Perfect matroids over hyperfields

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PERFECT MATROIDS OVER HYPERFIELDS

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Abstract. A hyperfield $H$ is stringent if $a \oplus b$ is a singleton unless $a = -b$, for all $a, b \in H$. By a construction of Marc Krasner, each valued field gives rise to a stringent hyperfield.

We show that if $H$ is a stringent skew hyperfield, then weak matroids over $H$ are strong matroids over $H$. Also, we present vector axioms for matroids over stringent skew hyperfields which generalize the vector axioms for oriented matroids and valuated matroids.

1. Introduction

In [Dre86], Dress defined \textit{matroids with coefficients in a fuzzy ring} as a common abstraction of matroids, oriented matroids, and linear spaces. In this general theory, each of these classes arises as matroids over a particular fuzzy ring. Dress and Wenzel later defined \textit{valuated matroids} within this framework [DW92b].

It seems fair to say that the theory of matroids with coefficients constantly aspires towards the condition of oriented matroids, with its broad variety of different, yet equivalent, axiom systems. Among the many axiom systems for oriented matroids offered in [BLVS+99, Ch.3], one can distinguish at least the following three types:

1. Grassmann-Plücker relations for chirotopes; orthogonality of circuits and cocircuits;
2. 3-term Grassmann-Plücker relations for chirotopes; local orthogonality of circuits and cocircuits; modular circuit elimination axioms;
3. vector axioms.

The Grassmann-Plücker relations for chirotopes generalize the symmetric base exchange axiom for ordinary matroids, and both are combinatorial shadows of the Grassmann-Plücker relations among the Plücker coordinates of a linear subspace. The second type of axioms are weaker, ‘local’ versions of the first type. The vector axioms of an oriented matroid closely resemble the definition of a linear subspace as a set of vectors closed under addition and scalar multiplication, and they refine the axiom system for the flats of an ordinary matroid. So the equivalence of these different axiomatizations holds true for matroids, oriented matroids and linear spaces, but also for valuated matroids. The equivalence of type (1) and (2) axioms was established for valuated matroids by Dress and Wenzel [DW92b], and Murota and Tamura showed that valuated matroids are characterized by type (3) vector axioms [MT01].

However, the equivalence of the type (1) and (2) axioms does not generalize to each fuzzy ring. Moreover, there seems to be no natural generalization of the vector axioms (3), if only because these axioms refer to a composition operation for signs which has no counterpart in general fuzzy rings. The main objective of the present paper is show that matroids over a broad class of coefficient domains are equivalently characterized by axioms of type (1), (2) and (3).

We use the framework of matroids over hyperfields rather than matroids over fuzzy fields. Hyperfields were defined by Marc Krasner in [Kra57] as variants of fields in which adding two elements may yield several elements rather than just one. Krasner used this construct to define extensions of the residue field of a valued field. Matroids over hyperfields were defined by Baker and Bowler in [BB17], as a special case of their more general theory of matroids over \textit{tracts}. In this theory, the hyperfields and tracts play a role which is very similar to that of the fuzzy fields of Dress. Baker and Bowler distinguish \textit{strong} and \textit{weak} matroids over hyperfields, based on axioms of type (1) and (2) respectively. It was shown by Giansiracusa, Jun, and Lorscheid [GJL17] that there are canonical functors between the class of fuzzy rings and the class of hyperfields. Via these functors, matroids with coefficients in a fuzzy ring and strong matroids over hyperfields are essentially equivalent notions.
A hyperfield $H$ is stringenent if the hypersum $a \oplus b$ is a singleton unless $a = -b$, for all $a, b \in H$. The hyperfields used to describe matroids, oriented matroids, and valuated matroids are stringent, as well as the hyperfields that Krasner derived from valued fields.

This paper has two main results, both stated in terms of vectors and covectors of matroids over hyperfields. If $M$ is a weak left $H$-matroid on ground set $E$, then $V \in H^E$ is vector of $M$ if $V \perp X$ for each circuit $X$ of $M$, and $U \in H^E$ is a covector of $M$ if $U \perp Y$ for each cocircuit $Y$ of $M$.

**Theorem 1.** Let $H$ be stringent skew hyperfield, and let $M$ be a weak left $H$-matroid on ground set $E$. If $V$ is a vector of $M$ and $U$ is a covector of $M$, then $V \perp U$.

In [DW92a], Dress and Wenzel explored the class of perfect fuzzy rings $R$, which they defined as those such that, for any strong matroid over $R$, all vectors are orthogonal to all covectors. They showed that these matroids have the property that type $(1)$ axioms are equivalent to type $(2)$. They showed perfection of a significant class of fuzzy rings, which includes the ones required for defining classical matroids, oriented- and valuated matroids. Adapting these results, Baker and Bowler argue that over doubly distributive hyperfields, weak matroids are equivalent to strong matroids.

Since if $R$ is perfect then any weak matroid over $R$ is strong, it follows that if $R$ is perfect then even the weak matroids over $R$ will have the property that all vectors are orthogonal to all covectors. In the current paper, we take this stronger statement as our definition of perfection. In particular, for us, in contrast to Dress and Wenzel, it is immediate from the definition that if $R$ is perfect then the type $(1)$ and type $(2)$ axioms are equivalent.

**Theorem 1** extends the existing results in two ways: to stringent hyperfields, which properly include the doubly distributive hyperfields; to stringent skew hyperfields even, which also generalize skew fields.

**Theorem 2.** Let $H$ be a stringent skew hyperfield. Let $E$ be a finite set, and let $V \subseteq H^E$. There is a left $H$-matroid $M$ such that $V = \mathcal{V}(M)$ if and only if

1. $0 \in V$.
2. if $a \in H$ and $V \in V$, then $aV \in V$.
3. if $V, W \in \mathcal{V}(M)$ and $V \oplus W = V \cup W$, then $V \circ W \in \mathcal{V}(M)$.
4. if $V, W \in \mathcal{V}, e \in E$ such that $V_e = -W_e \neq 0$, then there is a $Z \in \mathcal{V}$ such that $Z \in V \oplus W$ and $Z_e = 0$.

This theorem features a composition $\circ : H \times H \to H$, which we will define for all stringent hyperfields. For the tropical hyperfield, this composition is $a \circ b = \max\{a, b\}$, so that this theorem specializes to a similar characterization of Murota and Tamura [MT01]. If $H$ is the hyperfield of signs, then $a \circ b = a$ if $a \neq 0$ and $a \circ b = b$ otherwise, and then the theorem gives the oriented matroid vector axioms.

Anderson has proposed vector axioms for matroids over general hyperfields and tracts in [And19]. These axioms do not generalize the oriented matroid vector axioms. Her paper further discusses composition operators as in $(V2)'$, and elimination properties such as $(V3)$.

A main ingredient of our analysis is a recent classification of stringent skew hyperfields due to Bowler and Su [BS19]. If $H$ is a stringent skew hyperfield, then by their work there exists a linearly ordered group $(\Gamma, <)$ and a multiplicative group homomorphism $\psi : H^* \to \Gamma$ such that:

1. $\psi(x) > \psi(y) \Rightarrow x \parallel y = \{x\}$ for all $x, y \in H^*$; and
2. the restriction of $\psi(x)$ to $R := \{0\} \cup \{x : \psi(x) = 1\}$ is the Krasner hyperfield, the sign hyperfield, or a skew field.

One can think of the function $\psi$ as a (non-Archimedean) valuation of $H$, and the sub-hyperfield $R$ as its residue hyperfield.

We will show in this paper that if $M$ is a matroid over a stringent skew hyperfield $H$ with residue $R$, then there exists a matroid $M_0$ over $R$ whose bases are a subset of the bases of $M$, and whose coefficients are essentially an induced subset of the coefficients of $M$. This residue matroid $M_0$ generalizes the residue matroid of a valuated matroid as defined by Dress and Wenzel. We prove our main theorems for a matroid $M$ over a stringent skew hyperfield by applying well-known facts about matroids, oriented matroid and matroids over skew fields to residue matroids which arise form $M$.

The three cases for the residue of $H$ need similar, yet subtly different argumentation. We chose to present the case that the residue is the Krasner hyperfield separately in Section 3, and the general case in Section 4. The construction of the residue matroid is more involved for skew hyperfields compared to...
commutative hyperfields. We settle these difficulties in Section 3, so that the reader who is only interested in the commutative case could skip this section. Apart from the construction of the residue matroid, the proofs of the main theorems for the special case in Section 3 are easier than for the general case, and they may serve as a stepping stone for the general case in Section 4.

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2. Matroids over hyperfields

2.1. Hyperfields. A hyperoperation on $G$ is a map $\boxplus: G \times G \to 2^G$. A hyperoperation induces a map $\boxplus: 2^G \times 2^G \to 2^G$ by setting

$$X \boxplus Y := \bigcup \{x \boxplus y : x \in X, y \in Y\}.$$ We write $x \boxplus Y := \{x\} \boxplus Y$, $X \boxplus y := X \boxplus \{y\}$, and $X \boxplus Y := X \boxplus Y$. The hyperoperation $\boxplus$ then is associative if $x \boxplus (y \boxplus z) = (x \boxplus y) \boxplus z$ for all $x, y, z \in G$.

A hypergroup is a triple $(G, \boxplus, 0)$ such that $0 \in G$ and $\boxplus: G \times G \to 2^G \backslash \{\emptyset\}$ is an associative hyperoperation, and

1. \[ (H0) \quad x \boxplus 0 = \{x\} \]
2. \[ (H1) \quad \text{for each } x \in G \text{ there is a unique } y \in G \text{ so that } 0 \in x \boxplus y, \text{ denoted } -x := y \]
3. \[ (H2) \quad x \in y \boxplus z \text{ if and only if } z \in (-y) \boxplus x \]
4. \[ (H3) \quad \text{A hyperring is a tuple } (R, \cdot, \boxplus, 1, 0) \text{ so that } \]
   1. \[ (R0) \quad (R, \boxplus, 0) \text{ is a commutative hypergroup} \]
   2. \[ (R1) \quad (R^*, \cdot, 1) \text{ is a monoid, where we denote } R^* := R \backslash \{0\} \]
   3. \[ (R2) \quad 0 \cdot x = x \cdot 0 = 0 \text{ for all } x \in R \]
   4. \[ (R3) \quad \alpha(x \boxplus y) = \alpha x \boxplus \alpha y \text{ and } (x \boxplus y) \alpha = x \alpha \boxplus y \alpha \text{ for all } \alpha, x, y \in R \]

A skew hyperfield is a hyperring such that $0 \neq 1$, and each nonzero element has a multiplicative inverse. A hyperfield is then a skew hyperfield with commutative multiplication.

The following skew hyperfields play a central role in this paper:

**The Krasner hyperfield** $K = \{(0, 1), \cdot, \boxplus, 1, 0\}$, with hyperaddition $1 \boxplus 1 = \{0, 1\}$.

**The sign hyperfield** $S = \{(0, 1, -1), \cdot, \boxplus, 1, 0\}$, with

\begin{align*}
1 \boxplus 1 &= \{1\}, & -1 \boxplus -1 &= \{-1\}, & 1 \boxplus -1 &= \{0, 1, -1\}
\end{align*}

and the usual multiplication.

- Skew fields $K$, which can be considered as skew hyperfields with hyperaddition $x \boxplus y = \{x + y\}$.

