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# Postprocess Galerkin Method for Steady Navier-Stokes Equations \*

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## Abstract

A kind of postprocess Galerkin method for steady Navier-Stokes equations with nonsingular solution, which is based on the virtue of Inertial Manifold and Approximate Inertial Manifold, is presented in this paper. Different from other construction of approximate inertial manifolds, we use a kind of so-called  $G$ -decomposition to get the large and small eddy components of the true solution. Its attractive advantage is that one could get the large eddy components exactly. Then the error of the scheme only comes from the approximation of the small eddy components. We prove that the proposed postprocess Galerkin scheme can greatly improve the convergence rate of the approximate solution with lower computing effort. And we also give an numerical example to verify our result.

**Key words.** Navier-Stokes Equations, Galerkin method, convergence, nonsingular solution

## 1 Introduction

To improve the convergence rate of the standard Galerkin approximate solutions, many authors derived new techniques and methods in last decade. For example, Lin Qun[1], W. Layton[2], J. Xu[3] used extrapolation and two level meshes respectively. Especially, since 1988 when the concepts of Inertial Manifold (IM)[4] and Approximate Inertial manifold (AIM)[5] for dissipative evolutionary partial differential equation were given by C. Foias, G. R. Sell, R. Temam and C. Foias, O. Manley, R. Temam respectively, a kind of new numerical method called nonlinear Galerkin method is studied and developed by many authors, see [7], [8] and [9], etc. This method is designed for numerically simulating the long time behavior of the solution and its main advantage is that the related numerical scheme has better stability and convergence rate than that of standard Galerkin method. The main ideal of this kind of inertial scheme is to find some kind of interactive or approximate interactive rules between large eddy and small eddy components. Based upon the virtue of IM and AIM, we apply this ideal to simulating the nonsingular solution of steady Navier-Stokes equations and give a kind of postprocess Galerkin procedure in this paper. We will show that it can greatly improve the convergence rate of standard Galerkin solution by using this procedure.

Suppose that  $u \in H$  be a solution of Navier-Stokes equations, where  $H$  is a Hilbert space,  $H_m \subset H$  is a finite dimensional subspace,  $u_m$  is the standard Galerkin solution, then

$$\|u - u_m\| \leq c\lambda_{m+1}^{-\frac{1}{2}},$$

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and for general nonlinear Galerkin solution  $\tilde{u}_m$

$$\|u - \tilde{u}_m\| \leq c\lambda_{m+1}^{-1}.$$

The basic technique of our postprocess Galerkin method is to decompose the true solution  $u$  with respect to  $u_m$ , that is

$$u = \bar{u}_m + \hat{u}, \quad \text{where } \hat{u} = u - u_m, \quad \bar{u}_m = u_m.$$

We identify  $\bar{u}_m$  and  $\hat{u}$  with the large and small eddy components respectively. Then we can exactly derive  $\bar{u}_m$ , the large eddy components of  $u$ , by standard Galerkin method. For convenience, we will use  $u_m$  to denote  $\bar{u}_m$  directly. Under this presupposition, we construct an interactive rule between  $u_m$  and  $\hat{u}$ , that is a mapping  $\Phi$  from  $H_m$  to  $H$ , such that  $\Phi(u_m)$  can generate a suitable approximation of  $\hat{u}$ . Then we use the postprocess Galerkin solution  $u_m^* = u_m + \Phi(u_m)$  to approximate  $u$  and get

$$\|u - u_m^*\| \leq c\lambda_{m+1}^{-\frac{3-\varepsilon}{2}}.$$

## 2 Preliminary

Consider the Navier-Stokes equations in bounded domain  $\Omega \subset \mathcal{R}^d$ ,  $d = 2, 3$

$$\begin{cases} -\nu\Delta u + (u \cdot \nabla)u + \nabla p = F & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ \text{boundary conditions,} \end{cases} \quad (1)$$

where  $u : \Omega \rightarrow \mathcal{R}^d$  is the flow field,  $p : \Omega \rightarrow \mathcal{R}$  the pressure,  $F$  the exterior force which drives the flow and  $\nu = \frac{1}{Re}$ ,  $Re > 0$  is the Reynolds number.  $\Delta$  and  $\nabla$  denote the Laplace and gradient operator respectively. The boundary conditions can be either fixed boundary or periodic boundary conditions ( $\Omega$  is a cube under this circumstances).

