Post Galerkin method for the Navier-Stokes equations

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Abstract: A kind of post Galerkin method based on the virtue of inertial manifold and approximate inertial manifold for the two dimensional Navier-Stokes equations is constructed in this paper. This kind of post Galerkin method also leads to a kind of new construction of approximate inertial manifold. We investigate the property of this manifold and derive the error estimation of our scheme. According to our method, one can get a much more accuracy approximate solution at any time once the standard Galerkin approximate solution is at hand. Obviously, this method will yield a significant gain in computing time.
Key Words: Approximate inertial manifold, Galerkin method, error estimation, Navier-Stokes equations
AMS Subject Classification 65M15, 65M70, 76D05, 35Q30

1 Motivation

Although the computing facilities improved in the last decades, directly simulating the Navier-Stokes Equations(NSE) still remains an open problem because of its large computing scale and long time integrations. Therefore, how to construct high effective and high accuracy numerical scheme is still an important and practical problem attracting people. Many authors derived new techniques and methods. For example, Lin Qun[1], W. Layton[5] and J. Xu[6] used extrapolation and two level meshes respectively. Especially, it is worth mentioning the applications of Inertial Manifolds (IMs) and Approximate Inertial Manifolds (AIMs) theory which were firstly introduced in 1988 by C. Foias, G. R. Sell, R. Temam[7] and C. Foias, O. Manley, R. Temam[8]. Based upon the finite dimensional behavior of the solutions, they show that there must be at least some approximate interactive rules between large eddy components and small eddy components of the solutions of many dissipative partial differential equations. From then on, many papers were contributed to this subject on constructing related new algorithms, that is all sort of nonlinear Galerkin methods, and their numerical analysis. For example, we refer readers to [9], [10], [11],[12], [13] and references therein.

Suppose H be a Hilbert space and $H = H_m \oplus \hat{H}$ with $\dim(H_m) = m < +\infty$, $u(t) \in H$ be the solution of two dimensional NSE. Decomposing $u(t)$ as

$$u(t) = p(t) + q(t), \qquad \text{with } p(t) \in H_m,\ q(t) \in \hat{H},$$

AIMs believes that there must be some approximate interactive rule $\Phi : H_m \to \hat{H}$ such that $q(t) \approx \Phi(p(t))$. Then its related nonlinear Galerkin method aims to search the approximate solution of u in form of $\tilde{u}_m = \tilde{p} + \tilde{q}$ with $\tilde{q} = \Phi(\tilde{p})$ such that it can generate a more accuracy approximation of u than that of Galerkin approximation u_m. In fact, for some positive sequence

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$\{\lambda_m\}_{m\in\mathcal{N}}$ which tends to ∞ as $m\to\infty$, general nonlinear Galerkin solution $\tilde{u}_m$ admits

$$|u(t)-\tilde{u}_m(t)|_{H_0^1}\le C(t)\lambda_{m+1}^{-1}. \tag{1.1}$$

And the Galerkin solution u_m satisfies

$$|u(t)-u_m(t)|_{H_0^1}\le C(t)\lambda_{m+1}^{-\frac{1}{2}}. \tag{1.2}$$

Here $C(t)$ is some positive constant depending on various data. Obviously, nonlinear Galerkin method can greatly improve the convergence rate of Galerkin method. That is to say we could get more accuracy approximation of $u(t)$ with lower computing price compared with Galerkin method. But its defects is also obvious:

1 At each time step, nonlinear Galerkin method must solve $\tilde{p}$ and $\tilde{q}$ simultaneously, that is, nonlinear Galerkin method can not obtain $\tilde{p}$ without $\tilde{q}$. This leads to solving a coupled equations and increasing computing price.

2 Nonlinear Galerkin methods use large eddy component $\tilde{p}$ to correct small eddy component $\tilde{q}$. Noticing $\tilde{q}\in\hat{H}$ and $\tilde{p}\in H_m$, the final accuracy depends on both $p-\tilde{p}$ and $q-\tilde{q}$. Therefore, the final accuracy can not exceed $p-\tilde{p}$. This may restrict the high performance of nonlinear Galerkin method.