If $G, H$ are hypergroups, then a map $f: G \to H$ is a hypergroup homomorphism if $f(x \boxplus y) \subseteq f(x) \boxplus f(y)$ for all $x, y \in G$, and $f(0) = 0$. If $R, S$ are hyperrings, then $f: R \to S$ is a hyperring homomorphism if $f$ is a hypergroup homomorphism, $f(1) = 1$, and $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in R$. A (skew) hyperfield homomorphism is just a homomorphism of the underlying hyperrings.

2.2. Matroids over hyperfields. Let $H$ be a skew hyperfield, and let $E$ be a finite set. For any $X \in H^E$, let $X := \{e \in E : X_e \neq 0\}$ denote the support of $X$. A left $H$-matroid on $E$ is a pair $(E, C)$, where $C \subseteq H^E$ satisfies the following circuit axioms.

- \[ (C0) \quad 0 \notin C. \]
- \[ (C1) \quad \text{if } X \in C \text{ and } \alpha \in H^* \text{, then } \alpha \cdot X \in C. \]
- \[ (C2) \quad \text{if } X, Y \in C \text{ and } X \subseteq Y \text{, then there exists an } \alpha \in H^* \text{ so that } Y = \alpha \cdot X. \]
- \[ (C3) \quad \text{if } X, Y \in C \text{ are a modular pair in } C \text{ and } e \in E \text{ is such that } X_e = -Y_e \neq 0, \text{ then there exists a } Z \in C \text{ so that } Z_e = 0 \text{ and } Z \in X \boxplus Y. \]

In (C3), a pair $X, Y \in C$ is modular if $X \cup Y$ are modular in $C := \{X : X \in C\}$, in the sense that there are no two distinct elements $X'$ and $Y'$ of $C$ with $X' \cup Y'$ a proper subset of $X \cup Y$. A right $H$-matroid is defined analogously, with $\alpha \cdot X$ replaced by $X \cdot \alpha$ in (C1) and (C2). There is no difference between a left- and a right $H$-matroid if $H$ is commutative, and then we speak of $H$-matroids.
If $M = (E, C)$ is a left $H$-matroid, then $C$ is the set of circuits of a matroid in the traditional sense, the matroid $M$ underlying $M$. If $H$ is the Krasner hyperfield, then $M$ determines $M$.

If $N$ is a matroid on $E$ and $H$ is a skew hyperfield, then a collection $C \subseteq H^E$ is a left $H$-signature of $N$ if $C$ satisfies (C0), (C1), and (C2), and $\bar{C}$ is the collection of circuits of $N$. Then $M = (E, C)$ is a left $H$-matroid by definition if and only if $C$ satisfies (C3).

If $X, Y \in H^E$, then we say that $X$ is orthogonal to $Y$, notation $X \perp Y$, if $0 \in X \cdot Y := \sum_e X_e Y_e$. Sets $C, D \subseteq H^E$ are $k$-orthogonal, written $C \perp_k D$, if $X \perp Y$ for all $X \in C$ and $Y \in D$ such that $|X \cap Y| \leq k$, and they are simply orthogonal, written $C \perp D$, if $X \perp Y$ for all $X \in C$ and $Y \in D$.

Orthogonality gives an alternative way to characterize if a circuit signature determines a matroid.

**Theorem 3.** Let $N$ be a matroid on $E$, let $H$ be a skew hyperfield, an let $C$ be a left $H$-signature of $N$. Then $M = (E, C)$ is a left $H$-matroid if and only if there exist a right $H$-signature $D$ of $N^*$ so that $C \perp D$.

If $M = (E, C)$ is a left $H$-matroid, then there is exactly one set $D$ as in the theorem, and then $M^* := (E, D)$ is a right $H$-matroid, the dual of $M$. We say that $M$ has strong duality if $C \perp D$.

The circuits $D$ of the dual of $M$ are the cocircuits of $M$. We may write $C(M)$ and $D(M)$ for the circuits and cocircuits of $M$.

In one direction of Theorem 3 we may drop the assumptions that $C$ and $D$ satisfy (C0) and (C2). More precisely, let’s say that a collection $C \subseteq H^E$ is a weak left $H$-signature of $N$ if $C$ satisfies (C1) and $\bar{C}$ is the collection of circuits of $N$.

**Lemma 4.** Let $N$ be a matroid on $E$ and $H$ a skew hyperfield. Let $C$ be a weak left $H$-signature of $N$ and $D$ a weak right $H$-signature of $N^*$. If $C \perp_2 D$ then $C$ is a left $H$-signature. If $C \perp_3 D$ then $(E, C)$ is a left $H$-matroid.

**Proof.** Suppose first that $C \perp_2 D$. Now $C$ satisfies (C0) since no circuit of $N$ is empty. To show that it satisfies (C2), suppose that we have $X$ and $Y$ in $C$ with $X \subseteq Y$. Since both $X$ and $Y$ are circuits of $N$, they must be equal. Let $e_0$ be any element of $X$ and let $\alpha := Y(e_0) \cdot X(e_0)^{-1}$. Let $e$ be any other element of $X$. Then there is some cocircuit $D$ of $N$ with $X \cap D = \{e_0, e\}$. Let $Z \in D$ with $Z = D$. Then $0 \in X(e_0) \cdot Z(e_0) \equiv X(e) \cdot Z(e)$, so that $X(e_0) \cdot Z(e_0) = -X(e) \cdot Z(e)$. Similarly $Y(e_0) \cdot Z(e_0) = -Y(e) \cdot Z(e)$. Then

$$\alpha \cdot X(e) = Y(e_0) \cdot X(e_0)^{-1} \cdot X(e) = -Y(e_0) \cdot Z(e_0) \cdot Z(e)^{-1} = Y(e).$$

Since $e$ was arbitrary, we have $Y = \alpha \cdot X$, completing the proof of (C2). Thus $C$ is a left $H$-signature.

A dual argument shows that $D$ is a right $H$-signature. So if $C \perp_3 D$ then $(E, C)$ is a left $H$-matroid by Theorem 3.\[\square\]

If $\mu : H \to H'$ is a homomorphism and $X \in H^E$, then we denote

$$\mu^* X := \{\mu(X_e) : e \in E\}.$$

If $M$ is a left $H$-matroid on $E$, and $\mu^* C := \{\mu^* X : X \in C\}$ then $\mu^* M := (E, \mu^* C)$ is a left $H'$-matroid.

2.3. **Rescaling.** If $H$ is a skew hyperfield, $X \in H^E$ and $\rho : E \to H^*$, then right rescaling $X$ by $\rho$ yields the vector $X_{\rho} \in H^{E'}$ with entries $(X_{\rho})_e = X_e \rho(e)$. Similarly, left rescaling gives a vector $\rho X$. The function $\rho$ is called a scaling vector in this context, and we use the shorthand $\rho^{-1}$ for the function $E \to H^*$ such that $\rho^{-1}(e) = \rho(e)^{-1}$.

If $X, Y \in H^E$ and $\rho : E \to H^*$ is a rescaling vector, then clearly $X \perp Y$ if and only if $(X_{\rho^{-1}}) \perp (\rho Y)$. By extension, we have $C \perp_k D$ if and only if $(C_{\rho^{-1}}) \perp_k (\rho D)$ for any sets $C, D \subseteq H^E$, where we wrote

$$C_{\rho^{-1}} := \{X_{\rho^{-1}} : X \in C\} \text{ and } \rho D := \{\rho Y : Y \in D\}.$$

Hence if $M$ is a left $H$-matroid on $E$ with circuits $C$ and cocircuits $D$, then $C_{\rho^{-1}}$ and $\rho D$ are the circuits and cocircuits of a left $H$-matroid $M_{\rho}$. We say that $M_{\rho}$ arises from $M$ by rescaling.
24. **Minors.** Let $H$ be a skew hyperfield and let $E$ be a finite set. For any $Q \subseteq H^E$, we define

$$\operatorname{Min}(Q) := \{Q \in Q : \forall R \subseteq Q \Rightarrow R = Q \text{ for all } R \in Q\}.$$  

Since $\operatorname{Min}(Q) \subseteq Q$, it is evident that e.g. $Q \perp_k R \Rightarrow \operatorname{Min}(Q) \perp_k R$. For a $Q \in H^E$ and an $e \in E$, we write $Q \setminus e$ for the restriction of $Q$ to $E \setminus e$. For any $e \in E$, we write

$$Q_e := \{Q \in Q : Q_e = 0\} \text{ and } Q^e := \{Q \in Q : Q \setminus e = 0\}.$$  

Clearly, $Q \perp_k R \Rightarrow Q_e \perp_k R_e$.

Let $M$ be a left $H$-matroid on $E$ with circuits $C$ and cocircuits $D$. Since $C \perp D$, we have

$$C_e \perp_3 \operatorname{Min}(D^e) \text{ and } \operatorname{Min}(C^e) \perp_3 D_e$$

for any $e \in E$. By Theorem 8, $M \setminus e := (E \setminus e, C_e)$ and $M/e := (E/e, \operatorname{Min}(C^e))$ are both left $H$-matroids. We say that $M \setminus e$ arises from $M$ by deleting $e$ and $M/e$ arises from $M$ by contracting $e$. Evidently, $M \setminus e^* = (M/e)^*$ and $M/e^* = (M/e)^*$.

A matroid $M'$ is a *minor* of $M$ if $M'$ is be obtained by deleting and contracting any number of elements of $M$. If $M$ has strong duality, then this property is inherited by $M \setminus e$ and $M/e$, and hence by all minors of $M$.

25. **Vectors, covectors, and perfection.** A *vector* of a left $H$-matroid $M$ is any $V \in H^E$ so that $V \perp Y$ for all cocircuits $Y \in D$, and a *covector* is a $U \in H^E$ so that $X \perp U$ for all circuits $X \in C$. We write $\mathcal{V}(M), \mathcal{U}(M)$ for the sets of vectors and covectors of $M$.

**Lemma 5.** Let $M$ be a left $H$-matroid with strong duality. Then

$$\mathcal{C}(M) = \operatorname{Min}(\mathcal{V}(M) \setminus \{0\}) \text{ and } \mathcal{D}(M) = \operatorname{Min}(\mathcal{U}(M) \setminus \{0\}).$$

We say that a matroid $M$ is *perfect* if $\mathcal{V}(M) \perp \mathcal{U}(M)$, and that a hyperfield $H$ is perfect if each matroid $M$ over $H$ is perfect. Not all hyperfields are perfect.

**Theorem 6.** The Krasner hyperfield, the sign hyperfield, and skew fields are perfect.

Rescaling of a matroid $M$ has a straightforward effect on the vectors and covectors.

**Lemma 7.** If $M$ is a left $H$-matroid on $E$ and $\rho : E \to H^*$, then $\mathcal{V}(M^\rho) = \mathcal{V}(M)\rho$ and $\mathcal{U}(M^\rho) = \rho \mathcal{U}(M)$.

Matroids, oriented matroids, and linear spaces exhibit the following natural relation between the vectors of a matroid $M$ and its direct minors.

**Theorem 8.** Let $H$ be the Krasner hyperfield, the sign hyperfield, or a skew field, let $M$ be a $H$-matroid on $E$ and let $e \in E$. Then $\mathcal{V}(M/e) = \mathcal{V}(M)^e$ and $\mathcal{V}(M \setminus e) = \mathcal{V}(M)_e$.

For other hyperfields $H$ this statement may fail, but the following holds in general.

