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Incomplete Anger-Weber Functions: A Class of Special Functions for Electromagnetics

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Abstract A novel class of special functions for electromagnetics is presented. Formed by the incomplete Anger-Weber functions, this class conveniently allows solving electromagnetic problems involving truncated circular electromagnetic structures. The definition of these functions is here introduced, and the relevant analytical properties are derived. The definition is such that the interrelationships between the incomplete functions parallel, as far as is feasible, those for canonical Anger-Weber functions.

1. Introduction

Different applications of modern engineering and physical science require a thorough knowledge of applied mathematics, particularly special functions. These are frequently adopted in acoustics, thermodynamics, electromagnetics, and optics, to express the approximate or exact analytical solution of complex problems (Andrews et al., 2001; Olver et al., 2010).

In this study, attention is given to the applications of special functions in electromagnetic field theory. In this context, it is well-known that the general solution of the wave equation can be represented in terms of harmonic, Bessel, and associated Legendre functions, depending on the specific (rectangular, cylindrical, and spherical) symmetry of the problem (Jackson, 1999). Besides that, numerous other special functions commonly appear in electromagnetic applications. For instance, the Fresnel integral has been adopted by Sommerfeld to solve the half-plane problem (Sommerfeld, 1954) and plays a key role as transition function for ray field representations in the uniform theory of diffraction introduced by Kouyoumjian and Pathak (Kouyoumjian & Pathak, 1974). The Maliuzhinets function allows addressing wave diffraction problems involving wedges with different face impedances (Maliuzhinets, 1958). The approximate expression of the radiation resistance of dipole antennas involves sine and cosine integrals (Kraus, 1950). Hankel functions are extensively used to evaluate the spatial distribution of electromagnetic waves radiated by two-dimensional structures (Harrington, 1968), whereas the Airy function and its incomplete version enable the analytical description of the field in caustic regions of reflector antennas, as well as guiding and radiating optical structures (Borovikov, 1994; Levey & Felsen, 1969). More recently, conventional Anger-Weber functions have found application in the analytical solution of the spectral integral which describes reflected cylindrical waves at a planar interface (Tedeschi & Frezza, 2014).

The research of many authors has led to a large body of knowledge on the previously mentioned special functions, as well as on new ones. In particular, incomplete Hankel functions have been adopted to determine, in closed form, the spatial distribution of progressive and evanescent wave contributions excited in truncated cylindrical electromagnetic structures (Cicchetti et al., 2015), while incomplete modified spherical Bessel functions (Caratelli & Yarovoy, 2010a; Caratelli et al., 2015) have been used to analyze the transient electromagnetic field radiated in Fraunhofer region by a general antenna, thus providing a better understanding and meaningful insight into the physical mechanisms, which are responsible for the behavior of three-dimensional radiators (Caratelli & Yarovoy, 2010b). More recently, higher-order special functions have been introduced in Cicchetti et al. (2013) and used, in combination with incomplete Hankel functions, to derive the exact analytical solution of the electromagnetic field distribution supported by uniform currents excited on Cartesian quadrants. Said functions belong to the larger class of so-called incomplete cylindrical...
functions, which can be defined in different Poisson, Bessel, and Sonine-Schläfli forms featuring distinct properties useful to effectively tackle a wide variety of applications (Agrest & Maksimov, 1971).

In this paper, the class of incomplete Anger-Weber functions is presented. The properties of said class of functions, with particular attention on the relevant governing differential equation and recurrence formulae, as well as the uniform asymptotic expansions for small and large arguments, are investigated and discussed thoroughly. Whereas incomplete Bessel and Hankel functions have been employed to solve different electromagnetic field problems (see cited references), no mention or application concerning the incomplete Anger-Weber functions seem to be present in the scientific literature. As an example, the general theory of incomplete Anger-Weber functions is here adopted to solve, in a computationally effective manner, the integral equation which governs the antenna element positioning and illumination functions useful to synthesize a given radiation pattern mask by means of an isophoric sparse ring array.

Creating efficient algorithms for special functions may become problematic when several parameters are involved. In particular, problems arise when the considered functions suddenly change their behavior, let us say from monotonic to oscillatory behavior. For many special functions of mathematical physics uniform asymptotic expansions are available, which describe precisely how the relevant functions behave, which are valid for large domains of the parameters, and which provide tools for designing high-performance computational procedures. In this respect, the method of the steepest descent path (SDP; Morse & Feshbach, 1953; Wong, 1989) is applied here to the class of incomplete Anger-Weber functions for the effective and accurate numerical evaluation of their exact form, furthermore leading to uniform asymptotic representations which are suitable to describe the far-field radiation properties of ring arrays.

The paper is organized as follows. Section 2 details the general theory of the incomplete Anger-Weber functions. Section 3 discusses the application of the considered class of functions to the evaluation of the radiation pattern of equi-amplitude circular ring arrays. Finally, the concluding remarks are presented in section 4.

2. Theory

In this study, we consider the class of incomplete Anger-Weber functions stemming from the analysis of truncated circular electromagnetic structures. The general function $A_v(\xi, z)$ of order $v$ is defined by the following integral (see Figure 1):

$$A_v(\xi, z) = \frac{1}{\pi} \int_0^\xi e^{i(z \sin w - v w)} \, dw.$$  \hspace{1cm} (1)

The representation in Bessel form (1) shows that $A_v(\pi, z) = J_v(z) - i E_v(z)$, where

$$J_v(z) = \frac{1}{\pi} \int_0^\pi \cos (v w - z \sin w) \, dw,$$  \hspace{1cm} (2)

$$E_v(z) = \frac{1}{\pi} \int_0^\pi \sin (v w - z \sin w) \, dw,$$  \hspace{1cm} (3)

denote the classical Anger and Weber functions, respectively (Abramowitz & Stegun, 1972; Prudnikov et al., 1990). This property therefore explains why the terminology referencing $A_v(\xi, z)$ as an incomplete Anger-Weber function is pertinent. Indeed, a strong interrelationship occurs between these classes of special functions, with the analytical properties of the former generalizing those of the latter.
2.1. Differential Equation

To derive the differential equation that defines the general incomplete Anger-Weber function, let us first introduce the Bessel operator (Watson, 1962):

$$D_{\nu} = z^2 \frac{\partial^2}{\partial z^2} + z \frac{\partial}{\partial z} + z^2 - \nu^2,$$

as well as the auxiliary operator:

$$D_{A_{\nu}} = \frac{\partial^2}{\partial w^2} + 2j\nu \frac{\partial}{\partial w}.$$  

Therefore, applying (4) to (1) immediately yields the following identity:

$$D_{B_{\nu}} A_{\nu}(\xi, z) = -\frac{1}{\pi} \int_0^\xi D_{A_{\nu}} e^{i(z \sin \nu w - \nu w)} dw$$

$$= -\frac{1}{\pi} \left( \frac{\partial}{\partial w} + 2j\nu \right) e^{i(z \sin \nu w - \nu w)} \bigg|_{w=0}^{w=\xi}$$

$$= \frac{1}{j\pi} \left[ (v + z \cos \xi) e^{i(z \sin \xi - \xi)} - (v + z) \right] = \ell_\nu(\xi, z).$$