3 Nonlinear Galerkin solution $\tilde{u}_m$ takes no information from Galerkin solution u_m.

Being aimed at the above shortages, our paper intends to find a new Φ such that the related algorithms can overcome those defects.

2 NSE and Its Galerkin Approximation

Let us consider the following two dimensional NSE confined on a bounded domain $\Omega\subset\mathcal{R}^2$

$$\begin{cases} \dfrac{\partial u}{\partial t}-\nu\Delta u+(u\cdot\nabla)u+\nabla p=F, & (x,t)\in\Omega\times\mathcal{R}^+,\\ \nabla\cdot u=0, & (x,t)\in\Omega\times\mathcal{R}^+,\\ u(x,0)=a(x), & x\in\Omega,\\ u(x,t)=0, & (x,t)\in\partial\Omega\times\mathcal{R}^+. \end{cases} \tag{2.1}$$

Here u is fluid velocity and a the initial velocity satisfying $\nabla\cdot a=0$, p the pressure. $\nu>0$ stands for kinetic viscosity and F is external force which is assumed to be time independent. For the sake of simplicity, we also assume that $\partial\Omega$ is of class C^2.

Now we introduce a Hilbert space

$$H=\{u\in L^2(\Omega)^2,\nabla\cdot u=0 \text{ in weak sense },u\cdot n|_{\partial\Omega}=0\},$$

where n stands for unit out normal vector of Ω. If we denote P the Leray orthogonal projection from $L^2(\Omega)^2$ onto H, by projecting (2.1) onto H, we can obtain the abstract NSE

$$\begin{cases} \dfrac{du}{dt}+\nu Au+B(u,u)=f,\\ u(0)=a, \end{cases} \tag{2.2}$$

where $A=-P\Delta$ is Stokes operator, $B(u,u)=P[(u\cdot\nabla)u]$ and $f=PF$. It is well known that A is an unbounded, self-adjoint and positive definite operator with compact inverse. Thus we have that there exists two sequences

$$0<\lambda_1\le\lambda_2\le\cdots\le\lambda_n\le\cdots\to\infty, \text{ and } \phi_1,\phi_2,\cdots,\phi_n\cdots\in H,$$

such that

$$A\phi_i = \lambda_i \phi_i, \quad \forall i \in \mathcal{N}.$$

At the same time we can define its powers A^α for $\alpha \in \mathcal{R}$. In fact,

$$D(A^\alpha) = \{v \in H : v = \sum_{j=1}^{+\infty} v_j \phi_j, \sum_{j=1}^{+\infty} \lambda_j^{2\alpha} |v_j|^2 < +\infty, v_j \in \mathcal{R}\}$$

is a closed subspace of $H^{2\alpha}(\Omega)^2$ and $|A^\alpha \cdot|$ is an equivalent norm of it at least for $\alpha < \frac{5}{4}$, where $|\cdot|$ stands for the L^2- norm. In the rest, sometimes we denote $V = D(A^{\frac{1}{2}})$. In addition, $-\nu A$ generates an analytic semigroup on H, denoted by $\{e^{-\nu A t}\}_{t \geq 0}$, with following estimation

$$\|A^\alpha e^{-\nu A t}\|_{\mathcal{L}(H,H)} \leq c_0 (\nu t)^{-\alpha} e^{-\nu \delta t}, \quad t > 0,\ \alpha > 0, \tag{2.3}$$

where $\delta > 0$ is a constant related only on A, $c_0 > 0$ is a constant. For the sake of convenience, we always use $c_j > 0, j \in \mathcal{N}$, to denote constants which have different meaning in different places appearing in the analysis. We also introduce the notation $|||u|||_t = \sup_{0 \leq s \leq t} |u(s)|$, especially, $|||u||| = \sup_{s \in \mathcal{R}^+} |u(s)|$.