Now we introduce a Hilbert space  $H$ . For fixed boundary case, we take

$$H = \{u \in L^2(\Omega)^d, u \cdot n|_{\partial\Omega} = 0, \operatorname{div} u = 0 \text{ under weak sense}\}.$$

And for periodic boundary case, we take

$$H = \left\{ u = \sum_{k \in \mathbb{Z}^2, k \neq 0} u_k e^{ik \cdot \alpha(x)}, c_k = \overline{c_{-k}}, \sum_{k \in \mathbb{Z}^2} |u_k|^2 < +\infty, \operatorname{div} u = 0 \text{ under weak sense} \right\},$$

where  $\alpha(x)$  is a linear function of  $x$  related to the shape of  $\Omega$ . Denote by  $P : L^2(\Omega)^d \rightarrow H$  the Leray orthogonal projection, we can obtain the abstract Navier-Stokes equation by projecting (1) onto  $H$

$$\nu Au + B(u, u) = f, \quad (2)$$

where  $A = -P\Delta$ ,  $B(u, u) = P[(u \cdot \nabla)u]$ ,  $f = PF$ . As well known,  $A$  is a linear unbounded self-adjoint and positive definite operator which possesses bounded inverse  $A^{-1}$ , then its eigenvalues and related eigenfunctions are

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots \rightarrow +\infty, \quad m \in \mathcal{N},$$

and

$$\phi_1, \phi_2, \cdots, \phi_m, \cdots.$$

And we denote  $V = D(A^{\frac{1}{2}})$ , the closure of  $H$  under the sense of  $H^1$  endowed with the norm  $\|v\| = |A^{\frac{1}{2}}v|$  where  $|v| = (v, v)^{\frac{1}{2}} = \int_{\Omega} v \cdot v dx$ . For any  $s \in \mathcal{R}$ ,  $A^s$  is a power operator. Then the weak form of (2) reads

$$\begin{cases} \text{find } u \in V \text{ such that} \\ a(u, v) + b(u; u, v) = (f, v), \quad \forall v \in V. \end{cases} \quad (3)$$

Here

$$\begin{aligned} a(u, v) &= \nu(\nabla u, \nabla v), \\ b(u; u, v) &= ((u \cdot \nabla)u, v). \end{aligned}$$

From [6], we know

$$b(u; w, v) = -b(u; v, w) \quad \forall u, v, w \in V, \quad (4)$$

$$\begin{aligned} |b(u; v, w)| &\leq c_1 \|u\|_{s_1} \|w\|_{s_2+1} \|v\|_{s_3}, \\ \forall u &\in D(A^{\frac{s_1}{2}}), v \in D(A^{\frac{s_2+1}{2}}), w \in D(A^{\frac{s_3}{2}}), \end{aligned} \quad (5)$$

where  $s_1 + s_2 + s_3 \geq \frac{d}{2}$ ,  $s_1, s_2, s_3 \geq 0$  with  $(s_1, s_2, s_3) \neq (0, 0, \frac{d}{2}), (0, \frac{d}{2}, 0), (\frac{d}{2}, 0, 0)$ . If we denote by  $P_m$  the  $L^2$  orthogonal projection operator from  $H$  onto  $H_m = \{\phi_1, \dots, \phi_m\}$ , the standard Galerkin approximate solution  $u_m \in H_m$  satisfies

$$a(u_m, v) + b(u_m; u_m, v) = (f, v), \quad \forall v \in H_m. \quad (6)$$

Because of  $u_m \in H_m$ , the  $L^2$  error estimate can not exceed the  $L^2$  estimation of  $u_m - P_m u$ . In fact

$$|u - u_m| \leq c\lambda_{m+1}^{-1}, \quad \|u - u_m\| \leq c\lambda_{m+1}^{-\frac{1}{2}}, \quad (7)$$

where  $c > 0$  is a constant depending on  $f$  and  $\nu$ .

