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Entanglement-assisted Quantum Codes from Algebraic Geometry Codes

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Abstract. Quantum error correcting codes play the role of suppressing noise and decoherence in quantum systems by introducing redundancy. Some strategies can be used to improve the parameters of these codes. For example, entanglement can provide a way for quantum error correcting codes to achieve higher rates than the one obtained via traditional stabilizer formalism. Such codes are called entanglement-assisted quantum (QUENTA) codes. In this paper, we use algebraic geometry codes to construct several families of QUENTA codes via Euclidean and Hermitian construction. Two of the families created has maximal entanglement and have quantum Singleton defect equal to zero or one. Comparing the other families with the codes with the respective quantum Gilbert-Varshamov bound, we show that our codes has rate that surpass such bound. At the end, asymptotically good tower of linear complementary dual codes is used to obtain an asymptotically good families of maximal entanglement QUENTA codes with nonzero rate, relative minimal distance, and relative entanglement. Furthermore, a simple comparison with the quantum Gilbert-Varshamov bound demonstrates that from our construction it is possible to create an asymptotically family of QUENTA codes that exceed such this bound.⁴

Keywords: Quantum Codes · Algebraic Geometry Codes · Maximal Distance Separable · Maximal Entanglement · Asymptotically Good.

1 Introduction

It is generally accepted that the prospect of practical large-scale quantum computers and the use of quantum communication are only possible with the implementation of quantum error correcting codes. Suppressing noise and decoherence can be done via Quantum error correcting codes. The capability of correcting errors of such codes can be improved if it is possible to have pre-shared entanglement states. They are known as Entanglement-Assisted Quantum (QUENTA) codes, also denoted by EAQECC's in the literature. Additionally, this class of codes can achieve the hashing bound [26,14] and violate the quantum Hamming bound [15]. The first QUENTA codes were proposed by Bowen [1] followed by the work from Fattal, *et al.* [9]. The stabilizer formalism of QUENTA codes was created by Brun *et al.* [2], where they showed that QUENTA codes paradigm does not require the dual-containing constraint as the standard quantum error-correcting code does [16]. Wilde and

⁴ Some results in this paper were presented at the 2019 IEEE International Symposium on Information Theory.

Brun [25] proposed two methods to create QUENTA codes from classical codes, which is named in this paper as the Euclidean construction method and the Hermitian construction method. These methods were recently generalized by Galindo, *et al.* [10]

After these works of Brun *et al.*, many works have focused on the construction of QUENTA codes based on classical linear codes [25,5,19,13,18]. However, the analysis of q -ary QUENTA codes was taken into account only recently [8,5,21,6,13,17,12,18]. The majority of them utilized constacyclic codes [8,6,21] or negacyclic codes [5,21] as the classical counterpart. Since the length of the classical codes is normally proportional to the square of the size of the field, most of the quantum codes from the previous works have a length that is proportional to the square of the size of the finite field. Hence, there is no result in the literature with QUENTA codes having length proportional to a greater power of the cardinality of the finite field. In addition, it has not been shown previously that there exists a family of asymptotically good maximal entanglement QUENTA codes either attaining quantum Gilbert-Varshamov bound [10]. Such a family can be used to achieve the hashing bound. A possible approach to solve both questions is using algebraic geometry (AG) codes as the classical counterpart to construct QUENTA codes.

The AG codes were invented by Goppa [11] and have several properties. An important property of these codes is that its parameters can be calculated via the degree of a divisor, which allows a direct description of the code. The first result of this paper comes from these properties. We show two methods to create new AG codes from old ones via intersection and union of divisors. As will be shown, the former “new codes from old” construction is crucial when two AG codes are used to derive QUENTA codes. To derive the QUENTA codes in this paper, it is necessary to define some new mathematical tools and the relation between them and the parameters of QUENTA codes.

First, we introduce the idea of intersection and union of divisor and how these concepts can be used to construct new AG codes from old ones. In addition, it is shown that the amount of entanglement in the Euclidean construction method of QUENTA codes can be described via the intersection of two classical codes used. The practicality of such description is presented by applying our method to AG codes derived from three curves: projective curve (rational function field), Hermitian Curve, and Elliptic curve. The QUENTA codes derived from the first (third) curve are shown to be maximal distance separable (MDS) codes (almost MDS), i.e., the minimal distance of these codes achieve the quantum Singleton bound (differs from the quantum Singleton bound by at most one unit), and maximal entanglement. These codes can be employed to achieve entanglement-assisted quantum capacity of a depolarizing channel [1,7,20]. For the Hermitian curve, a comparative analysis with literature codes shows that our codes have better parameters.

The use of AG codes in the Hermitian construction method for QUENTA codes does not follow the same procedure of the Euclidean one. The reason for this is that there is no general characterization of the Hermitian dual code of an AG code. Thus, to determine the parameters of QUENTA codes derived from the AG codes used, it is computed the intersection of basis of two AG codes. The

curve used to construct the QUENTA codes is the projective curve. The QUENTA codes created are MDS codes and also have maximal entanglement.

Lastly, asymptotically good families of LCD codes are to construct asymptotically good families of QUENTA codes that have maximal entanglement. Using AG codes from tower of function fields that attain the Drinfeld-Vladut bound [24] we show that the QUENTA codes in this paper surpass the quantum Gilbert-Varshamov bound [10].

The paper is organized as follows. In Section 2, we describe what needs to be known about AG codes, so that they can be applied to the generalization of the construction methods of QUENTA codes from Wilde and Brun [25] proposed by Galindo, *et al.* [10]. It is also shown in this section two methods to construct new AG codes from old ones. Afterwards, several new families of QUENTA codes are derived from AG codes. These derivations comes from the Euclidean construction method and Hermitian construction method for QUENTA codes with the use of three different types of curves. In Section 4, we compare the codes in this paper with the quantum Singleton bound and with other quantum codes in the literature. In particular, it is shown that three families of QUENTA codes constructed are MDS or almost MDS. In Section 5, we show that there exists families of QUENTA codes that surpass the quantum Gilbert-Varshamov bound. Lastly, the conclusion is carried out in Section 6.

Notation. Throughout this paper, p denotes a prime number and q is a power of p . Let F/\mathbb{F}_q be an algebraic function field over \mathbb{F}_q of genus g , where \mathbb{F}_q denotes the finite field with q elements. A linear code C with parameters $[n, k, d]_q$ is a k -dimensional subspace of \mathbb{F}_q^n with minimum distance d . Lastly, an $[[n, k, d; c]]_q$ quantum code is a q^k -dimensional subspace of \mathbb{C}^{q^n} with minimum distance d that utilizes c pre-shared entanglement pairs.

2 Preliminaries

In this section, we introduce some ideas related to linear complementary dual (LCD) codes, algebraic geometry (AG) codes and entanglement-assisted quantum (QUENTA) codes. Before we give a description of LCD codes, a definition for the Euclidean and Hermitian dual of a code needs to be given.

Definition 1. *Let C be a linear code over \mathbb{F}_q with length n . The (Euclidean) dual of C is defined as*

$$C^\perp = \{ \mathbf{x} \in \mathbb{F}_q^n \mid \mathbf{x} \cdot \mathbf{c} = 0 \text{ for all } \mathbf{c} \in C \}. \quad (1)$$

If the finite field has cardinality equal to q^2 , an even power of a prime, then we can define the Hermitian dual of C . This dual code is defined by

$$C^{\perp_h} = \{ \mathbf{x} \in \mathbb{F}_{q^2}^n \mid \mathbf{x} \cdot \mathbf{c}^q = 0 \text{ for all } \mathbf{c} \in C \}, \quad (2)$$

where $\mathbf{c}^q = (c_1^q, \dots, c_n^q)$ for $\mathbf{c} \in \mathbb{F}_{q^2}^n$.

When the intersection between a code and its dual gives only the vector $\mathbf{0}$, the code is called LCD. A formal description can be seen below.

Definition 2. *The hull of a linear code C is given by $\text{hull}(C) = C^\perp \cap C$. The code is called linear complementary dual (LCD) code if the hull is trivial; i.e, $\text{hull}(C) = \{\mathbf{0}\}$. Similarly, it is defined $\text{hull}_H(C) = C^{\perp_h} \cap C$ and the idea of hermitian LCD code.*

The class of LCD codes is a possible way to construct QUENTA codes that have maximal entanglement and asymptotically good families (see Sections 3 and 5).