**Theorem 9** (Anderson [And19]). Let $H$ be a skew hyperfield, let $M$ be matroid over $H$ on $E$, and let $e \in E$. Then $\mathcal{V}(M/e) \supseteq \mathcal{V}(M)^e$ and $\mathcal{V}(M \setminus e) \supseteq \mathcal{V}(M)_e$.

**Proof.** Suppose first that $M$ is a left $H$-matroid.

$\mathcal{V}(M/e) \supseteq \mathcal{V}(M)^e$: Let $W \in \mathcal{V}(M)^e$, and let $V \in \mathcal{V}(M)$ be such that $V \setminus e = W$. If $W \notin \mathcal{V}(M/e)$, then there is a cocircuit $Z \in \mathcal{D}(M/e)$ so that $W \not\perp Z$. Then for the cocircuit $Y \in \mathcal{D}(M)$ so that $Y \setminus e = Z$ and $Y_e = 0$, we have $V \not\perp Y$, a contradiction.  

$\mathcal{V}(M \setminus e) \supseteq \mathcal{V}(M)_e$: Let $W \in \mathcal{V}(M)_e$, and let $V \in \mathcal{V}(M)$ be such that $V \setminus e = W$ and $V_e = 0$. If $W \notin \mathcal{V}(M/e)$, then there is a cocircuit $Z \in \mathcal{D}(M \setminus e)$ so that $W \not\perp Z$. Then for the cocircuit $Y \in \mathcal{D}(M)$ so that $Y \setminus e = Z$, we have $V \not\perp Y$ since $V_e = 0$, a contradiction.

A straightforward adaptation of this argument settles the case when $M$ is a right $H$-matroid. □
Example: Oriented matroids. Matroids over the hyperfield of signs $\mathbb{S}$ are exactly oriented matroids (see [BLVS+99] Thm. 3.6.1), and the above definitions of circuit, cocircuit, vector and covector generalize the oriented matroid definitions. All oriented matroids are perfect. For the hyperfield of signs, there is a single-valued composition $\circ : \mathbb{S} \times \mathbb{S} \to \mathbb{S}$ defined by

$$a \circ b := \begin{cases} a & \text{if } a \neq 0 \\ b & \text{otherwise} \end{cases}$$

The vector axiomatization of oriented matroids can be stated as follows (cf. [BLVS+99] Thm. 3.7.9)).

**Theorem 10.** Let $E$ be a finite set, and let $V \subseteq \mathbb{S}^E$. There is an $\mathbb{S}$-matroid $M$ such that $V = V(M)$ if and only if

(V0) $0 \in V$.

(V1) if $a \in \mathbb{S}$ and $V \in V$, then $aV \in V$.

(V2) if $V, W \in V$, then $V \circ W \in V$.

(V3) if $V, W \in V, e \in E$ such that $V_e = -W_e \neq 0$, then there is a $Z \in V$ such that $Z \in V \oplus W$ and $Z_e = 0$.

Then $C(M) = \text{Min}(\mathcal{V}\{0\})$.

These vector axioms have no obvious counterpart for matroids over general hyperfields, if only because there is no clear way to define a composition $\circ$ for all hyperfields.

3. Valuated matroids

3.1. Valuative skew hyperfields. Each totally ordered group $(\Gamma, \cdot, <)$ determines a skew hyperfield $\Gamma_{\text{max}} := (\Gamma \cup \{0\}, \cdot, \oplus, 1, 0)$ which inherits its multiplication on $\Gamma_{\text{max}}$ from $\Gamma$, and with a hyperaddition given by

$$x \oplus y = \begin{cases} \max(x, y) & \text{if } x \neq y \\ \{ z \in \Gamma : z \leq x \} \cup \{0\} & \text{if } x = y \end{cases}$$

for $x, y \in \Gamma_{\text{max}}$. The linear order $<$ of $\Gamma$ extends to $\Gamma_{\text{max}}$ by setting $0 < x$ for all $x \in \Gamma$.

If $\Gamma$ is abelian, then a $\Gamma_{\text{max}}$-matroid is exactly a valuroid as defined by Dress and Wenzel [DW92a].

**Lemma 11.** Let $M$ be a left $\Gamma_{\text{max}}$-matroid on $E$ with circuits $C$ and cocircuits $\mathcal{C}$. Then $C \subseteq \mathcal{C}$.

**Proof.** We will show that for all left $\Gamma_{\text{max}}$-matroids $M$, if $e \in (X \setminus Y)_e$, then consider the restrictions $X' := X \setminus \{e\} \in C(M\{e\})$ and $Y' := Y \setminus \{e\} \in \mathcal{C}(M\{e\})$. We have $|E(M\{e\})| < |E(M)|$, so by induction we obtain $X_e \cdot Y_e = X'_e \cdot Y'_e \neq 0$, and hence $X \perp Y$. Hence $X \subseteq Y$. By the dual argument, we also obtain $Y \subseteq X$.

By assumption $X \perp Y$ whenever $|X \cap Y| \leq 3$, hence $|X \cap Y| > 3$. Then $X \perp Y$, then there is an $e \in Y$ so that $X_e \cdot Y_e > X_f \cdot Y_f$ for all $f \in Y\{e\}$. Pick any $T \in C$ such that $X, T$ is a modular pair of circuits. Scaling $T$, we can make sure that $T_g \leq X_f$ for all $f \in T$ and $T_g = X_g$ for some $g \in T$.

If $T_e = X_e$, then fix $g \in T$ so that $T_g = X_g$. By modular circuit elimination, there exists an $e \in X \setminus \{g\}$ such that $Z_g = 0$ and $Z \in X \oplus T$. Then $Z_e = X_e$ and $Z_f \leq X_f$ for all $f \in Z \setminus \{e\}$. Hence $Z_e \cdot Y_e > X_e \cdot Y_f \geq X_f \cdot Y_f$ for all $f \in Z \cap \{e\}$, so that $Z \perp Y$. Since $Z_g = 0$, we also have $\lfloor Z \cap Y \rfloor < \lfloor X \cap Y \rfloor$, and hence by induction $Z \perp Y$, a contradiction.

If $T_e = X_e$, then $T_e \cdot Y_e = X_e \cdot Y_e > X_f \cdot Y_f$ for all $f \in T \cap \{e\}$, and hence $T \perp Y$. Since $X, T$ are a modular pair, $T$ is distinct from $X \setminus Y$. Hence $T \cap \{e\} < \lfloor X \cap Y \rfloor$ and by induction $T \perp Y$, a contradiction. \[\square\]

3.2. The residue matroid. For any vector $X \in \Gamma_{\text{max}}^E$, let $X^\dagger := \{ e \in E : X_e = \max_f X_f \}$. The following observation is key to our analysis of matroids over $\Gamma_{\text{max}}^E$.

**Lemma 12.** Let $E$ be a finite set and let $X, Y \in \Gamma_{\text{max}}^E$. Then $X \perp Y \Rightarrow X^\dagger \perp Y^\dagger$. Conversely, if $X^\dagger \cap Y^\dagger \neq \emptyset$, then $X^\dagger \perp Y^\dagger \Rightarrow X \perp Y$.

**Proof.** If $X^\dagger \cap Y^\dagger = \emptyset$, then $X^\dagger \perp Y^\dagger$ and we are done. If $X^\dagger \cap Y^\dagger \neq \emptyset$, then $X_e Y_e = \max_f X_f Y_f$ if and only if $e \in X^\dagger \cap Y^\dagger$. Then $X \perp Y$ if and only if

$$\emptyset \in \bigoplus_{e \in X^\dagger \cap Y^\dagger} X_e \cdot Y_e$$
if and only if $|X^\uparrow \cap Y^\downarrow| \neq 1$, that is if $X^\uparrow \cap Y^\downarrow$.

For a set $Q \subseteq \Gamma_{\text{max}}^E$, we put $Q_0 := \text{Min}\{Q^1 : Q \in Q\}$. By the Lemma, we have $X^\perp \cap Q \Rightarrow X^\perp \cap Q_0$. We next show that if $C$ is the set of circuits of a matroid over $\Gamma_{\text{max}}$, then $C_0$ is the set of circuits of a matroid. Our argument will make use of a theorem of Minty.

**Theorem 13** (Minty [Min66]). Let $E$ be a finite set and let $C$ and $D$ be sets of nonempty subsets of $E$. Then there is a matroid $M$ on $E$ with circuits $C$ and cocircuits $D$ if and only if:

1. $|C \cap D| = 1$.

2. for each partition of $E$ in parts $B, G, R$ so that $|G| = 1$, either
   - there is a $C \in C$ such that $G \subseteq C \subseteq R \cup G$; or
   - there is a $D \in D$ such that $G \subseteq D \subseteq B \cup G$.

We will need the following consequence of this characterisation, which is also easy to derive from Minty's Theorem 4.1.

**Theorem 14.** Let $E$ be a finite set and let $C$ and $D$ be sets of subsets of $E$ satisfying (M1) and (M2) from Theorem 13. Let $C_0$ be the set of minimal nonempty elements of $C$ and $D_0$ be the set of minimal nonempty elements of $D$. Then there is a matroid $M$ on $E$ with circuits $C_0$ and cocircuits $D_0$.

**Proof.** It suffices to show that $C_0$ and $D_0$ satisfy (M0)-(M2) from Theorem 13. By definition both (M0) and (M1) hold, so it suffices to check (M2). So suppose we have a partition of $E$ in parts $B, G, R$ with $|G| = 1$. Using (M2) for $C$ and $D$, we may suppose without loss of generality that there is some $C \in C$ such that $G \subseteq C \subseteq R \cup G$. Let $C'$ be chosen minimal in $C$ with $G \subseteq C' \subseteq C$. We now show that $C' \in C_0$. Suppose for a contradiction that it is not. Then there must be some $C'' \in C_0$ with $C'' \subseteq C'$. By the choice of $C'$ we cannot have $G \subseteq C''$. Let $e$ be any element of $C''$. Let $B' := (E \setminus C') \cup \{e\}$ and $C' := C' \setminus (G \cup \{e\})$, so that $B', G$ and $C'$ give a partition of $E$ with $|R| = 1$. Now we apply (D2) for $C$ and $D$ to this partition. The first possibility is that we obtain some $C'' \subseteq C'$ with $G \subseteq C'' \subseteq R \cup G = C \setminus \{e\}$, but this cannot happen since it would contradict our choice of $C'$. The other possibility is that there is some $D \in D$ with $G \subseteq D \subseteq B' \cup G$. In that case we have $G \subseteq C' \cap D \subseteq G \cup \{e\}$, so that by (M1) we have $e \in D$. But then $C'' \cap D = \{e\}$, contradicting (M1).

This contradiction shows that $C' \in C_0$, and since $G \subseteq C' \subseteq G \cup R$ it witnesses that (M2) holds for the partition of $E$ into $B, G$ and $R$.

**Lemma 15.** Let $M$ be a left $\Gamma_{\text{max}}$-matroid on $E$ with circuits $C$ and cocircuits $D$. Then there exists a matroid $M_0$ on $E$ with circuits $C_0$ and cocircuits $D_0$.

**Proof.** We prove the theorem by induction on $|E|$. If $|E| \leq 1$, then it is straightforward that $M_0 = M$ is as required. If $|E| > 1$, we prove that $C_0$ and $D_0$ are the circuits and cocircuits of a matroid $M_0$ by applying Theorem 14 to the sets $C_1 := \{X^\uparrow : X \in C\}$ and $D_1 := \{Y^\downarrow : Y \in C\}$. So what we must show is that $C_1$ and $D_1$ satisfy (M1) and (M2).