Now for any $m \in \mathcal{N}$, we introduce an orthogonal projection P_m

$$\forall v = \sum_{j=1}^{+\infty} v_j \phi_j \in H, \quad P_m v = \sum_{j=1}^{m} v_j \phi_j.$$

Meanwhile, we use Q_m to denote $I - P_m$. The following inequalities are classical and we just state them out.

$$|Q_m \phi| \leq \lambda_{m+1}^{-\frac{1}{2}} |A^{\frac{1}{2}} \phi|, \quad \forall \phi \in V,$$

$$|\phi|_{L^\infty} \leq L_m |A^{\frac{1}{2}} \phi|, \quad \forall \phi \in H_m,$$

where $L_m \sim (1 + \ln \lambda_m)^{\frac{1}{2}}$.

By using these two orthogonal projections, we could decompose H as

$$H = H_m \oplus \hat{H}, \quad H_m = P_m H, \hat{H} = Q_m H.$$

To numerically solve (2.2), we use following Galerkin scheme,

$$\begin{cases} \dfrac{du_m}{dt} + \nu A u_m + P_m B(u_m, u_m) = P_m f, \\ u_m(0) = P_m a. \end{cases} \tag{2.4}$$

Here we use $u_m(t)$ to represent the Galerkin approximation of $u(t)$. As well known, its stability and convergence results are classical. For example, we have the following convergence results

$$|u(t) - u_m(t)| \leq C_1(t) \lambda_{m+1}^{-1}, \quad |A^{\frac{1}{2}}(u(t) - u_m(t))| \leq C_2(t) \lambda_{m+1}^{-\frac{1}{2}}, \tag{2.5}$$

where $C_1(t)$ and $C_2(t)$ are positive constants depending on ν, f and a.

How to construct more accuracy approximate solution of $u(t)$ by taking advantages of known information of it, that is $u_m(t)$, is the main problem to be solved in the rest. Of course, nonlinear Galerkin method can partly solve the problem although it has some defects as we just said in section 1. Here, we aim to get a new method which can overcome those shortages as well as improve convergence rate by using the virtue of AIMs to construct a new finite dimensional mapping. We call it post Galerkin method. It should be able to correct u_m at any time without introducing any extra computation except for computing u_m.

3 Further Properties of u_m

In this section, we will give some properties of u_m which will be very important for our further discussion. First of all, let us give a kind of new decomposition of true solution u. In fact, after we get its Galerkin approximation u_m, the very nature decomposition is to decompose $u(t)$ as $u_m(t)$ and its residue $\hat{u}(t) = u(t) - u_m(t)$, that is

$$u(t) = u_m(t) + \hat{u}(t). \tag{3.1}$$

For the convenience of stating, we identify $u_m(t) \in H_m$ and $\hat{u}(t) \in H$ as large and small eddy components respectively. Obviously, the Galerkin approximation $u_m(t)$ reaches the large eddy component of true solution exactly. Then, how to approximate small eddy components is the only problem. According to the ideal of AIMs, we suppose there also exist some kind of approximate interactive rule between large and small eddies. That is, there should exist a finite dimensional mapping Φ from H_m into H. Before we begin to construct it, we need some further property of u_m.

Subtracting (2.4) from (2.2), we can easily get

$$\begin{cases} \dfrac{d\hat{u}}{dt} + \nu A\hat{u} + B(\hat{u}, u_m) + B(u_m, \hat{u}) + B(\hat{u}, \hat{u}) = Q_m[f - B(u_m, u_m)], \\ \hat{u}(0) = Q_m a. \end{cases} \tag{3.2}$$

This is a nonlinear evolutionary equation of $\hat{u}(t)$.

Now let us consider some properties of $\hat{u}$. To do so, we decompose $\hat{u}$ as

$$\hat{u} = P_m\hat{u} + Q_m\hat{u} \stackrel{\triangle}{=} p + q. \tag{3.3}$$

Then p satisfies

$$\begin{cases} \dfrac{dp}{dt} + \nu Ap + P_m B(u_m, p+q) + P_m B(p+q, u_m) + P_m B(p+q, p+q) = 0, \\ p(0) = 0. \end{cases} \tag{3.4}$$