It is apparent that $A_{\nu}(\xi, z)$ is the solution of the nonhomogeneous Bessel differential equation with the uniquely determined right-hand-side term $\ell_\nu(\xi, z)$ in (6) and subject to the initial conditions:

$$A_{\nu}(\xi, 0) = \frac{1 - e^{-j\xi}}{\pi j\nu} = -\frac{1}{\nu^2} \ell_\nu(\xi, 0) = \ell_{\nu}(\xi).$$  

$$\frac{\partial}{\partial \xi} A_{\nu}(\xi, 0) = \frac{1 - e^{-j\xi}}{\pi j(\nu^2 - 1)} = -\frac{1}{\nu^2 - 1} \frac{\partial}{\partial \xi} \ell_{\nu}(\xi, 0) = \eta_{\nu}(\xi).$$

This property can be readily used to derive an alternative integral representation of the incomplete Anger-Weber functions. After selecting the Bessel functions $J_\nu(z)$ and $Y_\nu(z)$ as a fundamental set of solutions of the homogeneous equation, the method of variation of parameters (or Duhamel’s principle; Simmons, 2017) can be used to verify, in the special case of interest for this research where $\xi$ is a real-valued variable and $\nu$ is an integer index, that (see Appendix A):

$$A_{\nu}(\xi, z) = \frac{1}{2} \left( \left\lfloor \frac{\xi}{\pi} \right\rfloor + 1 \right) \left( \left\lfloor \frac{\xi}{\pi} \right\rfloor + 1 \right) J_\nu(z)$$

$$\frac{\pi}{2} \left\lfloor \frac{\xi}{\pi} \right\rfloor \int_0^\xi \ell_{\nu}(\xi, \xi) J_\nu(\xi) \frac{d\xi}{\xi} + J_\nu(\xi) \int_0^\infty \ell_{\nu}(\xi, \xi) Y_\nu(\xi) \frac{d\xi}{\xi},$$

with $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denoting, respectively, the ceiling and floor functions (Graham et al., 1994). In particular, the endpoints of the integrals appearing on the right-hand side of (9) are chosen in such a manner that the resulting expression exhibits an asymptotic behavior for $z \to 0$, which is consistent with (7) and (8).

2.2. Recurrence Relations

The incomplete Anger-Weber functions satisfy the following recurrence equations:

$$A_{\nu-1}(\xi, z) - 2\frac{\partial}{\partial z} A_{\nu}(\xi, z) - A_{\nu+1}(\xi, z) = 0,$$

$$A_{\nu-1}(\xi, z) + 2\frac{\nu}{z} A_{\nu}(\xi, z) = 2 \frac{g_{\nu}(\xi, z)}{z},$$

where

$$g_{\nu}(\xi, z) = \frac{1}{\pi j} \left[ e^{i(z \sin \xi - \nu \xi)} - 1 \right].$$

Because both formulas are similarly obtained, only (11) is derived below for the sake of brevity. To this end, the integral representation of $A_{\nu}(\xi, z)$ in Bessel form (1) is used to determine that

$$A_{\nu-1}(\xi, z) + A_{\nu+1}(\xi, z) = \frac{2}{\pi} \int_0^\xi \cos w e^{i(z \sin \nu w - \nu w)} dw$$

$$= \frac{2}{\pi z} \int_0^\xi \left( -j \frac{\partial}{\partial w} + \nu \right) e^{i(z \sin \nu w - \nu w)} dw.$$
Thus, a trivial integration of the right-hand side of (13) results in the reported recurrence relation (11) with the function $g_n(\xi,z)$ satisfying the Nielsen’s condition (Watson, 1962):

$$g_{n-1}(\xi,z) - g_{n+1}(\xi,z) - 2\frac{\partial}{\partial z}g_n(\xi,z) = 0. \quad (14)$$

The expressions (10) and (11) can be verified to collapse to the well-known recurrence equations for the canonical Anger-Weber functions (Olver et al., 2010) as the angle parameter $\xi$ approaches $\pi$.

### 2.3. Bessel Expansion Series

For arbitrary values of the variable $z$, a series expansion of $A_\nu(\xi,z)$ in terms of Bessel functions of the first kind can be derived by using the following Jacobi-Anger identity (Colton & Kress, 1998):

$$e^{i\xi \sin w} = \sum_{m=-\infty}^{+\infty} J_m(z)e^{imw}. \quad (15)$$

In fact, under the hypothesis of integer order $\nu$, substituting (15) in (1) and integrating in a term-by-term manner readily yields the following:

$$A_\nu(\xi,z) = \frac{1}{\pi} \sum_{m=-\infty}^{+\infty} J_m(z) \int_0^\pi e^{im(\nu-\xi)w} dw = \frac{\xi}{\pi} \sum_{m=-\infty}^{+\infty} e^{im\xi} J_{\nu+m}(z) \sin \frac{m\xi}{2}, \quad (16)$$

with $\sin(-\cdot)$ being the unnormalized cardinal sine function (Olver et al., 2010). Thus, a computationally efficient representation of the general incomplete Anger-Weber function for small values of the argument $z$ can be obtained by truncating the bilateral series on the right-hand side of (16) to a sufficient number of terms, such that the error introduced in this process is maintained below a given threshold (see Figure 2).

### 2.4. Asymptotic Approximation for Small Argument

In this section, attention is put on the derivation of the asymptotic approximation for small argument of the general incomplete associated Anger-Weber function $A_\nu(\xi,z)$. To this end, let us notice that, for arbitrary values of $z$, the MacLaurin expansion of the exponential term $e^{i\xi \sin w}$ appearing in the integral representation (1) gives

$$A_\nu(\xi,z) = \frac{1}{\pi} \int_0^\pi e^{-j\xi w} \sum_{m=0}^{+\infty} \frac{(j\xi \sin w)^m}{m!} dw = \sum_{m=0}^{+\infty} \frac{Y_{\nu,m}(\xi)}{m!}, \quad (17)$$

where

$$Y_{\nu,m}(\xi) = \frac{\partial^m}{\partial \xi^m} A_\nu(\xi,0) = \frac{j^m}{\pi} \int_0^\pi \sin^m w e^{j\xi w} dw = \frac{1}{2^m \pi \xi} \sum_{n=0}^{m} (-1)^n \binom{m}{n} \frac{e^{i(m-2n)\frac{\xi}{2}}}{m - \nu - 2n}. \quad (18)$$

The interchange of the order of integration and summation in (17) and (18) is clearly legitimate. It is worth noting that the first two expansion coefficients $Y_{\nu,0}(\xi)$ and $Y_{\nu,1}(\xi)$ coincide with the initial values $\chi_\nu(\xi)$, $\eta_\nu(\xi)$ in (7) and (8), respectively.