To see (M1), let $C \in C_1$ and $D \in D_1$, and let $X \in C$ and $Y \in D$ be such that $C = X^\uparrow$ and $D = Y^\downarrow$. Since $X \perp Y$, it follows that $C = X^\uparrow \perp Y^\downarrow = D$ by Lemma 12.

Finally, we show (M2). Let $E$ be partitioned in parts $B, G, R$ so that $|G| = 1$. Since $|E| > 1$, there is at least one element $e \in E \setminus G$, so $e \in R$ or $e \in B$. Replacing $M$ with $M^e$ if $e \in B$, may assume that $e \in R$.

By the induction hypothesis, the statement of the theorem holds for $M \setminus e$. Applying Minty’s Theorem to the matroid $(M/e)_0$, there exists either

- a $C \in (C \setminus e)_0$ so that $G \subseteq C \subseteq (R \setminus e) \cup G$, or
- a $D \in (D \setminus e)_0$ so that $G \subseteq D \subseteq B \cup G$.

In the former case there is an $X \in C \setminus e$ so that $C = X^\uparrow$, and hence there is an $X' \in C$ with $X' \setminus e = X$ and $X' = 0$, so that $C = X' \subseteq C_1$. Then we are done, since $C$ satisfies $G \subseteq C \subseteq R \cup G$. In the latter case, there exists a $Y \in D \setminus e$ such that $D = Y^\downarrow$. Then there exists a $Z \in D$ so that $Y$ is the restriction of $Z$ to $E \setminus e$. If $Z^\uparrow = Y^\downarrow = D$, then we are done, and hence $Z \subseteq \text{max}_{f \neq e} Z_f$.

By the induction hypothesis, the statement of the theorem holds for $M/e$. Applying Minty’s Theorem to the matroid $(M/e)_0$, there exists either
• a $C \in (C/e)_0$ so that $G \subseteq C \subseteq (R\setminus e) \cup G$, or
• a $D \in (D/e)_0$ so that $G \subseteq D \subseteq B \cup G$.

In the latter case, there exists a $Y \in D/e$ so that $D = Y^\perp$, and hence there is a $Y' \in C$ with $Y' \setminus e = Y$ and $Y'_e = 0$, so that $D = Y'^\perp \in D_1$. Since $D$ satisfies $G \subseteq D \subseteq B \cup G$, we are done. In the former case, there is an $X \in C/e$ so that $C = X^\perp$. Then there exists a $T \in C$ so that $X$ is the restriction of $T$ to $E \setminus e$. If $T^\perp \subseteq \{e\} \cup X^\perp = \{e\} \cup D$, then we are done, and hence $T_e > \max_{f \neq e} T_f$.

Summing up, we have obtained a $Z \in D$ so that $Z_e \geq \max_{f \neq e} Z_f$ from considering $M \setminus e$ as well as a $T \in C$ so that $T_e > \max_{f \neq e} T_f$ from considering $M/e$. It follows that $T_e \cdot Z_e > T_f \cdot Z_f$ for all $f \neq e$, so that $\sum_{f \neq e} T_f \cdot Z_f = T_e \cdot Z_e \neq 0$ and hence $T \subseteq Z$. This contradicts Lemma 11.

If $M$ is a left $\Gamma_{\max}$-matroid, then the matroid $M_0$ of Lemma 15 is the residue matroid of $M$. If one assumes that $\Gamma$ is commutative, then the bases of $M_0$ are exactly the maximizers of the Grassmann-Plücker coordinates of $M$. Thus in the commutative case, our residue matroid coincides with a construct proposed by Dress and Wenzel for valuated matroids [DW92], and Lemma 15 generalizes Proposition 2.9(i) of that paper to non-commutative matroid valuations.

Residue matroids are well-behaved with respect to certain minors:

**Lemma 16.** Let $M$ be a left $\Gamma_{\max}$-matroid on $E$ with circuits $C$ and let $e \in E$. If $e$ is not a loop of $M_0$ then $(M/e)_0 = M_0/e$. If $e$ is not a coloop of $M_0$ then $(M \setminus e)_0 = M_0 \setminus e$.

**Proof.** The circuits of $(M/e)_0$ are the minimal nonempty sets of the form $(X\setminus e)^\perp$ and those of $M_0/e$ are the minimal nonempty sets of the form $X^\perp \setminus e$ with $X \in C$. But if $e$ is not a loop of $M_0$ then for any $X \in C$ we have $(X \setminus e)^\perp = X^\perp \setminus e$. The second statement can be proved with a dual argument.

Later we will need the following consequence of this fact:

**Lemma 17.** Let $M$ be a left $\Gamma_{\max}$-matroid on $E$ with circuits $C$. Let $C$ be any circuit of $M_0$ and $S$ any spanning set of $M_0$. Then there is $X \in C$ such that $X^\perp = C$ and $X \subseteq S \cup C$.

**Proof.** We repeatedly apply Lemma 16 to delete all the elements of $E \setminus (S \cup C)$, giving $M_0[(S \cup C) = (M|(S \cup C))_0$, from which the statement follows. None of the elements that we delete are coloops, since they are spanned by $S$.

**Lemma 18.** Let $M$ be a left $\Gamma_{\max}$-matroid. Then $V \subseteq \mathcal{V}(M)$ implies $V^\perp \subseteq \mathcal{V}(M_0)$ and $U \subseteq \mathcal{U}(M)$ implies $U^\perp \subseteq \mathcal{U}(M_0)$.

**Proof.** If $V \subseteq \mathcal{V}(M)$, then by definition $V \perp Y$ for all $Y \in D(M)$, hence $V^\perp \perp Y^\perp$ for all $Y \in D(M)$. Then by Lemma 12, we have $V^\perp \perp D$ for all $D \in D(M)_0$, so that by definition $V^\perp \subseteq \mathcal{V}(M_0)$. The argument for $U$ is analogous.

**Theorem 19.** $\Gamma_{\max}$ is perfect.

**Proof.** Let $M$ be a left $\Gamma_{\max}$-matroid, let $V \subseteq \mathcal{V}(M)$ and $U \subseteq \mathcal{U}(M)$. We need to show that $V \perp U$. Let $g \in \Gamma$ be such that $V_e \cdot U_e > g \cdot U_f = \rho U_f$ for all $e \in V \cap U$ and $f \in U \setminus V$. Let $\rho : E \rightarrow \Gamma$ be determined by

$$
\rho(e) = \begin{cases} 
V_e & \text{if } V_e \neq 0 \\
0 & \text{otherwise}
\end{cases}
$$

Then $V \rho^{-1} \in \{0,1\}^E$ and $\emptyset \neq (\rho U)^\perp \subseteq \mathcal{V} = (V \rho^{-1})^\perp$, so that $(V \rho^{-1})^\perp \cap (\rho U)^\perp \neq \emptyset$. Since $V \rho^{-1} \perp \rho U$ if and only if $V \perp U$, we may assume that $\rho \equiv 1$ by replacing $M$ with $M^\rho$ if necessary. Then $V^\perp \cap U \not= \emptyset$. We have $V^\perp \subseteq \mathcal{V}(M_0)$, $U^\perp \subseteq \mathcal{U}(M_0)$ by Lemma 12, and $\mathcal{V}(M_0) \perp \mathcal{U}(M_0)$ since $M_0$ is an ordinary matroid and the Krasner hyperfield is perfect. By Lemma 12, we have $V \perp U$, as required.

3.3. Vector axioms. For $X, Y \in \Gamma_{\max}^E$, let $X \circ Y \in \Gamma_{\max}^E$ be the vector so that $(X \circ Y)_e = \max\{X_e, Y_e\}$ for all $e \in E$. Clearly $(X \circ Y) \circ Z = X \circ (Y \circ Z)$, and we will omit parenthesis in such expressions in what follows.

**Lemma 20.** Let $M$ be a left $\Gamma_{\max}$-matroid and let $V, W \subseteq \mathcal{V}(M)$. Then $V \circ W \subseteq \mathcal{V}(M)$. 

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Proof. Let $V, W \in \mathcal{V}(M)$. If $V \circ W \notin \mathcal{V}(M)$, then there is a $Y \in \mathcal{D}(M)$ so that $(V \circ W) \notin Y$. Then there is an $e$ so that $(V \circ W)_e > (V \circ W)_f Y_f$ for all $f \in E \setminus e$. Without loss of generality, we have $(V \circ W)_e = V_e$, so that

$$V_e Y_e = (V \circ W)_e Y_e > (V \circ W)_f Y_f \geq V_f Y_f$$

for all $f \in E \setminus e$. It follows that $V \notin Y$, contradicting that $V \in \mathcal{V}(M)$.

\[ \square \]

**Theorem 21.** Let $M$ be a left $\Gamma_{\max}$-matroid. Then $\mathcal{V}(M) = \{X^1 \circ \cdots \circ X^k : X^i \in \mathcal{C}(M), k \leq r^*(M)\}$.

Proof. If $X^1, \ldots, X^k \in \mathcal{C}(M)$, then $X^1, \ldots, X^k \in \mathcal{V}(M)$, and hence $X^1 \circ \cdots \circ X^k \in \mathcal{V}(M)$ by Lemma 20. Conversely, consider a $V \in \mathcal{V}(M)$. If $V = 0$ then $V$ is a composition of $k = 0$ circuits. Otherwise, we show by induction on $|E|$ that $V = X^1 \circ \cdots \circ X^k$ for some $X^1 \in \mathcal{C}(M)$ and $k \leq r^*(M)$. If $V \neq E$, pick an $e \in E \setminus V$. Then $V \setminus e \in \mathcal{V}(M \setminus e)$ by Theorem 9 and by induction $V \setminus e = T^1 \circ \cdots \circ T^k$ for some $T^i \in \mathcal{C}(M \setminus e) = \mathcal{C}(M)_e$, with $k \leq r^*(M \setminus e) \leq r^*(M)$. Taking $X^i \in \mathcal{C}(M)$ so that $X^i_e = 0$ and $X^i \setminus e = T^i$, we obtain $V = X^1 \circ \cdots \circ X^k$ as required. Hence we may assume that $V = E$. Rescaling, we may assume that $V_e = 1$ for all $e \in E$. Then $E = V^+ \in \mathcal{V}(M_0)$ by Lemma 18 and hence there are circuits $C_1, \ldots, C_k$ of $M_0$ so that $V^+ = E = \bigcup_i C_i$, with $k \leq r^*(M_0) = r^*(M)$. Let $X^i, \ldots, X^k$ be the collection of circuits of $M$ so that $maxf X^i_f = 1$ and $C_i = (X^i)^+$ for $i = 1, \ldots, k$. The vector $Z = X^1 \circ \cdots \circ X^k$ clearly has $maxf Z_f = 1$ and hence $Z^+ = \bigcup_i C_i = E$. It follows that $V = Z = X^1 \circ \cdots \circ X^k$, as required.

\[ \square \]

**Theorem 22.** Let $M$ be a left $\Gamma_{\max}$-matroid on $E$, let $e \in E$. Then $\mathcal{V}(M/e) = \mathcal{V}(M)^e$ and $\mathcal{V}(M \setminus e) = \mathcal{V}(M)_e$.

