A Godunov-type Scheme for the Drift Flux Model with Variable Cross Section


Abstract

This paper presents a modification of a classical Godunov-type scheme for the numerical simulation of a two-phase flow in a pipe with a piecewise constant cross-sectional area. This type of flow can occur in wellbores during drilling for oil and gas as well as after well completion. Contrary to classical finite-volume schemes, the numerical scheme proposed in this paper captures the steady-state solution of the system without generating non-physical discontinuities in the numerical solution close to the locations of discontinuities in the cross-section. Moreover, the proposed scheme can be extended to problems with piecewise continuous cross-sectional area. This extension is achieved by discretization of the area along the spatial domain and converting the piecewise continuous area into a piecewise constant area. The proposed scheme reduces to the classical scheme when the cross-sectional area is constant along the spatial domain. For the purpose of computational efficiency, the modification to the classical scheme is only applied at the locations of area variation and the numerical solver reduces to the classical scheme where the cross-sectional area is constant. It is also shown that the proposed scheme can be effectively used to simulate two-phase flows arising from the perturbation of the steady-state solution. The effectiveness of the proposed scheme is shown through illustrative numerical simulations. Finally, it should be noted that the proposed scheme retains the same order of accuracy as the underlying classical scheme.

Keywords: Drift Flux Model, Finite-volume scheme, Variable cross section, Well-balanced scheme, Two-phase flow, Non-conservative PDE.

1. Introduction

Reliable models and accurate numerical solutions for single- and two-phase flows are necessary for many industrial applications, such as drilling for oil and gas and flow in fuel bundles and pipelines [1][2][3]. Modeling of the transient behavior of the flow dynamics in these industrial systems plays a crucial role in the design, decision making and control of such systems. For the simulation of two-phase flows, the one-dimensional Drift Flux Model (DFM), which is constituted by a set of first-order nonlinear hyperbolic partial differential equations, has gained attention [4][5] due to its balance between predictive capabilities and simplicity. Compared to the two-fluid model, the DFM is favorable from a numerical simulation perspective [6]. In addition, the DFM remains hyperbolic over a wider region of the variables and it is also more accurate than the two-fluid model in homogeneous two-phase regimes [6].

In many industrial applications, the computational domain, typically a pipe, frequently has a variable cross-sectional area along its length. In particular, a drilling well experiences discontinuities in the cross-sectional area [7] as schematically illustrated in Figure 1. These area discontinuities affect the resonance frequency of the wave propagation effects...
inside the system, especially the rapid pressure dynamics. If this phenomenon is not considered, the model may lose its predictive capacity. Moreover, in the scope of controller design for Managed Pressure Drilling (MPD), the system performance may seriously deteriorate when such effects are not appropriately represented in the model. Hence, a model that accounts for such phenomena is required.

Two-phase flow in a pipe with variable cross-sectional area increases the complexity of the governing model and, subsequently, its numerical solution as a non-conservative term is added to the governing equations [8]. This means that after adding this term, all derivatives over the spatial variable cannot be gathered into a single differential term.

Classical finite-volume schemes are suitable for numerical simulation of conservative hyperbolic PDEs, such as the DFM with a constant cross-sectional area [4, 9, 10]. However, these classical methods cannot be effectively used to solve non-conservative PDE models, such as the DFM with variable cross-sectional area [11]. A common approach to incorporate the area variation is to treat the non-conservative term as a source term [12]. This treatment leads to non-physical and numerical spikes in the numerical solution and, subsequently, this approach cannot be reliably used [11].

Addressing the issue of the presence of non-conservative terms in mathematical models of various systems in the scope of numerical implementation is an active research area. Different methods have been developed for the simulation of the behavior of a single-phase flow in a pipe with a variable cross-sectional area. Instead of treating the non-conservative term as an additional source term, a modification to the Rusanov scheme [13] has been proposed in [14] to capture the steady-state solution of the Euler equations. However, this method is not well-balanced in the presence of non-zero flow in the system, i.e., the numerical solution does not preserve the steady-state solution for non-zero flow scenarios. A model-based modification of the input arguments of the finite-volume scheme has also been introduced in [8, 11, 15]. All the mentioned works deal with the variable cross-sectional area in single-phase flow systems while two-phase flows frequently occur in many realistic industrial applications [1, 2, 3, 16].

To the best of the authors’ knowledge, the effect of non-conservative terms in two-phase flow models has been studied only to a rather limited extent for conservative shock capturing schemes. As an example, in [17], the non-conservativeness in the two-fluid model originating from the state variables is considered. However, the non-conservativeness originating from the variable cross-sectional area is not discussed. Therefore, this paper focuses on
developing a reliable numerical approach for the DFM capturing the effects of variations in the cross-sectional area by introducing a model-based scheme, inspired by [11].

The results of this paper can be used to simulate the flow of gas and liquid mixtures in pipelines. In particular, this kind of flow is common in the upstream, midstream and downstream sector of the oil and gas industry. As in the upstream sector, for any drilling well, the understanding of the flow (and pressure) dynamics in the drill-string and annulus is essential. Herein, the drill-string consists of a series of drill pipes and the bottom hole assembly (assembly of heavy weight drill pipes and mud motors). Together these system components represent a drastically varying flow path, both inside the drill-string and the annulus (refer to Figure 1). In drilling operations for oil and gas, multiphase fluid flow arises in several cases, such as gas influx into the annulus or during under-balanced drilling where the gas is also present in the drill-string in addition to the annulus. Moreover, the proposed method can be used to validate hydraulics models in a drilling well with the DFM, as an extension to [16]. In many studies for the DFM validation in a drilling well, such as e.g. the one in [16], the effect of area discontinuity has been ignored. However, dealing with the field data, the effects of these discontinuities should be taken into account and the developments in this paper support this. The work of this paper can then be used to support the operational design of MPD-based operations and controller design for MPD. In the midstream operations, multi-zone completion designs, using a liner or an open-hole in combination with tubings, pose a multi-phase flow scenario in a pipe with potentially variable cross-sectional area during the production phase. In the downstream sector, the refining and separation of the natural gas and crude oil can be simulated using the results of this paper.

Contributions of this paper are provided in two areas. First, this paper reviews the current techniques to deal with non-conservative terms within single-phase flow. These techniques, which are not applicable for the DFM, are adapted for the DFM. Secondly, new approaches within the context of the DFM are introduced to deal with the non-conservative term induced by the variable cross-sectional area. Since the goal of this paper is to evaluate the merit of the scheme in capturing the effects of area variation, the effect of source terms such as friction and gravity has not been considered. The evaluation of the scheme in the absence of the source term is a common practice in the finite-volume community [18, 19, 11]. In addition, considering source terms raises the issue of the well-balancedness [20], which is beyond the scope of this paper. Incorporating these additional source terms is the subject of future works.

The structure of this paper is as follows. In Section 2, an overview of the model is given together with a comparison between the conservative and non-conservative DFM and the corresponding eigenvalue analysis is performed. Various methodologies to deal with the non-conservative term are proposed in Section 3. In Section 4, the effectiveness of the different variable cross-section treatments in preserving the steady-state solution is evaluated through numerical tests and, subsequently, transient simulations are performed. Finally, conclusions and recommendations for future works are presented in Section 5.

2. Two-Phase Flow Model

In this section, the DFM in case of constant and variable cross-sectional area is presented. Eigenvalue analysis is performed for the non-conservative system with variable cross-sectional area. Since the dynamics originating from the perturbation to the steady states are of numerical interest in this study, the steady-state solution is also presented in this section. Next, the necessary boundary conditions for both the transient and the steady-state model are defined.

2.1. Introduction to the Drift Flux Model

The Drift Flux Model is widely used to describe the behavior of two-phase flow systems [5, 21, 22, 23]. It consists of two mass balance equations, one for each phase, and one combined momentum balance equation for the mixture of the phases. The governing equations for one-dimensional systems are given by

\[
\frac{\partial (\alpha \rho l)}{\partial t} + \frac{\partial (\alpha \rho l u_l)}{\partial x} = 0, \\
\frac{\partial (\alpha \rho g)}{\partial t} + \frac{\partial (\alpha \rho g u_g)}{\partial x} = 0, \\
\frac{\partial (\alpha \rho l u_l + \alpha \rho g u_g)}{\partial t} + \frac{\partial (\alpha \rho l u_l^2 + \alpha \rho g u_g^2 + p)}{\partial x} = 0,
\]

In this equation, \(\alpha\) is the volume fraction of the phase, \(\rho\) is the density, \(u\) is the velocity, and \(p\) is the pressure. The subscripts \(l\) and \(g\) refer to the liquid and gas phases, respectively.
where \( \alpha(t, x), \rho(t, x), u(t, x) \) and \( p(t, x) \) are, respectively, the volume fraction, density, velocity and pressure, which are functions of time \( t \) and the one-dimensional spatial coordinate \( x \). The subscripts \( l \) and \( g \) denote the liquid and gas, respectively. Noteworthy, the DFM is based on the assumption of mechanical equilibrium between the two phases, i.e., the pressure of the gas and the pressure of the liquid are equal. The DFM as in (1) contains seven variables while it is expressed in only three equations. Thus, four other equations, called closure relationships, are required to, potentially, uniquely solve the system of equations. The most widely used closure relationships are listed below \([24, 5]\):

\[
\begin{align*}
\alpha_l + \alpha_g - 1 &= 0, \quad \text{(2)} \\
u_g &= (Ku_{mix} + S) = 0, \quad \text{(3)} \\
p - \rho_g c^2_s &= 0, \quad \text{(4)} \\
p - \left((\rho_l - \rho_0)c^2_l + p_0\right) &= 0. \quad \text{(5)}
\end{align*}
\]

Equation (2) implies that every section of the pipe is filled up with a mixture of the liquid and gas. The slip law \([5]\), showing a static relation between the velocity of the gas and the liquid, compensates for the fact that only one momentum balance is included in the DFM. Here, \( K \) and \( S \) are, respectively, the distribution coefficient and the drift coefficient and the drift velocity of the gas relative to the liquid defined according to the flow regime \([25, 26]\) and \( u_{mix} = \alpha_l u_l + \alpha_g u_g \) is the velocity of the mixture. Equations (4) and (5) represent the equation of state (EOS) for the gas and liquid phases, respectively. \( c_l \) and \( c_s \) are the speeds of sound in the liquid and gas phases, which can be assumed to be constants or be functions of other variables, such as pressure. Finally, \( \rho_0 \) and \( p_0 \) are, respectively, the reference values for density and pressure around which the EOS for the liquid has been linearized.