Moreover, we state the interpolation inequality in Sobolev spaces which will be used in the following: for  $v \in D(A^{\frac{1}{2}})$ ,  $0 \leq \varepsilon \leq 1$

$$\|v\|_{\varepsilon} \leq c_2 \|v\|^{\varepsilon} |v|^{1-\varepsilon}. \quad (8)$$

### 3 $G$ -Decomposition

From the point of view of  $L^2$  orthogonal projection  $P_m$ , the approximate order of  $P_m u - u_m$  is restricted by (7). Due to the restriction of truncate error,  $u_m - P_m u$  has already been optimal. Alternatively, from other projection point of view, the results maybe quite different. As we said in introduction, we can decompose  $u$  according to  $u_m$  intuitively

$$u = u_m + \hat{u}, \quad \hat{u} = u - u_m. \quad (9)$$

Identifying  $u_m \in H_m$  and  $\hat{u} \in V$  with large and small eddy components respectively, the standard Galerkin approximate solution  $u_m$  reaches the large eddy components of the true solution exactly. According to the ideal of IM and AIM, there may exist an interactive rule or at least some approximate interactive rules between large and small eddies.

In this paper, we call (9) the *Galerkin*-decomposition of the true solution, simply the  $G$ -decomposition. Of course, when one considers the usual  $L^2$  orthogonal decomposition, the small eddy components under the sense of  $G$ -decomposition consists of small eddy as well as large eddy components. In order to understand this  $G$ -decomposition deeply, we introduce the following formal nonlinear "orthogonal projection"  $Q_m: V \rightarrow V_m$ , where  $V_m = P_m V$ .

Restricting (3) on  $V_m$  and subtracting (6) from it, we derive

$$\begin{aligned} a(u - u_m, v) + b(u - u_m; u - u_m, v) + b(u_m; u - u_m, v) \\ + b(u - u_m; u_m, v) = 0, \quad \forall v \in V_m. \end{aligned} \quad (10)$$

That is,  $\hat{u}$  satisfies the above equation if we substitute  $\hat{u}$  for  $u - u_m$ . It is a nonlinear equation of  $\hat{u}$ . Enlightened by (10), we can formally define the formal nonlinear "orthogonal projection" operator  $Q_m$  from  $V$  onto  $V_m$

$$\begin{cases} \forall \phi \in V, \text{ find } Q_m \phi \in V_m \text{ such that} \\ a(\phi - Q_m \phi, v) + b(\phi - Q_m \phi; \phi - Q_m \phi, v) \\ + b(u_m; \phi - Q_m \phi, v) + b(\phi - Q_m \phi; u_m, v) = 0, \quad \forall v \in V_m. \end{cases} \quad (11)$$

Under this "projection", we have

$$u_m = Q_m u, \quad \hat{u} = (I - Q_m)u,$$

if we take  $\phi = u$  in (11). And  $\hat{u}$  is "orthogonal" with  $V_m$  under the sense of (11).  $G$ -decomposition of  $u$  is just the result of this kind of "projection".

As to whether (11) can really define a "orthogonal projection" or not, it is unnecessary for our discussion in the rest. On the other hand,  $\hat{u}$  should be a small quantity compared with  $u_m$  if  $m$  is large enough and we could get another formal linear "projection" by omitting the higher order small item  $b(\phi - Q_m \phi; \phi - Q_m \phi, v)$  in (11). In fact, we ever studied this kind of projection elsewhere in detail. The disadvantage is that the related algorithm is very hard to be completed in computing. On the contrary, our algorithm developed in this paper by using the formal "projection"  $Q_m$  can overcome this disadvantage and obtain sufficiently high convergence rate.

Now based upon  $G$ -decomposition of  $u$ , the Navier-Stokes equations (3) can be rewritten as

$$\begin{aligned} a(\hat{u}, v) + b(\hat{u}; \hat{u}, v) + b(\hat{u}; u_m, v) + b(u_m; \hat{u}, v) \\ + a(u_m, v) + b(u_m; u_m, v) = (f, v), \quad \forall v \in V. \end{aligned} \quad (12)$$

Obviously, we could obtain the small eddy components of  $u$  under the sense of  $G$ -decomposition by solving (12). But this procedure is as difficult as to solving (3) when we consider the complex form of (12).

