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Sensor and Actuator Placement for Proportional Feedback Control in Advection-Diffusion Equations


Abstract—In this paper, advection-diffusion equations with constant coefficients on infinite one-dimensional and two-dimensional spatial domains are considered. Suitable sensor and/or actuator locations are determined for which high-gain and low-gain proportional feedback can effectively reduce the influence of a disturbance at a point of interest. These locations are characterized by simple analytic expressions which can be used as guidelines for control system design. The obtained analytic expressions are validated by numerical results.

Index Terms—Distributed parameter systems, Control system architecture, PID control

I. INTRODUCTION

The placement of sensors and actuators is an important aspect of control system design for which many techniques have been developed (see e.g. [1]). The simplest approach to this problem is to choose a controller design method and evaluate the resulting closed-loop performance for a large number of sensor and/or actuator locations. However, such an approach is computationally demanding when there are a large number of potential sensor and/or actuator locations. This is typically the case when the problem is governed by Partial Differential Equations (PDEs).

Several publications have considered the sensor and/or actuator placement problem for parabolic PDEs such as the heat equation. For example, Burns and Rubio [2] proposed a strategy for sensor placement based on the kernel representation of feedback operators in infinite-dimensional linear quadratic estimation problems. More recent results can be found in [3], [4]. Another approach proposed by Armaou and Demetriou [5] used modal observability and controllability measures to determine optimal sensor and actuator locations in parabolic PDEs. In the approach proposed by Vaidya et al. [6] the optimal area for sensing or actuation in parabolic PDEs is determined by maximizing the support of the observability or controllability Gramian, respectively. The idea to determine the optimal area for sensing was also considered by Privat et al. [7], who proposed a method to determine the optimal area for sensing by minimizing a randomized observability constant. A similar idea can be used to determine the optimal area for actuation [8].

These approaches do not explicitly address the design of the resulting feedback controller (and/or observer). This is an important aspect of the control system design which may affect the found optimal sensor and actuator locations. Such combined actuator location and controller optimization problems have been considered in [9], [10], [11], [12]. The dual problem in which the variance of the estimation error is minimized over sensor locations and observers has also been addressed in [13], [3], [4]. An additional problem is that practical implementation requires that the order of the controller (and/or observer) is sufficiently low. This seems to be a largely open problem, for which some potential solutions have been proposed by Demetriou [14], [15].

In many of the approaches that address the design of a feedback controller (and/or observer), the optimal sensor and/or actuator locations by searching through a discrete set of potential sensor and/or actuation locations, see e.g. [10], [13], [14]. Such an approach does not seem to use the connection to the underlying PDE fully.

In this paper, the placement of a single sensor and/or a single actuator in advection-diffusion equations with proportional feedback control is addressed. The problem is considered on one- and two-dimensional infinite spatial domains. Based on analytic expressions for the transfer functions, geometric rules that characterize the optimal sensor and actuator locations for high-gain and low-gain feedback are derived. Numerical experiments indicate that the derived rules accurately predict all (locally) optimal actuator and/or sensor locations.

The remainder of this paper is structured as follows. In Section II, the analysis and numerical results for the one-dimensional spatial domain are presented. In Section III, the two-dimensional spatial domain is considered. In Section IV, the conclusions are formulated and future work is discussed.

II. ONE-DIMENSIONAL SPATIAL DOMAIN

Consider the following PDE with constant coefficients on the one-dimensional spatial domain $x \in \mathbb{R}$

$$
\frac{\partial T}{\partial t} = v \frac{\partial T}{\partial x} + D \frac{\partial^2 T}{\partial x^2} - hT + \frac{1}{c} \delta(x - x_u)u(t) + \frac{1}{c} \delta(x - x_w)w(t),
$$

$$
y(t) = T(x_y, t), \quad z(t) = T(x_z, t),
$$

where $T$ is the temperature, $v$ is the velocity, $D$ is the diffusion coefficient, $h$ is the heat transfer coefficient, and $\delta$ is the Dirac delta function.
Remark 1. The \( H_{\infty} \)-norm of \( G(s,x) \) is bounded by
\[
\|G(\cdot,x)\|_{\infty} = G(0,x) \leq \frac{e^{-(x_w+|x|)/2D}}{c\sqrt{v^2 + 4D(h+s)/2D}}. \tag{5}
\]
Therefore, the advection dominated control problem (i.e., \( |v(x_w-x)| > 2D \)) is only relevant when \( x_w \) lies downstream of \( x \) (i.e., \( x_w < x \)). Otherwise, the bound on the \( H_{\infty} \)-norm in (5) shows that \( \|G_{zw}\|_{\infty} \approx 0 \) and control is not necessary.