Model (1) is based on the assumption that the cross-section is constant all along the spatial domain. When this does not hold, the DFM in (1) should be modified in order to take into account the cross-sectional variations. By including a variable cross-sectional area along the computational domain, \( A = A(x) \), as used in \([27]\), the system (1) changes to

\[
\begin{align*}
\frac{\partial(\alpha_l \rho_l A)}{\partial t} + \frac{\partial(\alpha_g \rho_g A)}{\partial x} &= 0, \quad \text{(6a)} \\
\frac{\partial(\alpha_l \rho_l A)}{\partial t} + \frac{\partial(\alpha_g \rho_g A)}{\partial x} &= 0, \quad \text{(6b)} \\
\frac{\partial}{\partial t} \left(\alpha_l \rho_l u_l + \alpha_g \rho_g u_g A\right) + \frac{\partial}{\partial x} \left(\alpha_l \rho_l u_l^2 + \alpha_g \rho_g u_g^2 + pA\right) &= \rho A \frac{\partial A}{\partial x}, \quad \text{(6c)} \\
\frac{\partial A}{\partial t} &= 0. \quad \text{(6d)}
\end{align*}
\]

Equation (6d) is trivial and it is only added to enable the eigenvalue analysis presented later. Notably, system (1) can be written in the conservative form while system (6) cannot be written in such a form due to the presence of the term \( p\partial A/\partial x \) in (6c), all terms concerning spatial derivatives of the state variables cannot be gathered in one single term and therefore the system (6) is non-conservative). Therefore, the area variation included in (6) significantly affects the solution, in particular the wave reflection pattern inside the domain of a pipe with a discontinuous area.

For this type of cross-sectional characteristics, the term \( p\partial A/\partial x \) becomes an impulsive force per unit length acting on a infinitesimally small spatial interval. This means that when a fluid particle passes a location with discontinuity in the area, it experiences an excessively large force (an impulsive force) for an infinitesimally short period of time that leads to a finite change in the momentum of the particle. The effect of such impulsive forces can not be captured appropriately by the classical finite-volume schemes. Therefore, other methods are needed to solve this system of equations.

In the following section, an eigenvalue analysis of the system (6) is carried out to explain the effect of a variable cross-sectional area on the solution.
2.2. Eigenvalue Analysis of the non-conservative DFM

To perform the eigenvalue analysis, \( Q := [\rho, u, p, A]^T \) is defined as the state variable vector and by inserting the closure laws into the PDEs (6), the system in (6) can be rewritten in a compact form as follows:

\[
J_1(\rho) \frac{\partial Q}{\partial t} + J_2(Q) \frac{\partial Q}{\partial x} = 0,
\]

where the concise representation of \( J_1 \) and \( J_2 \) (by partially inserting the closure laws) are as below

\[
J_1(Q) = \begin{bmatrix}
-\rho u_t & 0 & \frac{\alpha_f}{c_f^2} & 0 \\
\rho c_f & 0 & \frac{\alpha_g}{c_g^2} & 0 \\
\rho u_t - \rho u_t + \alpha_g \rho_s & \frac{K((K-1)u_t + S)}{1 - K \alpha_g} & \frac{\alpha_g}{c_g^2} & \frac{K \alpha_l}{(1 - K \alpha_g)} \\
0 & 0 & \frac{\alpha_g}{c_g^2} & 0 & 0 \\
\end{bmatrix},
\]

\[
J_2(Q) = \begin{bmatrix}
\rho u_t & \alpha_g \rho_s & \frac{\alpha_g \rho_s}{(1 - K \alpha_g)} & \frac{\alpha_g \rho_s}{A} \\
\rho - \rho u_t + \alpha_g \rho_s & \frac{\alpha_g \rho_l}{(1 - K \alpha_g)} & \frac{\alpha_g \rho_l}{A} & \frac{\alpha_g \rho_l}{c_g^2} \\
\alpha_g \rho_s & \frac{\alpha_g \rho_s}{c_g^2} & \frac{\alpha_g \rho_s}{c_g^2} & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

where

\[
a_{31} = \rho u_t u_s^2 - \rho u_t^2 + \alpha_g \rho_s \alpha_g \rho_s, \quad a_{32} = 2 \alpha_g \rho_u u_s + 2 \alpha_g \rho_s \alpha_g \rho_s \frac{K \alpha_l}{1 - K \alpha_g}.
\]

It should be noted that although the area is a known variable, it is considered to be one of the states of the system to facilitate the analyses and to enable writing system (6) in the format of (7). This is the main reason for keeping the last trivial equation in system (6). For the DFM, the eigenvalues of the matrix \( J^{-1}_1 J_2 \) have the following form [5]:

\[
\lambda_1 = u_t + \omega, \quad \lambda_2 = u_t, \quad \lambda_3 = u_t - \omega, \quad \lambda_4 = 0,
\]

where \( \omega \) is the speed of sound in the mixture of the gas and the liquid [5]. In case of no-slip, i.e., \( K = 1, S = 0 \), after computing the eigenvalues and casting the results into the format of (7), the speed of sound in the mixture can be analytically written as follows known as Wood or Wallis speed of sound [28-30]:

\[
\omega = c_c \sqrt{\frac{\beta \rho \rho_s}{\rho \rho_s (\alpha_g \rho_s c_f^2 + \alpha_g \rho_s c_g^2)}},
\]

with \( \beta = \alpha_g \rho_l + \alpha_g \rho_s \).

In the case when slip occurs between the two phases, computing the analytical speed of sound in the mixture is complex, if not impossible. Thus, simplified surrogates for the speed of sound have been suggested; for instance the surrogate in [5] is introduced for cases with \( \alpha_g \rho_s \ll \alpha_g \rho_l \) and \( 0 < \alpha_g < 1 \) as below:

\[
\omega \approx \sqrt{\frac{\rho}{\alpha_g \rho_s (1 - K \alpha_g)}},
\]

Notably, although the gas and the liquid phase play a symmetric role in [6], the contribution of the phases are non-symmetric in the eigenvalues [9] due to the closure laws [2]-[5]. For a detailed analysis of the eigenvalue problem of the conservative DFM, the reader may refer to [31]. The eigenvalue \( \lambda_4 \) shows that there is a stationary wave in the computational domain that becomes visible when the cross-sectional area is discontinuous and \( \partial A / \partial x \) becomes closer to the impulse function.
Remark 1. Equation (11) becomes ill-posed when \( \alpha_g \) tends to zero or \( K \alpha_g \to 1 \). When \( K > 1 \), the singular point even occurs for \( \alpha_g < 1 \). In these cases, other surrogate formulations should be used [23], which has not been studied in this paper.

Remark 2. In some special cases, the speed of sound \( \omega \) becomes very low, even less than \( c_g \), in the presence of both phases. However, in the special application of drilling for oil and gas, phase velocities are still lower than \( \omega \) and the flow is subsonic. Henceforth, we only consider subsonic flows.

In general, dynamics of the problems studied in this paper are the perturbation dynamics with respect to the steady-state solution. Thus, finding the steady-state solution is the first step towards solving this type of problems.

2.3. Steady-state solution

For obtaining the analytical steady-state solution of system (6), the partial derivatives of the variables with respect to time is set to zero and the following system of equations should be solved:

\[
\begin{align*}
\frac{\partial (\alpha_g \rho u_A)}{\partial x} &= 0, \quad (12a) \\
\frac{\partial (\alpha_s \rho u_A)}{\partial x} &= 0, \quad (12b) \\
\frac{\partial ((\alpha_g \rho u_g^2 + \alpha_s \rho u_g^2 + p)A)}{\partial x} &= \frac{\partial A}{\partial x}. \quad (12c)
\end{align*}
\]

By embedding (12a) and (12b) into (12c), simplified governing equations are obtained as in (13)

\[
\begin{align*}
\frac{\partial (\alpha_g \rho u_A)}{\partial x} &= 0, \quad (13a) \\
\frac{\partial (\alpha_s \rho u_A)}{\partial x} &= 0, \quad (13b) \\
\alpha_g \rho u_A \frac{\partial u_g}{\partial x} + \alpha_s \rho u_g \frac{\partial u_g}{\partial x} + A \frac{\partial p}{\partial x} &= 0. \quad (13c)
\end{align*}
\]

After inserting the closure laws and considering the slip law (3), the steady-state solution of the new state variable \( W := [u_l \ u_g \ \alpha_g \ p]^T \) is governed by the system of equations:

\[
M(W,x) \frac{\partial W}{\partial x} = E(W,x),
\]

where the concise representation of \( M \) and \( E \) (by partially inserting the closure laws) are as below

\[
M(W,x) = \begin{bmatrix}
-K(1-\alpha_g) & 1 - K \alpha_g & K(u_l - u_g) & 0 \\
(1-\alpha_g)\frac{p - p_0}{c_l^2} + \rho_0 A & 0 & -\frac{p - p_0}{c_l^2} + \rho_0 u_l A & 0 \\
0 & \alpha_s \frac{c_t}{c_g} p & \frac{p}{c_g} u_g A & \frac{u_g A}{c_g} \\
(1 - \alpha_g)\frac{p - p_0}{c_l^2} + \rho_0 u_l A & \alpha_s \frac{c_t}{c_g} u_g A & 0 & A
\end{bmatrix},
\]

\[
E(W,x) = \begin{bmatrix}
0 \\
-\frac{\alpha_s}{c_g} u_g \frac{\partial A}{\partial x} \\
-\alpha_g \frac{p}{c_g} u_g \frac{\partial A}{\partial x} \\
-(1-\alpha_g)\frac{p - p_0}{c_l^2} + \rho_0 u_l^2 \frac{\partial A}{\partial x}
\end{bmatrix}.
\]
Notably, the term $\partial A_1/\partial x$ represents an impulsive term at the discontinuities of $A(x)$, which leads to discontinuities in the steady-state solution of $W$. To solve this system for discontinuous function $A(x)$, left- or right-continuity of the function should be specified to be able to define the area at any location.