## 4 Construction of Finite Dimensional Mapping $\Phi$

As we said in introduction, the aim of this paper is to construct a finite dimensional mapping  $\Phi: V_m \rightarrow V$  such that it can form a suitable approximate interactive rule between the large and small eddy components of the solution. After we get the large eddy components  $u_m$  by standard Galerkin method, we could use it to obtain  $\Phi(u_m)$  which should be a high accuracy approximation of  $\hat{u}$ . We expect that  $u_m + \Phi(u_m)$  can generate a much better approximation of  $u$  than both standard Galerkin and nonlinear Galerkin approximation. We call this procedure the postprocess Galerkin procedure. Now, the key problem here is how to construct  $\Phi$  such that the computation of  $\Phi(u_m)$  is simple and  $\Phi$  can derive a high accuracy postprocess tool. Of course, we hope  $\Phi$  to be a linear mapping.

Based on the above consideration and noticing (12), we will use the linear part of (12) to determine  $\Phi$ , that is

$$\begin{cases} \forall \phi \in V_m, \text{ find } \Phi(\phi) \in V \text{ such that} \\ a(\Phi(\phi), v) + b(\Phi(\phi); u_m, v) + b(u_m; \Phi(\phi), v) \\ a(\phi, v) + b(\phi; \phi, v) = (f, v), \quad \forall v \in V. \end{cases} \quad (13)$$



$$\mu(u_m) = \|D\mathcal{F}(u) - D\mathcal{F}_m(u_m)\|_{\mathcal{L}(V, V^*)}.$$

If  $u_m$  is close to  $u$  so much such that

$$\sigma(u)\mu(u_m) < 1, \quad (20)$$

$D\mathcal{F}_m(u_m)$  is an isomorphism between  $V_m$  and  $V_m^*$ . Hence,  $u_m$  is a nonsingular point of  $\mathcal{F}_m$ .

**Proof** See [12].  $\square$

**Corollary 4.1** Suppose all conditions in lemma 4.1 hold. Take  $m$  large enough such that

$$\lambda_{m+1} \geq \frac{4c_1^2 c^2}{\alpha_0^2}, \quad (21)$$

then  $u_m$  is a nonsingular point of  $\mathcal{F}_m$ .

**Proof** From (18)~(19), we know

$$\sigma(u) = \|D\mathcal{F}^{-1}(u)\|_{\mathcal{L}(V^*, V)} \leq \alpha_0^{-1}.$$

On the other hand,

$$\begin{aligned} \mu(u_m) &= \|D\mathcal{F}(u) - D\mathcal{F}_m(u_m)\|_{\mathcal{L}(V, V^*)} \\ &= \sup_{w, v \in V} \frac{\langle (D\mathcal{F}(u) - D\mathcal{F}_m(u_m))w, v \rangle}{\|w\| \|v\|} \\ &\leq c_1 \|u - u_m\| \leq cc_1 \lambda_{m+1}^{-\frac{1}{2}}. \end{aligned}$$

Thanks to lemma 4.1, we could get the results.  $\square$

As a result of lemma 4.1 and its corollary, it holds that

$$\inf_{w \in V_m} \sup_{v \in V_m} \frac{\mathcal{L}_m(w, v)}{\|w\| \|v\|} \geq \frac{\alpha_0}{2}, \quad (22)$$

$$\inf_{v \in V_m} \sup_{w \in V_m} \frac{\mathcal{L}_m(w, v)}{\|w\| \|v\|} \geq \frac{\alpha_0}{2}. \quad (23)$$

Further more, by using (16), (18) and (19), we have

$$\inf_{w \in V} \sup_{v \in V} \frac{\mathcal{L}_m(w, v)}{\|w\| \|v\|} \geq \frac{\alpha_0}{2}, \quad (24)$$

$$\inf_{v \in V} \sup_{w \in V} \frac{\mathcal{L}_m(w, v)}{\|w\| \|v\|} \geq \frac{\alpha_0}{2}. \quad (25)$$

Now let us consider (17). For any given  $\phi \in V_m$ , we know  $\mathcal{F}(\phi) \in V^*$ . At the same time, (24)~(25) show that the bilinear form  $\mathcal{L}_m(\cdot, \cdot)$  is weak coercive on  $V \times V$ . Then, we can obtain the following uniqueness theorem of (17) by using the generalized Lax-Milgram theorem.