Under proportional control \( u(t) = -P_y(t) \), the closed loop transfer function from \( W(s) \) to \( Z(s) \) is given by
\[
M(s) := G_{zw}(s) - \frac{G_{zw}(s)PG_{yu}(s)}{1 + PG_{yu}(s)}. \tag{6}
\]
The objective is to find a sensor location \( x_y \) and/or an actuator location \( x_u \) for which there exists a (stabilizing) feedback gain \( P \) that makes \( \|M\|_{\infty} \) as small as possible.

Note that for high-gain feedback (\( \|PG_{yu}(s)\| \gg 1 \))
\[
M(s) \approx M_{\text{HG}}(s) := G_{zw}(s) - \frac{G_{zw}(s)G_{yu}(s)}{G_{yu}(s)}, \tag{7}
\]
and that for low-gain feedback (\( \|PG_{yu}(s)\| \ll 1 \))
\[
M(s) \approx M_{\text{LG}}(s) := G_{zw}(s) - G_{zw}(s)PG_{yu}(s). \tag{8}
\]

The following two lemmas describe which choices of sensor and actuator locations make \( M_{\text{HG}}(s) \) and \( M_{\text{LG}}(s) \) small.

**Lemma 1.** Consider the transfer functions in (3)–(4). The high-gain feedback approximation \( M_{\text{HG}}(s) \) in (7) is zero for all \( s \) precisely when
\[
|x_z - x_u| + |x_y - x_w| = |x_y - x_u| + |x_z - x_w|. \tag{9}
\]

**Proof.** Using (3)–(4), the second term on the RHS of (7) can be rewritten as
\[
\frac{G_{zw}(s)G_{yu}(s)}{G_{yu}(s)} = \frac{1}{c} e^{-((x_z-x_u)v/2D)} x e^{-(|x_z-x_u|+|x_y-x_w|)\sqrt{v^2+4D(s+h)/2D}}. \tag{10}
\]

To obtain \( M_{\text{HG}}(s) \equiv 0 \), this expression should be equal to \( G_{zw}(s) \). This is the case when the second exponential factor in (10) is equal to the second exponential factor of \( G_{zw}(s) \). These are equal precisely when (9) holds.

It is not possible to achieve \( M_{\text{LG}}(s) = 0 \) for all \( s \). Since the magnitude of \( G(i\omega,x) \) is maximal for \( \omega = 0 \) (see also Remark 1), it is most important to make \( M_{\text{LG}}(s) \) small near \( s = 0 \). Locations where this is the case are characterized by the following result. Here, ' denotes the derivative w.r.t. \( s \).

**Lemma 2.** Consider the transfer functions in (3)–(4). There exists a \( P > 0 \) such that \( M_{\text{LG}}(s) \) in (8) satisfies \( M_{\text{LG}}(0) = 0 \) precisely when
\[
|x_z - x_u| + |x_y - x_w| = |x_y - x_u| - \frac{2D}{\sqrt{v^2+4Dh}}. \tag{11}
\]

**Proof.** Since \( G(0,x) \) in (4) is positive, it is clear that
\[
P = G_{zw}(0)/(G_{zw}(0)G_{yu}(0)), \tag{12}
\]
is positive and that this choice for \( P \) makes \( M_{\text{LG}}(0) \) in (8) zero. To compute \( M_{\text{LG}}(0) \), note that
\[
G'(s,x) = -\left( \frac{2D}{\sqrt{v^2+4D(h+s)}} + |x| \right) \times \frac{G(s,x)}{\sqrt{v^2+4D(h+s)}}. \tag{13}
\]

Using (8) and the choice of \( P \) in (12) it now follows that
\[
M_{\text{LG}}(0) = G_{zw}'(0) - G_{zw}'(0)PG_{yu}(0) = G_{zw}'(0) \left( \frac{G_{zw}(0)}{G_{yu}(0)} + \frac{G_{yu}(0)}{G_{yu}(0)} \right)
= \frac{G_{zw}(0)}{\sqrt{v^2+4Dh}} \left( \frac{2D}{\sqrt{v^2+4Dh}} + |x_z - x_u| + |x_y - x_w| - |x_z - x_w| \right). \tag{14}
\]
This expression is zero precisely when (11) holds.