2.1 Algebraic-Geometry codes

Let F/\mathbb{F}_q be an algebraic function field of genus g . A place P of F/\mathbb{F}_q is the maximal ideal of some valuation ring \mathcal{O}_P of F/\mathbb{F}_q . We also define the set of all places by $\mathbb{P}_F = \{P | P \text{ is a place of } F/\mathbb{F}_q\}$.

A divisor of F/\mathbb{F}_q is a formal sum of places given by $D = \sum_{P \in \mathbb{P}_F} n_P P$, with $n_P \in \mathbb{Z}$, where almost all $n_P = 0$. The support and degree of D are defined as $\text{supp}(D) = \{P \in \mathbb{P}_F | n_P \neq 0\}$ and $\text{deg}(D) = \sum_{P \in \mathbb{P}_F} n_P \text{deg}(P)$, respectively, where $\text{deg}(P)$ is the degree of the place P . When a place has degree one, it is called a rational place.

The discrete valuation corresponding to a place P is written as ν_P . For every element f of F/\mathbb{F}_q , we can define a principal divisor of f by $(f) = \sum_{P \in \mathbb{P}_F} \nu_P(f) P$. For $f \in \mathcal{O}_P$, we define $f(P) \in \mathcal{O}_P/P$ to be the residue class of f modulo P ; for $f \in F \setminus \mathcal{O}_P$, we put $f(P) = \infty$. For a given divisor G , we denote the Riemann-Roch space associated to G by $\mathcal{L}(G) = \{f \in F^* | (f) \geq -G\} \cup \{0\}$. It is possible to associate to a divisor $G = mQ$ a numerical semigroup that it is called Weierstrass semigroup, which it is described in Definition 3. This semigroup will be important in the part that we construct QUENTA codes from the Hermitian construction method.

Definition 3. *Let F/\mathbb{F}_q be an algebraic function field and Q a rational place of F/\mathbb{F}_q . First of all, we define the space of rational functions having poles only at Q as $\mathcal{L}(\infty Q) = \cup_{m \geq 0} \mathcal{L}(mQ)$. Now, the Weierstrass semigroup of Q is given as*

$$S(Q) = \{-\nu_Q(f) | f \in \mathcal{L}(\infty Q)\} = \{0 = \rho_1 < \rho_2 < \dots\}. \quad (3)$$

The given description of Riemann-Roch spaces shows that when we are talking about such spaces we deal with functions that obey a set of rules which are described by the defining divisor. One natural question that could arise is the relation between the intersection of two Riemann-Roch spaces and the respective divisor that defines such a space. Such a result was shown by Munuera and Pellikaan [22]. Before showing it, we need to define the intersection and union of two divisors, which is done in the following.

Definition 4. Let G and H be divisors over F/\mathbb{F}_q . If $G = \sum_{P \in \mathbb{P}_F} \nu_P(G)P$ and $H = \sum_{P \in \mathbb{P}_F} \nu_P(H)P$, where $P \in \mathbb{P}_F$ is a place, then the intersection $G \cap H$ of G and H over F/\mathbb{F}_q is defined as follows

$$G \cap H = \sum_{P \in \mathbb{P}_F} \min\{\nu_P(G), \nu_P(H)\}P. \quad (4)$$

In addition, the union is given by

$$G \cup H = \sum_{P \in \mathbb{P}_F} \max\{\nu_P(G), \nu_P(H)\}P. \quad (5)$$

Proposition 1. [22, Lemma 2.6] Let G and H be divisors over F/\mathbb{F}_q . Then $\mathcal{L}(G) \cap \mathcal{L}(H) = \mathcal{L}(G \cap H)$.

In the Section 3 it will be shown that when AG codes are used to construct QUENTA codes, the amount of entanglement used is related to the dimension of the intersection of the two Riemann-Roch spaces.

For the exactly value of the dimension of a Riemann-Roch space and the construction of the dual code of a AG code, it is necessary to introduce the ideas of differential spaces and canonical divisors. $\Omega_F = \{\omega | \omega \text{ is a Weil differential of } F/\mathbb{F}_q\}$ be the differential space of F/\mathbb{F}_q . Given a nonzero differential ω , we denote by $(\omega) = \sum_{P \in \mathbb{P}_F} \nu_P(\omega)P$ the canonical divisor of ω . All canonical divisors are equivalent and have degree equal to $2g - 2$. Furthermore, for a divisor G we define $\Omega_F(G) = \{\omega \in \Omega_F | \omega = 0 \text{ or } (\omega) \geq G\}$, and its dimension as an \mathbb{F}_q -vector space is denoted by $i(G)$.

The dimension of a Riemann-Roch space can be calculated through its defining divisor, the divisor of a Weil differential and the genus of a curve.

Proposition 2. [24, Theorem 1.5.15](Riemann-Roch Theorem) Let W be a canonical divisor of F/\mathbb{F}_q . Then for each divisor G , the dimension of $\mathcal{L}(G)$ is given by $\ell(G) = \deg(G) + 1 - g + \ell(W - G)$, where $\deg(G)$ is the degree of the divisor G .

Now we can define the first AG code utilized in this paper, see Definition 5, and its parameters, see Proposition 3. The definition of such AG codes is given as an image of a linear map called evaluation map. About the parameters of the AG codes, they are related with the degrees of divisors, genus and number of rational places. Thus, with simple arithmetic we can create families of codes, even when the algebraic function field is fixed.

Definition 5. Let P_1, \dots, P_n be pairwise distinct rational places of F/\mathbb{F}_q and $D = P_1 + \dots + P_n$. Choose a divisor G of F/\mathbb{F}_q such that $\text{supp}(G) \cap \text{supp}(D) = \emptyset$. The algebraic-geometry (AG) code $C_{\mathcal{L}}(D, G)$ associated with the divisors D and G is defined as the image of the linear map $ev_D: \mathcal{L}(G) \rightarrow \mathbb{F}_q^n$ called evaluation map, where $ev_D(f) = (f(P_1), \dots, f(P_n))$; i.e., $C_{\mathcal{L}}(D, G) = \{(f(P_1), \dots, f(P_n)) | f \in \mathcal{L}(G)\}$.

Proposition 3. [24, Corollary 2.2.3] *Let F/\mathbb{F}_q be a function field of genus g . Then the AG code $C_{\mathcal{L}}(D, G)$ is a $[n, k, d]$ -linear code over \mathbb{F}_q with parameters $k = \ell(G) - \ell(G - D)$ and $d \geq n - \deg(G)$. If $2g - 2 < \deg(G) < n$, then $k = \deg(G) - g + 1$.*

The next two propositions present a way to construct new AG codes from old AG codes via intersection and union of divisors. Proposition 4 will be used in Section 3 in order to use AG codes to create QUENTA codes.

Proposition 4. *Let F/\mathbb{F}_q be a function field of genus g and let D be a divisor as in Definition 5. If G_1 and G_2 are two divisors such that $\text{supp}G_1 \cap \text{supp}D = \emptyset$, resp. $\text{supp}G_2 \cap \text{supp}D = \emptyset$, and $\deg(G_1 \cup G_2) < n$, then $C_{\mathcal{L}}(D, G_1) \cap C_{\mathcal{L}}(D, G_2) = C_{\mathcal{L}}(D, G_1 \cap G_2)$.*

Proof. First of all, let the evaluation map from F/\mathbb{F}_q to \mathbb{F}_q^n be the same as in Definition 5; i.e.,

$$ev_D(f) = (f(P_1), \dots, f(P_n)) \in \mathbb{F}_q^n. \quad (6)$$

Thus, if $c \in C_{\mathcal{L}}(D, G_1) \cap C_{\mathcal{L}}(D, G_2)$, then exist $g_1 \in \mathcal{L}(G_1)$ and $g_2 \in \mathcal{L}(G_2)$ such that $c = ev_D(g_1) = ev_D(g_2)$, which implies in $ev_D(g_1 - g_2) = 0$. Since that $g_1 - g_2 \in \mathcal{L}(G_1 \cup G_2)$ and $\deg(G_1 \cup G_2) < n$, then $g_1 = g_2$ and, consequently, $c \in C_{\mathcal{L}}(D, G_1 \cap G_2)$. The other inclusion is straightforward consequence of Proposition 1. \square