**Theorem 4.1** Suppose that  $u$  is a nonsingular solution of (3) and  $m$  is large enough such that (21) holds. Then, (17) is well-posed. Therefore, (17) can define a finite dimensional mapping  $\Phi$  from  $V_m$  to  $V$ . Moreover,  $\Phi$  is a local Lipschitz mapping, that is, for any bounded subset  $B_\rho \subset V_m$  with

$$B_\rho = \{\phi \in V_m : \|\phi\| < \rho\}, \text{ where } 0 < \rho < +\infty \text{ is any fixed constant,}$$

there exists a positive constant  $l_\rho > 0$  such that

$$\|\Phi(\phi_1) - \Phi(\phi_2)\| \leq l_\rho \|\phi_1 - \phi_2\|, \quad \forall \phi_1, \phi_2 \in V_m.$$

**Proof** The proof is quite simple and we omit it.  $\square$

## 5 Postprocess Galerkin Procedure

In this section, we will use the finite dimensional mapping  $\Phi$  derived in section 4 to construct the postprocess Galerkin procedure for numerically solving the Navier-Stokes equations. As the result of this kind of so called  $G$ -decomposition of the true solution,  $u$  can be decomposed as the sum of large eddy and small eddy components, that is

$$u = u_m + \hat{u},$$

where  $u_m$  is the large eddy components of  $u$ . The main ideal of our postprocess is to obtain a suitable approximation of  $\hat{u}$  by using  $u_m$  and  $\Phi$  which is considered as an approximate interactive rule between above two parts. The theoretical basis of this procedure is the concepts of IM and AIM which suppose that there at least exist some approximate interactive rules between the large and small eddy components of the solution of some dissipative systems, in our case, they are  $u_m$  and  $\hat{u}$ . After we get  $u_m$ , we want to use  $\Phi(u_m)$  to approximate  $\hat{u}$  and then use  $u_m^* = u_m + \Phi(u_m)$  to approximate  $u$ . We will show that  $u_m^*$  can approximate  $u$  much better than either  $u_m$  or  $\tilde{u}_m$ . We call it postprocess Galerkin procedure in this paper and we write it in following three steps.

$$\text{(Step 1)} \quad \begin{cases} \text{find } u_m \in V_m \text{ such that} \\ a(u_m, v) + b(u_m; u_m, v) = (f, v), \quad \forall v \in V_m; \end{cases}$$

$$\text{(Step 2)} \quad \begin{cases} \text{for } u_m \in V_m, \text{ find } \Phi(u_m) \in V \text{ such that} \\ \mathcal{L}_m(\Phi(u_m), v) + \langle \mathcal{F}(u_m), v \rangle = 0, \quad \forall v \in V; \end{cases}$$

$$\text{(Step 3)} \quad u_m^* = u_m + \Phi(u_m).$$

Following theorem describes the convergence rate of this procedure.

**Theorem 5.1** Assume  $u$  is the nonsingular solution of (3),  $m$  is large enough such that (21) is valid. Then the postprocess Galerkin solution  $u_m^*$  has the following convergence rate

$$\|u_m^* - u\| \leq \frac{2c_3}{\alpha_0} \lambda_{m+1}^{-\frac{3-\varepsilon}{2}}, \quad (26)$$

where  $\varepsilon = 0$  for  $d = 2$ ,  $\varepsilon = \frac{1}{2}$  for  $d = 3$  and  $c_3 = c^2 c_1 c_2$ .

**Proof** Because  $m$  is large enough such that (21) is valid,  $u_m$  is also a nonsingular solution of (6) because of lemma 4.1 and its corollary. As we know,

$$u = u_m + \hat{u}, \quad u_m^* = u_m + \Phi(u_m).$$

Therefore

$$u_m^* - u = \Phi(u_m) - \hat{u} \triangleq e. \quad (27)$$

Our main task is to estimate  $\|e\|$ . For convenience, the equation of (Step 2) can be rewritten as

$$\begin{aligned} a(\Phi(u_m), v) + b(\Phi(u_m); u_m, v) + b(u_m; \Phi(u_m), v) \\ + a(u_m, v) + b(u_m; u_m, v) - (f, v) = 0, \quad \forall v \in V. \end{aligned} \quad (28)$$