It is important to note that Lemma 1 and 2 consider approximations of the closed-loop transfer function \( M(s) \) based
on the assumption that $|PG_{yu}(s)| \gg 1$ or $|PG_{yu}(s)| \ll 1$, respectively. Whether these assumptions can be realized is not obvious. In particular, the high-gain feedback assumption $|PG_{yu}(s)| \gg 1$ typically leads to instability and it is not clear that the $P$ selected in Lemma 2 indeed satisfies the low-gain assumption $|PG_{yu}(s)| \ll 1$.

The following result demonstrates that the high-gain feedback assumption can be achieved for a particular sensor and actuator placement.

**Corollary 1.** Consider (1)–(2) and let $x_z, x_u$ be fixed. If $xy = x_u$, the proportional feedback controller $u(t) = -Py(t)$ is stabilizing for all $P > 0$. Furthermore, if $xy = x_u$ is chosen between $x_z$ and $x_w$ there exists a $P > 0$ such that $\|M\|_{\infty} < \varepsilon$ for any $\varepsilon > 0$.

**Proof.** Consider (1)–(2) with $w(t) = 0$ and $xy = x_u$. Then the Lyapunov function $V(t) = \int_{-\infty}^{t} T^2(x, t) \, dt$ has time derivative $\dot{V}(t) = u(t)y(t) = -Py^2(t)$. This implies that the closed loop is stable for all $P > 0$. Now note that for $P \to \infty$, $M(s) \to M_{HG}(s)$ and that Lemma 1 asserts that $M_{HG}(s) \equiv 0$ if (9) holds. For $xy = x_u$, (9) reduces to $|x_y - x_u| + |x_u - x_w| = |x_z - x_w|$. This equation is satisfied when $x_u$ is between $x_z$ and $x_w$. □

**Remark 2.** Similarly as in [17], it can be shown that if all stabilizing (dynamic) controllers are considered $\|M\|_{\infty}$ can be made arbitrarily small for all choices of $x_y$ and $x_u$.

**Remark 3.** When $xy \neq x_u$, the feedback $u(t) = -Py(t)$ will always be destabilizing for $P$ sufficiently large.

**Remark 4.** Note that the result in Corollary 1 applies to any $x_w$ and $x_z$ chosen on opposite sides of $xy = x_u$. A consequence of Corollary 1 is therefore that when the area in which the disturbance(s) are applied lies on the opposite side of $xy = x_u$ as the area in which the performance variable(s) are measured, the influence of the disturbances on the performance variables can be made arbitrarily small by a stabilizing proportional feedback controller.

Corollary 1 addresses a particularly effective sensor and actuator placement, which might not always be possible to achieve. Therefore, it is now assumed that the location of the actuator $x_u$ is already fixed, i.e. for given $x_z, x_w$, and $x_u$, the goal is to find the optimal $xy$ for high-gain and low-gain proportional feedback control.

The optimal choice of $xy$ for high-gain proportional feedback depends on the ordering of $x_z, x_w$, and $x_u$. If $x_u$ is between $x_z$ and $x_w$, Corollary 1 shows that the optimal choice for $xy$ is $xy = x_u$. If $x_u$ is between $x_z$ and $x_w$, it can be verified that (9) only holds when $xy = x_z$. If $x_u$ is between $x_z$ and $x_w$, it can be shown that (9) holds for all $xy$ that are on the same side of $x_w$ as $x_z$. Because a greater distance $|x_y - x_u|$ limits the range of stabilizing gains $P$, the optimal choice for $xy$ is $xy = x_w$ in this case.