The initial condition for the simulations in the presence of area variation is the solution of (14)-(15) unless otherwise mentioned. If a system starts from its unique steady-state solution, the numerical solution should remain on the same solution afterwards. Therefore, a significant discrepancy between the numerical simulation and the steady-state initial condition reveals the poor performance of the scheme, which may be hard to diagnose in dynamical simulations. Thus, this test is a powerful measure for assessing the necessary performance of a scheme, i.e., predicting the correct steady-state solution.

Equations (14)-(15) represent a two-point boundary value problem due to the boundary conditions specified at both ends, which are detailed in the next section.

2.4. Boundary conditions

As three PDEs are involved in the system of (14), three physical boundary conditions have to be specified. For subsonic flow, it is typical to set a specific mass flow rate of the liquid and the gas at the left boundary (at $x = 0$) and a pressure at the right boundary (at $x = L$) \[5, 32\]. Henceforth, the mass flow rate of the liquid and gas are, respectively, denoted by $m_l$ and $m_g$ (i.e., $m_l := \alpha_l \rho_l u_l A$ and $m_g := \alpha_g \rho_g u_g A$), and the pressure at the right boundary is denoted by $p_R$.

However, for finding the numerical solution of system (9), all conservative variables should be prescribed at the boundaries. Since the number of conservative variables at each boundary is more than the number of physical boundary conditions, additional conditions at the boundary are required to find the unique solution for the boundary variables. For instance, at the right boundary, only pressure is prescribed and other variables should be obtained by some compatibility equations. By following the approach described in \[32\], characteristic boundary conditions are combined with the physical boundary conditions in order to fulfill all the necessary conditions at the boundaries. The characteristic boundary equations can be found in \[32\]. Under the assumption of constant area only at the boundaries, the characteristic boundary equation corresponding to the pressure wave propagating in the downstream direction $\lambda_1 = u_l + \omega$ reads as:

$$\frac{d}{dt} p + \rho_l \omega (u_g - u_l) \frac{d}{dt} u_g - \rho_l \alpha_l (u_g - u_l - \omega) \frac{d}{dt} u_l = 0, \quad \text{with} \quad \frac{d}{dt} = \frac{\partial}{\partial t} + (u_l + \omega) \frac{\partial}{\partial x}. \quad (16)$$

Similarly, for the gas volume wave $\lambda_2 = u_g$, we have:

$$\frac{d}{dt} p + \frac{p}{\alpha_g (1 - Ka_g)} \frac{d}{dt} \alpha_g = 0, \quad \text{with} \quad \frac{d}{dt} = \frac{\partial}{\partial t} + u_l \frac{\partial}{\partial x}. \quad (17)$$

Finally, for the pressure wave propagating in the upstream direction $\lambda_3 = u_l - \omega$, we have:

$$\frac{d}{dt} p - \rho_l \omega (u_g - u_l) \frac{d}{dt} u_g - \rho_l \alpha_l (u_g - u_l + \omega) \frac{d}{dt} u_l = 0, \quad \text{with} \quad \frac{d}{dt} = \frac{\partial}{\partial t} + (u_l - \omega) \frac{\partial}{\partial x}. \quad (18)$$

The discrete version of equations (16) and (17) are solved at the right boundary and discrete version of equation (18) is solved at the left boundary.

Remark 3. Before going through the numerical solvers, it should be noted that, in this paper, the function $A(x)$ is piecewise continuous and it is discretized over the spatial domain. After such discretization, $A(x)$ becomes piecewise constant as it is constant within each grid cell and the discontinuities occur only at the interfaces. Then, in the case of discontinuous area, wherever the computation of $\partial A / \partial x$ is required, for instance in the steady-state calculations, the spatial derivative of $A(x)$ is approximated by the finite difference method.

Remark 4. Some existing methods to deal with the variable cross-section are based on adapting the Rusanov scheme \[18\]. For the sake of a fair comparison, we also consider the Rusanov scheme as our numerical scheme. However, the method introduced in this paper is a universal modification that can be used along with any numerical scheme such as the AUSMV scheme \[5\].
3. Numerical solvers for the DFM with piecewise continuous cross-section

In this section, different approaches are presented to deal with the non-conservative term in the DFM; some of which are proposed in this paper and some are extensions of existing methods for \([6]\). The effects of variable cross-sectional area in the DFM and Euler equations are similar; a stationary wave is added to the existing waves in both cases. Therefore, the strategies introduced in different works such as \([4, 8, 11, 12]\) to deal with non-conservative terms in the Euler equations with area variation are the main source of inspiration for this work.

For the sake of completeness, some terminologies widely used in the context of finite-volume method are introduced beforehand. For a general first-order PDE of the form
\[
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = S(u, t, x), \quad t \in I = [0, T], \quad x \in \Omega = [0, L],
\] (19)
u is called the conservative variable, \(f\) is called the mathematical flux function and \(S\) is the source term. Furthermore, primitive variables with physical interpretation are defined, denoted by \(v\). For instance, for the system (1), the conservative variables are \(u = [\alpha_l \rho_l A, \alpha_g \rho_g A, (\alpha_l \rho_l u_l + \alpha_g \rho_g u_g) A]^T\) and primitive variables are any combination of three independent variables, such as \(v = [\alpha_l \rho_l u_l, p A]^T\). Finite-volume discretization is commonly employed to solve such PDEs by a discretization of the spatial computational domain \(\Omega\) and the temporal computational domain \(I\), as shown in Figure 2. Assume that we are interested in the solution at the \(i\)-th spatial grid cell at the time step \(n+1\) (the green block in Figure 2). First-order Godunov-type schemes numerically solve (19) by
\[
U_{i+1}^n = U_i^n - \Delta t \Delta x \left( F(U_i^n, U_{i+1}^n) - F(U_{i-1}^n, U_i^n) \right) + \Delta t S(U_i^n, t^n, x_i),
\] (20)
where \(U_i^n\) is the spatial average of the conservative variable \(u\) over \(i\)-th cell at the time instant \(t^n = n\Delta t\), schematically shown at different grid cells by red lines in Figure 2. Similarly, \(V_i^n\) is the spatial average of the primitive variables \(v\), which will be used later. Also, \(\Delta t\) and \(\Delta x\) refer to temporal and spatial discretization step sizes, respectively.

The numerical flux function \(F\) is a scheme-dependent function of the conservative variables. The classical Rusanov scheme \([13]\) for the system (6) employs a flux function as below:
\[
F(U_i^n, U_{i+1}^n) = \frac{f(U_{i+1}^n) + f(U_i^n)}{2} - A_{i+1/2}^n (U_{i+1}^n - U_i^n),
\] (21)
with
\[
f = [\alpha_l \rho_l u_l A, \alpha_g \rho_g u_g A, (\alpha_l \rho_l u_l^2 + \alpha_g \rho_g u_g^2 + p) A]^T,
\] (22a)
\[
U = [\alpha_l \rho_l A, \alpha_g \rho_g A, (\alpha_l \rho_l u_l + \alpha_g \rho_g u_g) A]^T,
\] (22b)
In other words, the third entry of the following vector which appears in (24), inside the domain, we obtain with a higher accuracy. In the second approach, when evaluated at the steady-state solution at the presence of flow scheme in (25). The new modification, proposed here, is motivated by the need to capture the steady-state solution case of zero flow.

3.3. Third approach: novel, modified Rusanov scheme

The third method is inspired by the idea behind the second approach, by applying further modifications to the scheme in (25). The new modification, proposed here, is motivated by the need to capture the steady-state solution with a higher accuracy. In the second approach, when evaluated at the steady-state solution at the presence of flow inside the domain, we obtain \( U_i^{n+1} \neq U_i^n \), and therefore, the numerical solution deviates from the actual steady-state
solution. This approach is motivated by enforcing the last entry of \( U_i^n \), which is \( (\alpha \rho_i u_i + \alpha_3 \rho_i u_3) A_i^n \), to be well-balanced. By using the fact that the mass flow rates of both phases are constant at every location during a steady-state solution, the new modification of the scheme is introduced by the flux function as given below:

\[
F(U_i^n, U_{i+1/2}^n) = \frac{f(U_i^{n+1}) + f(U_i^n)}{2} - A_{i+1/2} \frac{\begin{vmatrix} (\alpha \rho_i u_i - (\alpha \rho_i)_{i+1}^n) \\ (\alpha_3 \rho_i u_3 - (\alpha_3 \rho_i)_{i+1}^n) \\ \end{vmatrix}}{A_{i+1/2}} + \frac{\begin{vmatrix} (\alpha \rho_i u_i + \alpha_3 \rho_i u_3) A_i^n \\ (\alpha \rho_i u_i + \alpha_3 \rho_i u_3) A_{i+1}^n \\ \end{vmatrix}}{A_{i+1/2}},
\]

with the same \( f \) and \( U \) mentioned in the previous section. In this case, while starting from the steady-state solution, the third entry of the flux function \( F \) computed by (27), i.e., \( (\alpha \rho_i u_i + \alpha_3 \rho_i u_3) A_i^n - (\alpha \rho_i u_i + \alpha_3 \rho_i u_3) A_{i+1}^n \), becomes zero; therefore, at least the deviation of the third entry of conservative variable at the first time-step is zero and this modification outperforms the second approach. Similar to the second approach, this scheme reverts to the original Rusanov scheme when there is no change in the cross-sectional area.