One can verify heuristically that the convergence rate of (17) is typically much smaller than that featured by the Bessel expansion series (16). This can be inferred easily by visual inspection of the numerical results reported in Figure 3.
2.5. Addition Formulae

By virtue of the general property proved in (Watson, 1962) for cylindrical functions which satisfy the recurrence (10), the following Neumann’s addition formula holds true for \( A_\nu(\xi, z) \):

\[
A_\nu(\xi, z + \zeta) = \sum_{n=-\infty}^{+\infty} J_n(\zeta) A_{\nu-n}(\xi, z).
\]

As a matter of fact, by properly interchanging order of integration and summation, one can easily find that

\[
\sum_{n=-\infty}^{+\infty} J_n(\zeta) A_{\nu+n}(\xi, z) = \frac{1}{\pi} \sum_{n=-\infty}^{+\infty} J_n(\zeta) \int_{0}^{\zeta} e^{iz \sin w - i(\nu+n)w} \, dw
\]

\[
= \frac{1}{\pi} \int_{0}^{\zeta} e^{iz \sin w - i(\nu+n)w} \left[ \sum_{n=-\infty}^{+\infty} J_n(\zeta) e^{i nw} \right] \, dw
\]

\[
= \frac{1}{\pi} \int_{0}^{\zeta} e^{iz \sin w + i \nu w} \, dw \equiv A_{\nu}(\xi, z + \zeta),
\]

where use has been made of (15).

The identity (20) can be further generalized in analogy with the theory of conventional Bessel functions (Watson, 1962). In this regard, let us consider the following expression:

\[
\sum_{n=-\infty}^{+\infty} A_{\nu+n}(\xi, z) J_n(\zeta) e^{i \nu \phi} = \frac{1}{\pi} \sum_{n=-\infty}^{+\infty} J_n(\zeta) e^{i \nu \phi} \int_{0}^{\zeta} e^{iz \sin w + i(\nu+n)w} \, dw.
\]

Once more changing order of integration and summation and using (15), we obtain

\[
\sum_{n=-\infty}^{+\infty} A_{\nu+n}(\xi, z) J_n(\zeta) e^{i \nu \phi} = \frac{1}{\pi} \int_{0}^{\zeta} e^{iz \sin w + \zeta \sin(\phi + w) - i \nu w} \, dw.
\]

Let us now introduce the auxiliary quantities \( \Omega \) and \( \Psi \) defined by

\[
\begin{align*}
\Omega \cos \Psi &= z - \zeta \cos \phi, \\
\Omega \sin \Psi &= \zeta \sin \phi,
\end{align*}
\]

so that (see Figure 4)

\[
\Omega = \sqrt{z^2 + \zeta^2 - 2z \zeta \cos \phi}.
\]
and

\[ \Psi = \frac{1}{2} \text{Arg} \left\{ \frac{z - \zeta e^{-j\phi}}{z - \zeta e^{j\phi}} \right\} , \quad (25) \]

with \( \text{Arg} \{ \cdot \} \) denoting the principal value of the argument function. Therefore, the substitution \( w + \Psi = t \) for the variable of integration in (22) gives the Graf’s addition formula for incomplete Anger-Weber functions:

\[
\sum_{n=\infty}^{+\infty} A_{\nu+n} (\xi, z) J_n (\zeta) e^{j\nu \phi} = \frac{1}{\pi} e^{j\nu \phi} \int_{\Psi}^{\Psi + \zeta} e^{j(\Omega \sin t - \nu \Theta)} dt \\
= e^{j\nu \phi} \left[ A_\nu (\Psi + \zeta, \Omega) - A_\nu (\Psi, \Omega) \right].
\]

which, on changing the signs of \( \phi \) and \( \Psi \), may be written as

\[
\sum_{n=\infty}^{+\infty} A_{\nu+n} (\xi, z) J_n (\zeta) e^{-j\nu \phi} = e^{-j\nu \phi} \left[ A_\nu (\Psi + \zeta, \Omega) - A_\nu (\Psi, \Omega) \right].
\]

Evidently, Neumann’s identity (19) derived earlier is a particular case of (26) and (27) for \( \phi = \pi \) since, under said assumptions, \( \Omega = z + \zeta \) and \( \Psi = 0 \). On the other hand, for \( \zeta = 2\pi \theta \) and integer index \( \nu \), (26) and (27) reduce to the well-known formulas for Bessel functions.

We shall finally consider the series expansion analogous to the Lommel’s addition formula for Bessel functions (Watson, 1962)

\[
J_\nu \left( z\sqrt{1+jk} \right) = \sum_{n=0}^{+\infty} \frac{1}{n!} (-j kz)^n J_{\nu+n} (z).
\]

To that end, we evaluate the right-hand side of (28) after replacing the general Bessel function term \( J_{\nu+n} (z) \) with \( A_{\nu+n} (\zeta, z) \). In this way, upon using the MacLaurin series of the exponential function, we have

\[
\sum_{n=0}^{+\infty} \frac{1}{n!} \left( -j kz \right)^n A_{\nu+n} (\zeta, z) = \frac{1}{\pi} \sum_{n=0}^{+\infty} \frac{1}{n!} \left( -j kz \right)^n \int_0^\zeta e^{j(z \sin w - (\nu+n)w)} dw \\
= \frac{1}{\pi} \int_0^\zeta e^{j(z \sin w - (\nu+n)w)} \left[ \sum_{n=0}^{+\infty} \frac{1}{n!} \left( -j kz \right)^n \right] dw \\
= \frac{1}{\pi} \int_0^\zeta e^{j(z \sin w - \frac{1}{2}j \nu w)} dw.
\]

This expression may be simplified by introducing the auxiliary quantity \( \Theta \) defined by the following relation:

\[
e^{j\theta} = 1 + jk,
\]

so that

\[
\begin{align*}
\sqrt{1+jk} \cos \Theta &= 1 + \frac{j}{2} \theta, \\
\sqrt{1+jk} \sin \Theta &= \frac{k}{2} \theta.
\end{align*}
\]

Then, after some algebraic manipulations, it is found that

\[
\sum_{n=0}^{+\infty} \frac{1}{n!} \left( -j kz \right)^n A_{\nu+n} (\zeta, z) = \frac{1}{\pi} \int_0^\zeta e^{j(z \sqrt{1+jk} \sin \Theta - \theta w)} dw \\
= \frac{1}{\pi} e^{-j\Theta} \int_{-\Theta}^{\zeta} e^{j\sqrt{1+jk} \sin \Theta} dw \\
= A_\nu \left( \zeta - \Theta, z \sqrt{1+jk} \right) - A_\nu \left( -\Theta, z \sqrt{1+jk} \right) \\
= A_\nu \left( 2\pi - \Theta, z \right) - A_\nu \left( -\Theta, z \right) = A_\nu \left( 2\pi, z \right) = 2J_\nu (z).\]

Clearly, for \( \zeta = 2\pi \theta \) and integer index \( \nu \), (32) reduces to (28) if use is made of the identity:

\[
A_\nu \left( 2\pi - \Theta, z \right) - A_\nu \left( -\Theta, z \right) = A_\nu \left( 2\pi, z \right) = 2J_\nu (z),
\]

which holds true for arbitrary parameters \( Z \) and \( \Theta \).
2.6. Asymptotic Approximation for Large Argument