If we set ϕ, v, w be any vectors in V which have the forms of

$$\phi = (\phi_1, \phi_2)^T, \quad v = (v_1, v_2)^T, \quad w = (w_1, w_2)^T,$$

we denote by b the trilinear form[2]

$$b(\phi, v, w) = (B(\phi, v), w) = \int_\Omega (\phi \cdot \nabla)v \cdot w dx,$$

which has the following estimations

$$b(\phi, v, w) \le c_b |A^{\frac{s_1}{2}}\phi|\,|A^{\frac{s_2+1}{2}}v|\,|A^{\frac{s_3}{2}}w|, \quad \forall \phi \in D(A^{\frac{s_1}{2}}), v \in D(A^{\frac{s_2+1}{2}}), w \in D(A^{\frac{s_3}{2}}). \tag{3.5}$$

Here, $s_1, s_2, s_3 \ge 0$ satisfies $s_1 + s_2 + s_3 \ge 1$ with $(s_1, s_2, s_3) \neq (1,0,0)$,$(0,1,0)$ and $(0,0,1)$.

On the other hand, we can alter the form of b. In fact

$$\begin{aligned} (\phi \cdot \nabla)v \cdot w &= \left[\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \cdot \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}\right] \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ &= \phi_1 \partial_x v_1 w_1 + \phi_2 \partial_y v_1 w_1 + \phi_1 \partial_x v_2 w_2 + \phi_2 \partial_y v_2 w_2 \\ &= (w \cdot \nabla v) \cdot \phi, \end{aligned}$$

where $w \cdot \nabla v$ means $w_1 \nabla v_1 + w_2 \nabla v_2$. Denoting $\mathcal{B}(w, v) = P(w \cdot \nabla v)$, then we have

$$b(\phi, v, w) = (B(\phi, v), w) = (\mathcal{B}(w, v), \phi). \tag{3.6}$$

We are familiar with the general properties of B,about which we refer readers to [2]. But those are not enough for our discussion, we still need some further properties of B and $\mathcal{B}$ which we state as following two lemmas.

Lemma 3.1 *For any $w \in D(A^{\frac{1}{2}})$, $v \in D(A)$ and $0 < r < \frac{1}{2}$, it holds*

$$|A^r B(w,v)| \le c_1 |A^{\frac{1}{2}} w|\, |Av|.$$

And for $r = \frac{1}{2}$ and $w \in D(A^{\frac{3}{4}})$, we have

$$|A^{\frac{1}{2}} B(w,v)| \le c_1 |A^{\frac{3}{4}} w|\, |Av|.$$

Here c_1 is some positive constant depending only on Ω and r.

This property of B was proven in [13]. Noticing the form of B and $\mathcal{B}$ are quite alike, we can easily get the similar property of $\mathcal{B}$ by the same method used in [13]. So we only state the property in the following without proving.

Lemma 3.2 *For any $w \in D(A^{\frac{1}{2}})$, $v \in D(A)$ and $0 < r < \frac{1}{2}$, it holds*

$$|A^r \mathcal{B}(w,v)| \le c_1 |A^{\frac{1}{2}} w|\, |Av|.$$

And for $r = \frac{1}{2}$ and $w \in D(A^{\frac{3}{4}})$, we have

$$|A^{\frac{1}{2}} \mathcal{B}(w,v)| \le c_1 |A^{\frac{3}{4}} w|\, |Av|.$$

Here c_1 has the same meaning as in lemma 3.1.

Now we are ready to study the important property of $\hat{u}$. By using the semi-group presentation, we can rewrite (3.4) as

$$\begin{aligned} p(t) = & -\int_0^t e^{-\nu A(t-s)} P_m \{B(u_m,p) + B(p,u_m) + B(p,p) + B(p,q) + B(q,p) \\ & + B(u_m,q) + B(q,u_m) + B(q,q)\} ds \\ = & -\int_0^t e^{-\nu A(t-s)} P_m B_1(p) ds - \int_0^t e^{-\nu A(t-s)} P_m B_2(q) ds, \end{aligned}$$

where

$$\begin{aligned} B_1(p) &= B(u_m,p) + B(p,u_m) + B(p,p) + B(p,q) + B(q,p), \\ B_2(q) &= B(u_m,q) + B(q,u_m) + B(q,q). \end{aligned}$$