Proposition 5. *Let F/\mathbb{F}_q be a function field of genus g and let D be a divisor as in Definition 5. If G_1 and G_2 are two divisors such that $\text{supp}G_1 \cap \text{supp}D = \emptyset$, resp. $\text{supp}G_2 \cap \text{supp}D = \emptyset$, and $\deg(G_1 \cap G_2) > 2g - 2$ and $\deg(G_1 \cup G_2) < n$, then $C_{\mathcal{L}}(D, G_1) + C_{\mathcal{L}}(D, G_2) = C_{\mathcal{L}}(D, G_1 \cup G_2)$.*

Proof. Lets begin with the inclusion $C_{\mathcal{L}}(D, G_1) + C_{\mathcal{L}}(D, G_2) \subseteq C_{\mathcal{L}}(D, G_1 \cup G_2)$. Since $G_i \leq G_1 \cup G_2$, for $i = 1, 2$, then $C_{\mathcal{L}}(D, G_i) \subseteq C_{\mathcal{L}}(D, G_1 \cup G_2)$, for $i = 1, 2$. Hence $C_{\mathcal{L}}(D, G_1) + C_{\mathcal{L}}(D, G_2) \subseteq C_{\mathcal{L}}(D, G_1 \cup G_2)$. On the other hand, notice that $\ell(G_1) + \ell(G_2) = \ell(G_1 \cap G_2) + \ell(G_1 \cup G_2)$, since that $\deg(G_1 \cap G_2) > 2g - 2$. This implies that $\mathcal{L}(G_1 \cup G_2) = \mathcal{L}(G_1) + \mathcal{L}(G_2)$ by Proposition 1. Now, the proof of the remaining inclusion follows from the hypothesis that $\deg(G_1 \cup G_2) < n$. \square

Another important type of AG code is given in the following.

Definition 6. *Let F/\mathbb{F}_q be a function field of genus g and let G and D be divisors as in Definition 5. Then we define the code $C_{\Omega}(D, G)$ as $C_{\Omega}(D, G) = \{(res_{P_1}(\omega), \dots, res_{P_n}(\omega)) \mid \omega \in \Omega_F(G - D)\}$, where $res_{P_i}(\omega)$ denotes the residue of ω at P_i , with parameters $[n, k', d']$, where $k' = i(G - D) - i(G)$ and $d' \geq \deg(G) - (2g - 2)$.*

Proposition 6. [24, Theorem 2.2.7] *Let $C_{\Omega}(D, G)$ be the AG code from Definition 6. If $2g - 2 < \deg(G) < n$, then $C_{\Omega}(D, G)$ is an $[n, k', d']$ -linear code over \mathbb{F}_q , where $k' = n + g - 1 - \deg(G)$ and $d' \geq \deg(G) - (2g - 2)$.*

The relationship between the codes $C_{\mathcal{L}}(D, G)$ and $C_{\Omega}(D, G)$ is given in the next proposition.

Proposition 7. [24, Proposition 2.2.10] *Let $C_{\mathcal{L}}(D, G)$ be the AG code described in Definition 5. Then $C_{\Omega}(D, G)$ is its Euclidean dual, i.e., $C_{\mathcal{L}}(D, G)^{\perp} = C_{\Omega}(D, G)$. Additionally, if we have a Weil differential η such that $\nu_{P_i}(\eta) = -1$ and $\eta_{P_i}(1) = 1$ for all $i = 1, \dots, n$, then $C_{\Omega}(D, G) = C_{\mathcal{L}}(D, G^{\perp})$, where $G^{\perp} = D - G + (\eta)$.*

2.2 Entanglement-assisted quantum codes

Definition 7. *A quantum code \mathcal{Q} is called an $[[n, k, d; c]]_q$ entanglement-assisted quantum (QUENTA) code if it encodes k logical qudits into n physical qudits using c copies of maximally entangled states and can correct $\lfloor (d-1)/2 \rfloor$ quantum errors. The rate of a QUENTA code is given by k/n , relative distance by d/n , and entanglement-assisted rate by c/n . Lastly, a QUENTA code is said to have maximal entanglement when $c = n - k$.*

Formulating a stabilizer paradigm for QUENTA codes gives a way to use classical codes to construct this quantum codes [3]. In particular, we have the next two procedures by Galindo, *et al.* [10].

Proposition 8. [10, Theorem 4] *Let C_1 and C_2 be two linear codes over \mathbb{F}_q with parameters $[n, k_1, d_1]_q$ and $[n, k_2, d_2]_q$ and parity check matrices H_1 and H_2 , respectively. Then there is an QUENTA code with parameters $[[n, k_1 + k_2 - n + c, d; c]]_q$, where $d = \min\{d_H(C_1 \setminus (C_1 \cap C_2^{\perp})), d_H(C_2 \setminus (C_1^{\perp} \cap C_2))\}$, with d_H as the minimum Hamming weight of the vectors in the set, and*

$$c = \text{rank}(H_1 H_2^T) = \dim C_1^{\perp} - \dim(C_1^{\perp} \cap C_2) \quad (7)$$

is the number of required maximally entangled states.

A straightforward application of LCD codes to the Proposition 8 can produce some interesting quantum codes. See Theorem 1 and Corollary 1.

Theorem 1. *Let C_1 and C_2 be two linear codes with parameters $[n, k_1, d_1]_q$ and $[n, k_2, d_2]_q$, respectively, with $C_1^{\perp} \cap C_2 = \{\mathbf{0}\}$. Then there exists an QUENTA code with parameters $[[n, k_2, \min\{d_1, d_2\}; n - k_1]]_q$.*

Proof. Since that $C_1^{\perp} \cap C_2 = \{\mathbf{0}\}$, from Proposition 8 we have that the QUENTA code constructed from C_1 and C_2 has parameters $[[n, k_2, \min\{d_1, d_2\}; n - k_1]]_q$. \square

Corollary 1. *Let C be a MDS LCD code with parameters $[n, k, d]_q$. Then there exists a MDS maximal entanglement QUENTA code with parameters $[[n, k, d; n - k]]_q$.*

Proof. Let $C_1 = C_2 = C$. Since C is LCD, then $\dim(\text{hull}(C)) = 0$. Then, from Theorem 1, we have that there exists an QUENTA code with parameters $[[n, k, d; n - k]]_q$. \square

It is shown in Corollary 1 that for any MDS LCD code in the literature we can construct a QUENTA code that is, simultaneously, MDS and has maximal entanglement.

Proposition 9. [10, Proposition 3 and Corollary 1] *Let C be a linear codes over \mathbb{F}_{q^2} with parameters $[n, k, d]_q$, H be a parity check matrices for C , and H^* be the q -th power of the transpose matrix of H . Then there is an QUENTA code with parameters $[[n, 2k - n + c, d'; c]]_q$, where $d' = d_H(C \setminus (C \cap C^{\perp_h}))$, with d_H as the minimum Hamming weight of the vectors in the set, and*

$$c = \text{rank}(HH^*) = \dim C^{\perp_h} - \dim(C^{\perp_h} \cap C) \quad (8)$$

is the number of required maximally entangled states.

In the same way as before, it possible to use hermitian LCD codes to derive QUENTA codes with interesting properties. See the following theorem.

Theorem 2. *Let C be a hermitian LCD code with parameters $[n, k, d]_{q^2}$. Then there exists a maximal entanglement QUENTA code with parameters $[[n, k, d; n - k]]_q$. In particular, if C is MDS, then the QUENTA code is also MDS.*

Proof. Since C is a hermitian LCD code, then $\dim(\text{hull}_H(C)) = 0$. Therefore, using C in the Proposition 9, we have that there exists an QUENTA code with parameters $[[n, k, d; n - k]]_q$. \square

A measurement of goodness for a QUENTA code is the quantum Singleton bound (QSB). Let $[[n, k, d; c]]_q$ be an QUENTA code, then the QSB is given by

$$d \leq \left\lfloor \frac{n - k + c}{2} \right\rfloor + 1. \quad (9)$$

The difference between the QSB and d is called quantum Singleton defect. When the quantum Singleton defect is equal to zero (resp. one) the code is called maximum distance separable quantum code (resp. almost maximum distance separable quantum code) and it is denoted MDS quantum code (resp. almost MDS quantum code).

