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Gaussian bounds for reduced heat kernels of subelliptic operators on nilpotent Lie groups

A.F.M. ter Elst\textsuperscript{1} and Humberto Prado\textsuperscript{2}

Abstract

We obtain Gaussian estimates for the kernels of the semigroups generated by a class of subelliptic operators $H$ acting on $L_p(\mathbb{R}^k)$. The class includes anharmonic oscillators and Schrödinger operators with external magnetic fields. The estimates imply an $H_\infty$-functional calculus for the operator $H$ on $L_p$ with $p \in (1, \infty)$ and in many cases the spectral $p$-independence. Moreover, we show for a subclass of operators satisfying a homogeneity property that the Riesz transforms of all orders are bounded.

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1 Introduction

In this paper we consider a class of subelliptic operators given by a composition of differential and multiplication operators acting on $L_p(\mathbb{R}^k)$. These operators generate holomorphic semigroups which are consistent on $L_p(\mathbb{R}^k)$ for $p \in [1, \infty]$. Moreover, the semigroup operators turn out to be integral operators with a smooth kernel on $\mathbb{R}^k \times \mathbb{R}^k$. Examples of such operators include the (an)harmonic oscillator and the Hamiltonian for curved magnetic fields. All these operators are naturally associated to subelliptic operators on a nilpotent Lie group.

If $H$ is a subelliptic operator affiliated to a continuous representation $U$ of a Lie group $G$ then the closure generates a holomorphic semigroup $S$ which has a representation independent kernel $K$ such that

$$S_t = \int_G dg K_t(g) U(g).$$

For the kernel $K$ one has Gaussian bounds (see [ELR3]). Henceforth we consider a class of representations of a nilpotent Lie group on $L_p(\mathbb{R}^k)$. Under suitable conditions we show that the semigroup $S$ has a reduced heat kernel $\kappa$ such that

$$(S_t \varphi)(x) = \int_{\mathbb{R}^k} dy \kappa_t(x; y) \varphi(y)$$

for all $t > 0$, $\varphi \in C_0^\infty(\mathbb{R}^k)$ and $x \in \mathbb{R}^k$. The aim of this work is to prove Gaussian bounds for $\kappa$. Previously, Gaussian bounds for reduced heat kernels have been deduced in [ELR2] and [ELS] for semigroups generated by strongly elliptic operators affiliated to irreducible unitary representations of nilpotent Lie groups and in [ELS] for strongly elliptic operators on homogeneous spaces $G/M$ with $G$ unimodular and $M$ compact. In [Sik] Sikora proved off- and on-diagonal bounds for the kernels of semigroups generated by (second-order) Schrödinger operators with magnetic field and a potential of polynomial growth, satisfying a Nash inequality. The novelty of this paper is that the operators are weighted subcoercive instead of strongly elliptic. As a consequence of the Gaussian bounds for the kernel we obtain that $H$ has a bounded $H^\infty$-functional calculus on all the $L_p$-spaces with $p \in (1, \infty)$ and also in many cases the $p$-independence of the spectrum of $H$. We also show that the Riesz transforms are bounded on $L_p$ for $p \in (1, \infty)$.

Typical examples for the second order operators are the spinless particles of mass $m$ in an external magnetic field $\mathbf{B}$ where $\mathbf{B}$ is a polynomial. Then the Hamiltonian is
given by $H = -\frac{1}{2m}(\mathbf{p}^2 - \mathbf{A} \mathbf{A})^2$ where $\mathbf{p} = -i\hbar \nabla$ and $\mathbf{A}$ is a polynomial vector potential satisfying $\mathbf{B} = \nabla \times \mathbf{A}$, see [JoK] and [Sim]. Other examples are the anharmonic oscillators $(-\partial^2 / \partial x^2)^j + x^{2n}$, with $j, n \in \mathbb{N}$.

Throughout the following let $G$ be a connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then the exponential mapping on $\mathfrak{g}$ is surjective. Let $a_1, \ldots, a_d$ be an algebraic basis of $\mathfrak{g}$, i.e., $a_1, \ldots, a_d$ are independent and together with their multi-commutators span $\mathfrak{g}$. Let $U$ be a continuous representation of $G$ in a Banach space $X$. For $i \in \{1, \ldots, d\}$ let $A_i = dU(a_i)$ be the infinitesimal generator of the one parameter group $t \mapsto U(\exp(-ta_i))$. We also need multi-index notation. Set $J(d') = \bigcup_{n=0}^\infty \{1, \ldots, d\}^n$. If $\alpha = (i_1, \ldots, i_n) \in J(d')$ define $A^\alpha = A_{i_1} \ldots A_{i_n}$. Generally we adopt the notation of [ELR3].

The representations that we consider in this paper are of the following type. First we assume that there exist $a_{d'+1}, \ldots, a_d \in \mathfrak{g}$ such that $a_1, \ldots, a_d$ is a basis for $\mathfrak{g}$ and
[g, g] ⊂ span{a_{d+1}, \ldots, a_d}. Secondly, let \( k \leq d \). For \( p \in [1, \infty] \) let \( U \) be a representation of \( G \) in \( L_p(\mathbb{R}^k) \) of the form
\[
(U(\exp a)\varphi)(x) = e^{iE(x,a)}\varphi(x + \xi_a^{(0)})
\]
for all \( \varphi \in C_c^\infty(\mathbb{R}^k) \), where \( E: g \times \mathbb{R}^k \to \mathbb{R} \) is a real polynomial and
\[
\xi_a^{(0)} = (\xi_1, \ldots, \xi_k)
\]
for all \( a = \sum_{i=1}^d \xi_i a_i \in g \). It is straightforward to see that the representation is a continuous representation acting on \( L_p(\mathbb{R}^k) \) for all \( p \in [1, \infty] \).

Fix \( n_1, \ldots, n_{d'} \in \mathbb{N} \) and set
\[
H = \sum_{i=1}^{d'} (-1)^{n_i} A_i^{2n_i}
\]
with domain
\[
D(H) = \bigcap_{\alpha \in J(d')} D(A^\alpha)
\]
where \( m = 2\text{lcm}(n_1, \ldots, n_{d'}) \), \( w_i = (2n_i)^{-1} m \) for all \( i \in \{1, \ldots, d'\} \) and \( \|\alpha\| = w_i + \ldots + w_{i_n} \) if \( \alpha = (i_1, \ldots, i_n) \in J(d') \). Define the modulus \( \| \cdot \|: \mathbb{R}^k \to [0, \infty) \) by
\[
\| x \|^{2w} = \sum_{i=1}^k |x_i|^{2w_i}
\]
where \( x = (x_1, \ldots, x_k) \) and \( w = \text{lcm}(w_1, \ldots, w_k) \). The main result of this paper is the next theorem.