Proof. $\mathcal{V}(M/e) = \mathcal{V}(M)^e$: By Theorem 9, it suffices to show that $\mathcal{V}(M/e) \subseteq \mathcal{V}(M)^e$. Suppose $W \in \mathcal{V}(M/e)$. By Theorem 21 applied to $M/e$, there exist circuits $T^1, \ldots, T^k \in \mathcal{C}(M/e)$ such that $W = T^1 \circ \cdots \circ T^k$. Let $X^i \in \mathcal{C}(M)$ be such that $X^i_e = T^i_i$, for each $i$. By Lemma 20 we have $V := X^1 \circ \cdots \circ X^k \in \mathcal{V}(M)$, and moreover $V \setminus e = T^1 \circ \cdots \circ T^k = W$, as required.

$\mathcal{V}(M \setminus e) = \mathcal{V}(M)_e$: By Theorem 9 it suffices to show that $\mathcal{V}(M \setminus e) \subseteq \mathcal{V}(M)_e$. Suppose $W \in \mathcal{V}(M \setminus e)$. By Theorem 21 applied to $M \setminus e$, there exist circuits $T^1, \ldots, T^k \in \mathcal{C}(M \setminus e)$ such that $W = T^1 \circ \cdots \circ T^k$. Let $X^i \in \mathcal{C}(M)$ be such that $X^i_e = T^i_i$ and $X^i_f = 0$ for each $i$. By Lemma 20 we have $V := X^1 \circ \cdots \circ X^k \in \mathcal{V}(M)$, and moreover $V \setminus e = T^1 \circ \cdots \circ T^k = W$ and $V_e = X^1_e \circ \cdots \circ X^k_e = 0$, as required.

\[ \square \]

**Theorem 23.** Let $E$ be a finite set, and let $\mathcal{V} \subseteq \Gamma_{\max}$. There is a left $\Gamma_{\max}$-matroid $M$ such that $\mathcal{V} = \mathcal{V}(M)$ if and only if

\begin{enumerate}
  \item[(V0)] $0 \in \mathcal{V}$.
  \item[(V1)] If $a \in \Gamma$ and $V \in \mathcal{V}$, then $aV \in \mathcal{V}$.
  \item[(V2)] If $V, W \in \mathcal{V}$, then $V \circ W \in \mathcal{V}$.
  \item[(V3)] If $V, W \in \mathcal{V}$, $e \in E$ such that $V_e = -W_e \neq 0$, then there is a $Z \in \mathcal{V}$ such that $Z \in V \oplus W$ and $Z_e = 0$.
\end{enumerate}

Then $\mathcal{C}(M) = \text{Min}(\mathcal{V} \setminus \{0\})$.

Proof. Sufficiency: Suppose $\mathcal{V}$ satisfies (V0),(V1),(V2),(V3). Let $C := \text{Min}(\mathcal{V} \setminus \{0\})$. Then $C$ satisfies (C1) by (V1). To see (C2), let $X, Y \in C$ be such that $X \subseteq Y$. If $Y \neq aX$ for all $a \in \Gamma$, then scaling $X$ so that $Y_e = X_e$ for some $e \in \mathcal{X}$, we have $X \neq Y$. By (V3), there is a $Z \in \mathcal{V}$ so that $Z_e = 0$ and $Z \in X \oplus Y$. Then $\emptyset \neq Z \subseteq Y \setminus C$, contradicting that $Y \in C$. We show that $C$ satisfies the modular circuit elimination axiom (C3). If $X, Y \in C$ are a modular pair, and $X_e = Y_e$, then by (V3) there exists a $Z \in \mathcal{V}$ such that $Z \in X \oplus Y$ and $Z_e = 0$. If $Z \notin C$, then there exists a $Z' \in C$ so that $Z' \oplus Z$ is a proper subset of $Z$. Applying (V3) to $Z$, $Z'$, $f \in Z' \subseteq Z$, then implies the existence of a $Z'' \in C$ such that $Z'' \subseteq Z \setminus f$. Then the existence of $Z'$, $Z'' \in C$ would contradict the modularity of the pair $X,Y$ in $C$, since $Z' \cup Z'' \subseteq Z \subseteq X \cup Y \setminus \{e\}$. Hence, we have $Z \in C$. This proves that $C$ also satisfies modular circuit elimination, so that $\mathcal{C} = \mathcal{C}(M)$ for some left $\Gamma_{\max}$-matroid $M$. We show that $\mathcal{V} = \mathcal{V}(M)$. We have $\mathcal{V}(M) = \{X^1 \circ \cdots \circ X^k : X^i \in \mathcal{C}(M), k \leq r^*(M)\}$ by Theorem 21. Since $\mathcal{V} \supseteq \mathcal{C} = \mathcal{C}(M)$, and $\mathcal{V}$ is closed under $\circ$ by (V2), we have $\mathcal{V} \supseteq \mathcal{V}(M)$. To show $\mathcal{V} \subseteq \mathcal{V}(M)$, suppose $V \in \mathcal{V}(M)$ and $V$ has minimal support among all such vectors. Let $X \in C$ be any vector with $X \subseteq V$. Scale $X$ so that $X \subseteq V$, with $X_e = V_e$ for some $e$. Then applying (V3) to $V$, $X, e$ yields a vector $Z$ such that $Z \in V \oplus X$. Then $V = X \circ Z$, since if $V_f = 0$ then $X_f = 0$ and hence $Z_f = 0$, and if $V_f > X_f$, then $V_f = Z_f$. We have $X \in \mathcal{V}(M)$ as $X \in \mathcal{C}(M)$ and $Z \in \mathcal{V}(M)$ by minimality of $V$. Hence $V \in \mathcal{V}(M)$ by Lemma 21, contradicting the choice of $V$.\[ 9 \]
In a recent paper, Bowler and Su [BS19] gave a constructive characterization of stringent skew hyperfields.

Theorem 24. Let \( E \) be finite set and let \( C \subseteq \Gamma_{\text{max}}^E \). Then \( M = (E,C) \) is a left \( \Gamma_{\text{max}} \)-matroid if and only if

\begin{enumerate}[(C0)]
    
    \item \( f \in E \) such that \( X_e = Y_e \neq 0 \) and \( X_f > Y_f \), there is a \( Z \in C \) such that \( Z_e = 0 \), \( Z_f = X_f \), and \( Z \leq X \cap Y \).
\end{enumerate}

Proof. Necessity: Suppose that \( M = (E,C) \) is a left \( \Gamma_{\text{max}} \)-matroid. Then (C0), (C1), (C2) hold by definition, and we show (C3). So assume that \( X,Y \in C, e,f \in E \) are such that \( X_e = Y_e \neq 0 \) and \( X_f > Y_f \). By the vector axiom (V3), there exists a \( V \in \mathcal{V}(M) \) such that \( V \subseteq X \cap Y \) and \( V_e = 0 \). As \( X_f > Y_f \), we have \( V_f = X_f \). By Theorem 20 there exist \( Z^1, \ldots, Z^k \in C \) so that \( V = Z^1 \circ \cdots \circ Z^k \). Pick \( i \) so that \( V_f = Z^i_f \) and put \( Z := Z^i \). Then \( Z \in C, Z_e = 0 \) since \( V_e = 0 \), \( Z_f = V_f = X_f \), and \( Z \leq V \subseteq X \cap Y \), as required.

Sufficiency: Suppose (C0), (C1), (C2), (C3) hold for \( M = (E,C) \). To show that \( M \) is a left \( \Gamma_{\text{max}} \)-matroid it suffices to show (C3). So let \( X,Y \in C \) be a modular pair of circuits so that \( X_e = Y_e \). Pick any \( f \in X \setminus Y \). By (C3), there exists a \( Z \in C \) such that \( Z_e = 0 \), \( Z_f = X_f \), and \( Z \leq X \cap Y \). If \( Z \subseteq X \cap Y \), there is a \( Z \in C \) such that \( Z_e = 0 \), \( Z_f = X_f \), and \( Z \leq X \cap Y \).

4. Matroids over stringent hyperfields

4.1. Stringent hyperfields. A skew hyperfield \( H \) is stringent if \( a \neq -b \) implies \( |a \oplus b| = 1 \) for all \( a,b \in H \). In a recent paper, Bowler and Su [BS19] gave a constructive characterization of stringent skew hyperfields. We next describe their characterization. Let \( R \) be a skew hyperfield with hyperaddition \( \oplus^R \), let \((U,\cdot)\) be a group and let \((\Gamma,\cdot,\cdot)\) be a (bi-)ordered group. Consider an exact sequence of multiplicative groups

\[
0 \rightarrow R^* \xrightarrow{\phi} U \xrightarrow{\psi} \Gamma \rightarrow 1
\]

where \( \phi \) is the identity map. Assume that this exact sequence has stable sums, that is, the map \( r \mapsto u^{-1}ru \) is an automorphism of the hyperfield \( R \) for each \( u \in U \).
Define a multiplication \( \cdot \) on \( U \cup \{0\} \) by extending the multiplication of the group \( U \) with \( 0 \cdot x = x \cdot 0 = 0 \), and define a hyperoperation \( \boxplus \) on \( U \cup \{0\} \) by setting

\[
x \boxplus y = \begin{cases} 
\{x\} & \text{if } \psi(x) > \psi(y) \\
\{y\} & \text{if } \psi(x) < \psi(y) \\
(1 \boxplus_R yx^{-1})x & \text{if } \psi(x) = \psi(y) \text{ and } 0 \not\in 1 \boxplus_R yx^{-1} \\
(1 \boxplus_R yx^{-1})x \cup \{z \in R : \psi(z) < \psi(x)\} & \text{if } \psi(x) = \psi(y) \text{ and } 0 \in 1 \boxplus_R yx^{-1}
\end{cases}
\]

for all \( x, y \in U \), and \( x \boxplus 0 = 0 \boxplus x = \{x\} \) for all \( x \in U \cup \{0\} \). Let \( R \times_{U,\psi} \Gamma := (U \cup \{0\}, \boxplus, 1, 0) \). In what follows, whenever we write \( R \times_{U,\psi} \Gamma \) we will implicitly assume the above conditions on \( R, U, \Gamma, \psi \), in particular that the exact sequence has stable sums. The following are two key results from [BS19].

**Lemma 25.** \( R \times_{U,\psi} \Gamma \) is a skew hyperfield. If \( R \) is stringent, then so is \( R \times_{U,\psi} \Gamma \).

**Theorem 26.** Let \( H \) be a stringent skew hyperfield. Then \( H \) is of the form \( R \times_{U,\psi} \Gamma \), where \( R \) is either the Krasner or sign hyperfield or a skew field.

Stringent hyperfields may arise, for example, from a construction due to Krasner [Kra83].

**Theorem 27** (Krasner,1983). Let \( R \) be a ring and let \( G \) be a normal subgroup of \( R^* \). Let

\[
R/G := \{rG : r \in R\}, \oplus, 0, 1G
\]

where \( rG \oplus sG := \{tG : tG \subseteq rG + sG\} \) and \( rG \oplus sG := (rs)G \). Then \( R/G \) is a hyperring and

\[
R \rightarrow R/G : r \mapsto rG
\]

is a hyperring homomorphism. Moreover, if \( R \) is a skew field then \( R/G \) is a skew hyperfield.

Krasner used this construction to derive hyperfields from valued fields, and we note that some hyperfields that arise this way are stringent.

**Lemma 28.** Let \( K \) be a field with valuation \( |.| : K \rightarrow \Gamma_{\text{max}} \), and let \( G := \{1 + k : |k| < 1\} \). Then \( K/G \) is a stringent hyperfield.