Then by using (2.3), we have

$$\begin{aligned} (3.7) \quad |A^{-\frac{1}{2}} p(t)| \le & \int_0^t |A^{-\frac{1}{2}} e^{-\nu A(t-s)} P_m B_1(p)| ds + \int_0^t |A^{-\frac{1}{2}} e^{-\nu A(t-s)} P_m B_2(q)| ds \\ = & \int_0^t |A^{\frac{1}{2}} e^{-\nu A(t-s)} A^{-1} P_m B_1(p)| ds + \int_0^t |A^{\frac{1}{2}} e^{-\nu A(t-s)} A^{-1} P_m B_2(q)| ds \\ \le & c_0 \nu^{-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} e^{-\delta(t-s)} |A^{-1} P_m B_1(p)| ds \\ & + c_0 \nu^{-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} e^{-\delta(t-s)} |A^{-1} P_m B_2(q)| ds \end{aligned}$$

Let us estimate each term of $|A^{-1}B_1(p)|$ and $|A^{-1}B_2(q)|$. Firstly, we consider each term of $|A^{-1}B_1(p)|$. For example, consider $|A^{-1}P_mB(u_m,p)|$. Noticing lemma 3.1, for any $v \in H_m$

$$
\begin{aligned}
|b(u_m,p,A^{-1}v)| =&|b(u_m,A^{-1}v,p)| = |(A^{\frac{1}{2}}B(u_m,A^{-1}v),A^{-\frac{1}{2}}p)| \\
\leq& c_1|A^{\frac{3}{4}}u_m|\,|A^{-\frac{1}{2}}p|\,|v|.
\end{aligned}
$$

Thus we have

$$
|A^{-1}P_mB(u_m,p)| \leq c_1|A^{\frac{3}{4}}u_m|\,|A^{-\frac{1}{2}}p|. \tag{3.8}
$$

Similarly, we can derive

$$
|A^{-1}P_mB(p,p)| \leq c_1|A^{\frac{3}{4}}u|\,|A^{-\frac{1}{2}}p|, \tag{3.9}
$$

$$
|A^{-1}P_mB(q,p)| \leq c_1|A^{\frac{3}{4}}u|\,|A^{-\frac{1}{2}}p|. \tag{3.10}
$$

For the other two terms, we will use lemma 3.2 to cope with them. For any $v \in H_m$

$$
\begin{aligned}
|b(p,u_m,A^{-1}v)| =&|b(p,A^{-1}v,u_m)| = |(A^{\frac{1}{2}}\mathcal{B}(u_m,A^{-1}v),A^{-\frac{1}{2}}p)| \\
\leq& c_1|A^{\frac{3}{4}}u_m|\,|A^{-\frac{1}{2}}p|\,|v|.
\end{aligned}
$$

So we get

$$
|A^{-1}P_mB(p,u_m)| \leq c_1|A^{\frac{3}{4}}u_m|\,|A^{-\frac{1}{2}}p|. \tag{3.11}
$$

Do the same thing to the last term, we have

$$
|A^{-1}P_mB(p,q)| \leq c_1|A^{\frac{3}{4}}u|\,|A^{-\frac{1}{2}}p|. \tag{3.12}
$$

Combining (3.8)$\sim$(3.12), we derive the first estimation

$$
|A^{-1}P_mB_1(p)| \leq 3c_1(|||A^{\frac{3}{4}}u||| + |||A^{\frac{3}{4}}u_m|||)|A^{-\frac{1}{2}}p|. \tag{3.13}
$$

For the estimation of $|A^{-1}P_mB_2(q)|$, the method is completely the same as the above one. We also use lemma 3.1 to deal with $B(u_m,q)+B(q,q)$ and lemma 3.2 to deal with $B(q,u_m)$. So we just give the result in the following

$$
|A^{-1}P_mB_2(q)| \leq 2c_1(|||A^{\frac{3}{4}}u||| + |||A^{\frac{3}{4}}u_m|||)|A^{-\frac{1}{2}}q|. \tag{3.14}
$$