**Theorem 1.1** Let \( p \in [1, \infty] \). Then the following are satisfied.

I. **The closure \( \overline{H} \) of \( H \) generates a semigroup \( S \) in \( L_p(\mathbb{R}^k) \), which is holomorphic in the right half-plane.**

II. **For all \( t > 0 \) the semigroup operator \( S_t \) has a smooth kernel \( \kappa_t \in C^\infty(\mathbb{R}^k \times \mathbb{R}^k) \) such that the maps \( x \mapsto \kappa_t(x; y_0) \) and \( y \mapsto \kappa_t(x_0; y) \) belong to the Schwartz space \( S(\mathbb{R}^k) \) for all \( x_0, y_0 \in \mathbb{R}^k \) and**
\[
(S_t \varphi)(x) = \int_{\mathbb{R}^k} dy \kappa_t(x; y) \varphi(y)
\]
for all \( \varphi \in L_p(\mathbb{R}^k) \) and (a.e.) \( x \in \mathbb{R}^k \).

III. **There exist \( c, \tau > 0 \) such that**
\[
|\kappa_t(x; y)| \leq ct^{-Q/m} e^{-\tau(\|x-y\|^{m-1})^{1/(m-1)}}
\]
for all \( t > 0 \) and \( x, y \in \mathbb{R}^k \), where \( Q = w_1 + \ldots + w_k \).

Moreover, if \( A_i \) and \( B_i \) denote the left derivative of \( \kappa_t \) with respect to the first and second variable, respectively, and \( A^\alpha \) and \( B^\beta \) the corresponding multi-derivatives, then for all \( \alpha, \beta \in J(d') \) there exist \( c, \tau > 0 \) such that
\[
|(A^\alpha B^\beta \kappa_t)(x; y)| \leq ct^{-Q/\|\alpha\| + \|\beta\|/m} e^{-\tau(\|x-y\|^{m-1})^{1/(m-1)}}
\]
for all \( t > 0 \) and \( x, y \in \mathbb{R}^k \).
IV. For all $p \in (1, \infty)$ the operator $H$ is closed and for all $\alpha \in J(d')$ one has $D(H^{\|\alpha\|/m}) \subseteq D(A^\alpha)$. Moreover, there exists a $c > 0$ such that
\[\|A^\alpha \varphi\|_p \leq c \|H^{\|\alpha\|/m} \varphi\|_p\]
for all $\varphi \in D(H^{\|\alpha\|/m})$.

Statement I is a direct consequence of [EIR3]. To be precise, it follows from Example 4.4, Proposition 11.3 and Theorem 1.1 of [EIR3]. The sketch of the proof of the other three statements is as follows. The directions $a_1, \ldots, a_{d'}$ with the weights $w_1, \ldots, w_{d'}$ make the operator $H$ homogeneous. Unfortunately these weights do not in general allow one to define a family of dilations on $\mathfrak{g}$. This problem, however, can be circumvented by lifting the representation to a free nilpotent group $\mathcal{G}$. Then the semigroup has a kernel $\tilde{K}$ on $\mathcal{G}$ and one can relate the reduced heat kernel $\kappa$ with $\tilde{K}$. The Gaussian bounds for $\kappa$ follow by a projection and a scaling argument from the Gaussian bounds for $\tilde{K}$. Finally the boundedness of the Riesz transforms follows from transference.

Although Theorem 1.1 is formulated for operators $H$ which are sums of even powers, the conclusions of the theorem are with small modifications also valid for a larger class of operators affiliated to representations of the form $(1)$ of the group $G$. In Section 2 we prove Statements II and III of Theorem 1.1 in the generalized theorem and in Section 3 we give applications and examples. Finally, in Section 4 we discuss the boundedness of the Riesz transforms in case the operator is homogeneous. In that section the representation can be any induced representation from a character and the representation does not have to be of the form $(1)$. In particular the bounds are valid for any basis realization of an irreducible unitary representation.

2 Gaussian bounds

Before we can define the operators for which the generalization of Theorem 1.1 is valid we have to introduce a suitable free Lie group.

Let $a_1, \ldots, a_{d'}$ be an algebraic basis of the Lie algebra $\mathfrak{g}$ of a connected nilpotent Lie group $G$ and let $w_1, \ldots, w_{d'} \in \mathbb{N}$ be weights. For $\alpha = (i_1, \ldots, i_n) \in J(d')$ set $\|\alpha\| = w_{i_1} + \ldots + w_{i_n}$. Let
\[r = \max\{\|\alpha\| : \alpha = (i_1, \ldots, i_n) \in J(d') \text{ and } [a_{i_1}, [\ldots [a_{i_{n-1}}, a_{i_n}] \ldots]] \neq 0\} .\]

Let $\tilde{\mathfrak{g}}$ be the nilpotent Lie algebra with $d'$ generators $\tilde{a}_1, \ldots, \tilde{a}_{d'}$ which is free of weighted step $r$. So $\tilde{\mathfrak{g}}$ is the quotient of the free Lie algebra with $d'$ generators by the ideal spanned by the commutators $[\tilde{a}_{i_1}, [\ldots [\tilde{a}_{i_{n-1}}, \tilde{a}_{i_n}]] \ldots]$ with $\|(i_1, \ldots, i_n)\| \geq r + 1$. We give $\tilde{a}_i$ the weight $w_i$ for all $i \in \{1, \ldots, d'\}$. Then there exists a family $(\gamma_t)_{t > 0}$ of dilations of $\tilde{\mathfrak{g}}$ such that $\gamma_t(\tilde{a}_i) = t^{w_i} \tilde{a}_i$ for all $t > 0$ and $i \in \{1, \ldots, d'\}$ (see [NRS] or [EIR3], Example 2.7). Moreover, there exist $\tilde{a}_{d'+1}, \ldots, \tilde{a}_{d} \in \tilde{\mathfrak{g}}$ and $w_{d'+1}, \ldots, w_d \in \mathbb{N}$ such that $\tilde{a}_1, \ldots, \tilde{a}_{d'}$ is a basis for $\tilde{\mathfrak{g}}$ and $\gamma_t(\tilde{a}_i) = t^{w_i} \tilde{a}_i$ for all $t > 0$ and $i \in \{d'+1, \ldots, d\}$. Then $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] = \text{span}\{\tilde{a}_{d'+1}, \ldots, \tilde{a}_d\}$ (cf. [EIR3] Example 2.6).