Now, subtract (12) from (28), it yields

$$a(e, v) + b(e; u_m, v) + b(u_m; e, v) = b(\hat{u}; \hat{u}, v).$$

Alternatively,

$$\mathcal{L}_m(e, v) = b(\hat{u}; \hat{u}, v). \quad (29)$$

Since  $u_m$  is the nonsingular solution of (6) and (24)~(25) are valid, we have

$$\sup_{v \in V} \frac{\mathcal{L}_m(e, v)}{\|v\|} \geq \frac{\alpha_0}{2} \|e\|. \quad (30)$$

On the other hand, by using (4)~(5) and (8), we know

$$b(\hat{u}; \hat{u}, v) \leq \begin{cases} c_1 \|\hat{u}\|_{\frac{1}{2}}^2 \|v\| \leq c_1 c_2 |\hat{u}| \|\hat{u}\| \|v\|, & d = 2, \\ c_1 \|\hat{u}\|_{\frac{3}{4}}^2 \|v\| \leq c_1 c_2 |\hat{u}|^{\frac{1}{2}} \|\hat{u}\|^{\frac{3}{2}} \|v\|, & d = 3, \end{cases} \quad (31)$$

Combining (30), (31) and (7), we have

$$\frac{\alpha_0}{2} \|e\| \leq c_1 c_2 |\hat{u}|^{1-\varepsilon} \|\hat{u}\|^{1+\varepsilon} \leq c_1 c_2 c^2 \lambda_{m+1}^{-\frac{3-\varepsilon}{2}} \leq c_3 \lambda_{m+1}^{-\frac{3-\varepsilon}{2}},$$

where  $\varepsilon = 0$  for  $d = 2$ ,  $\varepsilon = \frac{1}{2}$  for  $d = 3$  and  $c_3 = c^2 c_1 c_2$ .  $\square$

In practical manipulating, (Step 2) of the postprocess procedure must be completed in finite dimensional subspace of  $V$ . Generally, we choose another constant  $M \in \mathcal{N}$  such that  $M \gg m$  and restrict (Step 2) on  $V_M$ . That is, we alternate (Step 2) with

$$\text{(Step 2')} \quad \begin{cases} \text{for } u_m \in V_m, \text{ find } \Phi_M(u_m) \in V_M \text{ such that} \\ \mathcal{L}_m(\Phi_M(u_m), v) + \langle \mathcal{F}(u_m), v \rangle = 0, \quad \forall v \in V_M. \end{cases}$$

and alternate (Step 3) with

$$\text{(Step 3')} \quad u_{mM}^* = u_m + \Phi_M(u_m).$$

**Corollary 5.1** Under the assumption of theorem 5.1, we take  $M \in \mathcal{N}$  large enough, then

$$\|u_{mM}^* - u\| \leq \frac{2c_3}{\alpha_0} \lambda_{m+1}^{-\frac{3-\varepsilon}{2}} + c \lambda_{M+1}^{-\frac{1}{2}}. \quad (32)$$

**Proof** To prove this corollary, we only need to notice  $M$  is large enough and

$$u_{mM}^* - u = u_{mM}^* - u_M + u_M - u,$$

where  $u_M$  is the standard Galerkin solution of

$$a(u_M, v) + b(u_M; u_M, v) = (f, v), \quad \forall v \in V_M. \quad (33)$$

Then the remainder of the proof is very simple and we omit it.  $\square$

**Remark 1.** Noticing (32), to balance the two items on the right hand side, we should take  $M > m$  such that  $\lambda_{m+1} \sim \lambda_{M+1}^{\frac{1}{3-\varepsilon}}$ , that is,  $m \sim M^{\frac{1}{3-\varepsilon}}$ . Thus, the computing scale of (6) is much smaller than that of (33).