In this section, the asymptotic expansion of $A_\nu(\xi, z)$ is derived under the hypothesis of real-valued arguments $\xi, z,$ and integer order $\nu,$ as it is actually the case in a number of applications, such as conformal antenna array synthesis. To this end, let us first note that the following symmetry properties hold true:

$$A_\nu(-\xi, z) = -A_\nu^*(\xi, z),$$

(34)

$$A_\nu(\xi, -z) = A_\nu^*(-\xi, z),$$

(35)

where the superscript $^*$ denotes complex conjugation. The identities (34) and (35), whose proof is based on a direct application of (1), allow us to restrict our attention to the domains $\xi, z \in \mathbb{R}_+^\nu$, with $\mathbb{R}_+^\nu$ being the set of positive real numbers inclusive of the zero. Under this assumption, after introducing the auxiliary functions

$$\mu(\xi) = \max \{m = 0, 1, 2, \ldots \mid \xi - 2m\pi \geq 0\},$$

(36)

$$\Xi(\xi) = \xi - 2\mu(\xi)\pi \in [0, 2\pi),$$

(37)

it is not difficult to show that

$$A_\nu(\xi, z) = A_\nu(2\mu(\xi)\pi, z) + \frac{1}{2\pi} \int_0^\xi e^{(\xi \sin w - zw)} dw = 2\mu(\xi)J_\nu(z) + A_\nu(\Xi(\xi), z),$$

(38)

where the identity $A_\nu(2k\pi, z) = 2kJ_\nu(z)$, following from (16) for arbitrary integer values of $k$, has been judiciously used. In addition to (38), one can easily find that

$$A_\nu(2\pi - \xi, z) = 2J_\nu(z) - A_\nu^*(-\xi, z).$$

(39)

Therefore, in the following, we can restrict the variable $\xi$ to the closed interval $[0, \pi]$. Let us now rewrite the incomplete Anger-Weber function of order $\nu$ in the alternative form:

$$A_\nu(\xi, z) = \int_0^\xi \sigma_\nu(w)e^{\nu H(w)} dw,$$

(40)

where $H(w) = j\sin w$ and $\sigma_\nu(w) = e^{-\nu w}/\pi$. Based on Cauchy’s integral theorem (Arfken, 1985), the global analyticity of the integrand in (40) can be used to freely deform the relevant contour of integration while maintaining the value of $A_\nu(\xi, z)$ unaltered. In particular, following the SDP method (Morse & Feshbach, 1953; Wong, 1989), the original contour of integration in (40) can be replaced by a sum of different contours evolving along the valleys of the complex phase function $H(w)$ (see Figure 5) such that the resulting contour integrals are all of a Laplace type (Bleistein & Handelsman, 1986) and thus prone to a simple asymptotic evaluation for large values of the argument.

To distinguish and choose the appropriate integral paths for $A_\nu(\xi, z)$, we must identify the critical points of the integrand, which are the endpoints of integration $w = 0$ and $w = \xi$, as well as the extrema points of $H(w)$ determined by the following condition:

$$H'(w) = j\cos w = 0.$$

(41)

The extrema are obviously distributed along the real axis at the points $\pm\pi/2, \pm 3\pi/2, \ldots$ where $H''(w) \neq 0$. However, because we limit ourselves to the interval $0 \leq \xi \leq \pi$, only the saddle point of first order $w_s = \pi/2$ (see Figure 5) must be considered. The implicit equation describing the SDP $C_s$ originating from $w_s$ is obtained by equating the imaginary part of the complex phase function with that of $H(w)$:

$$\text{Im} \{H(w)\} = \sin w_c \cosh w_t = 1 = \text{Im} \{H(w_s)\}$$

(42)

with $w_c$ and $w_t$ denoting the real and imaginary parts of the complex variable $w$, respectively. In this manner, the selection of the pertinent branches along which the exponential term appearing in the integrand of (40)
decays rapidly to zero away from the saddle point yields the following explicit expression:

\[ w_{\text{SDP}} (w_i) = \frac{\pi}{2} \text{sign} (w_i) \cos^{-1} \frac{1}{\cosh w_i}, \quad (43) \]

where \( w_i \) spans the complete set of real numbers and \( \text{sign}(\cdot) \) is the sign function (Bracewell, 1999), which is defined in such a way that \( \text{sign}(0) = -1 \) in order to ensure continuity from the left. Similarly, the SDP \( C_E (\xi) \) originating from the endpoint \( w = \xi \in [0, \pi] \) is determined by enforcing the constant phase condition \( \text{Im} \{ H(w) \} = \text{Im} \{ H(\xi) \} \) and discarding the branches that evolve along the hills of \( H(w) \). After performing simple algebra, one can verify that

\[ w_{\text{SDP}} (w_i, \xi) = \frac{\pi}{2} - \text{sign} \left( \frac{\pi}{2} - \xi \right) \cos^{-1} \frac{\sin \xi}{\cosh w_i}, \quad (44) \]

where \( w_i > 0 \) for \( \xi \in [0, \pi/2] \) and \( w_i \leq 0 \) for \( \xi \in (\pi/2, \pi] \). The equation describing the SDP \( C_0 \) departing from the other endpoint of integration \( w = 0 \) directly follows from (44) for \( \xi \rightarrow 0 \) and corresponds to the positive imaginary semiaxis.

The SDPs from the critical points of the integrand in (40) are shown in Figure 6. By invoking Cauchy’s integral theorem, the original interval of integration \([0, \xi]\) can be replaced with the combination of oriented contours

\[ \Sigma(\xi) = C_0 + C_E (\xi) + u \left( \xi - \frac{\pi}{2} \right) C_s, \quad (45) \]

with \( u(\cdot) \) being the unit step function satisfying the following equation:

\[ u(x) = \frac{1 + \text{sign}(x)}{2}, \quad \forall x \in \mathbb{R}, \quad (46) \]

where the identity \( u(0) = 0 \) holds true because of the convention adopted above. Thus, we can conclude that the general incomplete Anger-Weber function has the alternative yet rigorous integral representation:

\[ A_s (\xi, \zeta) = \int_{\Sigma(\xi)} \sigma_s (w) e^{2iH(w)} \, dw = I_0 (v, \zeta) + I_E (v, \xi, \zeta) + I_s (v, \zeta) u \left( \xi - \frac{\pi}{2} \right), \quad (47) \]

with the different terms appearing on the right-hand side of the equation denoting the integrals relevant to the separate paths that form \( \Sigma(\xi) \). In particular, the theory of Hankel functions (Agrest & Maksimov, 1971) can be used to derive the following:

\[ I_s (v, \zeta) = \frac{1}{\pi j} \int_{0 + j \infty}^{e^{-j\pi}} e^{\zeta \sin w} w \, dw = H_s^{(1)}(\zeta). \quad (48) \]