3.4. Fourth approach: model-based modification

The next novel method, proposed here, is obtained through modifying the conservative variables before calculating the flux functions that allows one to treat the non-conservative term in an indirect way, regardless of the type of the Godunov scheme. The underlying idea is inspired by [8, 11] and consists of defining the solution as follows:

\[
U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x} (F(U_i^n, U_{i+1/2}^n) - F(U_{i-1/2}^n, U_i^n)),
\]

where \( U_{i+1}^* \) and \( U_{i-1}^* \) are the modified conservative variables, which are henceforth called starred conservative variables. All conservative variables \( U \) and the mathematical flux function \( f \) (which will be used in the computation of \( F \)) in this approach are defined according to (1), not (6), as the effect of area is included in the starred variables, meaning that

\[
U = [\alpha \rho_i, \alpha_3 \rho_i, \alpha \rho_i u_i + \alpha_3 \rho_i u_3]^T,
\]

\[
f = [\alpha \rho_i u_i, \alpha_3 \rho_i u_3, \alpha \rho_i u_i^2 + \alpha_3 \rho_i u_3^2 + p]^T.
\]

First of all, an explanation to this approach is provided, and then, the computational steps for computing \( U_{i+1}^* \) and \( U_{i-1}^* \) at the \( i \)-th spatial grid cell is established to obtain the numerical solution by (28).

As mentioned in the eigenvalue analysis in Section 2.2, a stationary time-independent wave (corresponding to \( \lambda_4 \) in (2)) lies in the system that should be captured by (28). To this end, the method proposed in this section is composed of two steps. In the first step, the effect of the stationary wave generated by the non-conservative source term is captured. As the stationary waves are time-independent, steady-state model is employed to reconstruct this type of waves. In the second step, the effect of the non-stationary waves is taken into account. The latter waves are governed by (1), as the effect of the stationary wave is already considered in the first step.

This approach is visualized in Figure 3 where the first step mentioned above is denoted by (1) and the second step is denoted by (2). Assume that \( U_{i+1}^{n+1} \) is required (the green block in Figure 3). This grid cell is surrounded by two interfaces at \( x_{i+1/2} \) and \( x_{i-1/2} \). The conservative variables at the right-hand and left-hand side of the interface \( i + 1/2 \) are shown by \( U_{i+1} \) and \( U_i \), respectively, in Figure 3. Similarly, the left- and right-hand side values for the interface \( i - 1/2 \) can be defined. In this approach, the conservative and primitive variables affecting the solution in the \( i \)-th cell, \( U_{i+1} \) and \( U_{i-1} \), are modified such that these variables contain the effect of the stationary waves at the inlet and outlet of the cell. The starred values are denoted by \( U_{i+1}^* \) and \( U_{i-1}^* \) in Figure 3 at level (1). Now, as the system has a constant area only over the \( i \)-th grid cell and its neighboring cells, classical finite-volume discretization can be applied on (1) at level (1) to obtain the solution at the \( i \)-th spatial grid cell. It should be noted that the area is assumed to be constant over only these three grid-cells while the area of other grid cells do not affect the solution of the \( i \)-th grid cell at the current time step \( n + 1 \). Therefore, the solution obtained by (28) contains both the effect of the stationary and non-stationary waves.

After providing the intuition how this method works, the framework for finding the starred values is established here. As the stationary waves are time-independent, the steady-state solution of the non-conservative system (6) is
Figure 3: Finite-volume modification for the fourth approach, model-based modification.

exploited to find algebraic constraints that capture the stationary waves. Recalling steady-state equations in (13), we have

\[ \frac{\partial (\alpha_l \rho_l u_l A)}{\partial x} = 0, \]
\[ \frac{\partial (\alpha_g \rho_g u_g A)}{\partial x} = 0, \]
\[ A (\alpha_l \rho_l u_l \frac{\partial u_l}{\partial x} + \frac{1}{\rho_l} \frac{\partial p}{\partial x}) + \frac{\partial p}{\partial x} = 0. \]

To find an algebraic relation over the area discontinuity, (13) should be integrated over the interface. However, (13c) is difficult to be integrated analytically over the spatial domain and needs more investigation. This differential equation can be simplified, as shown in the following Lemma 1.

**Lemma 1.** The equation (13c) is equivalent to the following equation if both phases are present in the system:

\[ \bar{m}_g (u_s \frac{\partial u_s}{\partial x} + \frac{1}{\rho^s} \frac{\partial p}{\partial x}) + \bar{m}_l (u_l \frac{\partial u_l}{\partial x} + \frac{1}{\rho^l} \frac{\partial p}{\partial x}) + (u_l - u_s) (\bar{m}_l \alpha_l \frac{\partial u_l}{\partial x} - \bar{m}_g \alpha_g \frac{\partial u_s}{\partial x}) = 0, \]
\[ \text{(29)} \]

where \( \bar{m}_l = A \alpha_l \rho_l u_l \) and \( \bar{m}_g = A \alpha_g \rho_g u_g \).

**Proof.** From (13c), we have:

\[ \bar{m}_g \frac{\partial u_s}{\partial x} + \bar{m}_l \frac{\partial u_l}{\partial x} + A \frac{\partial p}{\partial x} = 0. \]
As variables $\alpha_u u_l$ and $\alpha_g u_k$ are not constantly zero in general due to the presence of both phases, we can multiply the above equation by these two variables:

$$\alpha_u u_l (\tilde{m}_g \frac{\partial u_g}{\partial x} + \tilde{m}_l \frac{\partial u_l}{\partial x} + A \frac{\partial p}{\partial x}) = 0,$$

$$\alpha_g u_k (\tilde{m}_g \frac{\partial u_k}{\partial x} + \tilde{m}_l \frac{\partial u_l}{\partial x} + A \frac{\partial p}{\partial x}) = 0.$$ 

Summation of the above equations and using the identity $\alpha_g + \alpha_l = 1$ leads to:

$$\tilde{m}_g \alpha_u u_l \frac{\partial u_g}{\partial x} + \tilde{m}_l (1 - \alpha_g) u_l \frac{\partial u_l}{\partial x} + A \alpha_u u_l \frac{\partial p}{\partial x} + \tilde{m}_g (1 - \alpha_l) u_k \frac{\partial u_k}{\partial x} + \tilde{m}_l \alpha_g u_k \frac{\partial u_k}{\partial x} + A \alpha_g u_k \frac{\partial p}{\partial x} = 0.$$ 

Rewriting the above relation leads to the claimed equation (29).

The relation in (29) cannot be simplified further unless additional assumptions are made. In the following sections, we analyze the DFM in two categories.

3.4.1. DFM without slip

Assume that there is no slip between the two phases, i.e., $K = 1$ and $S = 0$ in (3) and subsequently

$$\hat{u} := u_l = u_k.$$ 

In this case, as $u_l - u_k = 0$ in (29), system (13) changes to:

$$\frac{\partial (\alpha_g \rho_l \hat{u})}{\partial x} = 0,$$

$$\frac{\partial (\alpha_g \rho_k \hat{u})}{\partial x} = 0,$$

$$A \alpha_g \rho_l \hat{u} \frac{\partial \hat{u}}{\partial x} + \frac{1}{\rho_l} \frac{\partial p}{\partial x} + A \alpha_g \rho_k \hat{u} \frac{\partial \hat{u}}{\partial x} + \frac{1}{\rho_k} \frac{\partial p}{\partial x} = 0.$$ 

As $\partial p/\partial x = c_l^2 \partial \rho_l/\partial x = c_k^2 \partial \rho_k/\partial x$, the above equation is integrable over $x$. Therefore, over the interface, the following set of functions are set to be constant:

$$\alpha_g \rho_l \hat{u} = \text{constant},$$ 

$$\alpha_g \rho_k \hat{u} = \text{constant},$$ 

$$A \alpha_g \rho_l \hat{u} \left( \frac{\hat{u}^2}{2} + c_l^2 \ln \rho_l \right) + A \alpha_g \rho_k \hat{u} \left( \frac{\hat{u}^2}{2} + c_k^2 \ln \rho_k \right) = \text{constant}.$$ 

Equations (31a)–(31b) represent the mass flow continuity at the interface. Equation (31c) governs the rate of energy exchange at the interface. In other words, the term $\hat{u}^2/2$ is the kinetic energy per unit mass of each phase. Also, $c_l^2 \ln \rho_l$ is the potential energy of the compressible liquid and gas per unit mass. Therefore, at the interface, the mass and energy continuity should be preserved.