The optimal choice of $xy$ for low-gain proportional feedback control follows from (11), which typically gives two potential choices of $xy$ at equal distance of $x_w$. Note however that if $|x_z - x_u| < |x_z - x_w|$, (11) cannot be satisfied for any $xy$. Furthermore, these potential optimal locations are only good choices if the low-gain feedback assumption $|PG_{yu}(s)| \ll 1$ is satisfied at these locations. To check this, note that for the value of $P$ in (12), $|PG_{yu}(s)|$ is equal to

$$e(|x_z - x_u| + |x_y - x_u| - |x_z - x_w| - |x_y - x_w|)/\pi^2 + 4D^2/2D.$$ (15)

For each of the two potential low-gain feedback locations, it should be checked whether this number is small.

The method to determine the optimal sensor locations is validated by designing the optimal proportional feedback controller for a range of sensor locations $xy$ while keeping the locations $x_z, x_w, x_u$ fixed. The controller was designed as follows. First, the transfer function $G_{yw}(s)$ is evaluated on a grid of 4,000 frequency points $s = i\omega_k$ logarithmically distributed between $2\pi \cdot 10^{-2}$ and $2\pi \cdot 10^3$. The maximal allowable gain $P_{\text{max}}$ for which the modulus margin is 0.5 is found by solving $\min_k |1 + PG_{yu}(i\omega_k)| = 0.5$ with fzero in MATLAB (version R2017b). The optimal gain $P$ is determined by minimizing $\max_k |M(i\omega_k)|$ over $P \in [0, P_{\text{max}}]$ using fminbnd.

Fig. 2 shows the closed-loop performance $\|M\|_{\infty}$ that can be obtained for the controller designed at each of the sensor locations. Note that $\|M\|_{\infty}$ is normalized w.r.t. open loop performance $\|G_{zw}\|_{\infty}$. The used parameter values are
Fig. 3: Closed-loop performance $\|M\|_{\infty}$ for varying sensor location $x_y$ in the advection-dominated problem (the red arrows indicate locations where (9) is satisfied and the green arrows indicate locations where (11) is satisfied).

Fig. 4: Closed-loop performance $\|M\|_{\infty}$ for varying actuator location $x_u$ in the advection-dominated problem (the red arrows indicate locations where (9) is satisfied and the green arrows indicate locations where (11) is satisfied).

III. TWO-DIMENSIONAL SPATIAL DOMAIN

Now consider the following parabolic PDE with constant coefficients on $x = (x_1, x_2) \in \mathbb{R}^2$

$$\frac{\partial T}{\partial t} = v \frac{\partial T}{\partial x_1} + D \frac{\partial^2 T}{\partial x_1^2} + D \frac{\partial^2 T}{\partial x_2^2} - hT + \frac{1}{c_2} \delta(x - x_u)u(t) + \frac{1}{c_2} \delta(x - x_w)w(t), \quad (16)$$

$$y(t) = T(x_y, t), \quad z(t) = T(x_z, t), \quad (17)$$

with initial condition $T(x_1, x_2, 0) = 0$. Here, $\delta(x)$ denotes the Dirac delta. Similarly as before, this PDE can be used to model the temperature field $T = T(x_1, x_2, t)$ [K] in a medium with heat capacity $c_2$ [J/K/m$^2$], thermal diffusivity $D$ [m$^2$/s], cooling rate $h$ [1/s], and advective transport with velocity $v \geq 0$ [m/s]. Without loss of generality, it can be assumed that the velocity is in the $x_1$-direction. The input and output locations $x_u = (x_1, u, x_2, w)$, $x_w = (x_1, w, x_2, w)$, $x_y = (x_1, y, x_2, y)$, and $x_z = (x_1, z, x_2, z)$ now consist of two components. The considered situation is shown in Fig. 5.

Following the method described in [16], it follows that finding the transfer function of (16)–(17) requires the solution of an elliptic PDE in two spatial dimensions which has
the closed-form solution derived in [19, Chapter 11.3]. The relations between inputs and outputs can now be written in the Laplace domain as

\[
\begin{bmatrix}
Z(s) \\
Y(s)
\end{bmatrix} = \begin{bmatrix}
G_{zw}(s) & G_{zu}(s) \\
G_{yw}(s) & G_{yu}(s)
\end{bmatrix}
\begin{bmatrix}
W(s) \\
U(s)
\end{bmatrix} =
\begin{bmatrix}
G(s, x_z - x_u) & G(s, x_z - x_u) \\
G(s, x_y - x_u) & G(s, x_y - x_u)
\end{bmatrix}
\begin{bmatrix}
W(s) \\
U(s)
\end{bmatrix},
\]

with

\[
G(s, x) = \frac{e^{-x_1v/2D}}{2\pi c_2 D} K_0 \left( \sqrt{v^2 + 4D(h + s)} \|x\|/(2D) \right),
\]

where \(K_0(\zeta)\) denotes the modified Bessel function of the second kind of order 0 and \(\|x\| := \sqrt{x_1^2 + x_2^2}\) is the Euclidean norm.