However, by adopting the change of variable \( w = jsinh^{-1} t \), the integral contribution from the endpoint \( w = 0 \) can be written as follows:

\[ I_0 (v, \zeta) = \frac{1}{\pi} \int_{C_0} e^{\zeta \sin w} \, dw = -\frac{1}{\pi j} \int_{0}^{\infty} \left( \frac{t + \sqrt{1 + t^2}}{\sqrt{1 + t^2}} \right)^v e^{-\gamma t} \, dt. \quad (49) \]
Using the definition of the generating function of Jacobi polynomials (Askey, 1975) yields
\[
\frac{(t + \sqrt{1 + t^2})^v}{\sqrt{1 + t^2}} = \sum_{k=0}^{+\infty} \frac{(2t)^k}{k!} \frac{\Gamma\left(\frac{1+k+1}{2}\right)}{\Gamma\left(\frac{1-k+1}{2}\right)}.
\]  
(50)

where \(\Gamma(\cdot)\) denotes the classical Gamma function. In this manner, (50) can be substituted into (49) and the resulting expression is integrated term by term to obtain the following:
\[
I_{t}(v, z) = -\frac{1}{\pi j} \sum_{k=0}^{+\infty} \left(\frac{2}{z}\right) \frac{k \Gamma\left(\frac{1+k+1}{2}\right)}{\Gamma\left(\frac{1-k+1}{2}\right)} \sim -\frac{1}{\pi j}.
\]  
(51)

As it can be verified numerically, the asymptotic approximation appearing on the right-hand side of (51) features an accuracy within 50% for any \(z \geq 2v\), with the relative deviation from the exact value decreasing monotonically as \(v/z\) for \(z \to +\infty\).

We turn to the consideration of the integral contribution relevant to the endpoint \(w = \xi\). Upon introducing the auxiliary terms
\[
\Lambda(t, \xi) = \sinh t \cos \beta(t, \xi),
\]  
(52)

\[
\tau_{t}(t, \xi) = \text{sign}(\cos \xi) e^{\text{sign} \cos \xi |t-j|/|\beta(t, \xi)-\beta(0, \xi)|} [1 + j \tan t \tan \beta(t, \xi)],
\]  
(53)

with \(\beta(t, \xi) = \sin^{-1}(\sin \xi / \cosh t)\), one can find that
\[
I_{\xi}(v, \xi, z) = \frac{1}{\pi} \int_{C_{\xi}(\xi)} e^{iz\sin w\sin\theta} dw = a_{v}(\xi, z) \tilde{I}(v, \xi, z),
\]  
(54)

where
\[
a_{v}(\xi, z) = \frac{1}{\pi j} e^{iz \sin \xi - \xi},
\]  
(55)

\[
\tilde{I}(v, \xi, z) = \int_{0}^{\infty} \tau_{t}(t, \xi) e^{-z \sin \xi / \cosh t} dt,
\]  
(56)

with the properly oriented contours of integration shown in Figure 6. It can be noticed that \(I_{\xi}(v, \xi, z)\) features a jump discontinuity across the saddle point \(w_{s} = \pi/2\), which compensates for the term relevant to the SDP curve \(C_{t}\) in the representation (47). By using the saddle-point technique (Felsen & Marcuvitz, 1994), the asymptotic approximation of \(\tilde{I}(v, \xi, z)\), which is valid uniformly as \(z \to +\infty\), is derived as follows:
\[
\tilde{I}(v, \xi, z) \sim \frac{1}{2} \left[ e^{-j\left(\xi - \xi\right)} F_{1}(jz(1 - \sin \xi)) - 1 + \frac{1}{\cos \xi} \right],
\]  
(57)

with \(F_{1}(\cdot)\) denoting the transition function used in the uniform geometrical theory of diffraction (Kildal, 1984) and here expressed in terms of the complementary error function (Gradshteyn & Ryzhik, 2007) as follows:
\[
F_{1}(\zeta) = \sqrt{\pi \zeta} e^{\zeta} \text{erfc} \left(\sqrt{\zeta}\right),
\]  
(58)

where the square root \(\sqrt{\cdot}\) is defined by means of its principal value with the branch cut chosen along the negative real semi-axis. Substituting (57), (54), 51, and (48), into (47) finally yields the expression of the uniform asymptotic approximant relevant to the general incomplete Anger-Weber function for \(0 \leq \xi < \pi\):
\[
A_{v}(\xi, z) \sim A_{v}^{(4)}(\xi, z) = \sqrt{\frac{\pi}{\zeta}} e^{j\left(\xi - \xi\right) / \zeta} u\left(\xi = \frac{\pi}{2}\right) + \frac{1}{\pi jz} \left\{ e^{j\sin \xi - \xi} \left[ \frac{1}{\cos \xi} + e^{-j\left(\xi - \xi\right)} F_{1}(jz(1 - \sin \xi)) - 1 \right] \right\}.
\]  
(59)
Figure 7. Normalized magnitude of the incomplete Anger-Weber function $A_{\nu}(\xi, z)$ of order $\nu = 2$ (a) and $\nu = 6$ (b), when $z = 10\pi$.

where judicious use has been made of the well-known expansion of the Hankel function of the first kind for large arguments (Watson, 1962). From the application of the property (Ren & MacKenzie, 2007):

$$e^{\zeta^2} \text{erfc}(\zeta) \approx 1 - \frac{2\zeta}{\sqrt{\pi}},$$

for $\zeta \ll 1$, one can conclude that

$$F_{\nu}(\zeta) \sim \sqrt{\pi \zeta} - 2\zeta,$$

as $\zeta \to 0$. Using (61) and performing simple algebraic calculations, we obtain

$$\lim_{\xi \to \frac{\pi}{2}} \frac{\partial}{\partial \xi} A_{\nu}(\xi, z) = \lim_{\xi \to \frac{\pi}{2}} \frac{\partial}{\partial \xi} A_{\nu}(\xi, z) = -1 \frac{\pi}{\zeta} \left[ 1 + (-j)^{\nu+1} \left( \nu + \frac{\pi \zeta}{2j} \right) \right].$$

Similarly, it is not difficult to derive the following expressions:

$$\frac{\partial}{\partial \xi} A_{\nu}(\xi, z) = \frac{1}{\pi} e^{\frac{j\zeta}{\pi} \sin(\xi-\nu \zeta)},$$

$$\frac{\partial}{\partial \xi} A_{\nu}(\xi, z) = \frac{e^{\frac{j\zeta}{2\pi} \sin(\xi)}}{2\pi \zeta \cos(\xi)} \left[ 2(\zeta \cos \xi - j \tan \xi - \nu) e^{-\nu \xi} + j \frac{1 + \sin \xi}{\sqrt{2(1 - \sin \xi)}} e^{-\nu \xi} \text{sign} \left( \frac{\pi}{2} - \xi \right) \right],$$

from which it follows immediately that

$$\lim_{\xi \to \frac{\pi}{2}} \frac{\partial}{\partial \xi} A_{\nu}(\xi, z) = \lim_{\xi \to \frac{\pi}{2}} \frac{\partial}{\partial \xi} A_{\nu}(\xi, z) = 1 + \frac{1}{2 \pi \zeta} \left( \nu^2 - \frac{1}{4} \right),$$

this proving that $A_{\nu}(\xi, z)$ is continuously differentiable at the saddle point $\xi = \pi/2$, exactly as $A_{\nu}(\xi, z)$.

