An analysis of variance for experiments with paired comparisons

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Chapter 1

Introduction

In paired comparison experiments, observations are made by presenting objects in pairs to one or more judges. The term object may represent something like item, stimulus, treatment, and the like. The judge has to state at each trial his preference for one of the two objects. In the simplest case the observation can be recorded as 0 or 1, depending on which object is preferred. In most general cases the preference may be recorded on some finer scale, for example, a 7-points scale, with observations recorded with integer values between -3 and $+3$.

Many models have been formulated with regard to paired comparison experiments. The most widely known are probably the Bradly-Terry Luce model (Bradly & Terry, 1952; Luce, 1959) and the range of models originally proposed by Thurstone (1927). For a bibliography on the method of paired comparisons the reader is referred to Davidson & Farquhar (1976).

The method of paired comparisons provides in itself a simple experimental technique. However, both the enormous amount of data that have to be gathered for even a small number of objects (Coombs, 1964), as well as the large variety in options available to analyse the data, seriously hamper the usefulness of the paired-comparison technique. It is for this reason that in this report a relatively old technique, proposed by Scheffé (1952), is brought to the readers' attention, and a program, called SCHEFFE, is developed which makes the analysis of the data relatively simple. The advantages of the model as developed by Scheffé mainly rests on his simplicity in interpretation, and its close alliance to familiar analysis of variance techniques.

This report is split into two sections. First the Scheffé model is fully discussed. Secondly, the use of the program is described. Also a possible way is discussed of interpreting the results SCHEFFE gives.

Chapter 2

An analysis of variance for paired comparisons

2.1 The experiment

As already explained, a method of analysing paired comparison experiments has been developed by Scheffé for experiments in which preferences are expressed on a scale of 3 or more points. This method falls under the general theory of least squares and linear hypothesis.

Assuming we want to classify m items, we form all possible M pairs, where: $M = \frac{1}{2} \cdot m \cdot (m-1)$. Every pair (i,j) is presented[1] to $2r$ subjects, to r subjects in the order (i,j), and to r in the order (j,i). It is assumed that the number of subjects per pair is at least four. Each subject states his preference and this is converted to a numerical score. The preference for item i over item j of the kth of the r subjects presented with the pair i and j in the order (i,j) will be denoted by x_{ijk}.
In a 7-point scoring system the subject presented with the pair (i,j) makes one of the following statements:

(3) I prefer i to j strongly.

(2) I prefer i to j moderately.

(1) I prefer i to j slightly

(0) No preference.

(-1) I prefer j to i slightly.

(-2) I prefer j to i moderately.

(-3) I prefer j to i strongly.

[1] See Appendix A for optimal presentation orders.

It is assumed that the numerical scores increase with the strengh of the preference for i over j.

With this method of expressing preferences, it is possible that some kind of distortion is introduced, viz. if the assumption that the items used in the experiment can be scaled in a linear scale turns out to be wrong. Fortunately, such distortions will invalidate the hypothesis of subtractivity discussed in section 2.2.1.

2.2 The mathematical model

The underlying assumptions of our mathematical model are:

- all x_{ijk} are independent random variables.
- for a fixed ordered pair (i,j) all r variables x_{ijk} have the same mean μ_{ij} and the same variance σ^2 which does not depend on (i,j).
- for some purposes we will want to add the normality assumption that the x_{ijk} are normal, which can only be satisfied approximately.

The score assigned by a subject to a fixed ordered pair (i,j) may be thought of as the sum of two components:

(a) a characteristic of the subject representing his own average taste.

(b) the chance deviation of the subject from his own average.

Component (a) is a random variable because the subject is sampled from a population. It is not assumed that for all subjects in the population the component (a) equals μ_{ij}, but rather the meaning of μ_{ij} would be the mean of component (a) in the population.

The question may be put forward whether the x_{ijk}'s in practice have the same variances. For this we can carry out a test for homogeneity of variance. In section 2.2.3 we will describe a possible test.

Also, we can not assume that the normality hypothesis is valid. However, in practice, this does not appears to be a serious problem.

The mean preference for item i over item j, when presented in order (i,j) is μ_{ij}, and the mean preference for i over j in the order (j,i) is $-\mu_{ji}$. The average of these two means wil be denoted by:

$$\pi_{ij} = \frac{1}{2}(\mu_{ij} - \mu_{ji}) \tag{2.1}$$

and their average difference by

$$\delta_{ij} = \frac{1}{2}(\mu_{ij} + \mu_{ji}) \tag{2.2}$$

Thus the parameter $2\delta_{ij}$ is the difference due to the order of presentation in the mean preference for i over j, while π_{ij} is the average preference for item i over j, averaged over the two orders.
If there is any interest attached to the order effects δ_{ij}, we can introduce the average order effect:

$$\delta = \sum_{i<j} \frac{\delta_{ij}}{M} \tag{2.3}$$

The parameter 2δ then measures the average advantage to an item i of being in the order (i,j) rather than (i,j), averaged over all $2M$ ordered pairs.
The program SCHEFFE investigates whether the order effects, as expressed in (2.3), are significant.