For instance, for finding $U_{M_1}$ from $U_{M_2}$, we should solve:

$$\left( \alpha_g \rho_l \hat{u} \left( \frac{\hat{u}^2}{2} + c_l^2 \ln \rho_l \right) \right)_{M_1} = \left( \alpha_g \rho_l \hat{u} \left( \frac{\hat{u}^2}{2} + c_l^2 \ln \rho_l \right) \right)_{M_2},$$

$$\left( \alpha_g \rho_k \hat{u} \left( \frac{\hat{u}^2}{2} + c_k^2 \ln \rho_k \right) \right)_{M_1} = \left( \alpha_g \rho_k \hat{u} \left( \frac{\hat{u}^2}{2} + c_k^2 \ln \rho_k \right) \right)_{M_2},$$

$$\left( \alpha_g \rho_l \hat{u} \left( \frac{\hat{u}^2}{2} + c_l^2 \ln \rho_l \right) \right)_{M_1} = \left( \alpha_g \rho_k \hat{u} \left( \frac{\hat{u}^2}{2} + c_k^2 \ln \rho_k \right) \right)_{M_2},$$

where $M_1$ and $M_2$ refer to two neighboring cells, see Algorithm 4 for more details.

However, when the slip law is not discarded, finding an algebraic relation becomes hard as the third term $(u_l - u_k)(\tilde{m}_g \alpha_u u_l/(\partial x) - \tilde{m}_l \alpha_g u_k/(\partial x))$ in (29) is not negligible. In order to extend the applicability of the fourth approach for cases with slip, extra physical assumptions should be made, which is dealt with below.
3.4.2. DFM with slip

In this case, the analytical integration of the momentum equation (34c) is challenging. In [33], the authors claim that the DFM cannot be endowed with an entropy pair unless restrictive assumptions such as no-slip are made. This fact that the integration of the momentum equation is challenging might be related to the lack of an entropy inequality. Alternatively, here, we try to impose some physical assumptions to be able to integrate the momentum equation.

It is assumed that at the interfaces of area variation, the ratio of volume and mass composition of the mixture do not change (in other words, still mass continuity is maintained). This assumption is valid when one of the phases is dominant in volumetric sense in the pipe or the area varies smoothly. Otherwise, in situations where both phases occupy the space rather equally and the area variation is sudden, this assumption is less accurate. These two assumptions result in constant volumetric fraction and mass fraction of each phase at the interface, respectively. Meaning that at each interface, the following conditions hold:

\[
\frac{\alpha_l}{\alpha_g + \alpha_l} = \text{constant} \implies \alpha_{l|g} = \text{constant}, \quad (33a)
\]

\[
\frac{\alpha_l \rho_l}{\alpha_l \rho_l + \alpha_g \rho_g} = \text{constant} \implies \frac{\rho_l}{\rho_g} = \text{constant}. \quad (33b)
\]

Then (33c) is rewritten as below:

\[
\mathcal{A} \rho_g (\alpha_l \frac{\rho_l}{\rho_g} \frac{\partial u_l}{\partial x} + \alpha_l u_g \frac{\partial u_l}{\partial x} + \frac{1}{\rho_g} \frac{\partial p}{\partial x}) = 0 \implies \alpha_l \frac{\rho_l}{\rho_g} \frac{u_l^2}{2} + \alpha_g \frac{u_g^2}{2} + c^2 \ln \rho_g = \text{constant}. \quad (34c)
\]

Finally, the set of algebraic constraints under the set of assumptions mentioned in (33) for \(\rho_g > 0\) is defined as:

\[
\alpha_l \rho_l u_l A = \text{constant}, \quad (34a)
\]

\[
\alpha_g \rho_g u_g A = \text{constant}, \quad (34b)
\]

\[
\frac{\alpha_l \rho_l}{\rho_g} \frac{u_l^2}{2} + \alpha_g \frac{u_g^2}{2} + c^2 \ln \rho_g = \text{constant}. \quad (34c)
\]

Still the mass continuity exactly holds. The kinetic energy of both phases and the potential energy of the gaseous phase can still be detected in (34c). This is due to the assumption that we consider that one of the phases is dominant in space. Therefore, the potential energy due to the expansion of the dominant phase, here gas, is only reflected in the algebraic relation. Moreover, due to the assumption of the prevalence of one phase, the area has been disappeared from (34c). We emphasize again if both phases are rather equally present in the pipe and the area variation is sudden, the relation (34c) is not accurate.

Similar to the previous set of assumptions, for finding \(U^*_M\) from \(U_{M,i}\), the following set of algebraic equations should be solved:

\[
(\alpha_l \rho_l u_l)_{M,i} A_{M,b} = (\alpha_l \rho_l u_l A)_{M,i}, \quad (35a)
\]

\[
(\alpha_g \rho_g u_g A)_{M,b} = (\alpha_g \rho_g u_g A)_{M,i}, \quad (35b)
\]

\[
(\alpha_l \frac{\rho_l}{\rho_g} \frac{u_l^2}{2} + \alpha_g \frac{u_g^2}{2} + c^2 \ln \rho_g)_{M,i} = (\alpha_l \frac{\rho_l}{\rho_g} \frac{u_l^2}{2} + \alpha_g \frac{u_g^2}{2} + c^2 \ln \rho_g)_{M,b}, \quad (35c)
\]

where \(M_1\) and \(M_2\) refer to the neighboring cells. See Algorithm 1 for more details.

All steps involved in the fourth approach are summarized in Algorithm 1. Now, a justification on the performance of this approach is presented in the following claims.

**Claim 1.** For any given \(U^n_i\) in \(x_i \in [0, L]\) that satisfies (13), then \(U^n_1 = U^0_1 \forall n \in \mathbb{N}\), if no-slip condition (30) is assumed to obtain the solution using Algorithm 1.
Algorithm 1: Fourth approach: model-based modification

**Input:** \( U^n_{i-1}, U^n_i, V^n_{i-1}, A_i, A_{i+1} \)

**Output:** \( U^{n+1}_{i+1} \) = \((\alpha \rho_1 \hat{u}_1 + \alpha \rho_2 \hat{u}_2, V^n_{i+1}) \)

1. Compute primitive variables, \( V^n_{i-1}, V^n_i, V^n_{i+1} \), from conservative variables, \( U^n_{i-1}, U^n_i, U^n_{i+1} \).

2. If \( A_{i+1} \neq A_i \), then
   3. Solve (32) or (35), based on the assumption taken in Section 3.4, with \( M_1 = i + 1 \) and \( M_2 = i \), obtain \( V^n_{i+1} \) and then \( U^n_{i+1} \).
4. Else
5. \( U^n_{i+1} = U^n_i \)
6. End if
7. If \( A_i \neq A_{i-1} \), then
   8. Solve (32) or (35), based on the assumption taken in Section 3.4, with \( M_1 = i - 1 \) and \( M_2 = i \), and obtain \( V^n_{i-1} \) and then \( U^n_{i-1} \).
9. Else
10. \( U^n_{i-1} = U^n_i \)
11. End if
12. Compute \( U^n_{i+1} \) via (28).

**Proof.** Starting from a steady-state solution \( U^n_i \), according to (13) and using (31), we have:

\[
\begin{align*}
(\alpha \rho \hat{u} A)_{M_1} &= (\alpha \rho \hat{u} A)_{M_2}, \\
(\alpha \rho \hat{u} A)_{M_1} &= (\alpha \rho \hat{u} A)_{M_2}, \\
(\alpha \rho \hat{u} A(\frac{\hat{u}^2}{2} + c_2^* \ln \rho_1) + \alpha \rho \hat{u} A(\frac{\hat{u}^2}{2} + c_2^* \ln \rho_2))_{M_2}^* &= (\alpha \rho \hat{u} A(\frac{\hat{u}^2}{2} + c_2^* \ln \rho_1) + \alpha \rho \hat{u} A(\frac{\hat{u}^2}{2} + c_2^* \ln \rho_2))_{M_1}^*.
\end{align*}
\]

Comparing equations (36) with (32) reveals that \( U^n_{M_1} = U_{M_2} \). Using this property in (28), by replacing \( M_1 \) and \( M_2 \) as described in Algorithm 1, we yield:

\[
U^n_{i+1} = U^n_i - \frac{\Delta t}{\Delta x} \left( F(U^n_i, U^n_{i+1}) - F(U^n_i, U^n_{i-1}) \right) = U^n_i.
\]

Solving (37) recursively yields \( U^n_i = U^n_0 \). Thus, the proposed scheme captures the steady-state solution exactly. For the case with slip and assumptions (33), the scheme captures the steady-state solution only approximately.

**Claim 2.** In case of constant cross-sectional area, i.e., \( \partial A/\partial x = 0 \), the solution \( U^n_i \) obtained from (28) is equal to that obtained from (20) if no-slip condition \( (30) \) is assumed.

**Proof.** For the case of constant area over two neighboring cells and assuming no-slip condition, (32) reduces to:

\[
\begin{align*}
(\alpha \rho \hat{u} A)_{M_1} &= (\alpha \rho \hat{u} A)_{M_2}, \\
(\alpha \rho \hat{u} A)_{M_1} &= (\alpha \rho \hat{u} A)_{M_2}, \\
(\alpha \rho \hat{u} A(\frac{\hat{u}^2}{2} + c_2^* \ln \rho_1) + \alpha \rho \hat{u} A(\frac{\hat{u}^2}{2} + c_2^* \ln \rho_2))_{M_2}^* &= (\alpha \rho \hat{u} A(\frac{\hat{u}^2}{2} + c_2^* \ln \rho_1) + \alpha \rho \hat{u} A(\frac{\hat{u}^2}{2} + c_2^* \ln \rho_2))_{M_1}^*.
\end{align*}
\]

where a candidate solution would be \( U^n_{M_1} = U_{M_2} \). Then, by following Algorithm 1, (28) reduces to:

\[
U^n_{i+1} = U^n_i - \frac{\Delta t}{\Delta x} \left( F(U^n_i, U^n_{i+1}) - F(U^n_i, U^n_{i-1}) \right).
\]

This is similar to (20) with zero source terms (since area is constant and \( \partial A/\partial x = 0 \)) and the modified scheme reverts back to the classical scheme. This feature is reflected in the Algorithm 1 to accelerate the numerical solution. Again we should mention that for the case with slip and assumptions (33), the scheme recovers the classical scheme only approximately when the area is constant.
Remark 5. After finding the starred values through solving the algebraic constraints, the numerical solution of (6) can be computed. If the starred-values are not found, one cannot compute the numerical solution by this method and other methods should be followed. Since these constraints exploit the steady-state equation (13), there is the possibility that these constraints are not satisfied either when starting at initial conditions far from the steady-state solution or when abrupt perturbations occur inside the domain. Notably, the algebraic constraints (32) and (35) may also have multiple solutions. Although the assumption of starting close to the steady-state solution is restrictive, there are many applications in the industry for which the analysis of perturbations with respect to the steady-state solution is important. In addition, studying the perturbations of steady-state solution of different systems has been the subject of many other studies [34, 35]. Moreover, this work can be the first step towards solving the non-conservative DFM by understanding the restrictions of the current approach. For a more in-depth discussion on features of the solution of the algebraic constraints, refer to Appendix A at the end of this paper, where the procedure to choose the feasible solution in case of multiple solutions is also explained.