Let $m \in \mathbb{N}$ be such that $m \geq 2w_1N$ for all $i \in \{1, \ldots, d'\}$ and for all $\alpha \in J(d')$ with $\|\alpha\| \leq m$ let $c_\alpha \in \mathbb{C}$. Moreover, let $\tilde{A}_i = dL_{\tilde{G}}(\tilde{a}_i)$ for all $i \in \{1, \ldots, d'\}$, where $\tilde{G}$ is
the connected simply connected Lie group with Lie algebra $\mathfrak{g}$ and $L_\mathfrak{g}$ is the left regular representation of $G$ on $L_2(G)$. Set
\[\mathcal{H} = \sum_{\|\alpha\| \leq m} c_\alpha \tilde{A}^\alpha\]
with domain $D(\mathcal{H}) = \bigcap_{\|\alpha\| \leq m} D(\tilde{A}^\alpha)$ and assume that $\mathcal{H}$ is a weighted subcoercive operator, i.e., there exist $\mu, \nu > 0$ such that
\[\text{Re}(\varphi, \mathcal{H} \varphi) \geq \mu \sum_{\|\alpha\| = m/2} \|\tilde{A}^\alpha \varphi\|_2^2 - \nu \|\varphi\|_2^2\]
for all $\varphi \in C^\infty_c(G)$, that is, $\mathcal{H}$ satisfies a Gårding inequality on $G$ (see [ElR3]). Here $\| \cdot \|_2$ is the norm on $L_2(G)$.

Let $U$ be a representation of $G$ in $L_p(\mathbb{R}^k)$ of the form (1). We consider the analogue operator
\[H = \sum_{\|\alpha\| \leq m} c_\alpha A^\alpha\]
with domain $D(H) = \bigcap_{\|\alpha\| \leq m} D(A^\alpha)$ and the same coefficients as the operator $\mathcal{H}$.

**Theorem 2.1** Let $U$ be a representation of the form (1) and
\[H = \sum_{\|\alpha\| \leq m} c_\alpha A^\alpha\]
as above. Let $p \in [1, \infty]$. Then the following are satisfied.

**I.** The closure $\mathcal{H}$ of $H$ generates a semigroup $S$ in $L_p(\mathbb{R}^k)$, which is holomorphic in a $p$-independent sector.

**II.** For all $t > 0$ the semigroup operator $S_t$ has a smooth kernel $\kappa_t \in C^\infty(\mathbb{R}^k \times \mathbb{R}^k)$ such that the maps $x \mapsto \kappa_t(x ; y_0)$ and $y \mapsto \kappa_t(x_0 ; y)$ belong to the Schwartz space $\mathcal{S}(\mathbb{R}^k)$ for all $x_0, y_0 \in \mathbb{R}^k$ and
\[(S_t \varphi)(x) = \int_{\mathbb{R}^k} dy \kappa_t(x ; y) \varphi(y)\]
for all $\varphi \in L_p(\mathbb{R}^k)$ and (a.e.) $x \in \mathbb{R}^k$.

**III.** For all $\alpha, \beta \in J(d)$ there exist $c, \tau > 0$ and $\omega \in \mathbb{R}$ such that
\[\|(A^\alpha B^\beta \kappa_t)(x ; y)\| \leq ct^{-Q + \|\alpha\| + \|\beta\|} \|\varphi\|_{L_p(\mathbb{R}^k)} e^{-\tau \|x - y\|^{m-1}}\]
for all $t > 0$ and $x, y \in \mathbb{R}^k$, where $Q = w_1 + \ldots + w_k$ and the modulus $\| \cdot \|$ on $\mathbb{R}^k$ is as in (2).

Moreover, if $H$ is a pure $m$-th order operator, i.e., $H = \sum_{\|\alpha\| = m} c_\alpha A^\alpha$, then $\omega$ can be taken equal to 0.

**IV.** The Schwartz space $\mathcal{S}(\mathbb{R}^k)$ is a core for $\mathcal{H}$. 

4
Proof  Statement I follows from Proposition 11.3 and Theorem 1.1 of [ElR3].

Next, since $\tilde{g}$ is free of weighted step $r$ there exists a unique Lie algebra homomorphism $\pi: \tilde{g} \to \mathfrak{g}$ such that $\pi(\tilde{a}_i) = a_i$ for all $i \in \{1, \ldots, d'\}$. The Lie algebra homomorphism $\pi$ lifts to a Lie group homomorphism $\Phi$ from $\tilde{G}$ onto $G$. For $\tilde{g} \in \tilde{G}$ define $\tilde{U}(\tilde{g}) = U(\Phi(\tilde{g}))$. Then $\tilde{U}$ is a continuous representation of $\tilde{G}$ in $L_p(\mathbb{R}^k)$. Define $\tilde{E}: \tilde{g} \times \mathbb{R}^k \to \mathbb{R}$ by

$$\tilde{E}(\tilde{a}, x) = E(\pi(\tilde{a}), x)$$

Let $\tilde{a} = \sum_{i=1}^{d'} \tilde{\xi}_i \tilde{a}_i \in \tilde{g}$. Since $\pi(\tilde{a}_i) = \pi([\tilde{g}, \tilde{g}]) = [\mathfrak{g}, \mathfrak{g}] \subset \text{span}\{a_{d'+1}, \ldots, a_d\}$ for all $i \in \{d'+1, \ldots, d\}$ and

$$\pi(\tilde{a}) = \sum_{i=1}^{d'} \tilde{\xi}_i a_i + \sum_{i=d'+1}^{d} \tilde{\xi}_i \pi(\tilde{a}_i)$$

it follows that $\tilde{\xi}_i^{(0)} = \xi_i^{(0)}$, where

$$\tilde{\xi}_i^{(0)} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_k)$$

Hence

$$\left(\tilde{U}(e^{\tilde{\text{exp}}} \tilde{a}) \varphi\right)(x) = e^{i\tilde{E}(\tilde{a}, x)} \varphi(x + \tilde{\xi}_i^{(0)})$$

for all $\varphi \in C_c(\mathbb{R}^k)$, $\tilde{a} \in \tilde{g}$ and $x \in \mathbb{R}^k$, where $\tilde{\text{exp}}$ is the exponential map on $\tilde{g}$. Thus the representation $\tilde{U}$ is of the same type as the representation $U$.