2.2.1 The hypothesis of subtractivity

The hypothesis of subtractivity, which can be statistically tested, is that there exist parameters $\alpha_1, \alpha_2, \ldots, \alpha_m$ characterizing the m items, such that the average preference π_{ij} for item i over item j is the difference of the corresponding parameters:

$$\pi_{ij} = \alpha_i - \alpha_j \tag{2.4}$$

Since only the differences of the parameters matter because we measure on an interval scale, we may add the assumption:

$$\sum_{i=1}^{m} \alpha_i = 0 \tag{2.5}$$

Basing ourselves on the assumption that the hypothesis of subtractivity is valid, we can classify the m items by estimating the main effects α_i :

$$\hat{\alpha}_i = \sum_{j=1}^{m} \frac{\hat{\pi}_{ij}}{m} \quad met \quad \hat{\pi}_{ii} = 0 \tag{2.6}$$

At this point we may ask ourselves if the calculated $\hat{\alpha}_i$'s give an accurate representation of the $\hat{\pi}_{ij}$'s, in other words do we accept the hypothesis of subtractivity.

Assuming the hypothesis is valid, we may write without loss of generality:

$$\pi_{ij} = \alpha_i - \alpha_j + \gamma_{ij} \tag{2.7}$$

where $\gamma_{ij} = -\gamma_{ji}$ and $\sum_{j=1}^{m} \gamma_{ij} = 0$ for $i = 1 \ldots m$

The γ_{ij}'s are given explicitly by the formula:

$$\gamma_{ij} = \pi_{ij} - \sum_{k=1}^{m} \frac{\pi_{ik}}{m} + \sum_{k=1}^{m} \frac{\pi_{jk}}{m} \tag{2.8}$$

If and only if all the γ_{ij}'s are negligibly small, the hypothesis of subtractivity is accepted and the $\hat{\alpha}_{ij}$'s are correct estimations. We then may use the $\hat{\alpha}_{ij}$'s for representing the main effects in the experiment on a linear scale. In the situation that the main effects are significant and the hypothesis of subtractivity is not accepted, the usefulness and interpretation of the estimates $\hat{\alpha}_{ij}$'s is not so clear.

The program SCHEFFE tests this hypothesis of subtractivy and the user can decide if the hypothesis can be validated.

It may be helpful at this point to bring together the different effects that have been introduced. If we write e_{ijk} for the 'error' in x_{ijk}, that is, $e_{ijk} = x_{ijk} - \mu_{ijk}$, then

$$x_{ijk} = (\alpha_i - \alpha_j) + \gamma_{ij} + \delta_{ij} + e_{ijk} \tag{2.9}$$

2.2.2 Determining the main effects

In this subsection we will give a discussion about determining the main effects.

If all the earlier mentioned conditions are satisfied, it is possible to determine the main effects and the user may draw conclusions whether the main effects are significant.

We can estimate the variance σ^2 of the variable x_{ijk} by means of calculating

$$\hat{\sigma}^2 = \frac{S_e}{2M(r-1)} \quad where \quad S_e = \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{r}(x_{ijk} - \hat{\mu}_{ij})^2 \tag{2.10}$$

The variance of the estimation $(\hat{\alpha}_i - \hat{\alpha}_j)$ can be given by

$$Var(\hat{\alpha}_i - \hat{\alpha}_j) = \frac{\sigma^2}{rm} \tag{2.11}$$

With this it is possible to derive a 'yardstick' Y_ϵ for making all the comparisons among the main efffects. This 'yardstick' can be written as:

$$Y_\epsilon = q_{1-\epsilon}\sqrt{\frac{\hat{\sigma}^2}{2rm}} \tag{2.12}$$

in which $(1-\epsilon)$ is defined as an interval of confidence. Using m objects and $\nu = 2 \cdot M \cdot (r-1)$ degrees of freedom, we can look up $q_{1-\epsilon}$ in a table of the Studentitized range.

After calculating the 'yardstick' Y_ϵ, we may compare the scale values which represent the main effects.

The experiment will be said to have demonstrated a difference for any two main effects α_i and α_j if and only if their estimates $\hat{\alpha}_i$ and $\hat{\alpha}_j$ differ by at least the 'yardstick' Y_ϵ. Thus, with a confidence coefficient of $(1-\epsilon)$ per cent, we may give our opinion upon the M differences $\alpha_i - \alpha_j$, by looking at

$$\hat{\alpha}_i - \hat{\alpha}_j - Y_{1-\epsilon} \leq \alpha_i - \alpha_j \leq \hat{\alpha}_i - \hat{\alpha}_j + Y_{1-\epsilon} \tag{2.13}$$

This allows us to determine the significance of the main effects.

