J| > t$. By Claims 6.4.1 and 6.4.2, $\lambda\left(\bigcup_{j \in J} A_j\right) \leq 2t - 2$. Recall that $|K| > |J| > t$. Let $K'$ be a $t$-element subset of $K$. Let $J' = [r] - K'$, and note that $J \subseteq J'$. So, by Claim 6.4.2,

$$\lambda\left(\bigcup_{j \in J} A_j\right) \geq \lambda\left(\bigcup_{j \in J'} A_j\right) = \lambda\left(\bigcup_{k \in K'} A_k\right) = 2t - 2.$$

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We deduce that $\lambda(\bigcup_{j \in I} A_j) = 2t - 2$, as required.

Given a $t$-spike $M$ with associated partition $(A_1, \ldots, A_r)$, suppose that $(P_1, \ldots, P_m)$ is a partition of $E(M)$ such that, for each $i \in [m]$, $P_i = \bigcup_{j \in I} A_i$ for some subset $I$ of $[r]$, with $|P_i| \geq 2t - 2$. Using the terminology of [1], it follows immediately from Lemma 6.4 that $(P_1, \ldots, P_m)$ is a $(2t - 1)$-anemone. (Note that a partition whose concatenations give rise to a flower in this way has previously appeared in the literature [3] under the name of "quasi-flowers."

**Lemma 6.5.** Let $M$ be a $t$-spike of order at least $4t - 4$, for $t \geq 2$. Then $M$ is $(2t - 1)$-connected.

**Proof.** Let $r$ be the order of the $t$-spike $M$, and let $(A_1, \ldots, A_r)$ be the associated partition of $M$. Towards a contradiction, suppose $M$ is not $(2t - 1)$-connected, and let $(P, Q)$ be a $k$-separation for some $k < 2t - 1$. Without loss of generality, we may assume that $|P| > |Q|$. Note, in particular, that $\lambda(P) < k \leq |Q|$ and $\lambda(P) < 2t - 2$.

Suppose $|P \cap A_j| \neq 1$ for all $j \in [r]$. Then, by Lemma 6.4, $\lambda(P) = |Q|$ if $|Q| < 2t$, otherwise $\lambda(P) = 2t - 2$; either case is contradictory. So $|P \cap A_j| = 1$ for some $j \in [r]$. Suppose $|Q| < 2t - 2$. Then, by Lemma 6.3 and its dual, $Q$ is independent and coindependent, so $\lambda(P) = |Q|$ by (6.1); a contradiction.

Now we may assume that $|Q| > 2t - 2$. Suppose $\bigcup_{j \in I} A_j \subseteq P$, for some $(t - 1)$-element set $I \subseteq [r]$. Then $A_j \subseteq cl(P)$ for each $j \in [r]$ such that $|P \cap A_j| = 1$. For such a $j$, it follows, by the definition of $\lambda$, that $\lambda(P \cup A_j) \leq \lambda(P)$; we use this repeatedly in what follows. Let $U = \{u \in [r] : |P \cap A_u| = 1\}$. For any subset $U' \subseteq U$, we have $\lambda(P \cup (\bigcup_{u \in U'} A_u)) \leq \lambda(P) < 2t - 2$. Let $P' = P \cup (\bigcup_{u \in U'} A_u)$, and let $Q' = E(M) - P'$. If $|Q'| > 2t - 2$, then $\lambda(P') = 2t - 2$ by Lemma 6.4, contradicting that $\lambda(P') \leq \lambda(P) < 2t - 2$. So $|Q'| \leq 2t - 2$. Now, let $d = |Q| - (2t - 2)$, and let $U'$ be a $d$-element subset of $U$. Then $\lambda(P) \geq \lambda(P \cup (\bigcup_{u \in U'} A_u)) = \lambda(Q - \bigcup_{u \in U'} A_u)$. Since $|Q - \bigcup_{u \in U'} A_u| = 2t - 2$, we have that $\lambda(Q - \bigcup_{u \in U'} A_u) = 2t - 2$, so $\lambda(P) \geq 2t - 2$; a contradiction. We deduce that $|\{i \in [r] : A_i \subseteq P\}| < t - 1$. Since $|Q| \leq |P|$, it follows that $|\{i \in [r] : A_i \subseteq Q\}| \leq |\{i \in [r] : A_i \subseteq P\}| < t - 1$.