Note that $d\tilde{U}(a_i) = dU(a_i)$ for all $i \in \{1, \ldots, d'\}$. Therefore we can just as well use the group $\tilde{G}$ with the representation $\tilde{U}$ instead of the group $G$ with the representation $U$.

According to Theorem 1.1 of [ElR3] for all $t > 0$ there exists a $\tilde{K}_t \in \mathcal{S}(\tilde{G})$ such that

$$S_t \varphi = \int_{\tilde{G}} d\tilde{g} \tilde{K}_t(\tilde{g}) \tilde{U}(\tilde{g}) \varphi$$

for all $\varphi \in L_p(\mathbb{R}^k)$. Moreover, there exist $c, \tau > 0$ and $\omega \in \mathbb{R}$ such that

$$|\tilde{K}_t(\tilde{\text{exp}}(\tilde{a}))| \leq c t^{-D/m} e^{\omega t} e^{-\tau(||\tilde{d}||_{m-1})^{1/(m-1)}}$$

(3)

for all $t > 0$ and $\tilde{a} \in \tilde{g}$, where $\tilde{D} = w_1 + \ldots + w_{d'}$ and the modulus $|\cdot|$ on $\tilde{g}$ is defined by

$$\left| \sum_{i=1}^{d} \tilde{\xi}_i a_i \right|^{2\tilde{\omega}} = \sum_{i=1}^{d} \tilde{\xi}_i^{2\tilde{\omega}/w_i}$$

and $\tilde{\omega} = \text{lcm}(w_1, \ldots, w_{d'})$. By scaling one can take $\omega = 0$ in case the operator $\tilde{\mathcal{H}}$ is homogeneous. Next, set $\mathfrak{h} = \text{span}\{\tilde{a}_{d'+1}, \ldots, \tilde{a}_d\}$ and for $y = (y_1, \ldots, y_k) \in \mathbb{R}^k$ define $\hat{y} \in \tilde{g}$ by

$$\hat{y} = y_1 \tilde{a}_1 + \ldots + y_k \tilde{a}_k$$

Since $\tilde{E}$ is real valued one can define for all $t > 0$ the function $\kappa_t \in C^\infty(\mathbb{R}^k \times \mathbb{R}^k)$ by

$$\kappa_t(x; y) = \int_{\mathfrak{h}} db \tilde{K}_t(\tilde{\text{exp}}(b + \hat{y})) e^{i\tilde{E}(\tilde{x} + b, y)}$$

Then it is easy to verify that Statement II of Theorem 2.1 is valid.
Now we prove Statement III. It follows from the Gaussian bounds (3) that

\[
|\kappa_t(x;y)| \leq \int \frac{d^\beta c t^{-D/m}}{\pi^m} e^{\omega(t)(|\tilde{b} - \tilde{x} + \tilde{y}|^{m-1})^{1/(m-1)}}
\]

\[
\leq c t^{-D/m} e^{\omega(t)} \int \frac{d^\beta c e^{-2t^2(|\tilde{b} - \tilde{x} + \tilde{y}|^{m-1})^{1/(m-1)}}}{\pi^m} e^{-2t^2(|\tilde{b} - \tilde{x} + \tilde{y}|^{m-1})^{1/(m-1)}}
\]

\[
= c t^{-Q/m} e^{\omega(t)} e^{-2t^2(|\tilde{b} - \tilde{x} + \tilde{y}|^{m-1})^{1/(m-1)}} \left( t^{-Q/m} \int \frac{d^\beta c e^{-2t^2(|\tilde{b} - \tilde{x} + \tilde{y}|^{m-1})^{1/(m-1)}}}{\pi^m} \right)
\]

for all \( t > 0 \) and \( x, y \in \mathbb{R}^k \). But the quantity between the brackets is independent of \( t \) (and also of \( x \) and \( y \)), by scaling. Moreover, there exists a \( \tau' > 0 \) such that \( ||z|| \leq \tau' |\tilde{z}| \) for all \( z \in \mathbb{R}^k \) with \( ||z|| \leq 1 \). Hence, again by scaling, it follows that \( ||z|| \leq \tau' |\tilde{z}| \) for all \( z \in \mathbb{R}^k \). Therefore the proof of the Gaussian bounds of Statement III is complete if \( ||\alpha|| = ||\beta|| = 0 \).

Next we consider derivatives of the reduced heat kernel \( \kappa_t \). If \( \alpha, \beta \in J(d') \) then

\[
(A^\alpha B^\beta \kappa_t)(x;y) = \int \frac{d^\beta}{\pi^m} (\tilde{A}^\alpha \tilde{B}^\beta \tilde{K}_t) \left( \tilde{e}x \tilde{p}(\tilde{b} - \tilde{x} + \tilde{y}) \right) e^{i\tilde{E}(\tilde{b} - \tilde{x} + \tilde{y},x)}
\]

for all \( t > 0 \) and \( x, y \in \mathbb{R}^k \), where \( \tilde{B}_i = dR_G(\tilde{a}_i) \) for all \( i \in \{1, \ldots, d'\} \) and \( R_G \) is the right regular representation on \( \tilde{G} \). Since one has Gaussian bounds

\[
|A^\alpha B^\beta \kappa_t| \leq c t^{-Q + ||\alpha|| + ||\beta||/m} e^{\omega(t)} e^{-2t^2(|\tilde{b} - \tilde{x} + \tilde{y}|^{m-1})^{1/(m-1)}}
\]

by [ELR3], Theorem 1.1, one can estimate \( A^\alpha B^\beta \kappa_t \) as in (4). Again \( \omega \) can be taken equal to 0 if \( H \) is homogeneous.

Finally, since \( A_i = \partial_1 / \partial x_i + M_i \) for all \( i \in \{1, \ldots, k\} \) with \( M_i \) a multiplication operator with a polynomial it follows from the Gaussian bounds that \( S_t \) maps \( \mathcal{S}(\mathbb{R}^k) \) into \( \mathcal{S}(\mathbb{R}^k) \) for all \( t > 0 \). Therefore \( \mathcal{S}(\mathbb{R}^k) \) is a core for \( \mathcal{H} \) (see [BrR] Corollary 3.1.7). This completes the proof of Theorem 2.1.

\[\Box\]

3 Applications and examples

The Gaussian bounds have several implications. The first is the \( p \)-independence of the spectrum if all weights equal one. Note that it will follow from Theorem 4.2 that the operator \( H \) is already closed on \( L_p \) for all \( p \in \{1, \infty\} \).