2.2.3 The homogeneity of variances

As we have already seen, one of the assumptions of Scheffé's model is that the variances for each pair, due to experimental error, are homogeneous. If we want to be sure that the results are correctly interpretable, we first have to validate the hypothesis of homogeneity. For $2M$ variances we define:

$$H_0 : \sigma_1^2 = \sigma_2^2 = \cdots = \sigma_{2M}^2 \tag{2.14}$$

$$H_1 : \sigma_1^2 \neq \sigma_2^2 \neq \cdots \neq \sigma_{2M}^2 \tag{2.15}$$

with

$$\sigma_i^2 = \frac{(\sum_s (x_{is}^2)) - (\frac{\sum_s x_{is}}{s})^2}{s-1} \tag{2.16}$$

in which s denotes the number of observations per pair and i one of the possible pairs.

A relatively simple test for homogeneity of variance, developed by Cochran (1951), uses the statistic

$$C = \frac{\sigma_{max}^2}{\sum_{i=1}^{2M} \sigma_i^2} \tag{2.17}$$

The parameters of the sampling distribution of this statistic are *df* (the number of degrees of freedom, equal to $s-1$) and $2M$ (the number of variances). This implies that the number of subjects is the same in each cell.

The program SCHEFFE calculates for all presented $2M$ pairs the matched variances, and also the Cochran factor. On the basis of these data the user can decide if the hypothesis of homogeneity in his experiment is validated.

In the literature (Winer, 1971) also other tests are mentioned. However, in most cases the results will lead to the same conclusion. Therefore we have implemented one simple test.

Chapter 3

The program SCHEFFE [1]

Following Scheffé's model, we have developed a program in which the proposed analysis of variance is implemented. In the next sections, the use of the program is elucidated.

3.1 Using SCHEFFE

3.1.1 Input file

After SCHEFFE has been invoked (on IPO's VAX 8750 computer once the command 'USE SCHEFFE' must be typed; then the program can be invoked with the command 'SCHEFFE'), the program asks to give the name of an input file. This input file is a file which contains the frequency matrix, representing the expressed preferences, given by the judges for each of the $2M$ pairs. The pair-order has been predefined by the programmer as follows: Using the objects $1 \ldots m$ the assumed input order is given in figure 3.1.

3.1.2 Output file

Also SCHEFFE asks the desired name of the output file. This output file is a textfile which contains the results, calculated by SCHEFFE. The used symbols in the output file have the same meaning as in the sections above.

The following results are listed in the output file:

input frequency matrix

preference-score matrix

estimated μ parameters per cell

[1] This program is available on request, please contact H.H. Ellermann, IPO, room 1.17.

pair	$scalevalue_{min}$	$\cdots$	0	$\cdots$	$scalevalue_{max}$
1 - 2					
2 - 1					
1 - 3					
3 - 1					
$\vdots$					
1 - m					
m - 1					
2 - 3					
3 - 2					
$\vdots$					
2 - m					
m - 2					
$\vdots$					
k - k+1					
k+1 - k					
$\vdots$					
k - m					
m - k					
$\vdots$					
m-1 - m					
m - m-1					

Figure 3.1: *The assumed input order for a preference matrix*

estimated π parameters per cell (2.1)

estimated δ parameters per cell (2.2)

estimated γ parameters per cell (2.8)

analysis of variance table

yardstick (2.12)

table with calculated scale values (2.6)

significance of main effects

table of pair variances (2.16)

cochran factor (2.17)

3.1.3 Other input data

Other input which must be entered, running SCHEFFE:

the number of categories of the used preference scale

the number of objects used in the experiment the number of judgements per pair, obtained in the experiment.

Then SCHEFFE gives the number of variances and the degrees of freedom, and the user must look up the appropriate value in the Studentitized range table (appendix B), so SCHEFFE can calculate the 'yardstick'.
Finally, after SCHEFFE has written the output file, the program stops.

3.1.4 Interpretation of the results

If SCHEFFE doesn't give any warning messages, the user afterwards can list the output file and look at the results. First the hypothesis of homogeneity is tested. The calculated Cochran factor must be compared with the appropriate value in an appropiate table (Appendix C). If the hypothesis has been validated, the user may look at the analysis of variance table.

- Main effects are significant if and only if : $p < 0.05$

- No deviations of subtractivity if and only if : $p > 0.05$

- No order effects if and only if : $p > 0.05$

If there are no deviations of subtractivity, then the α parameters are correct estimates for the investigated objects. Using the 'yardstick' SCHEFFE calculates whether the alfa comparisons are significant.

If the conditions mentioned above are not satisfied, the user must find out whether the used method is suitable for the purpose he wants to achieve. In that case we recommend the user to take a close look at the μ, π, δ and γ matrices for some severe deviations.