3 New Construction Method for QUENTA Codes

3.1 Euclidean Construction

In Proposition 8 is shown the connection between the entanglement in an QUENTA code and the relative hull of two classical codes. However, the computation of such hull can be difficult in some cases, however, as we are going to show in Theorem 3, this is not the case for AG codes.

The rank of a matrix that is the product of the two parity check matrices of the classical codes is utilized to construct such a quantum code. However, such rank can be difficult to calculate in some cases. As it will be shown, it is possible to, instead of calculating such rank, relate the entanglement with the relative hull between the two classical codes. However, we need first to present the connection between the rank in Proposition 8 and the relative hull.

Theorem 3. *Let P_1, \dots, P_n be pairwise distinct rational places of F/\mathbb{F}_q and $D = P_1 + \dots + P_n$. Choose divisors G_1, G_2 of F/\mathbb{F}_q such that $\text{supp}(G_1) \cap \text{supp}(D) = \emptyset$ and $\text{supp}(G_2) \cap \text{supp}(D) = \emptyset$. Assume that $C_1 = C_{\mathcal{L}}(D, G_1)$ and $C_2 = C_{\mathcal{L}}(D, G_2)$. If $\deg(G_1^\perp \cup G_2) < n$, then $\dim(C_1^\perp \cap C_2) = \ell(G_1^\perp \cap G_2)$.*

Proof. Since that $\deg(G_1^\perp \cup G_2) < n$, we can use Proposition 4 in the codes $C_{\mathcal{L}}(D, G_1)^\perp$ and $C_{\mathcal{L}}(D, G_2)$. Hence, it is easy to see from this proposition that $C_{\mathcal{L}}(D, G_1)^\perp \cap C_{\mathcal{L}}(D, G_2) = C_{\mathcal{L}}(D, G_1^\perp \cap G_2)$, which implies that $\dim(C_{\mathcal{L}}(D, G_1)^\perp \cap C_{\mathcal{L}}(D, G_2)) = \ell(G_1^\perp \cap G_2)$. \square

Theorem 3 allows us to use AG codes from any function field to construct QUENTA codes, which is given in detail in Theorem 4. In particular, as it will be shown, we can use AG codes to derive MDS quantum codes and asymptotically good QUENTA codes.

Theorem 4. *Let P_1, \dots, P_n be pairwise distinct rational places of F/\mathbb{F}_q and $D = P_1 + \dots + P_n$. Choose divisors G_1, G_2 of F/\mathbb{F}_q such that $\text{supp}(G_i) \cap \text{supp}(D) = \emptyset$ and $2g - 2 < \deg(G_i) < n$, for $i = 1, 2$. Assume that $C_1 = C_{\mathcal{L}}(D, G_1)$ and $C_2 = C_{\mathcal{L}}(D, G_2)$. If $\deg(G_1^\perp \cup G_2) < n$, then there exists a QUENTA code with parameters $[[n, \deg(G_1 + G_2) - 2g + 2 - n + c, d; c]]_q$, where $d \geq n - \max\{\deg(G_1), \deg(G_2)\}$ and $c = n + g - 1 - \deg(G_1) - \ell(G_1^\perp \cap G_2)$.*

Proof. First of all, notice that the parameters of the AG codes $C_{\mathcal{L}}(D, G_1)$ and $C_{\mathcal{L}}(D, G_2)$ are $[n, \deg(G_1) - g + 1, d_1 \geq n - \deg(G_1)]_q$ and $[n, \deg(G_2) - g + 1, d_2 \geq n - \deg(G_2)]_q$, respectively, and the dimension of the Euclidean dual of $C_{\mathcal{L}}(D, G_1)$ is $n + g - 1 - \deg(G_1)$, as can be seen in Proposition 7. From Theorem 3 we have that $\dim(C_1^\perp \cap C_2) = \ell(G_1^\perp \cap G_2)$. Hence, using Proposition 8 it is possible to derive the QUENTA code with the mentioned parameters. \square

Corollary 2. *Let P_1, \dots, P_n be pairwise distinct rational places of F/\mathbb{F}_q and $D = P_1 + \dots + P_n$. Choose divisors G_1, G_2 of F/\mathbb{F}_q such that $\text{supp}(G_i) \cap \text{supp}(D) = \emptyset$ and $2g - 2 < \deg(G_i) < n$, for $i = 1, 2$. If $\deg(G_1^\perp \cup G_2) < n$ and $\deg(G_1^\perp \cap G_2) < 0$, then there exists an QUENTA code with parameters $[[n, \deg(G_2) - g + 1, d; c]]_q$, where $d \geq n - \max\{\deg(G_1), \deg(G_2)\}$ and $c = n + g - 1 - \deg(G_1)$. In particular, if it is possible to have $G_1 = G_2 = G$, then the QUENTA code has parameters $[[n, \deg(G) - g + 1, n - \deg(G); n + g - 1 - \deg(G)]]_q$.*

The first explicit description of a family of QUENTA codes constructed in this paper is shown in the following theorem. The rational function field $\mathbb{F}_q(z)/\mathbb{F}_q$ is used to derive this family.

Theorem 5. *Let q be a power of a prime. Consider a_1, a_2, b_1, b_2 positive integers such that $b_1 \leq a_2$ and $b_2 \leq q - 2 - a_2$, with $a_1 + a_2 < q - 1$ and $b_1 + b_2 < q - 1$, then we have the following:*

– *If $b_2 \geq a_1 + 1$, then there exists a QUENTA code with parameters*

$$[[q - 1, a_1 + b_1 - 1, q - 1 - \max\{a_1 + a_2, b_1 + b_2\}; q - 2 - (a_2 + b_2)]]_q.$$

– *If $b_2 < a_1 + 1$, then there exists a QUENTA code with parameters*

$$[[q - 1, b_1 + b_2 + 1, q - 1 - \max\{a_1 + a_2, b_1 + b_2\}; q - 2 - (a_1 + a_2)]]_q.$$

Proof. Let $\mathbb{F}_q(z)/\mathbb{F}_q$ be the rational function field. The Weil differential $\eta = \frac{1}{x^q - x} dx$ satisfies the requirements of Proposition 7 and it has divisor given by $(\eta) = (q - 2)P_\infty - P_0 - D$, where P_∞ and P_0 are the place at infinity and the origin, respectively, and $D = \sum_{i=1}^{q-1} P_i$, with P_i being the remaining rational places. Assume that $G_1 = a_1 P_0 + a_2 P_\infty$ and $G_2 = b_1 P_0 + b_2 P_\infty$ and $C_1 = C_{\mathcal{L}}(D, G_1)$ and $C_2 = C_{\mathcal{L}}(D, G_2)$. Since that $b_2 \leq q - 2 - a_2$, we have that $\deg(G_1^\perp \cup G_2) = b_1 + q - 2 - a_2 < q - 1$, by the hypothesis $b_1 \leq a_2$, and $G_1^\perp \cap G_2 = (-1 - a_1)P_0 + b_2 P_\infty$. Thus, we can use Theorem 4. For the first case, we have that $c = q - 1 - 1 - (a_1 + a_2) - (b_2 - a_1) = q - 2 - (a_2 + b_2)$, since $\deg(G_1^\perp \cap G_2) \geq 0$. For the second case, when $b_2 < a_1 + 1$, we have that $\deg(G_1^\perp \cap G_2) < 0$, which implies in $\ell(G_1^\perp \cap G_2) = 0$ and $c = q - 2 - (a_1 + a_2)$. The remaining claims are derived from Theorem 4 and from the observation that $\deg(G_1) = a_1 + a_2$ and $\deg(G_2) = b_1 + b_2$. \square

Corollary 3. *If $b_1 \leq a_2$, $b_2 \leq \min\{q - 2 - a_2, a_1\}$, with $a_1 + a_2 = b_1 + b_2 < q - 1$, there are maximal entanglement almost MDS $[[q - 1, b_1 + b_2 + 1, q - 1 - (a_1 + a_2); q - 2 - (a_1 + a_2)]]_q$ QUENTA codes. In particular, if $a_1 \leq q - 2 - a_2$, then there exists a maximal entanglement MDS $[[q - 1, a_1 + a_2 + 1, q - 1 - (a_1 + a_2); q - 2 - (a_1 + a_2)]]_q$ QUENTA codes.*

Proof. Consider the second case of Theorem 5. Then considering $a_1 + a_2 = b_1 + b_2 < q - 2$, the result comes. \square

The following theorem shows a construction of QUENTA codes derived from the Hermitian function field. Next, the elliptic function field will be used to obtain maximal entanglement QUENTA codes with Singleton defect at most one.