**Remark 2.** Observing the postprocess procedure (Step 1), (Step' 2) and (Step' 3), the computational complexity mainly comes from (Step 1) which tends to solve a nonlinear system by some kind of iterative methods. Comparing with (Step 1), the computing of (Step' 2) is quite simple though the scale of the equation is larger than that of (Step 1), because it is a linear system and only need solving one time during the whole procedure. On the other hand, to get a standard Galerkin approximation  $u_M$  which has the same accuracy as  $u_{mM}^*$ , we must solve (33) by some iterative method for  $M \gg m$ . As well known, (33) is harder and need much more iterative steps to convergence than (Step 1) since  $M \gg m$ . At the same time, the computing effort in (Step' 2) is almost the same as that of one iterative step of (33). Obviously, the postprocess method presented in this paper is much cheaper to complete than standard Galerkin method.

$Re$	SGM ( $\ u_m - u\ $ ) ( $m = 9$ )	PGM ( $\ u_{mM}^* - u\ $ ) ( $M = 2m$ )
1.0	1.002E-06	1.005E-07
	1.100E-07	9.851E-09
	1.012E-08	0.0000000
2.0	3.129E-06	5.017E-07
	2.985E-07	4.111E-08
	2.917E-08	0.0000000
3.01	5.990E-07	3.501E-07
	5.827E-08	2.770E-08
	4.164E-09	0.0000000
3.02	5.991E-07	3.872E-07
	5.881E-08	3.219E-08
	4.105E-09	0.0000000
10.0	1.534E-06	1.773E-07
	1.491E-07	1.707E-08
	1.569E-08	0.0000000

## 6 Numerical Test

To shed light on the better convergence rate of the postprocess Galerkin method proposed in section 5, we will give some numerical tests by taking example of two-dimensional Kolmogorov flows on rectangle :  $\Omega = [-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}] \times [-\pi, \pi]$ ,

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u, v) + b(u; u, v) = (f, v), \quad \forall v \in V, \end{cases} \quad (34)$$

where

$$V = \left\{ \phi = \sum_{k \in \mathbb{Z}^2, k \neq 0} \phi_k e^{i(k_1 \alpha x + k_2 y)}, \phi_k = \overline{\phi_{-k}}, \sum_{k \in \mathbb{Z}^2, k \neq 0} |\phi_k|^2 (k_1^2 \alpha^2 + k_2^2) < +\infty, \text{div} \phi = 0 \right\},$$

$f = \frac{1}{Re} (\sin y, 0)^T$ ,  $Re > 0$  is the Reynolds number defined in [13]. From the literature of Kolmogorov flows (e.g. [13], [14]), the above equations has a trivial solution  $u = (\sin y, 0)^T$  for any Reynolds number, and when we take  $\alpha = 0.7$ , the bifurcation occurs at  $Re^* = 3.01119 \dots$ .

From (32), we know

$$\|u_{mM}^* - u\| \leq \frac{2c_3}{\alpha_0} \lambda_{m+1}^{-\frac{3}{2}} + c \lambda_{M+1}^{-\frac{1}{2}}. \quad (35)$$

As we said in remark 1, we should take  $M \sim m^3$ . On the other hand, we notice that the exterior force of the Kolmogorov flows has only two modes as well as the trivial solution has only two modes, and for any  $w \in (I - P_{2m})V$ ,  $b(u_m; u_m, w) = 0$ . That is, for  $M > 2m$  and some suitable iterative initial value, the solutions of standard Galerkin method (SGM) and the postprocess Galerkin method (PGM) will keep unchanged with the growth of  $M$ . Thus, for this concrete test model, we take  $M = 2m$ .

Following table gives the comparison of the errors of SGM solutions and PGM solutions for different  $Re$  which is not equal to  $Re^*$  with  $\alpha = 0.7$ .

Just as we discussed in previous sections, our postprocess Galerkin method in this paper is designed for treating the nonsingular case, that is  $Re \neq Re^*$ . For singular case, we will deal with it later.

When  $Re$  is very close to  $Re^*$ , we suppose PGM will lose its higher convergence rate because  $\alpha_0$  in (18)~(19) will tend to zero. From the numerical results corresponding to  $Re = 3.01$  and  $Re = 3.02$  which are quite close to  $Re^*$ , PGM has almost the same convergence rate as SGM. That is, PGM will lose its higher convergence rate near singular point. When  $Re$  is far from  $Re^*$ , the results show PGM can improve the convergence rate of SGM at about ten times.

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