3.2 A session

When a user has counted the preference scores, he is ready to make an input file, in which the obtained scores are listed. An example:

pair	-3	-2	-1	0	+1	+2	+3
1,2	2	1	3	2	1	0	0
2,1	0	2	1	3	1	2	0
1,3	2	0	1	4	0	1	1
3,1	1	1	1	5	0	1	0
2,3	0	1	0	1	3	3	1
3,2	1	2	0	5	0	1	0

Note that an input file containes only the scores, seperated by spaces.

Before invoking SCHEFFE (by typing the command 'SCHEFFE'), the user must first give the command 'USE SCHEFFE'.

When SCHEFFE has started up, the program gives some messages and asks a number of questions (for the above input matrix the answers are given between brackets):

Enter name of data file (e5eg.dat[2])

Enter name of outputfile. (e5eg.sch)

Enter number of categories in preference scale (7)

Enter number of objects to be compared (3)

Enter number of judges per pair (9)

Studentitized range (3.44)[3]

After the program has calculated the results, the user can list or print the output (text) file. For the input file given earlier, the output file is given below.

```
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file is e5eg.dat
Score matrix as input
For each cell the frequency for each category is indicated

i  j
1, 2 -    2    1    3    2    1    0    0
     -    0    2    1    3    1    2    0
1, 3 -    2    0    1    4    0    1    1
     -    1    1    1    5    0    1    0
2, 3 -    0    1    0    1    3    3    1
     -    1    2    0    5    0    1    0

Score matrix: Measures of preference.

     0 -10  -2
     0   0  10
    -4  -5   0

Mu Parameters per cell as estimated. Nr of judges involved:     9

    0.00 -1.11 -0.22
    0.00  0.00  1.11
   -0.44 -0.56  0.00

Pi estimates per cell (lower triangular)

    0.00 -0.56  0.11
    0.56  0.00  0.83
   -0.11 -0.83  0.00

Delta values per cell (lower triangular)

    0.00 -0.56 -0.33
   -0.56  0.00  0.28
   -0.33  0.28  0.00
```

[2]The names and extensions of the input and output files are arbitrary.

[3]This value can be found in appendix B (After Winer, 1971).

Gamma estimates per cell.

0.00	0.06	-0.06
-0.06	0.00	0.06
0.06	-0.06	0.00

Analysis of variance table

Source	SS	Df	MS	F	p
Main effects	18.111	2	9.056	3.754	0.03056
Dev. from subtractivity	0.167	1	0.167	0.069	0.79378
Average preferences	18.278	3			
Order effects	8.944	3	2.981	1.236	0.30694
Means	27.222	6			
Error	115.778	48	2.412		
Total	143.000	54			

Yardstick size 0.7270
Studentitized range statistics 3.4400
Degrees of freedom: 48

Values for alfa parameters.

1	-0.1481
2	0.4630
3	-0.3148

Significance of alfa comparisons
signif=1 means significant, signif=0 is not significant

i j	alfa(i)	alfa(j)	difference	signif
1 2	-0.1481	0.4630	-0.6111	0
1 3	-0.1481	-0.3148	0.1667	0
2 3	0.4630	-0.3148	0.7778	1

table of pair variances

i j	variance
1 2	1.861
2 1	2.250
1 3	3.944
3 1	2.028
2 3	2.111
3 2	2.278

Cochran factor = 0.273

Number of variances = 6

Degrees of freedom = 8

References

Bradley, R.A. & Terry, M.E.(**1952**)The rank analysis of incomplete block designs. I. The method of paired comparisons, *Biometrika 39*, 324-345.

Cochran, W.G. (**1947**)Some consequences when assumptions for the analysis of variance are not satisfied, *Biometrics 3*, 22-38.

Coombs, C.H. (**1964**)*A theory of data*, Academic Press, New York.

Davidson, R.R. & Farquhar, P.H. (**1976**)A Bibliography on the method of paired comparisons, *Biometrics 32*, 241-252.

Luce, R.D. (**1959**)*Individual Choice Behaviour*, Wiley, New York.

Phillips, J.P.N. (**1964**)On the presentation of stimulus objects in the method of paired comparisons, *American Journal of Psychology 77*, 660-664.

Scheffé, H. (**1952**)An analysis of variance for paired comparisons, *Journal of the Statistical Association of America 47*, 381-400.

Thurstone, L.L. (**1927**)A law of comparative judgement,*Psychological Review 34*, 273-286.

Winer, B.J. (**1971**)*Statistical principles in experimental design*, 2nd edition, McGraw-Hill, New York.

Appendix A: Optimal orders for presenting pairs, when using the method of paired comparisons[1].