Theorem 6. *Let q be a power of a prime and a_1, a_2, b_1, b_2 be positive integers such that $b_1 \leq a_2 - q(q - 1)$, $b_2 \leq q^3 + q(q - 1) - 2 - a_2$, with $b_1 + b_2 < q^3 - 1$ and $a_1 + a_2 < q^3 - 1$. Then we have the following:*

– *If $b_2 \geq a_1 + 1$, then there exists a QUENTA code with parameters*

$$[[q^3 - 1, a_1 + b_1 + 1, q^3 - 1 - \max\{a_1 + a_2, b_1 + b_2\}; q^3 - 2 + q(q - 1) - (a_2 + b_2)]]_{q^2}.$$

– If $b_2 < a_1 + 1$, then there exists a QUENTA code with parameters

$$\left[\left[q^3 - 1, b_1 + b_2 + 1 - \frac{q(q-1)}{2}, q^3 - 1 - \max\{a_1 + a_2, b_1 + b_2\}; q^3 - 2 + \frac{q(q-1)}{2} - (a_1 + a_2) \right] \right]_{q^2}.$$

Proof. Let F/\mathbb{F}_{q^2} be the Hermitian function field defined by the equation

$$y^q + y = x^{q+1}.$$

Then F/\mathbb{F}_{q^2} has $1 + q^3$ rational points and genus $g = q(q-1)/2$. Assume that $D = P_1 + \dots + P_{q^3-1}$, $G_1 = a_1P_0 + a_2P_\infty$, and $G_2 = b_1P_0 + b_2P_\infty$, where P_∞ and P_0 are rational places. Thus, one possible Weil differential satisfying Proposition 7 is given by $\eta = \frac{1}{x^{q^2-x}}dx$, which has divisor $(\eta) = -D - P_0 + (q^3 + q(q-1) - 2)P_\infty$. The fact that $b_2 \leq q^3 + q(q-1) - 2 - a_2$ implies in $G_1^\perp \cup G_2 = b_1P_0 + (q^3 + q(q-1) - 2 - a_2)P_\infty$. By the hypothesis $b_1 \leq a_2 - q(q-1)$, we have that $\deg(G_1^\perp \cup G_2) < q^3 - 1$, thus we can use Theorem 4. From this theorem, it is possible to derive that $G_1^\perp \cap G_2 = (-1 - a_1)P_0 + b_2P_\infty$. Hence, if $b_2 \geq a_1 + 1$ we have that $\deg(G_1^\perp \cap G_2) \geq 0$, which implies in $c = q^3 - 1 + \frac{q(q-1)}{2} - 1 - (a_1 + a_2) - (b_2 - a_1) + \frac{q(q-1)}{2} = q^3 - 2 + q(q-1) - (a_2 + b_2)$. On the other hand, if $b_2 < a_1 + 1$ we have that $\deg(G_1^\perp \cap G_2) < 0$, which implies in $\ell(G_1^\perp \cap G_2) = 0$ and $c = q^3 - 2 + \frac{q(q-1)}{2} - (a_1 + a_2)$. Since that $\deg(G_1) = a_1 + a_2$ and $\deg(G_2) = b_1 + b_2$, using Theorem 4 and the values of c computed, we can derive the mentioned parameters for the QUENTA codes. \square

Theorem 7. Let $q = 2^m$, with $m \geq 1$ an integer, and F/\mathbb{F}_q be the elliptic function field with e rational places and genus $g = 1$ defined by the equation

$$y^2 + y = x^3 + bx + c, \tag{10}$$

where $b, c \in \mathbb{F}_q$. Let a_1, a_2, b_1, b_2 be positive integers such that $b_1 \leq a_2$, $b_2 \leq e - 1 - a_2$, with $a_1 + a_2 < e - 2$ and $b_1 + b_2 < e - 2$. Then we have the following:

– If $b_2 \geq a_1 + 1$, then there exists a QUENTA code with parameters

$$\left[\left[e - 2, a_1 + b_1 + 1, e - 2 - \max\{a_1 + a_2, b_1 + b_2\}; e - 1 - (a_2 + b_2) \right] \right]_q.$$

– If $b_2 < a_1 + 1$, then there exists a QUENTA code with parameters

$$\left[\left[e - 2, b_1 + b_2, e - 2 - \max\{a_1 + a_2, b_1 + b_2\}; e - 2 - (a_1 + a_2) \right] \right]_q.$$

Proof. First of all, let $S = \{\alpha \in \mathbb{F}_q \mid \text{there exists } \beta \in \mathbb{F}_q \text{ such that } \beta^2 + \beta = \alpha^3 + b\alpha + c\}$. For each $\alpha \in S$, there are two $\beta \in \mathbb{F}_q$ satisfying the equation $\beta^2 + \beta = \alpha^3 + b\alpha + c$. Thus, for each $\alpha \in S$, there are two places corresponding to x -coordinate equal to α . Hence, the set of all rational places is given

by these x and y coordinates and the place at infinity, P_∞ . The number of rational places is denoted by e . So $e = |S| + 1$. Now, assume that $D = \sum_{i=1}^{e-2} P_i$, $G_1 = a_1 P_0 + a_2 P_\infty$, and $G_2 = b_1 P_0 + b_2 P_\infty$, where P_0, P_1, \dots, P_{e-1} are pairwise distinct rational places. Additionally, let $\eta = \frac{dx}{\prod_{\alpha_i \in S} (x + \alpha_i)}$, then the divisor of the Weil differential η is given by $(\eta) = (e-1)P_\infty - P_0 - D$. The fact that $b_2 \leq e-1-a_2$ implies in $G_1^\perp \cup G_2 = b_1 P_0 + (e-1-a_2)P_\infty$. By the hypothesis $b_1 \leq a_2$, we have that $\deg(G_1^\perp \cup G_2) < e-2$, thus we can use Theorem 4. From this theorem, it is possible to derive that $G_1^\perp \cap G_2 = (-1-a_1)P_0 + b_2 P_\infty$. Hence, if $b_2 \geq a_1 + 1$ we have that $\deg(G_1^\perp \cap G_2) \geq 0$, which implies in $c = e-1-(a_2+b_2)$. On the other hand, if $b_2 < a_1 + 1$ we have that $\deg(G_1^\perp \cap G_2) < 0$, which implies in $\ell(G_1^\perp \cap G_2) = 0$ and $c = e-2-(a_1+a_2)$. Since that $\deg(G_1) = a_1 + a_2$ and $\deg(G_2) = b_1 + b_2$, using Theorem 4 and the values of c computed, we can derive the mentioned parameters for the QUENTA codes. \square

Corollary 4. *Consider that there exists an elliptic curve with e rational places. Then for $b_1 \leq a_2$, $b_2 \leq \min\{e-1-a_2, a_1\}$, with $a_1 + a_2 = b_1 + b_2 < e-2$, there are maximal entanglement almost MDS $[[e-2, b_1 + b_2, e-2-(a_1+a_2); e-2-(a_1+a_2)]_q$ QUENTA codes. In particular, if $a_1 + a_2 < e-2$, then there exists a maximal entanglement almost MDS $[[e-2, a_1 + a_2, e-2-(a_1+a_2); e-2-(a_1+a_2)]_q$ QUENTA codes.*

Proof. Consider the second case of Theorem 7. Then considering $a_1 + a_2 = b_1 + b_2 < e-2$, the result comes. \square

In Table 1 is shown the numbers of rational points of several elliptic curves are given depending on the value of s , the degree of the extension \mathbb{F}_{2^s} .