**Corollary 3.1** Assume the notation and conditions of Theorem 2.1. Moreover, suppose that \( w_i = 1 \) for all \( i \in \{1, \ldots, d'\} \). Then for all \( p \in \{1, \infty\} \) the spectrum \( \sigma_p(\mathcal{H}) \) of the operator \( \mathcal{H} \) on \( L_p(\mathbb{R}^k) \) is independent of \( p \).

**Proof** This follows from [Kun], or [LiV]. \[\Box\]

Note that the spectrum \( \sigma(\mathcal{H}) \) is independent of \( p \in [1, \infty] \) if the representation \( U \) is irreducible, by the arguments given in the proof of Theorem 2.5 of [ELR2].

The second implication of the Gaussian bounds is that the bounded \( H_\infty \)-functional calculus on \( L_2 \) extends to all \( L_p \) spaces.
Corollary 3.2 Assume the notation and conditions of Theorem 2.1. Then for all \( p \in (1, \infty) \) and large enough \( \lambda > 0 \) the operator \( \mathcal{H} + \lambda I \) has a bounded \( H_\infty \) functional calculus on \( L_p(\mathbb{R}^k) \). If the operator \( \mathcal{H} \) is homogeneous then one can take \( \lambda = 0 \).

Proof By [EIR3] Theorem 9.2.111 the semigroup generator \( \mathcal{H} \) satisfies a Gårding inequality on \( L_2 \). Therefore \( \mathcal{H} + \lambda I \) is maximal accretive if \( \lambda > 0 \) is large enough. Hence it follows from Theorem G of [ADM] that the operator \( \mathcal{H} + \lambda I \) has a bounded \( H_\infty \) functional calculus on \( L_2 \). Then the corollary is a consequence of the Gaussian bounds of Theorem 2.1.111 and [DuR] Theorem 3.4. \( \Box \)

Quadrature of the Gaussian bounds gives semigroup bounds.

Corollary 3.3 Assume the notation and conditions of Theorem 2.1. Then there exist \( c > 0 \) and \( \omega \in \mathbb{R} \) such that for all \( p, q \in [1, \infty] \) with \( p \leq q \) one has

\[
\|S_t\|_{p \to q} \leq c t^{-Q(1/p-1/q)/m} e^{\omega t}
\]

uniformly for all \( t > 0 \).

Proof Obviously \( \|S_t\|_{1 \to \infty} = \|\kappa_t\|_{\infty} \leq c t^{-Q/m} e^{\omega t} \) by the bounds of Theorem 2.1.111. Next, let \( c, \tau \) be as in Theorem 2.1.111 with \( \|\alpha\| = \|\beta\| = 0 \). For \( t > 0 \) define \( G_t : \mathbb{R}^k \to \mathbb{R} \) by \( G_t(x) = c t^{-Q/m} e^{-\tau(\|x-y\|^{m-1})^{1/(m-1)}} \). Then

\[
|\langle S_t \varphi, \psi \rangle| \leq e^{\omega t} \int_{\mathbb{R}^k} dy \ G_t(x-y) |\varphi(y)| = e^{\omega t} (G_t * |\varphi|)(x)
\]

for all \( \varphi \in C_c^\infty(\mathbb{R}^k) \) and \( x \in \mathbb{R}^k \), where * denotes the convolution on the commutative group \( \mathbb{R}^k \). Therefore, \( \|S_t\|_{p \to q} \leq \|G_t\|_{1 \to 1} e^{\omega t} = \|G_t\|_{1 \to 1} e^{\omega t} \) for all \( t > 0 \) and \( p \in [1, \infty] \), by scaling. Now the corollary follows by interpolation. \( \Box \)

If the spectrum \( \sigma(\mathcal{H}) \) of \( \mathcal{H} \) is a subset of \( (0, \infty) \) then one also has exponential decay for \( t \to \infty \) in the Gaussian bounds. Furthermore, the decay at infinity of the kernel is almost equal to the growth bound of the semigroup on \( L_2 \).

Proposition 3.4 Assume the notation and conditions of Theorem 2.1. Let

\[
\lambda_1 = \inf \{ \Re(\varphi, H\varphi) : \varphi \in C_c^\infty(\mathbb{R}^k) \}.
\]

Then for all \( \varepsilon > 0 \) there exist \( c, \tau > 0 \) such that

\[
|\kappa_t(x ; y)| \leq c t^{-Q/m} e^{-\tau(\|x-y\|^{m-1})^{1/(m-1)}}
\]

for all \( t > 0 \) and \( x, y \in \mathbb{R}^k \).

Proof It follows from semigroup theory that \( \|S_t\|_{2 \to 2} \leq e^{-\lambda_1 t} \) for all \( t > 0 \). Let \( \varepsilon > 0 \). Then by Corollary 3.3 there exist \( c, \omega > 0 \) such that

\[
\|\kappa_t\|_{\infty} = \|S_t\|_{1 \to \infty} \leq \|S_{t/2}\|_{1 \to 2} \|S_{1-t/2}\|_{2 \to 2} \|S_{t/2}\|_{2 \to \infty} \leq c t^{-Q/m} e^{-\lambda_1 (1-\varepsilon) t} e^{\omega t}
\]

for all \( t > 0 \). Hence interpolation with the Gaussian estimates of Theorem 2.1.111 gives

\[
|\kappa_t(x ; y)| = |\kappa_t(x ; y)| \leq c t^{-Q/m} e^{-\tau(\|x-y\|^{m-1})^{1/(m-1)}} t^{-Q/m} e^{-\lambda_1 (1-\varepsilon) t} e^{\omega t}
\]

for all \( t > 0 \) and \( x, y \in \mathbb{R}^k \), from which the proposition follows. \( \Box \)
Example 3.5 Let $j, n \in \mathbb{N}$. Then the anharmonic oscillator is the operator

$$H_0 = \left(-\frac{d^2}{dx^2}\right)^j + x^{2n}$$

on $L_2(\mathbb{R})$ and domain the Schwartz space. This operator is a special example for which Theorem 1.1 applies in the following way. Let $G$ be the connected simply connected Heisenberg group with Lie algebra $\mathfrak{g}$ and let $a_1, a_2, a_3$ be a basis of $\mathfrak{g}$ such that $[a_1, a_2] = a_3$. Then the standard irreducible representation $U$ of $G$ is given by