For a number of reasons, it is important to chose special presentation orders. Below, some optimal presention orders for various numbers of objects are given.

n = 3: 1-2, 3-1, 2-3

n = 4: 1-2, 4-3, 3-1, 4-2, 1-4, 2-3

n = 5: 1-2, 3-5, 4-1, 2-3, 5-4, 1-3, 4-2, 5-1, 3-4, 2-5

n = 6: 1-2, 6-3, 5-4, 3-1, 4-2, 6-5, 1-4, 5-3, 2-6, 5-1, 6-4, 3-2, 1-6, 2-5, 4-3

n = 7: 1-2, 3-7, 4-6, 5-1, 2-3, 7-4, 6-5, 1-3, 4-2, 5-7, 6-1, 3-4, 2-5, 7-6, 1-4, 5-3, 6-2, 7-1, 4-5, 3-6, 2-7

n = 8: 1-2, 8-3, 7-4, 5-6, 3-1, 4-2, 8-5, 6-7, 1-4, 5-3, 2-6, 7-8, 5-1, 6-4, 3-7, 8-2, 1-6, 7-5, 4-8, 2-3, 7-1, 8-6, 5-2, 3-4, 1-8, 2-7, 6-3, 4-5

n = 9: 1-2, 3-9, 4-8, 5-7, 6-1, 2-3, 9-4, 8-5, 7-6, 1-3, 4-2, 5-9, 6-8, 7-1, 3-4, 2-5, 9-6, 8-7, 1-4, 5-3, 6-2, 7-9, 8-1, 4-5, 3-6, 2-7, 9-8, 1-5, 6-4, 7-3, 8-2, 9-1, 5-6, 4-7, 3-8, 2-9

n = 10: 1-2, 10-3, 9-4, 5-8, 7-6, 3-1, 4-2, 10-5, 6-9, 8-7, 1-4, 5-3, 2-6, 7-10, 9-8, 5-1, 6-4, 3-7, 8-2, 10-9, 1-6, 7-5, 4-8, 9-3, 2-10, 7-1, 8-6, 5-9, 10-4, 3-2, 1-8, 9-7, 6-10, 2-5, 4-3, 9-1, 10-8, 7-2, 3-6, 5-4, 1-10, 2-9, 8-3, 4-7, 6-5

n = 11: 1-2, 3-11, 4-10, 5-9, 6-8, 7-1, 2-3, 11-4, 10-5, 9-6, 8-7, 1-3, 4-2, 5-11, 6-10, 7-9, 8-1, 3-4, 2-5, 11-6, 10-7, 9-8, 1-4, 5-3, 6-2, 7-11, 8-10, 9-1, 4-5, 3-6, 2-7, 11-8, 10-9, 1-5, 6-4, 7-3, 8-2, 9-11, 10-1, 5-6, 4-7, 3-8, 2-9, 11-10, 1-6, 7-5, 8-4, 9-3, 10-2, 11-1, 6-7, 5-8, 4-9, 3-10, 2-11

n = 12: 1-2, 12-3, 11-4, 5-10, 9-6, 7-8, 3-1, 4-2, 12-5, 6-11, 10-7, 8-9, 1-4, 5-3, 2-6, 7-12, 11-8, 9-10, 5-1, 6-4, 3-7, 8-2, 12-9, 10-11, 1-6, 7-5, 4-8, 9-3, 2-10, 11-12, 7-1, 8-6, 5-9, 10-4, 3-11, 12-2, 1-8, 9-7, 6-10, 11-5, 4-12, 2-3, 9-1, 10-8, 7-11, 12-6, 5-2, 3-4, 1-10, 11-9, 8-12, 2-7, 6-3, 4-5, 11-1, 12-10, 9-2, 3-8, 7-4, 5-6, 1-12, 2-11, 10-3, 4-9, 8-5, 6-7

n = 13: 1-2, 3-13, 4-12, 5-11, 6-10, 7-9, 8-1, 2-3, 13-4, 12-5, 11-6, 10-7, 9-8, 1-3, 4-2, 5-13, 6-12, 7-11, 8-10, 9-1, 3-4, 2-5, 13-6, 12-7, 11-8, 10-9, 1-4, 5-3, 6-2, 7-13, 8-12, 9-11, 10-1, 9-11, 10-1, 4-5, 3-6, 2-7, 13-8, 12-9, 11-10, 1-5, 6-4, 7-3, 8-2, 9-13, 10-12, 11-1, 5-6, 4-7, 3-8, 2-9, 13-10, 12-11, 1-6, 7-5, 8-4, 9-3, 10-2, 11-13, 12-1, 6-7, 5-8, 4-9, 3-10, 2-11, 13-12, 1-7, 8-6, 9-5, 10-4, 11-3, 12-2, 13-1, 7-8, 6-9, 5-10, 4-11, 3-12, 2-13

n = 14: 1-2, 14-3, 13-4, 5-12, 11-6, 7-10, 9-8, 3-1, 4-2, 14-5, 6-13, 12-7, 8-11, 10-9, 1-4, 5-3, 2-6, 7-14, 13-8, 9-12, 11-10, 5-1, 6-4, 3-7, 8-2, 14-9, 10-13, 12-11, 1-6, 7-5, 4-8, 9-3, 2-10, 11-14, 13-12, 7-1, 8-6, 5-9, 10-4, 3-11, 12-2, 14-13, 1-8, 9-7, 6-10, 11-5, 4-12, 13-3, 2-14, 9-1, 10-8, 7-11, 12-6, 5-13, 14-4, 3-2, 1-10, 11-9, 8-12, 13-7, 6-14, 2-5, 4-3, 11-1, 12-10, 9-13, 14-8, 7-2, 3-6, 5-4, 1-12, 13-11, 10-14, 2-9, 8-3, 4-7, 6-5, 13-1, 14-12, 11-2, 3-10, 9-4, 5-8, 7-6, 1-14, 2-13, 12-3, 4-11, 10-5, 6-9, 8-7