Similarly as on the one-dimensional spatial domain, conditions under which the high-gain feedback approximation \(M_{HG}(s)\) is zero for all \(s\) can be determined.

**Lemma 3.** Consider the transfer functions in (18)–(19). The high-gain feedback approximation \(M_{HG}(s)\) in (7) is zero when either

\[
\|x_y - x_u\| = \|x_y - x_w\| \land \|x_z - x_u\| = \|x_z - x_w\|,
\]

or

\[
\|x_y - x_u\| = \|x_z - x_u\| \land \|x_y - x_w\| = \|x_z - x_w\|.
\]

**Proof.** Note that if (20) holds, it follows that \(G_{yu}(s) = G_{yw}(s)\) and that \(G_{zu}(s) = G_{zw}(s)\), so that (7) shows that \(M_{HG}(s) = 0\). Similarly, (21) implies that \(G_{yw}(s) = G_{zw}(s)\) and that \(G_{yu}(s) = G_{zu}(s)\), so that \(M_{HG}(s) = 0\).

To study the low-gain feedback approximation, the following asymptotic expansion of \(K_0(\zeta)\) will be used

\[
K_0(\zeta) \approx \sqrt{\frac{\pi}{2\zeta}} e^{-\zeta}. \tag{22}
\]

The error in the approximation can be bounded by

\[
\frac{1}{\zeta |\zeta|} \sqrt{\frac{\pi}{2\zeta}} e^{-\zeta}, \quad \text{when } \text{Re}(\zeta) \geq 0
\]

The relative error in the approximation will therefore be small for \(|\zeta|\) large enough. Using this approximation, \(G(s, x)\) in (19) can be approximated by

\[
\tilde{G}(s, x) = \frac{e^{-x_1v/2D} e^{\sqrt{v^2 + 4D(h + s)} \|x\|/2D}}{c_2 \sqrt{2\pi D} \|x\| \sqrt{v^2 + 4D(h + s)}}. \tag{23}
\]

Now (18) can be approximated by

\[
\begin{bmatrix}
\tilde{Z}(s) \\
\tilde{Y}(s)
\end{bmatrix} = \begin{bmatrix}
\tilde{G}_{zw}(s) & \tilde{G}_{zu}(s) \\
\tilde{G}_{yw}(s) & \tilde{G}_{yu}(s)
\end{bmatrix}
\begin{bmatrix}
W(s) \\
U(s)
\end{bmatrix} =
\begin{bmatrix}
\tilde{G}(s, x_z - x_u) & \tilde{G}(s, x_z - x_u) \\
\tilde{G}(s, x_y - x_u) & \tilde{G}(s, x_y - x_u)
\end{bmatrix}
\begin{bmatrix}
W(s) \\
U(s)
\end{bmatrix}, \tag{24}
\]

where \(\tilde{Z}(s)\) and \(\tilde{Y}(s)\) are good approximations of \(Z(s)\) and \(Y(s)\) when \(4\sqrt{v^2 + 4Dh} \|x\|/D \gg 1\).

The transfer functions in (23)–(24) can be used to find conditions under which \(M_{LG}(s)\) is small.

**Lemma 4.** Consider the transfer functions in (23)–(24). There exists a \(P > 0\) such that \(M_{LG}(s)\) in (8) satisfies \(M_{LG}(0) = M_{LG}(0) = 0\) precisely when

\[
\|x_z - x_u\| + \|x_y - x_w\| = \|x_z - x_w\| - \frac{D}{\sqrt{v^2 + 4Dh}}. \tag{25}
\]

**Proof.** Apart from the observation that now

\[
\tilde{G}'(s, x) = \left( -\frac{D}{\sqrt{v^2 + 4D(h + s)}} + \|x\| \right) \times \tilde{G}(s, x),
\]

the proof is analogous to the proof of Lemma 2.