$$\left(U(\exp(\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3))\varphi\right)(x) = e^{i\xi_3} e^{ix\xi_2} \varphi(x + \xi_1)$$

for all $\varphi \in C_0^\infty(\mathbb{R})$. So $U$ is of the form (1) and if one takes $k = 1$ and $d = 2$. Then $A_1 = -iP$ and $A_2 = iQ$, where $P$ and $Q$ are the self-adjoint operators in $L_2(\mathbb{R})$ given by $(Pf)(x) = x f'(x)$ and $(Qf)(x) = x f(x)$ for all $f \in C_0^\infty(\mathbb{R})$ and $x \in \mathbb{R}$. If

$$H = (-1)^j A_1^{2j} + (-1)^n A_2^{2n}$$

then the operator $H_0$ is the restriction of the self-adjoint operator $H$ to the Schwartz space, which is a core for $H$ (see [ElR1], Example 7.1). Let $d_0 = \gcd(j, n)$. Then $w_1 = n/d_0$, $w_2 = j/d_0$, $Q = n/d_0$ and the weighted order of $H$ equals $m = 2j n / d_0$. Moreover, $\|x\| = |x|$ for all $x \in \mathbb{R}$.

Let $\kappa$ be the reduced heat kernel of the semigroup generated by $H$. Then it is a consequence of Theorem 1.1 that there are $c, \tau > 0$ such that

$$|\kappa_t(x, y)| \leq c t^{-1/2j} e^{-\tau(|x-y|^2 / d_0^n x^{2j})^{1/2j}}$$

for all $t > 0$ and $x, y \in \mathbb{R}$. Moreover, if $j = n = 1$ then the smallest eigenvalue of $H$ equals $\lambda_1 = 1$ and it follows from Proposition 3.4 that for all $\varepsilon > 0$ there are $c, \tau > 0$ such that

$$|\kappa_t(x, y)| \leq c t^{-1/2} e^{-(1-\varepsilon) t} e^{-\tau |x-y|^2 / 4}$$

for all $t > 0$ and $x, y \in \mathbb{R}$. Note that these bounds are consistent with the explicit expression for $\kappa$ by Mehler’s formula;

$$\kappa_t(x, y) = \left(\pi(1 - e^{-4t})\right)^{-1/2} e^{-(x+y)^2(tanht)/4} e^{-(x-y)^2(cotht)/4} e^{-t}$$

for all $t > 0$ and $x, y \in \mathbb{R}$ (see [Dav] Theorem 7.13).

Lower order terms are also allowed. If, for example, $H$ is an operator of the form

$$H = \left(-\frac{d^2}{dx^2}\right)^j + \mu x^{2n} + \sum_{i, k \geq 0 \atop l + kn < 2jn} c_{ik} x^l \frac{d^k}{dx^k}$$

with $\mu > 0$, $c_{ik} \in \mathbb{C}$ and domain $D(H) = D(P^{2j}) \cap D(Q^{2n})$ then the semigroup generated by $H$ has a smooth reduced heat kernel $\kappa$. Moreover, there are $c, \tau, \omega > 0$ such that

$$|\kappa_t(x, y)| \leq c t^{-1/2j} e^{\omega t} e^{-\tau(|x-y|^2 / d_0^n x^{2j})^{1/2j}}$$

for all $t > 0$ and $x, y \in \mathbb{R}$. A typical example of an operator $H$ in (5) is the operator

$$H = -\frac{d^2}{dx^2} + \mu x^4 + \mu' x^2$$

with $\mu > 0$ and $\mu' \in \mathbb{R}$. 

8
Example 3.6 Let \( \mathfrak{m} \) be an ideal in the Lie algebra \( \mathfrak{g} \) of a connected simply connected Lie group \( G \), let \( M \) be the connected (and simply connected) subgroup of \( G \) with Lie algebra \( \mathfrak{m} \) and let \( \chi \) be a one dimensional representation of \( M \). Set \( k = d - \dim \mathfrak{m} \). Let \( a_1, \ldots, a_d \) be a basis for \( \mathfrak{g} \) such that \( a_{k+1}, \ldots, a_d \) is a basis for \( \mathfrak{m} \). Finally, assume that \( d' \geq k \) is such that \( a_1, \ldots, a_{d'} \) is an algebraic basis for \( \mathfrak{g} \) and \( [\mathfrak{g}, \mathfrak{g}] \subseteq \{a_{d'+1}, \ldots, a_d\} \). Then the basis realization of the induced representation \( \text{Ind}(M \uparrow G, \chi) \) is of the form (1). This follows immediately from the description in [CoG] p. 125 and the fact that \( \mathfrak{m} \) is an ideal.

As an example reconsider the Hamiltonian with polynomial vector field \( \tilde{A} \) and magnetic field \( \tilde{B} = \tilde{\nabla} \times \tilde{A} \). In order to avoid confusion between the components of the vector field \( \tilde{A} \) and the infinitesimal generators \( A_j \) which we introduce below, we denote the components of the components of the vector field \( \tilde{A} \) by \( A_j^{(M)} \). Set \( X_j = \hbar \partial_j - \imath c e^{-1} A_j^{(M)} \) with domain \( \mathcal{S}(\mathbb{R}^3) \) for all \( j \in \{1, 2, 3\} \). Then \( [X_i, X_j] = -\imath \hbar c e^{-1} \sum_{k=1}^3 \varepsilon_{ijk} B_k \) for all \( i, j \in \{1, 2, 3\} \). Since multiplication operators commute and the \( B_j \) are polynomials it follows that \( X_1, X_2, X_3 \) generate a finite dimensional Lie subalgebra \( \mathfrak{g} \) of operators in \( \text{Hom}(\mathcal{S}(\mathbb{R}^3)) \), the space of all linear operators acting on the Schwartz space \( \mathcal{S}(\mathbb{R}^3) \). Extend \( X_1, X_2, X_3 \) to a basis \( X_1, X_2, X_3, \ldots, X_d \) for \( \mathfrak{g} \) such that \( X_1, \ldots, X_d \) are all polynomial multiplication operators, say with polynomials \( \psi_1, \ldots, \psi_d \). Then \( \mathfrak{m} = \text{span}\{X_1, \ldots, X_d\} \) is an (Abelian) ideal in \( \mathfrak{g} \). Moreover, there exists a unique linear map \( i: \mathfrak{g} \to \mathfrak{c} \) such that \( i(X_j) = -\imath \hbar c e^{-1} A_j^{(M)}(0) \) for all \( j \in \{1, 2, 3\} \) and \( i(X_j) = -\imath \psi_j(0) \) for all \( j \in \{4, \ldots, d\} \). Define \( \chi: M \to \mathfrak{c} \) by \( \chi(\exp a) = \exp(i l(a)) \) for all \( a \in \mathfrak{m} \). Then it follows from [HeN] Proposition II.1.6.1 and its proof that \( \chi \) is a one dimensional representation of \( M \) and that the basis realization \( U \) of the induced representation \( \text{Ind}(M \uparrow G, \chi) \) with respect to the basis \( a_1, \ldots, a_d \) given by \( a_j = \hbar^{-1} X_j \) is of the form (1). For \( j \in \{1, 2, 3\} \) let \( A_j = \imath dU(a_j) \) be the associated infinitesimal generator. Note that \( X_j \psi = \hbar A_j \psi \) for all \( \psi \in \mathcal{S}(\mathbb{R}^3) \). One can take \( d = 3 \) and set \( H = -\frac{\hbar^2}{2m}(A_1^2 + A_2^2 + A_3^2) \). By Theorem 2.1.IV it follows that \( \mathcal{H} \) is the closure of the operator