Now $|\{i \in [r] : A_i \cap Q \neq \emptyset\}| \geq r - (t - 2)$, so $r(Q) \geq r - (t - 1)$ by Lemma 6.3. Similarly, $r(P) \geq r - (t - 1)$. So

$$\lambda(P) = r(P) + r(Q) - r(M)$$

$$\geq (r - (t - 1)) + (r - (t - 1)) - r$$

$$\geq (4t - 4) - 2(t - 1) = 2t - 2;$$

a contradiction. This completes the proof.

**Constructions.** We first describe a construction that can be used to obtain a $(t + 1)$-spike of order $r$ from a $t$-spike of order $r$, when $r \geq 2t + 1$. We then show that every $(t + 1)$-spike can be constructed from some $t$-spike in this way.

Recall that $M_1$ is an elementary quotient of $M_0$ if there is a single-element extension $M_0^+$ of $M_0$ by an element $e$ such that $M_1 = M_0^+/e$. A matroid $M_1$ is an elementary lift of $M_0$ if $M_0^+$ is an elementary quotient of $M_0$. Note also that if $M_1$ is an elementary quotient of $M_0$, then $M_0$ is an elementary lift of $M_1$.

Let $M_0$ be a $t$-spike of order $r \geq 2t + 1$ with associated partition $\pi$. Let $M_0^+$ be an elementary quotient of $M_0$ such that none of the $2t$-element cocircuits are preserved (that is, extend $M_0$ by an element $e$ that blocks all of the $2t$-element cocircuits, and then contract $e$). Now, in $M_0^+$, the union of any $t$ cells of $\pi$ is still a $2t$-element circuit, but, as $r(M_0^+) = r(M_0) - 1$, the union of any $t + 1$ cells of $\pi$ is a $2(t + 1)$-element
cocircuit. We then repeat this in the dual; that is, let \( M_1 \) be an elementary lift of \( M_0' \) such that none of the \( 2t \)-element circuits are preserved. Then \( M_1 \) is a \((t+1)\)-spike. Note that \( M_1 \) is not unique; more than one \((t+1)\)-spike can be constructed from a given \( t \)-spike \( M_0 \) in this way.

Given a \((t+1)\)-spike \( M_1 \), for some positive integer \( t \), we now describe how to obtain a \( t \)-spike \( M_0 \) from \( M_1 \) by a specific elementary quotient, followed by a specific elementary lift. This process reverses the construction from the previous paragraph. The next lemma describes the single-element extension (or coextension, in the dual) that gives rise to the elementary quotient (or lift) we desire. Intuitively, the extension adds a “tip” to a \( t \)-echidna. In the proof of this lemma, we assume knowledge of the theory of modular cuts (see [6, section 7.2]).

**Lemma 6.6.** Let \( M \) be a matroid with a \( t \)-echidna \( \pi = (S_1, \ldots, S_n) \). Then there is a single-element extension \( M^+ \) of \( M \) by an element \( e \) such that \( e \in \text{cl}_{M^+}(X) \) if and only if \( X \) contains at least \( t-1 \) spines of \( \pi \) for all \( X \subseteq E(M) \).

**Proof.** Let

\[
F = \left\{ \bigcup_{i \in I} S_i : I \subseteq [n] \text{ and } |I| = t-1 \right\}.
\]

By the definition of a \( t \)-echidna, \( F \) is a collection of flats of \( M \). Let \( \mathcal{M} \) be the set of all flats of \( M \) containing some flat \( F \in F \). We claim that \( \mathcal{M} \) is a modular cut. Recall that, for distinct \( F_1, F_2 \in \mathcal{M} \), the pair \((F_1, F_2)\) is modular if \( r(F_1) + r(F_2) = r(F_1 \cup F_2) + r(F_1 \cap F_2) \). It suffices to prove that for any \( F_1, F_2 \in \mathcal{M} \) such that \((F_1, F_2)\) is a modular pair, \( F_1 \cap F_2 \in \mathcal{M} \).