**Proof.** That \( K/G \) is a normal subgroup of \( K^* \) was established by Krasner [Kra83]. Hence \( K/G \) is a hyperfield. To see that \( K/G \) is stringent, consider two elements \( xG, yG \) of \( K/G \). If \( |x| > |y| \), then \( zG \subseteq xG + yG \) implies \( z = x(1 + k) + y(1 + k') = x + y + kx + ky' \) where \( |k|, |k'| < 1 \), so that \( z = x(1 + k'') \) with \( |k''| < 1 \) and hence \( zG \subseteq xG \). It follows that \( zG \subseteq xG \), so that \( xG \oplus yG = \{xG\} \). Similarly, if \( |x| < |y| \) then \( xG \oplus yG = \{yG\} \). If \( |x| = |y| \) and \( x + y \neq 0 \), then \( zG \subseteq xG + yG \) implies \( z = x(1 + k) + y(1 + k') = (x + y)(1 + k'') \) with \( |k''| < 1 \), so that \( xG \oplus yG = \{(x + y)G\} \). The remaining case \( x = -y \), and hence \( xG = -yG \), as required.

In the context of Lemma 28, we can write \( K/G = R \times_{U,\psi} \Gamma \). Then \( R \) coincides with the residue field of the valued field \( K \) in the usual sense. In general, we will refer to the hyperfield \( R \) as the residue of a stringent hyperfield \( H = R \times_{U,\psi} \Gamma \).

If \( R \) is the Krasner hyperfield, then \( \psi \) is an isomorphism and \( R \times_{U,\psi} \Gamma = \Gamma_{\text{max}} \). In this section, we will generalize the results of the previous section on matroids over \( \Gamma_{\text{max}} \) to matroids over stringent hyperfields.

**4.2. The residue matroid.** We next extend the notation which we introduced for valuative hyperfields \( \Gamma_{\text{max}} = K \times_{U,\psi} \Gamma \) to more general hyperfields of the form \( H := R \times_{U,\psi} \Gamma \). For any \( Q \in H_E \), define \( Q_I \in H_E \) by

\[
Q_I = \begin{cases} 
Q_e & \text{if } |Q_e| = \max_f |Q_f| \\
0 & \text{otherwise}
\end{cases}
\]

For a set \( Q \subseteq H_E \) we put \( Q_0 := \text{Min}\{Q_I : Q \in Q\} \cap R_E \).

**Lemma 29.** Let \( E \) be a finite set and let \( X, Y \in H_E \). Then \( X \perp Y \Rightarrow X^\perp \perp Y^\perp \). Conversely, if \( X^\perp \cap Y^\perp \neq \emptyset \), then \( X^\perp \perp Y^\perp \Rightarrow X \perp Y \).

If \( \nu : H \rightarrow H' \) is a hyperfield homomorphism and \( M \) is a left or right matroid over \( H \), then we would ordinarily denote the push-forward as \( \nu_* M \). For the valuation \( |.| : H \rightarrow \Gamma_{\text{max}} \), we write \( |M| := |.|_* M \).

**Lemma 30.** Let \( H = R \times_{U,\psi} \Gamma \), and let \( M \) be a left \( H \)-matroid on \( E \) with circuits \( C \) and cocircuits \( D \). There exists a left \( R \)-matroid \( M_0 \) on \( E \) with circuits \( C_0 \) and cocircuits \( D_0 \), and we have \( M_0 = |M|_0 \).
Proof. By induction on $|E|$. The case $|E| = 0$ is trivial, so we may suppose that $|E| \geq 1$.

We begin by showing that $C_0 \perp_2 D_0$. Suppose that $X \in C_0$ and $Y \in D_0$ with $[X \cap Y] \leq 2$. If $[X \cap Y] = 0$ then clearly $X_0 \perp Y_0$, and we cannot have $[X \cap Y] = 1$ since $|M_0|$ is a matroid. So we may assume that $[X \cap Y] = 2$. Call its two elements $x$ and $y$.

Let $X \in C$ and $Y \in D$ be such that $X^\perp = X$ and $Y^\perp = Y$. If $[X \cap Y] \leq 3$ then $X \perp Y$ and so $X \perp Y$ by Lemma [23]. So we may assume that $[X \cap Y] > 2$. Let $z$ and $t$ be distinct elements of $([X \cap Y] \setminus x) \times y$.

First we consider the case that $X \neq Y$. In this case, by loss of generality there is some $e \in X \setminus Y$. Applying the induction hypothesis to $M/e$ yields the desired result.

Now consider the case that $X = Y$. Let $k = |X|$. Then the rank and corank of $M$ are both at least $k - 1$, so $M$ has at least $k - 2 \geq 2$ elements outside of $X$. Let $e$ be such an element. Let $B$ be a basis of $|M_0|$ including $X \setminus x$ but disjoint from $Y \setminus y$. By dualising if necessary, we may suppose without loss of generality that $z \notin B$.

If $z$ is not a coloop of $|M_0|/e$ then by Lemma [17] there is a circuit $Z$ of $M \setminus e$ such that $Z^\perp = X^\perp = X$. But by the induction hypothesis applied to $M \setminus e$ its set of circuits satisfies $(C2)$, so by rescaling if necessary we may suppose that $Z^\perp = X^\perp = X$. Since $Z \neq Y$, we are done as in the above case that $X \neq Y$.

So we may suppose that $z$ is a coloop of $|M_0|/e$. Since it is not a $B$, it cannot be a coloop of $|M_0|$. So $z$ and $e$ are in series in $|M_0|$. Thus $t$ and $e$ cannot be in parallel in $|M_0|$. If $t \notin B$ then it cannot be a loop in $|M_0|/e$, so we are done by a dual argument to that above. Thus $t \notin B$ and in particular $t \notin X$. By an argument like that in the previous paragraph we may suppose that $t$ and $e$ are in series in $|M_0|$.

Since $z$ is not a coloop of $M/t$, by Lemma [17] there is a circuit $Z$ of $M/t$ such that $Z^\perp = (X \setminus t)^\perp = X$. By the induction hypothesis applied to $M \setminus t$ its set of circuits satisfies $(C2)$, so by rescaling if necessary we may suppose that $Z^\perp = X^\perp = X$. Let $Z$ be a circuit of $M$ with $Z^\perp = Z$. Then $Z^\perp = X$ is a vector of $|M_0|$, so it cannot meet the cocircuit $\{t, e\}$ of $|M_0|$ in only the element $t$. Thus $t \notin Z^\perp$ and so $Z^\perp = X^\perp = X$. Since $Z \neq Y$, we are done as in the above case that $X \neq Y$.

This completes the proof that $C_0 \perp_2 D_0$. Furthermore $C_0$ satisfies $(C1)$ since $C$ does. So $C_0$ is a weak left $R$-signature for $|M_0|$. Similarly $D_0$ is a weak right $R$-signature for $|M_0|^\perp$. By Lemma [4] $C_0$ is a left $R$-signature for $|M_0|$.

Next we show that $C_0 \perp_3 D_0$. Suppose that $X \in C_0$ and $Y \in D_0$ with $[X \cap Y] \leq 3$. Let $B$ be any basis of $|M_0|$. By Lemma [17] there is $\hat{X} \in C$ with $\hat{X}^\perp = X$ and $\hat{X} \subseteq B \cup X$. Using $(C2)$, by rescaling if necessary we may suppose that $\hat{X}^\perp = X$. By a dual argument there is some $\hat{Y} \in D$ with $\hat{Y}^\perp = Y$ and $\hat{Y} \subseteq (E \setminus B) \cup Y$. Then $[\hat{X} \cap \hat{Y}] = [X \cap Y] \leq 3$. So $\hat{X} \perp \hat{Y}$ and by Lemma [23] we have $X \perp Y$.

Thus $C_0 \perp_3 D_0$. Applying Lemma [3] again shows that $M_0 := (E, C_0)$ is a left $R$-matroid.

The matroid $M_0$ of Lemma [30] is the residue matroid of $M$. Lemma [30] generalizes Lemma 14 of [Pen18]. The proof of that lemma, which makes use of quasi-Plücker coordinates, extends to the general case.

Lemma 31. Let $H = R \times U, \emptyset, \Gamma$, let $M$ be a left $H$-matroid, and let $V, U \in H^E$. If $V \in \mathcal{V}(M)$ and $V^\perp \in R^E$, then $V^\perp \in \mathcal{V}(M_0)$ and if $U \in \mathcal{U}(M)$ and $U^\perp \in R^E$, then $U^\perp \in \mathcal{U}(M_0)$.

Proof. Suppose $V \in \mathcal{V}(M)$ and $V^\perp \in R^E$. If $V^\perp \notin \mathcal{V}(M_0)$, then there is a $T \in \mathcal{D}(M_0)$ so that $V^\perp \perp T$. Since $\mathcal{D}(M_0) = \text{Min}\{Y^\perp : Y \in \mathcal{D}(M) \cap R^E\}$ by definition of $M_0$, there exists a $Y \in \mathcal{D}(M)$ with $Y^\perp = T$. Then $V^\perp \perp Y^\perp$, and it follows that $V \perp Y$ by Lemma [31]. This contradicts that $V \in \mathcal{V}(M)$.

The argument for $U \in \mathcal{U}(M)$ is analogous.

Theorem 32. Let $H = R \times U, \emptyset, \Gamma$. If $R$ is perfect, then $H$ is perfect.

Proof. Suppose $R$ is perfect. Let $M$ be a left $H$-matroid, let $V \in \mathcal{V}(M)$ and $U \in \mathcal{U}(M)$. We need to show that $V \perp U$. Pick any $h \in H^*$ such that $|V_e| \cdot |U_e| > |h| \cdot |U_f|$ for all $e \in V \setminus U$ and $f \in U \setminus V$. Let $\rho : E \to H^*$ be determined by $\rho(e) = V_e$ for all $e \in V$ and $\rho(e) = h$ otherwise. Since $|\rho U_e| = |V_e \cdot U_e| = |V_e| \cdot |U_e| > |h| \cdot |U_f| = |\rho U_f|$ for all $e \in V \setminus U$ and $f \in U \setminus V$ it follows that $(\rho U)^\perp \subseteq (V \rho^{-1})^\perp$. Since $U$ is nonzero, $(\rho U)^\perp$ is nonempty, we have $(V \rho^{-1})^\perp \cap (\rho U)^\perp \neq \emptyset$. Replacing $M$ with $M^\rho$, we may assume that $\rho = 1$ and hence $U \perp V \neq \emptyset$. Scaling $V, U$, we may assume that $\max_e |V_e| = 1$ and $\max_e |U_e| = 1$, so that $V^\perp, U^\perp \in R^E$. By Lemma [31] we
have $V^\dagger \in \mathcal{V}(M_0)$ and $U^\dagger \in \mathcal{U}(M_p)$. Since $M_0$ is a left $R$-matroid and $R$ is perfect, we have $V^\dagger \perp U^\dagger$. Since $U^\dagger \cap V^\dagger \neq \emptyset$, it follows that $V \perp U$ by Lemma 29 as required.

Using the classification of stringent skew hyperfields and the fact that the Krasner hyperfield, the sign hyperfield, and skew fields are perfect (Theorem 6), we obtain:

**Corollary 33.** Let $H$ be a stringent skew hyperfield. Then $H$ is perfect.

4.3. Vector axioms. Let $H$ be a stringent hyperfield. By the classification of Bowler and Su, we have $H = R \times_{U,\emptyset} \Gamma$, where $R = \mathbb{K}$ or $R = \mathbb{S}$ or $R$ is a skew field. Let $\circ : H \times H \to H$ be defined as

\[
a \circ b = \begin{cases} 
  c & \text{if } a \oplus b = \{c\} \\
  a & \text{if } a = -b \text{ and } R = \mathbb{K} \text{ or } R = \mathbb{S} \\
  0 & \text{if } a = -b \text{ and } R \text{ is a skew field}
\end{cases}
\]

It is then straightforward that $a(b \circ c) = ab \circ ac$ and $(a \circ b)c = ac \circ bc$ for all $a, b, c \in H$, irrespective of $R$. However,

1. $\circ$ is associative if and only if $R$ is not a skew field;
2. $\circ$ is commutative if and only if $R$ is not the sign hyperfield.