In the following section, numerical results associated to each approach are discussed.

4. Numerical Results

Numerical results in this section are divided into five categories. First, in order to find the best scheme to solve system (6), the methods introduced in Section 3 have tested their ability to preserve the steady-state solution of system (6) with a piecewise constant cross-sectional area. Second, the most accurate approach, in the sense of capturing the steady-state solution, is used to compute the solution of (6) by imposing no variation in cross-sectional area in order to compare its performance with the solution obtained from the classical scheme of (21) applied to (1). Third, the selected approach has been tested for a transient flow modeling in a horizontal pipe with piecewise constant cross-sectional area. Fourth, the method is tested on a piecewise continuous cross-sectional area, both for capturing the steady-state solution and performing transient simulation. Finally, an error convergence study is performed.

The values of the parameters involved in system (6) are summarized in Table 1. The steady-state solution of (14) is acquired by the bvp4c solver of MATLAB. This solver approximates the solution to (13) in an iterative way while considering the boundary values at both ends of the computational domain [36].

Remark 6. In Sections 4.1 and 4.2 for the fourth approach, lines 2, 4 – 7, 9 – 11 of Algorithm 1 are ignored to test the performance of the new model-based modified scheme to automatically recover the classical scheme in locations where the area is constant. After becoming assured of the performance at constant area locations, the entire Algorithm 1 is used in other sections.

Remark 7. As mentioned before, to make fair comparisons between this approach and the second and the third approaches introduced in previous sections, we use the Rusanov scheme as the case study. However, the modifica-

| Table 1: Test case parameters. |
|-----------------|----------------|
| \( \rho_0 \)     | 1000 kg/m³     |
| \( p_0 \)        | 1 bar          |
| \( \dot{m}_l \)   | 0.3 kg/s       |
| \( \dot{m}_g \)   | 0.003 kg/s     |
| \( c_g \)        | 316 m/s        |
| \( c_l \)        | 1000 m/s       |
tion introduced in the fourth approach can be applied to other finite-volume schemes in a similar manner; only the numerical flux function \( F \) in \([27]\) should be changed according to the finite-volume scheme.

**Remark 8.** The Rusanov scheme is subject to the CFL condition

\[
\Delta t = \frac{\Delta x}{\text{CFL} \max(|\lambda_1|, |\lambda_2|, |\lambda_3|, |\lambda_4|)},
\]

where \( \lambda_i, i \in \{1, 2, 3, 4\} \) are given by \([9]\). For all simulations in this section, we estimate \( \max(|\lambda_1|, |\lambda_2|, |\lambda_3|, |\lambda_4|) \approx c_l \) and set \( \text{CFL} = 1 \). Then, according to the chosen \( \Delta x \), the temporal discretization \( \Delta t \) is specified.

**Remark 9.** To the best of authors’ knowledge, no Riemann solution for the DFM with variable cross-sectional area has been published. Before doing any transient simulations, the performance of the scheme at the steady state is evaluated. For transients, the performance of the scheme is assessed against the classical Rusanov scheme for the case of constant cross sectional area along the pipe. For a reference solution for piecewise constant area, refer to Appendix B. For a general piecewise continuous cross-sectional area, no reference solution exists for transient simulations.

### 4.1. Preservation of the steady-state solution

This section is dedicated to check the steady-state preservation of the numerical approaches proposed in Section 3. Since for the set of algebraic relations \([11]\) in the fourth approach, no slip between the phases is considered, i.e., \( K = 1 \) and \( S = 0 \), we apply the same condition in this section to perform a fair comparison between different approaches. Figure 4 shows the computational domain for this case study that is a horizontal pipe with one discontinuity in diameter along its length. Moreover, the time horizon is 1 s with discretization steps \( \Delta x = 1 \text{ m} \). The pressure at the right boundary is also \( p_R(t) = 1 \text{ bar} \). It should be noted as the simulation is stopped at \( t = 1 \text{ s} \), there are still some transient effects in the solution.

#### 4.1.1. First approach: source term approximation

The numerical solution of \([6]\) obtained by the first approach mentioned in Section 3.1 in comparison with solution of system \([13]\) is shown in Figure 5. Clearly, this kind of non-conservative remedy is afflicted by non-physical peaks at the locations of the discontinuities in the cross-sectional area where the impulsive force lies. As this method fails to capture the steady-state solution, it is not selected for further investigation by transient simulations.

#### 4.1.2. Second approach: modified Rusanov scheme

After modifying the Rusanov scheme as explained in Section 3.2, the solution of this test cases is shown in Figure 6. Clearly, this modification suffers from non-physical jumps at the locations of the area discontinuities. This should not be a surprise since this modification is proved in \([14]\) to be well-balanced only in the case of zero flow inside the domain. However, in the presence of flow, the performance of this scheme is not necessarily satisfactory in the sense that it is incapable of preserving the steady-state solution, similar to the presented results. Therefore, this method is also not studied further in this paper.

#### 4.1.3. Third approach: novel, modified Rusanov scheme

The results for the third type of modification is shown in Figure 7. Apparently, the results of the new modified Rusanov are better than the first and second approach in preserving the steady-state solution except for relatively small discontinuity jumps in the mass flow rate. These jumps are related to the approximation of the integral \( \int_{x_{i-1/2}}^{x_{i+1/2}} p \frac{\partial A}{\partial x} dx \) by \( p_i(A_{i+1} - A_{i-1})/2 \). This numerical deficiency pollutes the numerical solutions. Another weakness of this approach is that by increasing the spatial rate of cross-section variation in the discretized sense at a certain location (i.e., higher jumps in the cross section and a larger impulsive force), the solution deviates from the actual steady-state solution even further. Therefore, this method is also not selected for further investigation.
4.1.4. Fourth approach: model-based modification

The numerical results obtained by using the fourth approach with the set of algebraic constraints in (31) are demonstrated in Figure 8. The numerical results show a significant accuracy in the preservation of the steady-state.
solution of the PDEs (6). Compared to the previous results, pressure and mass flow rate are preserved with significantly higher accuracy. The small deviation from steady-state is due to the error in solving the algebraic relations (31).

Figure 7: Liquid phase variables of system (6) by using Rusanov scheme on variable area and the third approach: novel, modified Rusanov scheme in Section 3.3.

Figure 8: Liquid phase variables of system (6) by using Rusanov scheme on variable area and the fourth approach, model-based modification, together with (31).
The simulation results for the set of algebraic constraints in (34) are depicted in Figure 9. This set of assumptions also performs well in capturing the steady-state solution. As obvious from the top-left side of Figure 9, the gas phase is dominant here and \( \alpha_l \), and subsequently \( \alpha_g \), change negligibly over the interface. In addition, due to the very small change of pressure apparent from bottom-right side of Figure 9, the assumption of constant \( \rho_l/\rho_g \) over the interface is also valid.

Comparing all the results, the last method outperforms the other methods in capturing the steady-state solution. Therefore, the proposed modification in Section 3.4 together with the set of algebraic constraints (31) and (34) has been used for transient simulations. In these simulations, if the no-slip condition is imposed, the set of algebraic constraints (31) is used; otherwise, the algebraic constraints at the location of area variation are governed by (34). Since this approach uses a model-based modification, it does not suffer from any non-physical discontinuities in the solution of the state variables.

4.2. Comparisons in case of constant area

In this section, the performance of the proposed scheme to accurately characterize the dynamic behavior of the system in the case of constant area is analyzed. In order to do so, the result of the classical Rusanov scheme (21) applied to (1) is compared with the result of the model-based modified Rusanov scheme (31) or (34) applied to (6) in case of constant cross-sectional area. It should be noted again that the full Algorithm 1 is not implemented here as mentioned in Remark 6.

For the case of constant cross-sectional area, various benchmark tests for the DFM have been introduced. One well-known benchmark test is the DFM shock-tube problem, where the parameters of the simulation are taken from [5]. A horizontal pipe with the constant diameter of 0.1 m and length of 100 m is divided into two sections at the middle length of the pipe. For the left half of the pipe, the initial data are as below:

\[
\alpha_g = 0.55, \; u_l = 10.37 \text{ m/s}, \; p = 80450 \text{ Pa}.
\]

For the right half of the pipe, we set

\[
\alpha_g = 0.55, \; u_l = 0.561 \text{ m/s}, \; p = 24282 \text{ Pa}.
\]
In addition, discretization steps are $\Delta x = 1 \text{ m}$, and $K = 1.07$ and $S = 0.216$.