As shown in Figure 7, the consistency between the exact and the asymptotic expressions of $A_{\nu}(\xi, z)$ tends to degrade as the order $\nu$ becomes larger for a fixed value of the parameter $z$, but heuristically, that is found to be typically excellent under the condition $z \gg |\nu^2 - \frac{1}{4}|$. It is worth noting here that the domain in which $A_{\nu}(\xi, z)$ is defined can be extended by analytical continuation for $\nu \in \mathbb{R}$, similarly to the rigorous integral representation in (1); this can be readily inferred by visual inspection of Figure 8.

3. Application

Let us consider the analytical expression of the radiation pattern relevant to an isophoric circular ring array of radius $R_{\alpha}$ and consisting of $N_{\alpha}$ identical antennas excited with phases $\theta_{\alpha}(n = 1, 2, \ldots, N_{\alpha})$:

$$F(\psi_0, \varphi) = \sum_{n=1}^{N_{\alpha}} f_n(\psi_0, \varphi) e^{i[(\varphi - \Phi_{\alpha}) + \theta_{\alpha}]}.$$

where

$$\psi_0 = 2\pi \frac{R_{\alpha}}{\lambda_0} \sin \theta_0$$

denotes the normalized array size with respect to the wavelength $\lambda_0$ in free space, and dependent on the beam steering angle $\theta_0$ along the elevation plane. The array elements are assumed to be located at arbitrary
azimuthal angles $\Phi_n$ (see Figure 9). In (66), $f_n (\psi_0, \varphi)$ is the general element pattern which, by virtue of the symmetry of the radiating structure under analysis, can be expressed in the following form:

$$ f_n (\psi_0, \varphi) = f (\psi_0, \varphi - \Phi_n), $$

and, typically, includes the effect of the curvature, as well as of the electromagnetic interaction with the host platform.

As it can be readily inferred, the expression in (66) may be regarded as the Riemann’s sum (Ostbee & Zorn, 2001), which approximates, except for an unessential scaling constant, the auxiliary array pattern defined as follows (Caratelli et al., 2014):

$$ F_A (\psi_0, \varphi) = \int_0^Q f (\psi_0, \varphi - \Phi(q)) e^{[\psi_0 \cos(\varphi - \Phi(q)) + a(q)]} dq, $$

with $\Phi(q)$ and $a(q)$ denoting, respectively, the continuous element angle and phase distributions which generalize the discrete quantities $\Phi_n$ and $\Phi_n$ appearing in (66). Similarly, $q$ is the continuous version of the index $n$ relevant to the general antenna element forming the array ranging from 0 to the maximum value $Q$ that can be specified to be the unity for the purpose of normalization. A trivial physical reasoning leads immediately to the conclusion that $\Phi$ is a nondecreasing function of $q$, which implies that $\Phi(q) > 0$ for all $q \in [0, Q]$, with $\Phi(0) = 0$ and $\Phi(Q) = 2\pi$. Therefore, the inverse function $\Phi^{-1}(\cdot)$ is uniquely determined globally.

Upon performing the change of variable $\Phi(q) = \nu$, the auxiliary array pattern can be readily rewritten, by virtue of the inverse function theorem (Nijenhuis, 1974), as

$$ F_A (\psi_0, \varphi) = \sum_{m=1}^M F_{A_m} (\psi_0, \varphi), $$

where

$$ F_{A_m} (\psi_0, \varphi) = \int_{v_{m-1}}^{v_m} f (\psi_0, \varphi - \nu) e^{[\psi_0 \cos(\varphi - \Phi^{-1}(\nu))]} d\nu \Phi'(\Phi^{-1}(\nu)), $$

is the contribution pertinent to the $m$th interval $[v_{m-1}, v_m]$ ($m = 1, 2, \ldots, M$) in which the domain of integration extending from 0 to $2\pi$ is subdivided. A natural choice for the value of the general endpoint in (71) is $v_m = 2\pi m / M$. In each interval, the angular element density and the excitation tapering functions are assumed to be piecewise approximated according to the following expressions:

$$ \begin{cases} 
\Phi'(\Phi^{-1}(\nu)) \approx \Phi'_m, \\
\Phi'(\Phi^{-1}(\nu)) \approx \Phi'_m,
\end{cases} \quad (72) $$

with $\nu \in (v_{m-1}, v_m)$. Furthermore, since the element patterns are evidently $2\pi$–periodic functions of the azimuth angle, it is convenient to use, in this context, a Fourier series representation, namely, the following amplitude mode expansion (Rudge et al., 1983):

$$ f (\psi_0, \varphi) = \sum_{k=-\infty}^{+\infty} c_k (\psi_0) e^{-jk\varphi}, $$

where the coefficients $c_k (\psi_0)$ are given by

$$ c_k (\psi_0) = \frac{1}{2\pi} \int_0^{2\pi} f (\psi_0, \varphi) e^{jk\varphi} d\varphi. $$

**Figure 8.** Magnitude of the incomplete Anger-Weber function $A_n (\xi, \zeta)$ with real-valued index $\nu$ for $\zeta = 10\pi$, as computed by the rigorous integral representation in Bessel form (a), and by the relevant uniform asymptotic expansion for large argument (b).
In this way, upon combining (72) and (73) with (71), the analytical expression of the term $F_{A_m} (\psi_0, \varphi)$ in (71) can be derived, after trivial mathematical manipulations, as

$$F_{A_m} (\psi_0, \varphi) \simeq \gamma_m \Delta E_m (\psi_0, \varphi), \quad (75)$$

with

$$\gamma_m = \pi e^{i \alpha_m} / \Phi_m,$$

and

$$\Delta E_m (\psi_0, \varphi) = \frac{1}{\pi} \int_{\varphi - \psi_m}^{\varphi + \psi_m - 1} f (\psi_0, w) e^{i \psi_0 \cos w} dw =$$

$$= \sum_{k=-\infty}^{+\infty} j^k c_k (\psi_0) \left[ A_k \left( \frac{\pi}{2} + \varphi - \psi_{m-1}, \psi_0 \right) - A_k \left( \frac{\pi}{2} + \varphi - \psi_m, \psi_0 \right) \right]. \quad (76)$$

In typical applications, the condition $\psi_0 \gg 1$ holds true and, therefore, the incomplete Anger-Weber function terms $A_k (\cdot, \psi_0)$ in (76) can be evaluated in a computationally effective way through the uniform asymptotic expression in (59). From an electromagnetic theory point of view, it is worth noting that the far-field quantity $F_{A_m} (\psi_0, \varphi)$ clearly includes the effects of the wave radiation and edge diffraction arising from the $m$th subdomain of the array aperture.