n = 15: 1-2, 3-15, 4-14, 5-13, 6-12, 7-11, 8-10, 9-1, 2-3, 15-4, 14-5, 13-6, 12-7, 11-8, 10-9, 1-3, 4-2, 5-15, 6-14, 7-13, 8-12, 9-11, 10-1, 3-4, 2-5, 15-6, 14-7, 13-8, 12-9, 11-10, 1-4, 5-3, 6-2, 7-15, 8-14, 9-13, 10-12, 11-1, 4-5, 3-6, 2-7, 15-8, 14-9, 13-10, 12-11, 1-5, 6-4, 7-3, 8-2, 9-15, 10-14, 11-13, 12-1, 5-6, 4-7, 3-8, 2-9, 15-10, 14-11, 13-12, 1-6, 7-5, 8-4, 9-3, 10-2, 11-15, 12-14, 13-1, 6-7, 5-8, 4-9, 3-10, 2-11, 15-12, 14-13, 1-7, 8-6, 9-5, 10-4, 11-3, 12-2, 13-15, 14-1, 7-8, 6-9, 5-10, 4-11, 3-12, 2-13, 15-14, 1-8, 9-7, 10-6, 11-5, 12-4, 13-3, 14-2, 15-1, 8-9, 7-10, 6-11, 5-12, 4-13, 3-14, 2-15

[1] Copied from Phillips (1964).

Appendix B: Distribution of the Studentitized Range Statistic[1].

Note: In this table r represents the number of objects used in the experiment. The number of degrees of freedom is equal to $2 \cdot M \cdot (r-1)$ (see page 7).