First, we apply the modified Rusanov scheme (28) along with (34) on system (6) and compare the results with the original Rusanov scheme (21) applied to system (1). Both solutions should correspond exactly with each other, as seen in Figure 10. As area is constant, the assumptions made for the DFM with slip as in (33) are valid. Reference solution is obtained on a fine discretization of $\Delta x = 0.1 \text{ m}$ with the classical Rusanov scheme.

Thus, in order to reduce the computational cost, the algebraic relations of (31) and (34) are enforced only at the locations where the cross-section varies. At the other locations, we set $U^*_{i+1} = U_{i+1}^+$ and $U^*_{i-1} = U_{i-1}^-$ as already noted in Algorithm [1].

4.3. Wave reflection in the presence of piecewise constant cross section

In this section, transients near the steady-state solution for a flow inside a pipe with piecewise constant cross-section are analyzed. At the location of discontinuous cross-section, any pressure wave is partially reflected back. Therefore, presence of the discontinuous cross-section dramatically affects the frequency response and the natural frequency of the system, which highly depends on the location and number of area variations. This kind of behavior has to be captured by the numerical simulation. In this section, the wave reflection behavior in the model-based modified Rusanov scheme together with (31) in both cases of discontinuous and constant cross-section of area are compared as another criterion for assessing the performance of the scheme.

The wave propagation can be excited by variation of a mass flow at the left boundary or a pressure change at the right boundary. In this simulation, the pressure at the right boundary $p_R(t)$ is increased from 1 bar to 1.2 bar at $t = 1 \text{ s}$ to initiate a propagating pressure wave inside the domain. Other simulation parameters are shown in Table [1] with

![Figure 10: Performance of model-based modified and classical Rusanov scheme for shock tube test, constant area.](image-url)
Figure 11: Effect of discontinuous cross section in pressure reflection (the variable cross-section is defined in Figure 4 for the top case and the constants cross section is with diameter of 7.5 cm for the bottom case).

\[K = 1 \text{ and } S = 0\] with the pipe shown in Figure 4. The comparison of pressure wave propagation and reflection between two boundaries and area discontinuities can be observed in Figure 11 and 12.

The reference solution is obtained as explained in Appendix B. In Figure 11, the reference solution is obtained on a coarse with \( \Delta x = 1 \text{ m} \) (the same spatial resolution as the solution with the modified scheme) and a fine grid with \( \Delta x = 0.1 \text{ m} \). Result of the modified scheme and the reference solution are very close to each other. It can be observed that the pressure front generated at the location of discontinuous area (in this case at \( x = 25 \text{ m} \) which is shown by dashed black line in Figure 12) opposes the incoming pressure waves and this front partially reflects back the pressure waves as obvious from the top part of Figure 12 at around \( t = 3 \text{ s} \) and similarly at the other time instants. This feature, which changes the wave propagation pattern inside the domain, has been well predicted by the scheme. In case of constant area, no pressure reflection occurs until the pressure wave reaches the other boundary.

Remark 10. This example resembles a choke plugging scenario within managed pressure drilling operations. When the mud gets stuck in the choke installed at the top of the annulus, the choke pressure experiences a sudden increase. Due to this pressure increase, a pressure wave propagates inside the annulus, which can be analyzed similar to the example in this section by adding relevant friction and gravity source terms into the governing equations.

4.4. Piecewise continuous cross section

Here, a two-phase flow inside a converging-diverging nozzle is investigated. The dimensions of the pipe are shown in Figure 13. The simulation parameters are the same as in Section 4.3. Figure 14 depicts the gas velocity at different time instants.

The top-left plot in Figure 14 shows that the steady-state solution is captured well by the numerical scheme. Then, by increasing the pressure at the right boundary, the opposition force at the right boundary lowers the velocity inside the domain. This test case shows the capability of the proposed scheme both in capturing the steady-state solution and in the transient simulation of two-phase flow inside a pipe with piecewise continuous cross section.
4.5. Error convergence analysis

To accurately study the error convergence of the scheme, a test case should be introduced such that other aspects of the numerical solution does not affect the accuracy. For instance, $\partial A/\partial x$ and the mixture speed of sound $\omega$ should be analytically known. Therefore, a pipe with continuous cross section is selected as shown in Figure 13 which defines $\partial A/\partial x$ analytically. To know the analytical speed of sound, no slip condition is set, i.e., $K = 1$ and $S = 0$; then, $\omega$ is governed by (10). Other parameters are taken from Table 1. Boundary conditions do not change ($p_R(t) = 1$ bar) and the system remains on its steady-state.

In order to study convergence properties of the scheme, the number of grid-cells ($N$) are varied, the problem is solved with the model-based modified Rusanov scheme at other values for $\Delta x$ (i.e., other number of grid-cells) and the solution (here the gas velocity) is compared with the reference solution at the last time instant, $t = 1$ s. The error is defined as the relative difference between numerical gas velocity at the last time instant and the initial gas velocity.
at all locations, as in (39),

\[ e = \frac{\|u_g(x, 1) - u_g(x, 0)\|_{L_r}}{\|u_g(x, 0)\|_{L_r}}, \]  

where \( \|\cdot\|_{L_r} \) is the \( L_r \)-norm of its argument over the spatial domain. The values for the error indicator (39) with increasing the number of the cells are reported in Table 2 and in Figure 16 for \( r = 2, \infty \). The absolute value of the slope of each line is also depicted in this figure, confirming that the proposed scheme does not affect order of accuracy of the underlying scheme (it is well-known that the classical Rusanov scheme is first order accurate [18]).

The small discrepancy between the order of accuracy at different level of the number of grid cells is due to the inaccuracy in solving the nonlinear algebraic constraints (31).

All discussions in this paper focused on first-order schemes; the extension to higher-order scheme can be done, which is beyond the scope of this paper. This will be the topic of future works.

Generally, this approach of dealing with variable cross-sectional area is problematic when the system (5) starts far from steady-state solution or when it is used for simulating abrupt dynamics. In this setting, an analysis has been presented in Appendix A.

Figure 14: Gas velocity at different time instants.

Figure 15: Dimension of the pipe with continuous cross-section to study the error convergence.
Table 2: Relative error convergence for the model-based, modified Rusanov scheme.

<table>
<thead>
<tr>
<th>Number of cells</th>
<th>$r = 2$</th>
<th>$r = \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0.012</td>
<td>0.0289</td>
</tr>
<tr>
<td>100</td>
<td>0.0061</td>
<td>0.0143</td>
</tr>
<tr>
<td>200</td>
<td>0.0031</td>
<td>0.0070</td>
</tr>
<tr>
<td>400</td>
<td>0.0015</td>
<td>0.0033</td>
</tr>
<tr>
<td>800</td>
<td>0.00076</td>
<td>0.0016</td>
</tr>
</tbody>
</table>

5. Conclusions

This paper studied numerical solvers for the non-conservative Drift Flux Model in the presence of variable cross-sectional area. Different numerical approaches have been proposed and compared to the existing approaches in the sense of accurate preservation of the steady-state solution. It has been shown that one of the new proposed schemes, the model-based modified scheme, indeed captures the physical steady-state solution with an acceptable accuracy. The model-based modified scheme can be applied to piecewise continuous cross-sectional areas as well. This modified scheme reduces to the classical scheme in case of constant area and it is also shown that the modified scheme enables simulation of the wave reflection in case of discontinuous cross-sectional area. It has been numerically proved that the proposed modification retains the first order of accuracy of the underlying scheme. Based on the performance of the proposed scheme, it can be used for simulation of industrial applications such as the hydraulics of two-phase flow occurring in drilling for oil and gas in a well with discontinuous cross-sectional area.

Appendix A. Initial conditions away from steady state

As mentioned in Section 3.4, for initial conditions far from the steady-state solution or simulating dynamics including abrupt changes in the input variables, the model-based modification (the fourth approach proposed in this
paper) does not perform well. Due to the fact that equations in the DFM are rather complicated, it is hard to find a framework for defining the applicability region of this method. Below, a brief explanation and an example are given for the case when the model-based modification cannot be applied.

Different from the results presented in the main text, here we analyze the algebraic constraints in more detail and provide a qualitative insight why the algebraic constraints do not have a solution when the initial conditions for system are far away from the steady-state solution. Similar analysis can be performed on the algebraic constraints.

To find the solution of the algebraic constraints, seven equations (three from and four from closure laws of ) should be solved simultaneously to find the seven unknowns of primitive variables. Due to the nonlinearity of equations, these may have more than one solution or may not have any solution. This is investigated in this appendix by applying the following steps:

1. Since there are seven unknown variables in the equations, it is challenging to analyze the solutions. As a first step, we express the equations only in one variable, particularly in pressure.

2. The resulting single equation is a nonlinear function of pressure, which is hard to be analyzed. Therefore, the solution of that equation is investigated numerically.