The array synthesis is subsequently carried out by enforcing, in the azimuthal plane, the following condition:

$$F_A (\psi_0, \varphi) = F_O (\varphi), \quad (77)$$

where $F_O (\cdot)$ is the desired radiation pattern mask. To this end, a point-matching approach can be used conveniently, setting $N$ testing points $0 \leq \phi_1 < \phi_2 < \ldots < \phi_{N-1} < \phi_N < 2\pi$ in such a way that, by projecting (77) on the Dirac delta distributions (Spanier & Oldham, 1987) centered at $\varphi = \phi_n (n = 1, 2, \ldots, N)$, one can obtain the following set of linear algebraic equations:

$$\left\langle F_A (\psi_0, \varphi), \delta (\varphi - \phi_n) \right\rangle = \sum_{m=1}^{M} \Delta E_m (\psi_0, \phi_n) \gamma_m = F_O (\phi_n) = \left\langle F_O (\varphi), \delta (\varphi - \phi_n) \right\rangle, \quad (78)$$

which, setting for shortness $\Delta \mathbf{E} = [\Delta E_m (\psi_0, \phi_n)] \in \mathbb{C}^{N \times M}$, $\gamma = [\gamma_m] \in \mathbb{C}^M$, and $\mathbf{F}_O = [F_O (\phi_n)] \in \mathbb{C}^N$, may be readily recast into matrix form as $\Delta \mathbf{E} \cdot \gamma = \mathbf{F}_O$, whence

$$\gamma = (\Delta \mathbf{E})^+ \cdot \mathbf{F}_O, \quad (79)$$

with the superscript $^+$ denoting the Moore-Penrose pseudo-inversion (Penrose, 1955). In this way, upon introducing the Lagrange interpolation operator $L$ acting on the $M$–dimensional real-valued data vectors $\mathbf{x} = [x_m]$ and $\mathbf{y} = [y_m]$ (Powell, 1981):

$$L_A [\mathbf{y}] (x) = \sum_{m=1}^{M} y_m \ell_m (x), \quad (80)$$
the so-called array taper function can be evaluated as

\[ \pi e^{i\Phi(\psi)} \frac{\gamma'(\psi)}{\Phi'\left[\Phi^{-1}(\psi)\right]} = \gamma(\psi) \approx \mathcal{L}_v \left[ \gamma \left( \psi \right) \right]. \]  

(82)

Applying the inverse function theorem to (82) yields immediately

\[ \frac{1}{\pi} \int_0^\psi |\gamma(w)| \, dw = \Phi^{-1}(\psi). \]  

(83)

Hence, the application of the Lagrange interpolation operator \( \mathcal{L} \) to the vector \( \Phi^{-1}(\psi) = \left[ \frac{1}{\pi} \int_0^\psi |\gamma(w)| \, dw \right] \) allows for the evaluation of the array element positioning function as follows:

\[ \Phi(q) \approx \mathcal{L}_{\Phi^{-1}(\psi)} \left[ \gamma \left( \psi \right) \right](q). \]  

(84)
Figure 11. Topology (a) and nonuniform phase tapering (b) of the isophoric aperiodic ring antenna array featuring the radiation pattern shown in Figure 10.

with $0 \leq q \leq Q = \Phi^{-1}(2\pi)$. By making use of (81), one can readily determine the phase distribution as

$$a(q) = \arg \left\{ \gamma(\Phi(q)) \right\}.$$  \hfill (85)

Finally, once $\Phi(q)$ and $a(q)$ are known, the quantities $\bar{\Phi}_n$ and $\bar{a}_n$ appearing in the expression of the array pattern (66) can be computed, in a straightforward way, by uniform sampling at the indicial centroids:

$$\bar{q}_n = \left(n - \frac{1}{2}\right) \frac{Q}{N_a},$$  \hfill (86)

with $n = 1, 2, \ldots, N_a$.

As an example, using the developed technique, a uniform-amplitude aperiodic ring array architecture has been designed to mimic the end-fire ($\theta_0 = \pi/2$) radiation pattern mask shown in Figure 10. As it can be
noticed, the reference mask is characterized by a Gaussian-like beam centered at \( \hat{\phi}_1 = 84^\circ \) and featuring a width \( \Psi_1 = 50^\circ \) at the −10-dB reference level. In addition, a second shaped beam is synthesized in such a manner as to supply a logarithmically sloped illumination in a separate angular region of width \( \Psi_2 = 86^\circ \) around \( \hat{\phi}_2 = 253^\circ \). In particular, the array beams are required to display a well-defined phase behavior in accordance with the following mask:

\[
\arg \{ F_0(\varphi) \} = \begin{cases} 
\varphi, & |\varphi - \hat{\phi}_1| < \Psi_1, \\
-\varphi, & |\varphi - \hat{\phi}_2| < \Psi_2.
\end{cases}
\]  

The considered array synthesis problem has been addressed using antennas featuring cardioid-like radiation characteristics, namely, \( f(\psi_0, \varphi) \propto 1 + \cos \varphi \). In this case, the normalized elementary radiated field contribution from the general \( m \)th subdomain of the array is given by

\[
\Delta E_m(\psi_0, \varphi) = \frac{1}{\pi} \left( 1 - \frac{j}{\pi} \frac{\partial}{\partial \psi_0} \right) \int_{\psi_0-v_m}^{\psi_0-v_{m-1}} e^{j \psi \cos \varphi} d\psi = \frac{1}{\pi} \left( 1 - \frac{j}{\pi} \frac{\partial}{\partial \psi_0} \right) \left[ A_0 \left( \frac{\pi}{2} - v_{m-1} + \varphi, \psi_0 \right) - A_0 \left( \frac{\pi}{2} - v_m + \varphi, \psi_0 \right) \right],
\]

where the first derivative of the incomplete Anger-Weber functions can be evaluated by means of the relevant recurrence formula (10) in combination with the asymptotic expansion \( A^A(\cdot, \cdot) \) reported in (59). Thus, applying the point-matching procedure (78) and (79), the complex array illumination \( \gamma(v) \), which is useful in synthesizing the considered radiation pattern mask, can be readily determined. Thence, by making use of (84) and (85), antenna locations and excitation phases are computed as

\[
\begin{align*}
\Phi_n & = \Phi(\hat{q}_n), \\
\alpha_n & = \alpha(\hat{q}_n),
\end{align*}
\]  

for \( n = 1, 2, \ldots, N_a \). Under the assumption that \( N_a = 70 \), the array topology is found to be as sketched in Figure 11a, whereas the relevant nonuniform excitation phase distribution is shown in Figure 11b. In this context, we note how the phase tapering supplies the necessary degrees of freedom in the design to match the antenna specifications (Caratelli & Viganò, 2011a, 2011b; Caratelli, Viganó, & Yarovoy 2013; Caratelli, Viganó, Toso et al., 2013; Viganó et al., 2010; Viganó & Caratelli, 2010).

4. Conclusion

A novel class of special functions for electromagnetics and mathematical physics has been presented. This class, formed by the incomplete Anger-Weber functions, can be usefully employed to describe electromagnetic wave phenomena involving truncated circular electromagnetic structures.

The general properties of the considered functions, including the relevant uniform asymptotic expansions for small and large arguments, have been analyzed and discussed thoroughly. In particular, the governing differential equation, as well as the addition and recurrence formulae of the incomplete Anger-Weber functions, presents additional terms compared to the classical Anger-Weber function theory. Using the SDP method, an effective procedure has been developed for the accurate computation of the integral form of the considered functions.

As an example, the incomplete Anger-Weber functions of integer order have been used in the synthesis of general conformal ring antenna arrays, thus gaining meaningful insight into the physics of wave radiation from such structures. The application of the theory presented in this study to the solution of canonical electromagnetic problems involving loop antennas is currently under investigation.