Thus $H$ is a semi-ring with addition $\circ$ only if $R = \mathbb{K}$.

In the proof of the main theorem of this section, we will rely on the following property of $\circ$.

**Lemma 34.** Let $H$ be stringent hyperfield and let $a, b, c \in H$. If $|a| \geq |b|$, then $c \in a \oplus (-b)$ implies $a = c \circ b$.

**Proof.** Suppose $|a| \geq |b|$. If $a = 0$, then $b = 0$ and hence $c \in a \oplus (-b)$ implies $c = 0$ implies $a = c \circ b$. If $a \neq 0$, then there are two cases. If $|a| > |b|$, then $c \in a \oplus (-b)$ implies $c = a$ implies $a = c \circ b$. If $|a| = |b|$, then we may assume $|a| = |b| = 1$. Then $c \in a \oplus (-b)$ implies $|c| \leq \max\{|a|, |b|\}$. If $|c| < |a| = |b|$, then $c \in a \oplus (-b)$ implies $a = b$ implies $a = c \circ b$. If $c = |a| = |b|$, then if $R = \mathbb{K}$, then $a = b = c$, so $a = c \circ b$. If $R = \mathbb{S}$, then since $a \neq 0$, we have $c \in a \oplus (-b)$ implies $a = c \circ b$. If $R$ is a skew field, then since $c \neq 0$, we have $c \in a \oplus (-b)$ implies $c = a - b$ implies $a = c \circ b$. $\square$

**Lemma 35.** Let $H$ be a stringent hyperfield, and let $M$ be a matroid over $H$. If $V, W \in \mathcal{V}(M)$ and $V \circ W = V \cup W$, then $V \circ W \in \mathcal{V}(M)$.

**Proof.** Let $V, W \in \mathcal{V}(M)$ be such that $V \circ W = V \cup W$. Suppose $V \circ W \notin \mathcal{V}(M)$. Then $(V \circ W) \not\subseteq Y$ for some $Y \in \mathcal{D}(M)$. Rescaling the elements of $M$, we may assume that $V \circ W \in \{0, 1\}^E$ and $|Y_\epsilon| \geq |Y_\eta|$ for all $\epsilon \in V \circ W$ and $\eta \notin V \circ W$. Scaling $W$, we may assume that $\max_\epsilon |Y_\epsilon| = 1$ and hence $Y^\dagger \in R^E$. Since $V \circ W^\dagger = V \circ W^\dagger$, it follows that $\max_\epsilon |V_\epsilon| \leq |V_\epsilon| = 1$ for each $\epsilon$. If $|Y_\epsilon| < 1$ for all $\epsilon$ then $V \circ W = W \in \mathcal{V}(M)$ and we are done. So we have $V \circ W \in R^E$ and similarly $W^\dagger \in R^E$. It follows that $(V \circ W)^\dagger = V^\dagger \circ W^\dagger$ for all $\epsilon$. By Lemma 31, we have $V^\dagger \in \mathcal{U}(M_0)$ and $V^\dagger, U^\dagger \in \mathcal{V}(M_0)$. Since the statement of the lemma holds true if $H$ is the Krasner hyperfield, the sign hyperfield or a skew field, it follows that $(V \circ W)^\dagger = V^\dagger \circ W^\dagger \notin \mathcal{V}(M_0)$. Since $R$ is perfect, we have $(V \circ W)^\dagger \perp Y^\dagger$. By our rescaling, we have $V \circ W^\dagger \cap Y^\dagger \neq \emptyset$. Then $(V \circ W)^\dagger \perp Y^\dagger$ by Lemma 29, a contradiction. $\square$

**Lemma 36.** Let $H$ be a skew hyperfield, and let $M$ be a left $H$-matroid. If $V^1, \ldots, V^k \in \mathcal{V}(M)$, and $V^1 \boxplus \cdots \boxplus V^k = \{V\}$, then $V \in \mathcal{V}(M)$.

**Proof.** Let $V^1, \ldots, V^k \in \mathcal{V}(M)$, and suppose that $V^1 \boxplus \cdots \boxplus V^k = \{V\}$. Consider any $Y \in \mathcal{D}(M)$. By definition of vector, we have $V^\dagger \perp Y$ so that

\[
\bigoplus_\epsilon V^\dagger_\epsilon Y_\epsilon = \bigoplus_\epsilon V^\dagger_\epsilon Y_\epsilon = \bigoplus_\epsilon V^\dagger_\epsilon Y_\epsilon \geq \bigoplus_\epsilon 0 \ni 0,
\]

and hence $V \perp Y$. Then $V \in \mathcal{V}(M)$. $\square$

**Lemma 37.** Let $H$ be a stringent hyperfield whose core is the sign hyperfield or a skew field, and let $M$ be a left $H$-matroid. If $V \in \mathcal{V}(M)$, then there are $X^1, \ldots, X^k \in \mathcal{C}(M)$ such that $X^1 \boxplus \cdots \boxplus X^k = \{V\}$. 13
Proof. In the special case that $H$ is itself the sign hyperfield, then the lemma is equivalently stated as Proposition 3.7.2 of [BLVS+99]. If $H$ is a skew field, then lemma follows by induction on $V$: take any circuit such that $X \subseteq V$ and scale $X$ so that $V_x = X_e \neq 0$ for some $e$. Taking $V' := V - X$, we have $V' \subseteq V \setminus e$, and hence $V = X' + \cdots + X' + X$, as required.

In the general case, let $V \in \mathcal{V}(M)$. Rescaling $M$, we may assume that $V \in \{0,1\}^E$, and deleting any $e \in E \setminus V$ we may assume that $V = E$. Then $V' \in \mathcal{V}(M_0)$ by Lemma 31. Since the core $F$ of $H$ is the sign hyperfield or a skew field and $M_0$ is a left-$F$-matroid, there are circuits $T^1 \in \mathcal{C}(M_0)$ so that $T^1 \uplus \cdots \uplus T^k = \{V'\}$. Let $X' \in \mathcal{C}(M)$ be such that $(X')^\uparrow = T'$.

For each $e \in E$ we have $\max_i \nu(X_e^i) = 1$, so that $\bigoplus_i X_e^i = \bigoplus_i T_e^i = \{V_e\} = \{V_e\}$.
Hence $X' \uplus \cdots \uplus X^k = \{V\}$ as required. □

Lemma 38. Let $H$ be a stringent hyperfield, and let $x_1, \ldots, x_k \in H$. Then $\bigoplus_i x_i = 1$ unless $0 \in \bigoplus_i x_i$.

Theorem 39. Let $H$ be a stringent hyperfield, let $M$ be a left $H$-matroid on $E$ and let $e \in E$. Then $\mathcal{V}(M \setminus e) = \mathcal{V}(M)_e$ and $\mathcal{V}(M \setminus e) = \mathcal{V}(M)_e$.

Proof. If the core of $H$ is the Krasner hyperfield, then the theorem follows Theorem 22. We may therefore assume that the core of $H$ is the sign hyperfield or a skew field.

$\mathcal{V}(M \setminus e) = \mathcal{V}(M)_e$: By Theorem 9 it suffices to show that $\mathcal{V}(M \setminus e) \subseteq \mathcal{V}(M)_e$. So let $W \in \mathcal{V}(M \setminus e)$. By Lemma 37 there exist $T^1, \ldots, T^k \in \mathcal{C}(M)$ so that $T^1 \uplus \cdots \uplus T^k = \{W\}$. Let $X' \in \mathcal{C}(M)$ be such that $X' \setminus e = T'$. If $X^1 \uplus \cdots \uplus X^k = \{V\}$ for some $V \in H^E$, then $V \in \mathcal{V}(M)$ by Lemma 38. If not, then we must have $0 \in X^1 \uplus \cdots \uplus X^k$ by Lemma 38. Consider the vector $V \in H^E$ such that $V \setminus e = W$ and $V_e = 0$. For any $Y \in \mathcal{D}(M)$, we have $X^i \uparrow Y$, so that $-X^i Y_e \in \bigoplus_{f \neq e} X^j Y_f$. Then

$$
\bigoplus_f V_f Y_f = \bigoplus_f V_f Y_f \subseteq \bigoplus_i X^i Y_f \subseteq -\bigoplus_i X^i Y_e = -(\bigoplus_i X^i) Y_e \neq 0,
$$
so that $V \uparrow Y$. It follows that $V \in \mathcal{V}(M)$.

$\mathcal{V}(M \setminus e) = \mathcal{V}(M)_e$: By Theorem 9 it suffices to show that $\mathcal{V}(M \setminus e) \subseteq \mathcal{V}(M)_e$. So let $W \in \mathcal{V}(M \setminus e)$. By Lemma 37 there exist $T^1, \ldots, T^k \in \mathcal{C}(M)$ so that $T^1 \uplus \cdots \uplus T^k = \{W\}$. Let $X' \in \mathcal{C}(M)$ be such that $X' \setminus e = T'$ and $X_e^k = 0$. Then $X^1 \uplus \cdots \uplus X^k = \{V\}$ for some $V \in H^E$ with $V_e = 0$, and hence $V \in \mathcal{V}(M)$ by Lemma 38. □

Lemma 40. Let $H = R \rtimes_{\psi} \Gamma$, where $R = \mathbb{S}$ or $R$ is a skew field, let $M$ be a left $H$-matroid on $E$, let $X^1, \ldots, X^k \in \mathcal{V}(M)$ and $e \in E$. If $\bigoplus_i X^i \neq 1$, then there exist $Y \in \mathcal{V}(M)$ such that $\max_i |Y^i| < \max_i |X^i|$ and $\bigoplus_i Y^i \subseteq \bigoplus_i X^i$.

Proof. If $H = R$, then $\Gamma$ is a trivial group, and then the condition that $\max_i |Y^i| < \max_i |X^i|$ amounts to $Y^i = 0$ for all $i$. If $H = R$ is the sign hyperfield, we may assume $k = 2$ by omitting all $X^i$ except one with $X^i = 1$ and one with $X^i = -1$. Then the lemma follows by applying (V3) for oriented matroids to $V = X^i, W = X^2, e$. If $H = R$ is a skew field, then $Y = \sum_i X^i$ satisfies the condition of the Lemma.

We use induction on $|E|$. If $H$ is an $f \in \mathcal{V}(M \setminus f)$. Then $T^i := (X^i \uplus f) \in \mathcal{V}(M \setminus f)$, and by induction there are $Z^i \in \mathcal{V}(M \setminus f)$ so that $\bigoplus_i Z^i \subseteq \bigoplus_i T^i$ and $\max_i |Z^i| < \max_i |T^i|$. By Theorem 39 there are vectors $Y^i \in \mathcal{V}(M)$ with $Y^i = 0$ and $Y^i \uplus f = Z^i$. Then $\bigoplus_i Y^i \subseteq \bigoplus_i X^i$.