Assume we aim to find the starred values from by substituting and into and keeping pressure as the only variable. For reducing the number of variables and simplifying the equations, no slip between the phases, i.e., , is considered (it should be noted this is just an assumption for simplifying the computational procedure while the algebraic constraints are derived for general case of slip). Then, the simplified algebraic constraints of changes to:

\[
\begin{align*}
(\alpha_l \rho_l \hat{u})_{M_1} &= B, \\
(\alpha_g \rho_g \hat{u})_{M_1} &= C, \\
\left(\frac{\alpha_l \rho_l \hat{u}^2}{\rho_g} + \alpha_g \hat{u}^2 + c_g^2 \ln \rho_g\right)_{M_1} &= D,
\end{align*}
\]

with

\[
\begin{align*}
B &:= \frac{(\alpha_l \rho_l \hat{u} A)_{M_1}}{A_{M_1}}, \\
C &:= \frac{(\alpha_g \rho_g \hat{u} A)_{M_1}}{A_{M_1}}, \\
D &:= \frac{(\alpha_l \rho_l \hat{u}^2}{\rho_g} + \alpha_g \hat{u}^2 + c_g^2 \ln \rho_g)_{M_1}.
\end{align*}
\]

By embedding into and using the closure laws of and keeping only pressure as the variable (denoting by ), a nonlinear function of pressure is obtained as follows:

\[
\Phi(p) := \frac{1}{2} \varphi_1(p) \varphi_2^2(p) + c_g^2 \ln \frac{p}{c_g^2} - D,
\]

where

\[
\begin{align*}
\varphi_1(p) &:= \frac{p}{c_l^2} + F + \frac{C}{B \left(\frac{p}{c_l^2} + F\right)} + \frac{1}{p \left(\frac{c_g^2}{c_l^2} + F\right)} - \left(1 - \frac{p}{c_l^2} + F\right), \\
\varphi_2(p) &:= \frac{p}{c_g^2} \left(\frac{p}{c_l^2} + F\right).
\end{align*}
\]

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with $F := \rho_0 - p_0/c_l^2$. Now, the solution of the algebraic constraints of (35) correspond to the root of the function $\Phi$ in (A.3). Although function $\Phi$ is highly nonlinear and analytical investigation of the roots is challenging, analyzing the roots of the function $\Phi$ is easier compared to investigating the solution to the constraints (35). It can be shown that this equation either has one root, two roots or no root at all. Analytical investigation of the condition of having a root, however, is difficult, if not impossible. As the only variable in the function $\Phi$ is pressure, the roots can be found numerically by varying the pressure from zero to infinity and then we can analyze how many roots the function $\Phi$ has under different conditions.

This function tends to infinity as the pressure tends to zero or infinity when $B, C, D > 0$. In other words, for $p \to 0$ or $p \to \infty$, it holds that $\Phi \to \infty$. Thus, for $\Phi$ to have any real roots, the minimum of function $\Phi$ should be less than zero, which is highly dependent on the constant values of $B, C, D$, apparent from the dependence of the function on these constants. Hence, the necessary and sufficient condition for having solution(s) to the introduced algebraic constraints is $\Phi(p_{\text{min}}) < 0$ with $\Phi'(p_{\text{min}}) = 0$, where $p_{\text{min}}$ is the unique value at which the minimum of $\Phi$ is attained.

As mentioned previously, the analytical assessment of the roots of $\Phi$ is challenging, thus we tackle this numerically as follows with an example.

Consider the following test case as an example to show the fact that those algebraic constraints sometimes have multiple solutions and sometimes do not have any solution. If starting from the initial condition of shock tube problem presented in Section 4.2 accompanied with discontinuous area reduction at the middle cell of the pipe, say at the $i$-th cell, from $A_i = 1.5 \, \text{m}^2$ to $A_{i+1} = 1 \, \text{m}^2$, the evolution of function $\Phi$ with respect to $p$ (as pressure varies from zero to infinity) for finding $U^*_i$ and $U^*_{i+1}$ values can be shown in Figure A.1.

It can be noted that for finding the $U^*_i$, we do not have any solution, therefore the model-based modification fails to predict the solution (see the left plot in Figure A.1 that function $\Phi$ does not intersect the horizontal axis). In case of DFM, due to the complexity of the equations, it is hard to find a generic condition under which the constraint equations have solutions; unlike the case for Euler equations for which the condition is derived in [11]. However, a qualitative insight in the condition for the existence of a numerical solution can be provided as below. If we start exactly from steady-state solution, since the three algebraic constraints of (A.1) are defined by the steady-state equations, these equations are already satisfied. If starting close to steady states, the residuals of these equations are small and we are still in the region where the algebraic constraints can be satisfied by small changes in the primitive variables. In
contrary, if we start from an initial condition far from steady-states, the residual of the algebraic constraints are large and it might be impossible to satisfy all seven equations simultaneously (in other words, $\Phi(p_{\text{min}}) > 0$ due to the values of $B, C, D$ in (A.3)). Therefore, it is probable not to obtain solution if we start from inconsistent initial condition, i.e., initial conditions far away from the steady-state solution. Analogous analyses can be carried out for abrupt dynamic changes in the simulation.

For $U^*_i + 1$, we have two solutions (see right plot in Figure A.1). In case of having two solutions, one solution corresponds to the subsonic flow and the other to the supersonic flow. Following the same admissibility criterion of [11], a root which lies in the same side of $p_{\text{min}}$ is chosen, meaning that

$$ (p_{\text{correct}} - p_{\text{min}})(p_{i+1} - p_{\text{min}}) > 0 $$

where $p_{\text{correct}}$ is the correct root of function $\Phi$ that should be selected. Relying on this analysis, the second root should be selected in this simulation and the first root is ruled out.

Since in some cases, especially when the imposed assumptions (33) are not valid, we may not have the solutions of those algebraic relations, it is highly recommended to use the numerical method proposed in this paper when we start close to steady-state solution of the PDE. In industrial applications, it is often the case that system is simulated from steady states (or from rest) and the underlying dynamics are some perturbations to the steady-state solution. So, this is not a restrictive assumption in many cases. However, satisfying those algebraic relations to simulate the correct transient behavior comes at the expense of losing the solution in problems starting far from steady state. We leave further analysis of finding a better approach to future works. One tentative remedy could be a hybrid approach, combining the third and fourth approach proposed in this paper.

Appendix B. Reference solution for the transient simulation

As there is no Riemann solution for the DFM with variable cross-section, another method for having a reference solution for transient case is used. In this method, at the location of area discontinuity, the pipe is divided into two different pipes that are connected to each other by some boundary conditions. For instance, the pipe in Figure 4 is broken into two different connected pipes, as shown in Figure B.2.

Within each smaller pipe, classical schemes with high resolution can be applied as each pipe has a constant area. At the interconnected boundary, at least six equations should be defined to obtain three primitive variables for each pipe. Then, by using the closure laws (2)-(5), all primitive variables can be obtained.

From the left pipe $L^p$, two characteristic-based boundary conditions corresponding to the waves moving downstream similar to (16) and (17) are written. From the right pipe $R^p$, one characteristic-based boundary condition
moving upstream can be written similar to (18). Three more equations are required, which are obtained by integrating system (6) over the area discontinuity, 

\[ \int_{\delta x} \frac{\partial (\alpha \rho_1 u_1)}{\partial t} dx + \int_{\delta x} \frac{\partial (\alpha \rho_1 u_1 A)}{\partial x} dx = 0, \tag{B.1a} \]

\[ \int_{\delta x} \frac{\partial (\alpha \rho_2 u_2)}{\partial t} dx + \int_{\delta x} \frac{\partial (\alpha \rho_2 u_2 A)}{\partial x} dx = 0, \tag{B.1b} \]

\[ \int_{\delta x} \frac{\partial ((\alpha \rho_1 u_1 + \alpha_\epsilon \rho_0 u_\epsilon) A)}{\partial t} dx + \int_{\delta x} \frac{\partial ((\alpha \rho_1 u_1^2 + \alpha_\epsilon \rho_0 u_\epsilon^2 + p) A)}{\partial x} dx = \int_{\delta x} p \frac{\partial A}{\partial x} dx, \tag{B.1c} \]

where \( \delta x \) is a very narrow band around each area discontinuity. For \( \delta x \to 0 \), the integrals related to the time derivations vanish and (B.1) simplifies to the following algebraic relations

\[ (\alpha \rho_1 u_1)_{R^0} = (\alpha \rho_1 u_1)_{L^0}, \tag{B.2a} \]

\[ (\alpha_\epsilon \rho_0 u_\epsilon)_{R^0} = (\alpha_\epsilon \rho_0 u_\epsilon)_{L^0}, \tag{B.2b} \]

\[ \int_{\delta x} \left( \frac{\partial ((\alpha \rho_1 u_1^2 + \alpha_\epsilon \rho_0 u_\epsilon^2) A)}{\partial x} + A \frac{\partial p}{\partial x} \right) dx = 0, \tag{B.2c} \]

where the subscript \( R^0 \) and \( L^0 \) refer to the right and left pipe. For no-slip condition \( u_l = u_r = \hat{u} \), based on Lemma [1] and the algebraic equation (31c), we have

\[ (\alpha \rho_1 \hat{u} A)_{R^0} = (\alpha \rho_1 \hat{u} A)_{L^0}, \tag{B.3a} \]

\[ (\alpha_\epsilon \rho_0 \hat{u} A)_{R^0} = (\alpha_\epsilon \rho_0 \hat{u} A)_{L^0}, \tag{B.3b} \]

\[ (\alpha \rho_1 \hat{u} A(\hat{u}^2/2 + c_l^2 \ln \rho_l) + \alpha_\epsilon \rho_0 \hat{u} A(\hat{u}^2/2 + c_\epsilon^2 \ln \rho_\epsilon))_{R^0} = (\alpha \rho_1 \hat{u} A(\hat{u}^2/2 + c_l^2 \ln \rho_l) + \alpha_\epsilon \rho_0 \hat{u} A(\hat{u}^2/2 + c_\epsilon^2 \ln \rho_\epsilon))_{L^0}. \tag{B.3c} \]

The other three equations have also been obtained. Then, the six boundary equations are complete and the simulation in each pipe can be solved with a high resolution Rusanov scheme to deliver the reference solution.

### Acknowledgment

The authors would like to gratefully acknowledge the contribution of Dr. A.J.E.M. Janssen of TU Eindhoven, for his help with the analytical investigations of the function \( A(x) \).

### Funding

This research has been carried out in the HYDRA project, which has received funding from the European Union’s Horizon 2020 research and innovation program under grant agreement No 675731.

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