df	1 − α	r = 2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	.95	18.0	27.0	32.8	37.1	40.4	43.1	45.4	47.4	49.1	50.6	52.0	53.2	54.3	55.4
	.99	90.0	135	164	186	202	216	227	237	246	253	260	266	272	277
2	.95	6.09	8.3	9.8	10.9	11.7	12.4	13.0	13.5	14.0	14.4	14.7	15.1	15.4	15.7
	.99	14.0	19.0	22.3	24.7	26.6	28.2	29.5	30.7	31.7	32.6	33.4	34.1	34.8	35.4
3	.95	4.50	5.91	6.82	7.50	8.04	8.48	8.85	9.18	9.46	9.72	9.95	10.2	10.4	10.5
	.99	8.26	10.6	12.2	13.3	14.2	15.0	15.6	16.2	16.7	17.1	17.5	17.9	18.2	18.5
4	.95	3.93	5.04	5.76	6.29	6.71	7.05	7.35	7.60	7.83	8.03	8.21	8.37	8.52	8.66
	.99	6.51	8.12	9.17	9.96	10.6	11.1	11.5	11.9	12.3	12.6	12.8	13.1	13.3	13.5
5	.95	3.64	4.60	5.22	5.67	6.03	6.33	6.58	6.80	6.99	7.17	7.32	7.47	7.60	7.72
	.99	5.70	6.97	7.80	8.42	8.91	9.32	9.67	9.97	10.2	10.5	10.7	10.9	11.1	11.2
6	.95	3.46	4.34	4.90	5.31	5.63	5.89	6.12	6.32	6.49	6.65	6.79	6.92	7.03	7.14
	.99	5.24	6.33	7.03	7.56	7.97	8.32	8.61	8.87	9.10	9.30	9.49	9.65	9.81	9.95
7	.95	3.34	4.16	4.69	5.06	5.36	5.61	5.82	6.00	6.16	6.30	6.43	6.55	6.66	6.76
	.99	4.95	5.92	6.54	7.01	7.37	7.68	7.94	8.17	8.37	8.55	8.71	8.86	9.00	9.12
8	.95	3.26	4.04	4.53	4.89	5.17	5.40	5.60	5.77	5.92	6.05	6.18	6.29	6.39	6.48
	.99	4.74	5.63	6.20	6.63	6.96	7.24	7.47	7.68	7.87	8.03	8.18	8.31	8.44	8.55
9	.95	3.20	3.95	4.42	4.76	5.02	5.24	5.43	5.60	5.74	5.87	5.98	6.09	6.19	6.28
	.99	4.60	5.43	5.96	6.35	6.66	6.91	7.13	7.32	7.49	7.65	7.78	7.91	8.03	8.13
10	.95	3.15	3.88	4.33	4.65	4.91	5.12	5.30	5.46	5.60	5.72	5.83	5.93	6.03	6.11
	.99	4.48	5.27	5.77	6.14	6.43	6.67	6.87	7.05	7.21	7.36	7.48	7.60	7.71	7.81
11	.95	3.11	3.82	4.26	4.57	4.82	5.03	5.20	5.35	5.49	5.61	5.71	5.81	5.90	5.99
	.99	4.39	5.14	5.62	5.97	6.25	6.48	6.67	6.84	6.99	7.13	7.26	7.36	7.46	7.56
12	.95	3.08	3.77	4.20	4.51	4.75	4.95	5.12	5.27	5.40	5.51	5.62	5.71	5.80	5.88
	.99	4.32	5.04	5.50	5.84	6.10	6.32	6.51	6.67	6.81	6.94	7.06	7.17	7.26	7.36
13	.95	3.06	3.73	4.15	4.45	4.69	4.88	5.05	5.19	5.32	5.43	5.53	5.63	5.71	5.79
	.99	4.26	4.96	5.40	5.73	5.98	6.19	6.37	6.53	6.67	6.79	6.90	7.01	7.10	7.19
14	.95	3.03	3.70	4.11	4.41	4.64	4.83	4.99	5.13	5.25	5.36	5.46	5.55	6.64	5.72
	.99	4.21	4.89	5.32	5.63	5.88	6.08	6.26	6.41	6.54	6.66	6.77	6.87	6.96	7.05
16	.95	3.00	3.65	4.05	4.33	4.56	4.74	4.90	5.03	5.15	5.26	5.35	5.44	5.52	5.59
	.99	4.13	4.78	5.19	5.49	5.72	5.92	6.08	6.22	6.35	6.46	6.56	6.66	6.74	6.82
18	.95	2.97	3.61	4.00	4.28	4.49	4.67	4.82	4.96	5.07	5.17	5.27	5.35	5.43	5.50
	.99	4.07	4.70	5.09	5.38	5.60	5.79	5.94	6.08	6.20	6.31	6.41	6.50	6.58	6.65
20	.95	2.95	3.58	3.96	4.23	4.45	4.62	4.77	4.90	5.01	5.11	5.20	5.28	5.36	5.43
	.99	4.02	4.64	5.02	5.29	5.51	5.69	5.84	5.97	6.09	6.19	6.29	6.37	6.45	6.52
24	.95	2.92	3.53	3.90	4.17	4.37	4.54	4.68	4.81	4.92	5.01	5.10	5.18	5.25	5.32
	.99	3.96	4.54	4.91	5.17	5.37	5.54	5.69	5.81	5.92	6.02	6.11	6.19	6.26	6.33
30	.95	2.89	3.49	3.84	4.10	4.30	4.46	4.60	4.72	4.83	4.92	5.00	5.08	5.15	5.21
	.99	3.89	4.45	4.80	5.05	5.24	5.40	5.54	5.56	5.76	5.85	5.93	6.01	6.08	6.14
40	.95	2.86	3.44	3.79	4.04	4.23	4.39	4.52	4.63	4.74	4.82	4.91	4.98	5.05	5.11
	.99	3.82	4.37	4.70	4.93	5.11	5.27	5.39	5.50	5.60	5.69	5.77	5.84	5.90	5.96
60	.95	2.83	3.40	3.74	3.98	4.16	4.31	4.44	4.55	4.65	4.73	4.81	4.88	4.94	5.00
	.99	3.76	4.28	4.60	4.82	4.99	5.13	5.25	5.36	5.45	5.53	5.60	5.67	5.73	5.79
120	.95	2.80	3.36	3.69	3.92	4.10	4.24	4.36	4.48	4.56	4.64	4.72	4.78	4.84	4.90
	.99	3.70	4.20	4.50	4.71	4.87	5.01	5.12	5.21	5.30	5.38	5.44	5.51	5.56	5.61
∞	.95	2.77	3.31	3.63	3.86	4.03	4.17	4.29	4.39	4.47	4.55	4.62	4.68	4.74	4.80
	.99	3.64	4.12	4.40	4.60	4.76	4.88	4.99	5.08	5.16	5.23	5.29	5.35	5.40	5.45

[1]Copied from Winer (1971).