Proof. By Theorem 22 it suffices to prove the lemma for the case that the residue $R$ of $H$ is the sign hyperfield or a skew field. Let $V, W \in \mathcal{V}(M)$ be such that $V_e = -W_e \neq 0$. We show that there is a $Z \in \mathcal{V}(M)$ such that $Z \in V \uplus W$ and $Z_e = 0$, by induction on $|E|$.

Lemma 41. Let $H$ be a stringent hyperfield, and let $M$ be a left $H$-matroid on $E$. If $V, W \in \mathcal{V}(M), e \in E$ such that $V_e = -W_e \neq 0$, then there is a $Z \in \mathcal{V}(M)$ such that $Z \in V \uplus W$ and $Z_e = 0$.

Proof. By Theorem 22 it suffices to prove the lemma for the case that the residue $R$ of $H$ is the sign hyperfield or a skew field. Let $V, W \in \mathcal{V}(M)$ be such that $V_e = -W_e \neq 0$. We show that there is a $Z \in \mathcal{V}(M)$ such that $Z \in V \uplus W$ and $Z_e = 0$, by induction on $|E|$.
A good collection is a finite sequence $(X^i)_i$ such that
\[ X^i \in \mathcal{C}(M) \text{ for each } i, \quad \bigoplus_i X^i \subseteq V \cup W \text{ and } 0 \in \bigoplus_i X^i. \]

Good collections exist. By Lemma 37, there exist $X^i \in \mathcal{C}(M)$ so that $\{V\} = X^1 \oplus \cdots \oplus X^k$ and $\{W\} = X^{k+1} \oplus \cdots \oplus X^k$ and then $(X^i)_{i=1}^k$ is a good collection. For later use, we fix $Y := (X^2)^{-1} X^j$ for any $j$ so that $|X^2| = |V_e|$. Then $Y_e = 1$ and $|V_e| |Y_e| = \max |V_f|, |W_f|$ for all $f \in E$.

We first show that there exists a good collection $(X^i)$ so that $X^i_e = 0$ for all $e$. Consider the set of values
\[ S := \{|X_e| : X \in \mathcal{C}(M), \ |X_f| = \max |V_f|, |W_f| \text{ for some } f \in V \cup W|. \]

Then $S$ is finite, as by (C2) each circuit of the underlying matroid $M$ contributes at most one value to $S$. We claim that for each good collection $(X^i)$, there is a good collection $(Y^i)$ so that $\max_i |X^i_e| \geq \max_i |Y^i_e| \in S$. Suppose $(X^i)_{i=1}^k$ is a shortest sequence for which this fails. Then $\max_i |X^i_e| \notin S$. Rearranging the $X^i$, we may assume that $|X^i_2| = \max_i |X^i_e|$ if and only if $j > t$. Then for each $j > t$, we have $|X^j_2| = \max_i |V_f|, |W_f|$ for all $f \in V \cup W$. Since $0 \in X^i$, we have $|X^i_0| = X^i$ for all $i \geq 2$. Pick any $y \in \bigoplus_{i=1}^k X^i$. Then $|y| \leq \max_i |X^i_2| = |V_e|$ and hence $|y| Y_e < \max_i |V_f|, |W_f|)$ for all $f \in E$, so that the sequence $(Y^2) := (X^1, \ldots, X^t, -y)$ is a good collection of length $t + 1 < k$. By our choice of $(X^i)$, there is a good collection $(Y^i)$ so that $\max_i |X^i_e| > |y| = \max_i |Z^i_e| \geq |y| Y_e \in S$, a contradiction.

Hence, the minimum of $\max_i |X^i_e|$ over all good collections $(X^i)$ takes value in the finite set $S$. Let $(X^i)$ attain the minimum. If $\bigoplus_{i=1}^k X^i \neq 0$, then by Lemma 30 there is a collection $(Y^i)$ with $Y^i \in \mathcal{V}(M)$ such that $\max_i |Y^i_e| < \max_i |X^i_e|$ and $\bigoplus_{i=1}^k X^i$. Using Lemma 37 we may assume that each $Y^i \in \mathcal{C}(M)$. By our choice of $(X^i)$, we cannot have $0 \in \bigoplus_{i=1}^k Y^i$, and so $\bigoplus_{i=1}^k X^i \in \{y\}$. Then extending $(Y^i)$ with $-y$ yields a good collection as before, which violates the choice of $(X^i)$. So $X^i_e = 0$ for all $i$, as required.

Let $(X^i)_{i=1}^k$ be a shortest good collection with $X^i_e = 0$ for all $i$. We claim that $k = 1$. If not, consider $X^{k-1}$ and $X^k$. If $X^{k-1} \oplus X^k = \{Z\}$, then $Z \in \mathcal{V}(M)$ by Lemma 30 and otherwise there is an $f \notin e$ so that $X^{k-1} = -X^k_f$. Then by our induction hypothesis, there exists a $T \in \mathcal{V}(M \setminus e)$ so that $T_f = 0$ and $T \in X^{k-1}(\mathcal{V}(X^k \setminus e))$. By Theorem 39 there is a $Z \in \mathcal{V}(M)$ so that $Z_e = 0$ and $Z \setminus e = T$. In either case, we have $X^1 \oplus \cdots \oplus X^{k-2} \oplus Z \subseteq X^i \subseteq V \cup W$, so that $(X^1, \ldots, X^{k-2}, Z)$ is a shorter good collection, a contradiction. Hence $k = 1$, and taking $Z = X^1$ we have $Z \in \mathcal{V}(M), Z_e = 0$ and $Z \in V \cup W$, as required. \(\square\)

**Theorem 42.** Let $H$ be a stringent skew hyperfield. Let $E$ be a finite set, and let $V \subseteq H^E$. There is a left $H$-matroid $\mathcal{M}$ such that $\mathcal{V} = \mathcal{V}(M)$ if and only if

(V0) \[ 0 \in V. \]

(V1) \[ \text{if } a \in H \text{ and } V \in \mathcal{V}, \text{ then } aV \in \mathcal{V}. \]

(V2)' \[ \text{if } V, W \in \mathcal{V}(M) \text{ and } V \cup W = V \cup W, \text{ then } V \cup W \in \mathcal{V}(M). \]

(V3) \[ \text{if } V, W \in \mathcal{V}, e \in E \text{ such that } V_e = -W_e \neq 0, \text{ then there is a } Z \in \mathcal{V} \text{ such that } Z \in V \cup W \text{ and } Z_e = 0. \]

Then $\mathcal{C}(M) = \text{Min}(\mathcal{V}\setminus\{0\})$. \(\text{Proof.} \)

Sufficiency: Suppose $\mathcal{V}$ satisfies (V0),(V1),(V2)', and (V3). Let $\mathcal{C} := \text{Min}(\mathcal{V}\setminus\{0\})$. Then $\mathcal{C}$ satisfies (C1) by (V1). To see (C2), let $X, Y \in \mathcal{C}$ be such that $\sum X \subseteq \sum Y$. If $Y = aX$ for all $a \in H^*$, then scaling $X$ so that $Y_e = -X_e$ for some $e \in X$ we have $X \neq -Y$. By (V3), there is a $Z \in \mathcal{V}$ so that $Z_e = 0$ and $Z \in X \cup Y$, and since $X \neq -Y$ we have $Z \neq 0$. Then $\bigoplus X \neq \sum Y \setminus e$, contradicting that $Y \in \mathcal{C}$. We show that $\mathcal{C}$ satisfies the modular circuit axiom (C3). If $X, Y \in \mathcal{C}$ are a modular pair, $X_e = -Y_e$, then by (V3) there exists a $Z \in \mathcal{V}$ such that $Z_e = X \cup Y$ and $Z_e = 0$. If $Z \notin \mathcal{C}$, then there exists a $Z' \in \mathcal{C}$ so that $Z'$ is a proper subset of $Z$. Applying (V3) to $Z$, $Z'$, $f \in Z' \subseteq Z$, then implies the existence of a $Z'' \in \mathcal{C}$ such that $Z''$ is contained in $Z \setminus f$. Then the existence of $Z', Z'' \in \mathcal{C}$ would contradict the modularity of the pair $X, Y$ in $\mathcal{C}$, since $Z' \cup Z'' \subseteq Z \subseteq \sum X \cup \sum Y \setminus e$. Hence, we have $Z \in \mathcal{C}$. This proves that $\mathcal{C}$ also satisfies modular circuit axiom (C3).

Then $\mathcal{C} = (\mathcal{C}(M) \text{ for some left } H\text{-matroid } \mathcal{M}$.) We show that $\mathcal{V} = \mathcal{V}(M)$. To see $\mathcal{V} \subseteq \mathcal{V}(M)$, suppose $V \in \mathcal{V}\setminus\mathcal{V}(M)$ and $\mathcal{V}$ has minimal support among all such vectors. Let $X \in \mathcal{C}$ be any vector with $\sum X \subseteq V$. Scale $X$ so that $|X_f| \leq |V_f|$ for all $f \in E$, with $X_e = V_e$ for some $e$. Then applying (V3) to $V, -X, e$ yields a vector $Z$ such that $Z \in V \cup (X \setminus -X)$. Then $V = Z \cup X$ by Lemma 34. We have $X \in \mathcal{V}(M)$ as $X \in \mathcal{C}(M)$ and $Z \in \mathcal{V}(M)$ by minimality of $V$. Then $V = Z \cup X \in \mathcal{V}(M)$ as $\text{(V2)'}$ holds for $\mathcal{V}(M)$ by Lemma 35. That $\mathcal{V}(M) \subseteq \mathcal{V}$ follows in the same way, since $\text{(V2)'}$ holds for $\mathcal{V}$ by assumption and (V3) holds for $\mathcal{V}(M)$ by Lemma 31.
Necessity: If $V = V(M)$, then (V0),(V1) are clear, (V2)' is Lemma 35 and (V3) is Lemma 41. □

We note that if the residue $R$ of $H$ is the Krasner or sign hyperfield, then $V \circ W = V \cup W$ for all $V, W \in H^E$, so that then condition (V2)' may be simplified to (V2) as in Theorem 23. If $R$ is a skew field, then (V2)' is equivalent to

(V2)" if $V, W \in V(M)$ and $V \boxplus W = \{U\}$, then $U \in V(M)$.

A minor adaptation of the proof of Theorem 24 yields the following characterization.

**Theorem 43.** Let $H$ be a stringent skew hyperfield with residue $S$. Let $E$ be finite set and let $C \subseteq H^E$. Then $M = (E, C)$ is a left $H$-matroid if and only if (C0), (C1), (C2) and

(C3)' for any $X, Y \in C, e, f \in E$ such that $X_e = -Y_e \neq 0$ and $|X_f| > |Y_f|$, there is a $Z \in C$ such that $Z_e = 0, Z_f = X_f$, and $|Z_g| < |X_g \circ Y_g|$ or $Z_g \in X_g \boxplus Y_g$ for all $g \in E$.

For the circuits $C$ of a matroid over a stringent skew hyperfield $H$ whose residue is a skew field, combining (V3) and Lemma 37 evidently yields:

(C3)'' for any $X, Y \in C, e \in E$ such that $X_e = -Y_e \neq 0$, there is a $V \in X \boxplus Y$ and $Z^i \in C$ such that $V_e = 0$ and $Z^1 \boxplus \cdots \boxplus Z^k = \{V\}$.

For such hyperfields $H$, we could not imagine an axiom which is sufficiently strong to characterize matroids over $H$, but also more like (C3)' in that it claims the existence of just a single circuit $Z$.

**References**