Table 1. Some elliptic Curves over \mathbb{F}_q ($q = 2^s$) and their number of rational places

Elliptic curve	s	Number of rational places (e)
$y^2 + y = x^3$	odd s	$q + 1$
	$s \equiv 0 \pmod{4}$	$q + 1 - 2\sqrt{q}$
	$s \equiv 0 \pmod{2}$	$q + 1 + 2\sqrt{q}$
$y^2 + y = x^3 + x$	$s \equiv 1, 7 \pmod{8}$	$q + 1 + \sqrt{2q}$
	$s \equiv 3, 5 \pmod{8}$	$q + 1 - \sqrt{2q}$
$y^2 + y = x^3 + x + 1$	$s \equiv 1, 7 \pmod{8}$	$q + 1 - \sqrt{2q}$
	$s \equiv 3, 5 \pmod{8}$	$q + 1 + \sqrt{2q}$
$y^2 + y = x^3 + \delta x$ ($Tr(\delta) = 1$)	s even	$q + 1$
$y^2 + y = x^3 + \delta$ ($Tr(\delta) = 1$)	$s \equiv 0 \pmod{4}$	$q + 1 + 2\sqrt{q}$
	$s \equiv 2 \pmod{4}$	$q + 1 - 2\sqrt{q}$

Remark 1. In this section, it has been used two-point AG codes to construct QUENTA codes. The reason for this is that the QUENTA codes derived from one-point AG codes have trivial parameters;

i.e., they have either zero entanglement, which it are codes that are derived from the standard quantum stabilizer construction (e.g. quantum codes from the CSS construction [23]), or zero dimension, what make them not interesting for this paper.

3.2 Hermitian Construction

In opposition to the Euclidean dual of an AG code, there is no general formula to describe the Hermitian dual of an AG code as in Definition 5. However, describing an AG code via a basis of evaluated elements that belong to a Riemann-Roch space, we can obtain the information that we need from the Hermitian dual code. Before doing that, it is shown below our approach to calculate the dimension of the intersection between an AG code and its Hermitian dual.

Proposition 10. *Let C be a linear code over \mathbb{F}_{q^2} with length n and C^{\perp_h} its dual. Then $\dim(C \cap C^{\perp_h}) = \dim(C^{\perp} \cap C^q)$.*

Proof. Although it is well know that $C^{\perp_h} = (C^{\perp})^q$, we present this result here for completeness

$$\begin{aligned}
 \mathbf{x} \in C^{\perp_h} &\iff \mathbf{x} \cdot \mathbf{c}^q = 0, & \forall \mathbf{c} \in C, \\
 &\iff \sum_{i=1}^n x_i c_i^q = 0, & \forall \mathbf{c} \in C, \\
 &\iff \sum_{i=1}^n x_i^q c_i = 0, & \forall \mathbf{c} \in C, \\
 &\iff \mathbf{x}^q \in C^{\perp}, \\
 &\iff \mathbf{x} \in (C^{\perp})^q.
 \end{aligned}$$

Thus, we can see that

$$\begin{aligned}
 \dim(C \cap C^{\perp_h}) &= \dim(C \cap (C^{\perp})^q) \\
 &= \dim(C^q \cap C^{\perp}).
 \end{aligned}$$

Hence, we have $\dim(C \cap C^{\perp_h}) = \dim(C^{\perp} \cap C^q)$. □

Proposition 10 shows a new way to compute the dimension of $C \cap C^{\perp_h}$. To be able to use it for AG codes, we need to describe the linear code C^q . Proposition 11 approaches this by showing that it is possible to compute a basis to C^q from a basis of C . And Theorem 1 describes how to compute the intersection of two vector space (in particular, linear codes) when the basis of each one belongs to the same larger set which it is also a basis.

Proposition 11. *Let C be a linear code over \mathbb{F}_{q^2} with length n and dimension k . If $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ is a basis of C , then a basis of C^q is given by the set $\{\mathbf{x}_1^q, \dots, \mathbf{x}_k^q\}$.*

Proof. First of all, notice that for any $\mathbf{c}' \in C^q$ there is an $\mathbf{c} \in C$ such that $\mathbf{c}' = \mathbf{x}^q$. Since that $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ is a basis for C , then $\mathbf{c} = \sum_{i=1}^k c_i \mathbf{x}_i$, for $c_i \in \mathbb{F}_{q^2}$. Therefore, $\mathbf{c}' = \mathbf{c}^q = \sum_{i=1}^k c_i^q \mathbf{x}_i^q = \sum_{i=1}^k c'_i \mathbf{x}_i^q$. Since that C and C^q are isomorphic, then they have the same dimension which implies that $\{\mathbf{x}_1^q, \dots, \mathbf{x}_k^q\}$ is a basis for C^q . \square

For self-completeness of this paper, we present the following lemma.

Lemma 1. *Let B be a basis for $\mathbb{F}_{q^2}^n$ and B_1 and B_2 be two subsets of B . Denoting by V_1 and V_2 the subspaces generated by B_1 and B_2 , respectively, then we have that $\dim(V_1 \cap V_2) = |B_1 \cap B_2|$.*

Proof. The claim that any element in $B_1 \cap B_2$ gives a vector in $V_1 \cap V_2$ is trivial. In order to prove the reverse inclusion we consider $\mathbf{v} \in V_1 \cap V_2$. Thus, we can represent it as

$$\mathbf{v} = \sum_{v_{0i} \in B_1 \cap B_2} c_{0i} v_{0i} + \sum_{v_{1i} \in B_1 \setminus (B_1 \cap B_2)} c_{1i} v_{1i} \quad \text{and} \quad \mathbf{v} = \sum_{v_{0i} \in B_1 \cap B_2} d_{0i} v_{0i} + \sum_{v_{2i} \in B_2 \setminus (B_1 \cap B_2)} d_{2i} v_{2i},$$

which implies that

$$\sum_{v_{0i} \in B_1 \cap B_2} (c_{0i} - d_{0i}) v_{0i} + \sum_{v_{1i} \in B_1 \setminus (B_1 \cap B_2)} c_{1i} v_{1i} - \sum_{v_{2i} \in B_2 \setminus (B_1 \cap B_2)} d_{2i} v_{2i} = 0.$$

Since that v_{0i}, v_{1i} , and v_{2i} belong to the basis B , then we have that every coefficient in the previous equation needs to be equal to zero, which results in $\mathbf{v} = \sum_{v_{0i} \in B_1 \cap B_2} c_{0i} v_{0i}$ and the reverse inclusion is proved. \square

Now, we can derive QUENTA codes from the Hermitian construction using AG codes. To illustrate this, we are going to apply the result from Lemma 1 to AG codes derived from rational function field. See Theorem 8.

Theorem 8. *Let q be a prime power and m an integer which is written as $m = qt + r < q^2$, where $t \geq 1$ and $0 \leq r \leq q - 1$. Then we have the following:*

- *If $t \geq q - r - 1$, then there exists an MDS QUENTA code with parameters*

$$[[q^2, (t+1)^2 + 2r + 1 - 2q, q^2 - (qt+r); (q-t-1)^2]]_q.$$

- *If $t < q - r - 1$, then there exists an MDS QUENTA code with parameters*

$$[[q^2, t^2 - 1, q^2 - (qt+r); (q-t)^2 - 2(r+1)]]_q.$$

Proof. Let $F(z)/\mathbb{F}_{q^2}$ be the rational function field, $D = \sum_{i=0}^{q^2-1} P_i$ and $G = mP_\infty$, where $m = qt + r$. Let $C_{\mathcal{L}}(D, G)$ be the AG code derived from D and G with parameters $[q^2, m+1, q^2-m]_q$. Consider $\mathbf{x}^i = ev_D(x^i)$. Let $B = \{\mathbf{x}^i | 0 \leq i \leq n-1\}$. Then B is a basis of \mathbb{F}_q^n . A basis for $C_{\mathcal{L}}(D, G)$ is given by a similar set, $B' = \{\mathbf{x}^i | 0 \leq i \leq m\}$. Thus, a basis of $C_{\mathcal{L}}(D, G)^q$ can be described as $B_1 = \{\mathbf{x}^{qi} | 0 \leq i \leq m\}$. Now, notice that $\mathbf{x}^{q^2+a} = \mathbf{x}^{a+1}$ for all $a \geq 0$. Therefore, $B_1 = \{\mathbf{x}^{qi+j} | 0 \leq i \leq q-1, 0 \leq j \leq t-1\} \cup \{\mathbf{x}^{qi+t} | 0 \leq i \leq r\}$. On the other hand, a basis of $C_{\mathcal{L}}(D, G)^\perp$ is given by the set $B_2 = \{\mathbf{x}^i | 0 \leq i \leq q^2-2-m\} = \{\mathbf{x}^{qi+j} | 0 \leq i \leq q-t-2, 0 \leq j \leq q-1\} \cup \{\mathbf{x}^{(q-t-1)q+j} | 0 \leq j \leq q-r-2\}$. Thus, the powers of \mathbf{x} in the bases B_1 and B_2 can be represented by the sets