Appendix C: Critical Values for Cochran's Test for Homogeneity of Variance[1].

df	1 − α	k = number of variances										
		2	3	4	5	6	7	8	9	10	15	20
1	.95	.9985	.9669	.9065	.8412	.7808	.7271	.6798	.6385	.6020	.4709	.3894
	.99	.9999	.9933	.9676	.9279	.8828	.8376	.7945	.7544	.7175	.5747	.4799
2	.95	.9750	.8709	.7679	.6838	.6161	.5612	.5157	.4775	.4450	.3346	.2705
	.99	.9950	.9423	.8643	.7885	.7218	.6644	.6152	.5727	.5358	.4069	.3297
3	.95	.9392	.7977	.6841	.5981	.5321	.4800	.4377	.4027	.3733	.2758	.2205
	.99	.9794	.8831	.7814	.6957	.6258	.5685	.5209	.4810	.4469	.3317	.2654
4	.95	.9057	.7457	.6287	.5441	.4803	.4307	.3910	.3584	.3311	.2419	.1921
	.99	.9586	.8335	.7212	.6329	.5635	.5080	.4627	.4251	.3934	.2882	.2288
5	.95	.8772	.7071	.5895	.5065	.4447	.3974	.3595	.3286	.3029	.2195	.1735
	.99	.9373	.7933	.6761	.5875	.5195	.4659	.4226	.3870	.3572	.2593	.2048
6	.95	.8534	.6771	.5598	.4783	.4184	.3726	.3362	.3067	.2823	.2034	.1602
	.99	.9172	.7606	.6410	.5531	.4866	.4347	.3932	.3592	.3308	.2386	.1877
7	.95	.8332	.6530	.5365	.4564	.3980	.3535	.3185	.2901	.2666	.1911	.1501
	.99	.8988	.7335	.6129	.5259	.4608	.4105	.3704	.3378	.3106	.2228	.1748
8	.95	.8159	.6333	.5175	.4387	.3817	.3384	.3043	.2768	.2541	.1815	.1422
	.99	.8823	.7107	.5897	.5037	.4401	.3911	.3522	.3207	.2945	.2104	.1646
9	.95	.8010	.6167	.5017	.4241	.3682	.3259	.2926	.2659	.2439	.1736	.1357
	.99	.8674	.6912	.5702	.4854	.4229	.3751	.3373	.3067	.2813	.2002	.1567
16	.95	.7341	.5466	.4366	.3645	.3135	.2756	.2462	.2226	.2032	.1429	.1108
	.99	.7949	.6059	.4884	.4094	.3529	.3105	.2779	.2514	.2297	.1612	.1248
36	.95	.6602	.4748	.3720	.3066	.2612	.2278	.2022	.1820	.1655	.1144	.0879
	.99	.7067	.5153	.4057	.3351	.2858	.2494	.2214	.1992	.1811	.1251	.0960
144	.95	.5813	.4031	.3093	.2513	.2119	.1833	.1616	.1446	.1308	.0889	.0675
	.99	.6062	.4230	.3251	.2644	.2229	.1929	.1700	.1521	.1376	.0934	.0709

[1] Copied from Winer (1971).

Appendix D: Possible messages from SCHEFFE.

In this appendix, all possible messages which SCHEFFE can give, will be explained. Running SCHEFFE, the following messages can appear:

Message: Enter name of data file

The name of the input file must be entered.

Message: File does not exist
Continue to re-enter new file name

SCHEFFE can't find the input file. 'CONTINUE' must be entered, otherwise the program is terminated. After entering 'CONTINUE' SCHEFFE asks again the name of the input file with the message: Enter name of data file

Message: Enter name of output file

The desired name of the output file must be entered.

Message: File already exists
Continue (and overwrite file)

When this message appeares, there is already a file with the same name as just has been entered. If this file may be overwritten, enter 'CONTINUE'. The program then continues. Otherwise the program terminates.

Message: Enter number of categories in preference scale:

The number of categories in the used preference scale must be entered. This number must be positive, odd and smaller than 15. If these conditions are not satisfied, the following message appeares:

Message: Warning: No. of categories must be positive or
No. of categories must be less than 15
No. of categories must be uneven.
Re-enter data !!

Re-enter the data satisfying the conditions above.

Message: Enter number of objects to be compared:

The number of objects, used in the experiment, must be entered. This number must be smaller than 50 and greater as 2. Otherwise the following message appears:

Message: Warning: No. of objects must be more than 2
No. of objects must be less than 50
Re-enter data!!

This message appeares if the entered number of objects is not more than 2 and not less than 50. The program asks to re-enter data.

Message: Enter number of judges per pair:

The number of judges per pair must be entered. This number must be at least four, otherwise the model may not be valid. If this condition is not met, the program reports:

Message: No. of judges per pair must be at least 4
Re-enter data !!

The number of judges per pair must be more than three. The program asks to re-enter new data.

Message: Number of objects: <displayed by program>
Degrees of freedom: <displayed by program>
Give Studentitized range for Yardstick:

The program displays the number of objects and the numbers of the degrees of freedom and the user must look up in the studentitized range table an appropriate value.

Message: -Normal End Of Program-

An output file has been written and SCHEFFE has stopped without any problems.

The following reports may appear when the input data has been processed succesfully and the program is reading or writing from memory.

Message: Error while opening or writing to: < *filename* >
ABNORMAL END OF PROGRAM

Message: Error while opening or reading: < *filename* >
ABNORMAL END OF PROGRAM

Message: Error while closing: < *filename* >
ABNORMAL END OF PROGRAM

These messages point out that there has been a read or write error.