\[
H_0 = \frac{1}{2m}(\vec{p} - \frac{e}{c} \vec{A})^2 = -\frac{1}{2m}(X_1^2 + X_2^2 + X_3^2)
\]

with domain \( \mathcal{S}(\mathbb{R}^3) \). Moreover, Theorem 2.1 states that the semigroup generated by \( \mathcal{H} \) has a reduced heat kernel \( \kappa \) and there are \( c, \tau > 0 \) such that

\[
|\kappa_t(x, y)| \leq c t^{-3/2} e^{-\tau|x-y|^2} t^{-i}
\]

for all \( t > 0 \) and \( x, y \in \mathbb{R}^3 \), where \( |x-y| \) is the Euclidean modulus of \( x - y \).

Remark 3.7 It can be proved as above that any operator associated with a representation of the form (1) equals an operator associated with an induced representation as described in the first part of Example 3.6 on a possibly different Lie group.

4 Riesz transforms

If \( H = \sum_{\|a\|_m} c_a A^a \) is acting on \( L_p(\mathbb{R}^k) \) and is such that the comparable operator \( \mathcal{H} = \sum_{\|a\|_m} c_a A^a \) is a homogeneous weighted subcoercive operator then in this section we show that the Riesz transforms of all orders are bounded on \( L_p(\mathbb{R}^k) \) for all \( p \in (1, \infty) \). The result relies on an application of the transference theorem in [CoW], which holds naturally for kernels in \( L_1(\hat{G}) \), and a technique that has been used in the the study of the Riesz.
transforms of all orders for homogeneous subcoercive operator with complex coefficients in [ERS] Section 4. We stress that in the present context the representation of the nilpotent group $G$ can be any representation induced from a character, including the basis realization of a unitary irreducible representation (see [Kir], [CoG] and [ElR2], Lemma 2.1).

**Theorem 4.1** Let $(\mathcal{M}, \mu)$ be a $\sigma$-finite measure space, $p \in (1, \infty)$ and $U$ a continuous bounded representation of $G$ in $L_p(\mathcal{M})$. Suppose $\overline{H} = \sum_{\|\alpha\|=m} c_\alpha A_\alpha$ is a homogeneous weighted subcoercive operator of order $m$ on $\hat{G}$ and set

$$H = \sum_{\|\alpha\|=m} c_\alpha A_\alpha$$

with domain $D(H) = \cap_{\|\alpha\|=m} D(A_\alpha)$. Then $H$ is closed, generates a bounded semigroup and for all $\alpha \in J(d')$ one has $D(H^{\|\alpha\|/m}) \subset D(A_\alpha)$. Moreover, there exists a $c > 0$ such that

$$\|A_\alpha \varphi\|_p \leq c \|H^{\|\alpha\|/m} \varphi\|_p$$

for all $\varphi \in D(H^{\|\alpha\|/m})$.

**Proof** Let $\Phi: \hat{G} \to G$ be as in the proof of Theorem 2.1. Let $\tilde{U} = U \circ \Phi$. Then $\tilde{U}$ is a continuous representation of $\hat{G}$ in $L_p(\mathcal{M})$. If $\tilde{K}$ is the kernel of the semigroup $\tilde{S}$ generated by $\tilde{H}$ and $S$ the semigroup generated by $\Pi$ then $S_t \varphi = \int_G \tilde{k}(\tilde{g}) \tilde{U}(\tilde{g}) \varphi.$ for all $t > 0$ and $\varphi \in L_p(\mathcal{M})$. Then the transference method of [CoW], Theorem 2.4, together with a density argument, gives the bounds

$$\|S_t\|_{p-p} \leq c^2 \|\tilde{S}_t\|_{\tilde{p}-\tilde{p}} \leq c^2 \|\tilde{K}_t\|_1 = c^2 \|\tilde{K}_1\|$$

uniformly for all $t > 0$, where $\|\cdot\|_{\tilde{p}}$ is the norm on $L_p(\hat{G})$ and $c = \sup_{\tilde{g} \in G} \|U(\tilde{g})\|_{p-p}$. So $\Pi$ generates a bounded semigroup.

If $n \in \mathbb{N}$ is large enough then for all $\nu, \varepsilon > 0$ the convolution kernel $\tilde{k}_{\nu;\varepsilon}$ of the operator

$$\tilde{R}_{\nu;\varepsilon} = \tilde{A}_\nu (\nu I + \tilde{H})^{-\|\nu\|/m} (I + \varepsilon \tilde{H})^{-n}$$

is in $L_1(\hat{G})$. Since $A_i = dU(a_i) = d\tilde{U}(\tilde{a}_i)$ for all $i \in \{1, \ldots, d\}$ it follows that

$$R_{\nu;\varepsilon} = A_\nu (\nu I + \Pi)^{-\|\nu\|/m} (I + \varepsilon \Pi)^{-n} = \int_G d\tilde{g} \tilde{k}_{\nu;\varepsilon} \tilde{U}(\tilde{g}) \tilde{U}(\tilde{g}).$$