$$\left\{ \begin{array}{cccccc} 0 & 1 & 2 & \cdots & t-1 & t \\ q & q+1 & q+2 & \cdots & q+t-1 & q+t \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ rq & rq+1 & rq+2 & \cdots & rq+t-1 & rq+t \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ (q-1)q & (q-1)q+1 & (q-1)q+2 & \cdots & (q-1)q+t-1 & \end{array} \right\}.$$

and

$$\left\{ \begin{array}{cccccc} 0 & 1 & \cdots & q-r-2 & \cdots & q-1 \\ q & q+1 & \cdots & q+q-r-2 & \cdots & 2q-1 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ (q-t-2)q & (q-t-2)q+1 & \cdots & (q-t-2)q+q-r-2 & \cdots & (q-t-2)q+q-1 \\ (q-t-1)q & (q-t-1)q+1 & \cdots & (q-t-1)q+q-r-2 & \cdots & \end{array} \right\},$$

respectively. Using this description, we can see that these bases satisfy the hypothesis in Lemma 1, so it is possible to compute the intersection of the codes related with B_1 and B_2 via the computation of the intersection of the sets. To do so, we have to consider two cases separately, $t \geq q-r-1$ and $t < q-r-1$. For the first case, the intersection is given by the following set $B_1 \cap B_2 = \{\mathbf{x}^{qi+j} | 0 \leq i \leq q-t-2, 0 \leq j \leq t\} \cup \{\mathbf{x}^{(q-t-1)q+j} | 0 \leq j \leq q-r-2\}$. Thus, $\dim(C_{\mathcal{L}}(D, G)^q \cap C_{\mathcal{L}}(D, G)^\perp) = |B_1 \cap B_2| = (q-t-1)(t+1) + q-r-1$. Using the same description for the case $t < q-r-1$, we see that $B_1 \cap B_2 = \{\mathbf{x}^{qi+j} | 0 \leq i \leq q-t-1, 0 \leq j \leq t-1\} \cup \{\mathbf{x}^{(q-t-1)q+j} | 0 \leq j \leq r\}$, which implies in $\dim(C_{\mathcal{L}}(D, G)^q \cap C_{\mathcal{L}}(D, G)^\perp) = |B_1 \cap B_2| = (q-t)t + r + 1$. Applying the previous computations, and using the fact that $C_{\mathcal{L}}(D, G)$ has parameters $[q^2, m+1, q^2-m]_q$, to Proposition 9, we have that there exists a QUENTA code with parameters

- $[[q^2, (t+1)^2 + 2r + 1 - 2q, q^2 - (qt+r); (q-t-1)^2]]_q$, for $t \geq q-r-1$; and
- $[[q^2, t^2 - 1, q^2 - (qt+r); (q-t)^2 - 2(r+1)]]_q$, for $t < q-r-1$.

□

4 Code Comparison

In Tables 2 and 3, we present some optimal QUENTA codes obtained from Theorems 5, 7 and 8. The QUENTA codes derived from the Euclidean construction are presented in Table 2. It has been used AG codes obtained from projective curves and Elliptic curves to construct these codes. As can be seen, the codes in the first column of Table 2 are MDS, and the ones in the second are almost MDS, i.e., the minimal distance differ of the Singleton bound in one unit. For the Table 3, the QUENTA codes are derived from the Hermitian construction, where rational AG codes was used as the classical code. These codes have also optimal combination of parameters, since that they are MDS. Additionally, since that QUENTA codes use entanglement, then we can conclude that these quantum codes from Tables 2 and 3 have better or equal minimal distances than any quantum code with the same length and dimension derived from quantum stabilizer codes [23].

Table 2. Some new maximal entanglement MDS and maximal entanglement almost MDS QUENTA codes from Euclidean construction

New QUENTA codes – Theorem 5 [[$q - 1, a_1 + a_2 + 1, q - 1 - (a_1 + a_2); q - 2 - (a_1 + a_2)$]] _{q} $a_1 + a_2 \leq q - 2$	New QUENTA codes – Theorem 7 [[$e - 2, a_1 + a_2, e - 2 - (a_1 + a_2); e - 2 - (a_1 + a_2)$]] _{q} $a_1 + a_2 < e - 2$, and e as in Table 1
Examples	
[[3, 2, 2; 1]] ₄	[[7, 5, 2; 2]] ₄
[[4, 2, 3; 2]] ₅	[[11, 6, 5; 5]] ₈
[[6, 4, 3; 2]] ₇	[[11, 8, 3; 3]] ₈
[[7, 4, 4; 3]] ₈	[[23, 13, 10; 10]] ₁₆
[[10, 7, 4; 3]] ₁₁	[[23, 18, 5; 5]] ₁₆
[[12, 7, 6; 5]] ₁₃	[[39, 25, 14; 14]] ₃₂
[[15, 10, 6; 5]] ₁₆	[[39, 18, 11; 11]] ₃₂

The remaining QUENTA codes to compare with the literature are the ones derived from Hermitian curve. The first analysis of goodness of our codes is via Singleton defect, which is the difference between the quantum Singleton bound (QSB) presented in Eq. 9 and the minimal distance of the code. Recall that an $[[n, k, d; c]]_q$ quantum code satisfies $k + 2d \leq n + c + 2$ (QSB). Hence, the codes derived from Theorem 6 have maximum Singleton defect equals to $q(q - 1)/2$. Some examples of parameters that are possible to be derived are $[[7, 2, 4; 3]]_4$, $[[26, 10, 11; 10]]_9$, and $[[63, 19, 32; 31]]_{16}$ which have Singleton defect 1, 3, and 6, respectively. Comparing these examples with quantum stabilizer codes, we can see that our codes have minimal distance unreachable for the same length and dimension. This can be seen from the quantum Singleton bound for stabilizer codes (a more general case of quantum codes). Thus, even though the codes from Theorem 6 are not MDS with respect to its quantum Singleton bound, they can be used to attain parameters that are unreachable by quantum

Table 3. Some new MDS QUENTA codes from Hermitian construction

New QUENTA codes – Theorem 8	
$[[q^2, (t+1)^2 + 2r + 1 - 2q, q^2 - (qt+r); (q-t-1)^2]]_q$	
$m = qt + r < q^2, t \geq q - r - 1$ and $0 \leq r \leq q - 1$	
Examples	
$[[16, 6, 6; 1]]_4$	
$[[49, 25, 13; 1]]_7$	
$[[49, 11, 24; 9]]_7$	
$[[64, 29, 20; 4]]_8$	
$[[64, 25, 22; 4]]_8$	
$[[81, 33, 29; 9]]_9$	
$[[81, 16, 41; 16]]_9$	
$[[256, 141, 66; 16]]_{16}$	

stabilizer codes. Adopting entanglement defect as been equal to the difference between the actual amount of entanglement in the QUENTA code and $n - k$, we can see that the entanglement defect in this family of QUENTA code is equal to $2g$, where g is the genus of the Hermitian function field. Lastly, Table 4 shows some examples of QUENTA codes that has rate higher than the asymptotically Gilbert-Varshamov bound presented in Section 5.