For any \( F \in \mathcal{M} \), since \( F \) contains at least \( t-1 \) spines of \( \pi \), and the union of any \( t \) spines is a circuit (by the definition of a \( t \)-echidna), it follows that \( F \) is a union of spines of \( \pi \). So let \( F_1, F_2 \in \mathcal{M} \) such that \( F_1 = \bigcup_{i \in I_1} S_i \) and \( F_2 = \bigcup_{i \in I_2} S_i \), where \( I_1 \) and \( I_2 \) are distinct subsets of \([n]\) with \( u_1 = |I_1| \geq t-1 \) and \( u_2 = |I_2| \geq t-1 \). Then

\[
\begin{align*}
    r(F_1) + r(F_2) &= (t-1 + u_1) + (t-1 + u_2) \\
    &= 2(t-1) + u_1 + u_2.
\end{align*}
\]

Suppose that \( |I_1 \cap I_2| < t-1 \). Let \( s = |I_1 \cap I_2| \). Then \( F_1 \cup F_2 \) is the union of \( u_1 + u_2 - s \geq t-1 \) spines of \( \pi \). So

\[
\begin{align*}
    r(F_1 \cup F_2) + r(F_1 \cap F_2) &= (t-1 + (u_1 + u_2 - s)) + 2s \\
    &= (t-1) + s + u_1 + u_2.
\end{align*}
\]

Since \( s < t-1 \), it follows that \( r(F_1 \cup F_2) + r(F_1 \cap F_2) < r(F_1) + r(F_2) \). So, for every modular pair \((F_1, F_2)\) with \( F_1, F_2 \in \mathcal{M} \), we have \(|I_1 \cap I_2| \geq t-1 \), in which case \( F_1 \cap F_2 \) is a flat containing the union of \( t-1 \) spines of \( \pi \), and hence \( F_1 \cap F_2 \in \mathcal{M} \) as required.

Now, there is a single-element extension corresponding to the modular cut \( \mathcal{M} \), and this extension satisfies the requirements of the lemma (see, for example, [6, Theorem 7.2.3]).

Let \( M \) be a \( t \)-spike with associated partition \( \pi = (A_1, \ldots, A_r) \), for some integer \( t \geq 2 \), where \( r \geq 2t-1 \) by Lemma 6.1. Let \( M^+ \) be the single-element extension of \( M \) by an element \( e \) described in Lemma 6.6.

Consider \( M^+/e \). We claim that \( \pi \) is a \((t-1)\)-echidna and a \( t \)-coechidna of \( M^+/e \). Let \( X \) be the union of any \( t-1 \) spines of \( \pi \). Then \( X \) is independent in \( M \), and \( X \cup \{e\} \) is a circuit in \( M^+ \), so \( X \) is a circuit in \( M^+/e \). So \( \pi \) is a \((t-1)\)-echidna of \( M^+/e \).
Now let $C^*$ be the union of any $t$ spines of $\pi$, and let $H = E(M) - C^*$. Then $H$ is the union of at least $t - 1$ spines, so $e \in \cl_{M^+}(H)$. Now $H \cup \{e\}$ is a hyperplane in $M^+$, so $C^*$ is a cocircuit in $M^+$. Hence $\pi$ is a $t$-coechidna of $M^+/e$.

We now repeat this process on $N = (M^+/e)^*$. In $N$, the partition $\pi$ is a $t$-echidna and $(t - 1)$-coechidna. By Lemma 6.6, there is a single-element extension $N^+$ of $N$ (a single-element coextension of $M^+/e$) by an element $e'$. By the same argument as in the previous paragraph, $\pi$ is a $(t - 1)$-echidna and $(t - 1)$-coechidna of $N^+/e$, so $N^+/e$ is a $(t - 1)$-spike. Let $M' = (N^+/e)^*$.

Note that $M^+/e$ is an elementary quotient of $M$, so $M$ is an elementary lift of $M^+/e$ where none of the $2(t - 1)$-element circuits of $M^+/e$ are preserved in $M$. Similarly, $M^+/e$ is an elementary quotient of $M'$ where none of the $2(t - 1)$-element cocircuits are preserved. So the $t$-spike $M$ can be obtained from the $(t - 1)$-spike $M'$ using the earlier construction.

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