The effectiveness of the proposed approach is validated by the numerical results in Fig. 6. These figures show the minimal value of \(\|M\|_\infty\) that can be obtained for a proportional feedback controller with a modulus margin of at least 0.5 as a function of the sensor location \(x_y = (x_{1,y}, x_{2,y})\), where \(x_z, x_u, \) and \(x_y\) are chosen rather arbitrarily. The color scale indicates the logarithm of the reduction in closed-loop performance \(\|G\|_\infty\) relative to the open-loop performance \(\|G\|_\infty\). The parameter values are \(c_2 = 1150 \ [J/K/m^2]\), \(v = 0.33 \ [m/s]\), \(D = 91 \cdot 10^{-6} \ [m^2/s]\), and \(h = 1 \ [s]\).

Fig. 6(a) shows that the sensor locations where \(\|M\|_\infty\) is minimal are accurately predicted by the red dashed circle around \(x_u\) described by (25). Note that \(\|G_{yu}(s)\|_\infty\) is negligible when \(x_{1,y} > x_{1,u}\) because the problem is advection-dominated. Because \(\|x_z - x_u\| \neq \|x_z - x_w\|\), (20) does not apply. Condition (21) applies when \(x_y = x_u\) and in the location indicated by the red cross. No significant decrease in \(\|M\|_\infty\) is visible at these locations, because the high-gain feedback assumption \(|PG_{yu}(s)| \gg 1\) cannot be achieved.

Fig. 6(b) shows a situation where \(\|x_z - x_w\| < \|x_z - x_u\|\). In this case, there are no solutions that satisfy (25). The optimal sensor location is now \(x_y = x_w\). This is the location for which the ‘mismatch’ in (25) is as small as possible. Note that reduction in \(\|M\|_\infty\) is significantly smaller than in Fig. 6(a). Just as in Fig. 6(a), condition (20) cannot be satisfied and no significant decrease in \(\|M\|_\infty\) is observed at the locations where (21) is satisfied.

Fig. 6(c) shows again a situation where \(\|x_z - x_u\| < \|x_z - x_w\|\) for which (25) cannot be satisfied. The global minimum is again at \(x_y = x_u\). Furthermore, \(\|x_z - x_u\| = \|x_z - x_w\|\) so that (20) is satisfied on the line \(x_{1,y} = 0\). Indeed, \(\|M\|_\infty\) decreases on this line, especially at locations \(x_y\) near \(x_u\) at which the high-gain feedback assumption \(|PG_{yu}(s)| \gg 1\) can be approximated better.

**IV. CONCLUSIONS AND FUTURE WORK**

A proportional feedback control system design problem for one- and two-dimensional advection-diffusion equations with constant coefficients has been studied. Simple geometric rules have been derived describing the sensor and actuator locations for which high-gain and low-gain proportional feedback control can best reduce the influence a disturbance applied at a given point. The derivation is based on high-gain and low-gain
approximations of the analytic expression for the closed-loop transfer function. Numerical experiments indicate that the rules predict all locally optimal locations.

On the 1D spatial domain, there are either one, two, or three locally optimal sensor or actuator locations depending on the ordering of the other input and output locations. The three possible orderings of the other input and output locations are shown in Fig. 2. The optimal locations are accurately predicted by the conditions in Lemma 1 and 2. On the 2D spatial domain, the locally optimal sensor locations are mainly characterized by Lemma 4.

For the wafer heating problem in [18] it would be interesting to extend the results to thermomechanical systems in which the performance variable \( z(t) \) does not represent the temperature, but the displacement. Furthermore, extensions to spatially distributed inputs, noisy measurements, and dynamic feedback controllers are interesting topics for future research.

Fig. 6: Closed-loop performance \( \| M \|_{2} \) for varying sensor location \( x_{y} = (x_{y,1}, x_{y,2}) \) for three choices of \( x_{u}, x_{w}, x_{w} \) (condition (21) is satisfied at \( x_{y} = x_{u} \) and at the red cross, condition (20) is satisfied at the dashed red line).

References