Table 4. Some new QUENTA codes from Hermitian Curve

New QUENTA codes – Theorem 6	
$[[q^3 - 1, b_1 + b_2 + 1 - q(q-1)/2, q^3 - 1 - \max\{a_1 + a_2, b_1 + b_2\};$	
$q^3 + q(q-1)/2 - (a_1 + a_2) - 2]]_{q^2}$	
$b_1 \leq a_2 - q(q-1), b_2 \leq \min\{q^3 + q(q-1) - 2 - a_2, a_1\},$ and $a_1 + a_2, b_1 + b_2 < q^3 - 1$	
Examples	
$[[26, 15, 6; 5]]_9$	
$[[64, 39, 11; 10]]_{16}$	
$[[125, 51, 54; 53]]_{25}$	
$[[343, 179, 122; 121]]_{49}$	

5 Asymptotically Good Maximal Entanglement QUENTA Codes

In this section, we show that from any family of (classical) asymptotically good AG codes, we can construct a family of asymptotically good maximal entanglement QUENTA codes. This is a consequence of the use of the result from Carlet, *et al.* [4] applied to the Corollary 1. Before showing it, we need to define the concept of (classical) asymptotically good codes.

Definition 8. Let q be a prime power and $\alpha_q := \sup\{R \in [0, 1] : (\delta, R) \in U_q\}$, for $0 \leq \delta \leq 1$. Here U_q denotes the set of all ordered pair $(\delta, R) \in [0, 1]^2$ for which there is a family of linear codes that are indexed as C_i , with parameters $[n_i, k_i, d_i]_q$, such that $n_t \rightarrow \infty$ as $t \rightarrow \infty$ and $\delta = \lim_{i \rightarrow \infty} d_i/n_i$, $R = \lim_{i \rightarrow \infty} k_i/n_i$. If $\delta, R > 0$, then the family is called asymptotically good.

Proposition 12. [4, Corollary 5.5] Let $q \geq 3$ be a power of a prime and $A(q) = \limsup_{g \rightarrow \infty} \frac{N_q(g)}{g}$, where $N_q(g)$ denotes the maximum number of rational places that a global function field of genus g with full constant field \mathbb{F}_q can have. Then there exists a family of LCD codes with

$$\alpha_q^{LCD}(\delta) \geq 1 - \delta - \frac{1}{A(q)}, \text{ for } \delta \in [0, 1]. \quad (11)$$

Theorem 9. Let $q \geq 3$ be a power of a prime and $A(q)$ as defined in Proposition 12. Then there exists a family of asymptotically good maximal entanglement QUENTA codes with parameters $[[n_t, k_t, d_t; c_t]]_q$, such that

$$\lim_{t \rightarrow \infty} \frac{d_t}{n_t} \geq \delta, \quad \lim_{t \rightarrow \infty} \frac{k_t}{n_t} \geq 1 - \delta - \frac{1}{A(q)},$$

and

$$\lim_{t \rightarrow \infty} \frac{c_t}{n_t} \in [\delta, \delta + 1/A(q)].$$

for all $\delta \in [0, 1 - 1/A(q)]$.

Proof. Let $\mathcal{C} = \{C_1, C_2, \dots\}$ be a family of asymptotically good LCD codes as the ones in Proposition 12, where each C_t has parameters $[n_i, k_i, d_i]_q$. If we apply the family \mathcal{C} to construct QUENTA codes, it follows from Corollary 1 that we can construct maximal entanglement QUENTA codes with parameters $[[n_t, k_t, d_t; c_t]]_q$, such that

$$\lim_{t \rightarrow \infty} \frac{d_t}{n_t} = \lim_{i \rightarrow \infty} \frac{d_i}{n_i} \geq \delta, \quad \lim_{t \rightarrow \infty} \frac{k_t}{n_t} = \lim_{i \rightarrow \infty} \frac{k_i}{n_i} \geq 1 - \delta - \frac{1}{A(q)}.$$

Moreover, we have

$$\lim_{t \rightarrow \infty} \frac{c_t}{n_t} = \lim_{i \rightarrow \infty} \frac{n_i - k_i}{n_i} = \lim_{i \rightarrow \infty} 1 - \frac{k_i}{n_i} \leq \delta + \frac{1}{A(q)}$$

and

$$\lim_{t \rightarrow \infty} \frac{c_t}{n_t} = \lim_{i \rightarrow \infty} \frac{n_i - k_i}{n_i} \geq \lim_{i \rightarrow \infty} \frac{d_i - 1}{n_i} \geq \delta,$$

for $\delta \in [0, 1 - 1/A(q)]$. Thus, since that the families in Proposition 12 are asymptotically good, then the family of QUENTA codes is asymptotically good maximal entanglement. \square

Remark 2. In a recent paper, Galindo, *et al.* [10] derived the quantum Gilbert-Varshamov bound for QUENTA codes. Using AG codes derived from tower of function fields that attain the Drinfeld-Vladut bound [24] and the previous theorem, we can show that there is a family of QUENTA codes with parameters that exceed the mentioned bound (see Figure 1).

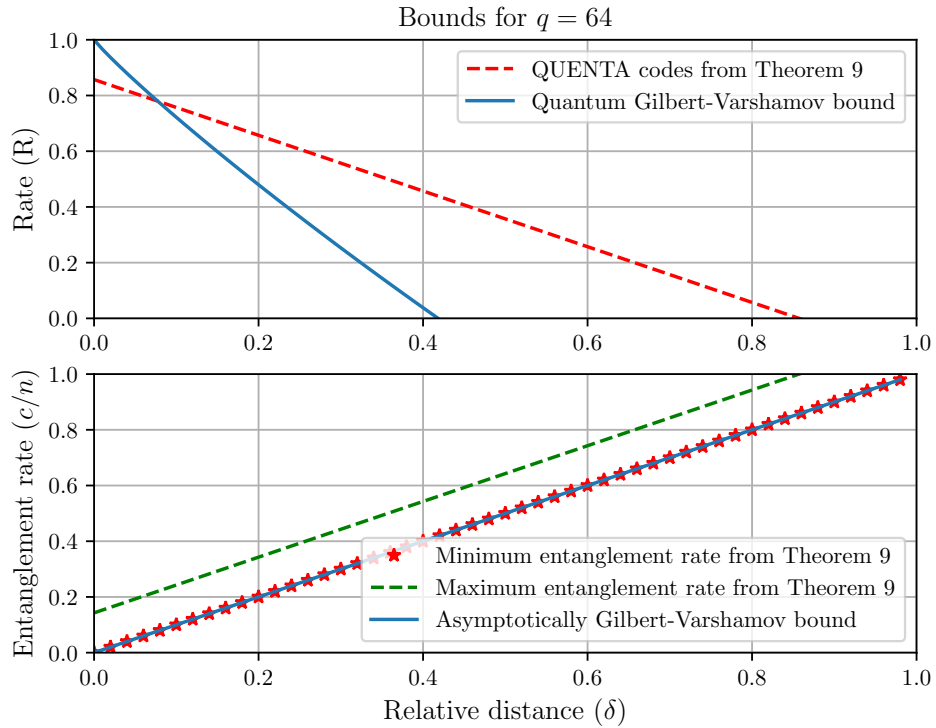


Fig. 1. Comparison between QUENTA codes derived from Theorem 9 and quantum Gilbert-Varshamov bound of [10] via analysis of rate and relative entanglement when $q = 64$.

6 Conclusion

This paper has been devoted to the use of AG codes in the construction of QUENTA codes. We firstly showed two methods to create new AG codes from old ones via intersection and union of divisors. Afterwards, the former method is applied to construct quantum codes via the Euclidean construction method for QUENTA codes. Two of the families derived in this part are MDS or almost MDS and, for some particular range of parameters, have maximal entanglement. For the QUENTA codes constructed from the Hermitian function field, we have shown that it is possible to achieve higher rates when compared with standard quantum stabilizer codes and the entanglement-assisted quantum Gilbert-Varshamov bound. In the following, using the Hermitian construction method for QUENTA codes, we have constructed one more family of QUENTA codes from AG codes, which it was shown to be also MDS. Lastly, it was shown that for any asymptotically good family of classical codes, there is a family asymptotically good maximal entanglement QUENTA codes. In addition, it is demonstrated that there are QUENTA codes surpassing the quantum Gilbert-Varshamov bound.

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