The transference method then gives the estimates

$$\|R_{\nu;\varepsilon}\|_{p-p} \leq c^2 \|\tilde{R}_{\nu;\varepsilon}\|_{\tilde{p}-\tilde{p}}$$

uniformly for all $\nu, \varepsilon > 0$. But the right hand side of (6) is bounded uniformly for all $\nu, \varepsilon > 0$ by scaling on $\hat{G}$ (cf. [ERS] Lemma 4.1). Hence there exists an $M > 0$ such that

$$\|R_{\nu;\varepsilon}\|_{p-p} \leq M$$

uniformly for all $\nu, \varepsilon > 0$. Then

$$\|A_\nu \varphi\|_p \leq M \|(\nu I + \Pi\|\nu\|/m (I + \varepsilon \Pi)^n \varphi\|_p$$

for all $\varphi \in D_\infty(\Pi) = \cap_{\beta \in J(d')} D(A^\beta)$. Taking the limit $\varepsilon \downarrow 0$ it follows that

$$\|A_\nu \varphi\|_p \leq M \|(\nu I + \Pi\|\nu\|/m \varphi\|_p$$

(7)
for all $\varphi \in D_{\infty}(\mathcal{H})$.

Now let $\nu > 0$, $N \in \mathbb{N}$ and $\varphi \in D(\mathcal{H}^{N/m}) = D((\nu I + \mathcal{H})^{N/m})$. Since $D_{\infty}(\mathcal{H})$ is a core for the operator $(\nu I + \mathcal{H})^{N/m}$ there are $\varphi_1, \varphi_2, \ldots \in D_{\infty}(\mathcal{H})$ such that $\lim \varphi_n = \varphi$ and $\lim(\nu I + \mathcal{H})^{N/m} \varphi_n = (\nu I + \mathcal{H})^{N/m} \varphi$. Then $\lim (\nu I + \mathcal{H})^{j/m} \varphi_n = (\nu I + \mathcal{H})^{j/m} \varphi$ for all $j \in \{0, 1, \ldots, N\}$. Hence by induction on the number of indices of the multi-index $\alpha$ and the closedness of the $A_i$ it follows from the estimates (7) that $\lim A^\alpha \varphi_n = A^\alpha \varphi$ for all $\alpha$ with $|\alpha| \leq N$. So $D(\mathcal{H}^{\alpha/m}) \subset D(A^\alpha)$ for all $\alpha \in J(d')$ and the estimates (7) are valid uniformly for all $\varphi \in D(\mathcal{H}^{\alpha/m})$ and $\nu > 0$. Taking the limit $\nu \downarrow 0$ yields $\|A^\alpha \varphi\|_p \leq M \|\mathcal{H}^{\alpha/m} \varphi\|_p$ for all $\varphi \in D(\mathcal{H}^{\alpha/m})$.

Finally, one has as a special case that $D(\mathcal{H}) \subset D(A^\alpha)$ for all $\alpha$ with $|\alpha| = m$. Therefore the operator $H$ is closed.

In the unweighted case, i.e., if $w_1 = \ldots = w_d = 1$, then one can prove as in Corollary 4.3 of [ERS] that

\[
D(H^{n/m}) = \bigcap_{|\alpha| = nw} D(A^\alpha)
\]

for all $n \in \mathbb{N}$ and that the seminorms on the two spaces are equivalent. It is unclear whether the equality (8) is also valid in the weighted case.

Finally, for non-homogeneous operators we prove optimal regularity for any weighted sub coercive operator.

**Theorem 4.2** Let $(\mathcal{M}, \mu)$ be a $\sigma$-finite measure space, $p \in (1, \infty)$ and $U$ a continuous bounded representation of $G$ in $L_p(\mathcal{M})$. Suppose $\mathcal{H} = \sum_{|\alpha| \leq m} c_\alpha A^\alpha$ is a weighted sub coercive operator of order $m$ on $G$, where $L_G$ is the left regular representation on $L_p(G)$ and $\hat{A}_i = dL_G(a_i)$ for all $i \in \{1, \ldots, d\}$. Let

\[
H = \sum_{|\alpha| \leq m} c_\alpha A^\alpha
\]

be the corresponding operator on $L_p(\mathcal{M})$. Then $H$ is closed and for all $\alpha \in J(d')$ one has $D((H + \lambda I)^{\alpha/m}) \subset D(A^\alpha)$ if $\lambda > 0$ is large enough. Moreover, there exists a $c > 0$ such that

\[
\|A^\alpha \varphi\|_p \leq c \|(H + \lambda I)^{\alpha/m} \varphi\|_p
\]

for all $\varphi \in D((H + \lambda I)^{\alpha/m})$.

**Proof** We may assume that $\mathcal{H}$ generates an exponentially decreasing semigroup $\hat{S}$ on $L_p(G)$. The proof of the theorem is similar to the proof of Theorem 4.1. If $n \in \mathbb{N}$ is large enough then for all $\varepsilon > 0$ the convolution kernel of the operator

\[
\hat{R}_{\alpha; \varepsilon} = \hat{A}^\alpha \hat{H}^{-\alpha/m} (I + \varepsilon \hat{H})^{-n}
\]

is in $L_1(G)$. Moreover, $D(\mathcal{H}^{\alpha/m}) \subset D(\hat{A}^\alpha)$ and the embedding is continuous in $L_p(G)$-sense by [EIR3], Section 9. Next,

\[
\|\hat{R}_{\alpha; \varepsilon}\|_{L_p(G) \to L_p(G)} \leq \|\hat{A}^\alpha \hat{H}^{-\alpha/m}\|_{L_p(G) \to L_p(G)} \|\hat{H}^{-n}\|_{L_p(G) \to L_p(G)}
\]

and if $M = \sup_{t > 0} \|\hat{S}_t\|_{L_p(G) \to L_p(G)}$ then

\[
\|\hat{H}^{-n}\|_{L_p(G) \to L_p(G)} \leq (n - 1)!^{-1} \int_0^\infty dt \ e^{-t} t^{n-1} \|\hat{S}_t\|_{L_p(G) \to L_p(G)} \leq M
\]

\[11\]
uniformly for $\varepsilon > 0$. Therefore the operators $\hat{R}_{n,\varepsilon}$ are bounded on $L_p(G)$ uniformly for $\varepsilon > 0$. The rest of the proof is by the same arguments as in the proof of Theorem 4.1. It relies on the transference method. \[\square\]

Note that in fact the above argument can be applied to any continuous bounded representation of an amenable Lie group $G$ in $L_p(M)$ where $(M, \mu)$ is a $\sigma$-finite measure space.

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**References**