Appendix A

Upon introducing the differential operator,

\[
\mathcal{L}_{z,v} = \frac{\partial^2}{\partial z^2} + \frac{1}{z} \frac{\partial}{\partial z} + \left( 1 - \frac{v^2}{z^2} \right),
\]

the governing equation of the incomplete Anger-Weber functions may be rewritten as

\[
\mathcal{L}_{z,\rho_v}(\xi, z) = \frac{1}{z^2} \rho_v(\xi, z).
\]
A particular solution of (A2) is given by

\[ \rho_\nu (\xi, z) = U_\nu (\xi, z) J_\nu (z) + V_\nu (\xi, z) Y_\nu (z), \]

where the Bessel functions \( J_\nu (z) \), \( Y_\nu (z) \) obviously satisfy the homogeneous equation \( \mathcal{L}_\nu \rho_\nu (\xi, z) = 0 \). In (A3), \( U_\nu (\xi, z) \) and \( V_\nu (\xi, z) \) denote differentiable functions subject to the following condition:

\[ J_\nu (z) \frac{\partial}{\partial z} U_\nu (\xi, z) + Y_\nu (z) \frac{\partial}{\partial z} V_\nu (\xi, z) = 0. \]

Using (A4), it is straightforward to derive the following expressions:

\[ \frac{\partial}{\partial z} \rho_\nu (\xi, z) = U_\nu (\xi, z) J_\nu' (z) + V_\nu (\xi, z) Y_\nu' (z), \]

\[ \frac{\partial^2}{\partial z^2} \rho_\nu (\xi, z) = U_\nu (\xi, z) J_\nu'' (z) + V_\nu (\xi, z) Y_\nu'' (z) + J_\nu' (z) \frac{\partial}{\partial z} U_\nu (\xi, z) + Y_\nu' (z) \frac{\partial}{\partial z} V_\nu (\xi, z). \]

from which, after simple algebra, it follows that

\[ \frac{1}{z^2} \ell_\nu (\xi, z) = \mathcal{L}_{\nu \xi} \rho_\nu (\xi, z) = J_\nu' (z) \frac{\partial}{\partial z} U_\nu (\xi, z) + Y_\nu' (z) \frac{\partial}{\partial z} V_\nu (\xi, z). \]

The combination of (A4) and (A7) results in the system of linear equations:

\[ \begin{bmatrix} J_\nu (z) & Y_\nu (z) \\ J_\nu' (z) & Y_\nu' (z) \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial z} U_\nu (\xi, z) \\ \frac{\partial}{\partial z} V_\nu (\xi, z) \end{bmatrix} = \begin{bmatrix} 0 \\ \ell_\nu (\xi, z) / z^2 \end{bmatrix}, \]

whose solution can be written in matrix form as

\[ \frac{\partial}{\partial z} \begin{bmatrix} U_\nu (\xi, z) \\ V_\nu (\xi, z) \end{bmatrix} = \frac{1}{W \{ J_\nu, Y_\nu \} (z)} \begin{bmatrix} Y_\nu' (z) - Y_\nu (z) \\ - J_\nu' (z) - J_\nu (z) \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \ell_\nu (\xi, z) / z^2 \end{bmatrix}, \]

with

\[ W \{ J_\nu, Y_\nu \} (z) = J_\nu (z) Y_\nu' (z) - J_\nu' (z) Y_\nu (z) = \frac{2}{\pi z} \]

being the Wronskian of the Bessel functions of the first and second kind. The integration of (A9) readily yields

\[ U_\nu (\xi, z) = S_\nu (\xi) + \frac{\pi}{2} \int_z^{+\infty} \ell_\nu (\xi, \zeta) Y_\nu (\zeta) \frac{d\zeta}{\zeta}, \]

\[ V_\nu (\xi, z) = T_\nu (\xi) + \frac{\pi}{2} \int_0^z \ell_\nu (\xi, \zeta) J_\nu (\zeta) \frac{d\zeta}{\zeta}, \]

where \( S_\nu (\xi) \) and \( T_\nu (\xi) \) are quantities independent of the variable \( z \).

In order to ensure the boundedness of the solution, we have to assume \( T_\nu (\xi) \equiv 0 \). Then, by enforcing for \( \rho_\nu (\xi, z) \) the same behavior of \( A_\nu (\xi, z) \) as \( z \to +\infty \), in combination with the property \( \rho_\nu (2k\pi, z) = A_\nu (2k\pi, z) = 2k J_\nu (z) \) for \( k \in \mathbb{Z} \), we obtain

\[ S_\nu (\xi) = S (\xi) = \frac{1}{2} \left( \left[ \frac{\xi}{\pi} + \frac{1}{2} \right] + \left[ \frac{\xi}{\pi} + \frac{1}{2} \right] - 1 \right). \]

Furthermore, with the particular choice of the endpoints of integration in (A11) and (A12), under the hypothesis of positive integer order \( \nu \), one can find that

\[ \int_0^{+\infty} \ell_\nu (\xi, \zeta) Y_\nu (\zeta) \frac{d\zeta}{\zeta} = \frac{(\nu - 1)!}{\pi} \left( \frac{2}{z} \right)^\nu \left[ \frac{1}{\nu} \ell_\nu (\xi, 0) + \frac{z}{\nu - 1} \frac{\partial}{\partial z} \ell_\nu (\xi, 0) + o(z) \right] \]

\[ = \frac{\nu!}{\pi} \left( \frac{2}{z} \right)^\nu \left[ \ell_\nu (\xi) + 2 \frac{1}{\nu} \eta_\nu (\xi) + o(z) \right], \]

where \( o(z) \) denotes a quantity that approaches zero more rapidly than \( z \).
\[
\int_0^z \left( \frac{d}{dz} J_\nu(z) \right) \frac{dz}{z} = \frac{1}{\nu!} \left( \frac{z}{2} \right) \left[ \frac{1}{\nu} J_\nu(\xi, 0) + \frac{z}{\nu + 1} \frac{d}{d\xi} J_\nu(\xi, 0) + o(z) \right] \\
- \frac{1}{(\nu - 1)!} \left( \frac{z}{2} \right) \left[ J_\nu(\xi) + z \frac{\nu - 1}{\nu} \eta_\nu(\xi) + o(z) \right].
\]

as \( z \to 0^+ \), with \( J_\nu(\xi) \), \( \eta_\nu(\xi) \) denoting the initial values of \( A_\nu(\xi, z) \) as reported in (7) and (8), respectively. In this way, after combining (A14) and (A15) with (A11) and (A12), the following expansion for \( z \to 0^+ \) is derived

\[
U_\nu(\xi, z) J_\nu(\xi, z) + V_\nu(\xi, z) Y_\nu(\xi, z) = \chi_\nu(\xi) + 2\eta_\nu(\xi) + o(z),
\]

which in turn yields

\[
\lim_{z \to 0^+} \rho_\nu(\xi, z) = \chi_\nu(\xi),
\]

\[
\lim_{z \to 0^+} \frac{d}{dz} \rho_\nu(\xi, z) = \eta_\nu(\xi),
\]

this proving that (A3) indeed provides an alternative representation of the incomplete Anger-Weber functions defined in (1).

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