Obviously,

$$
\sup_{t\geq 0}\int_0^t (t-s)^{-\frac{1}{2}}e^{-\delta(t-s)}ds < \delta^{-1/2}\gamma_{\frac{1}{2}} < +\infty
$$

where

$$
\gamma_\alpha = \int_0^\infty s^{-\alpha}e^{-s}ds = \Gamma(-1-\alpha).
$$

By introducing the following constants

$$
c_2 = 2c_0c_1\delta^{-\frac{1}{2}}\gamma_{\frac{1}{2}}(|||A^{\frac{3}{4}}u||| + |||A^{\frac{3}{4}}u_m|||),
$$

$$
c_3 = 3c_0c_1(|||A^{\frac{3}{4}}u||| + |||A^{\frac{3}{4}}u_m|||),
$$

we can get a new integration inequality from (3.7). That is

$$|A^{-\frac{1}{2}}p(t)| \le c_3\nu^{-\frac{1}{2}}\int_0^t (t-s)^{-\frac{1}{2}}e^{-\delta(t-s)}|A^{-\frac{1}{2}}p|ds + c_2\nu^{-\frac{1}{2}}|||A^{-\frac{1}{2}}q|||_t.$$

Set

$$g(s) = |A^{-\frac{1}{2}}p(s)|e^{\delta s},$$

we have

$$g(t) \le c_2\nu^{-\frac{1}{2}}e^{\delta t}|||A^{-\frac{1}{2}}q|||_t + c_3\nu^{-\frac{1}{2}}\int_0^t (t-s)^{-\frac{1}{2}}g(s)ds.$$

To give the estimation of g, we must introducing an inequality. Many inequalities of this type can be found in Henry[4]. The following special version, lemma 3.3, was proven in [15].

Lemma 3.3 *Let T, α, β and θ be positive constants, $0 < \theta < 1$. Then for any continuous function $f : [0, T] \to [0, +\infty)$ that satisfies*

$$f(t) \le \alpha + \beta\int_0^t (t-s)^{-\theta}f(s)ds, \quad 0 \le t \le T,$$

we have

$$f(t) \le c_4\alpha\exp\{c_4\beta^{1/(1-\theta)}t\}, \quad 0 \le t \le T,$$

with a positive constant c_4 that depends only on θ.

Now by using lemma 3.3, we can immediately obtain

$$g(t) \le c_2c_4\nu^{-\frac{1}{2}}e^{\delta t}\exp\{c_4c_3^2\nu^{-1}t\}|||A^{-\frac{1}{2}}q|||_t.$$

Denoting by $T_1(t) > 0$ the constant $c_2c_4\exp\{c_4c_3^2\nu^{-1}t\} + \nu^{\frac{1}{2}}$, we have

$$|A^{-\frac{1}{2}}p(t)| \le (\nu^{-\frac{1}{2}}T_1(t) - 1)|||A^{-\frac{1}{2}}q|||_t. \tag{3.15}$$

Now we summarize the above deducing into the following

Theorem 3.1 *For any given data $a \in D(A)$ and $f \in H$, we know the Navier-Stokes equations (2.6) and its Galerkin approximate equations (2.4) have unique solutions*

$$u(t) \in L^\infty(\mathcal{R}^+, D(A)), \quad u_m(t) \in L^\infty(\mathcal{R}^+, D(A)).$$

And there also exist some positive constants M_0 and M_1 related on ν, a and f such that

$$|||A^{\frac{1}{2}}u|||, |||A^{\frac{1}{2}}u_m||| \le \frac{M_0}{2}, \quad |||Au|||, |||Au_m||| \le M_1.$$

Then for any $t > 0$, we have

$$|A^{-\frac{1}{2}}(u - u_m)| \le \nu^{-\frac{1}{2}}\lambda_{m+1}^{-\frac{1}{2}}T_1(t)|||u - u_m|||_t.$$

Proof From [14] and (3.15), we can immediately get the result. □

4 Finite Dimensional Mapping Φ

As we said before, the main task in our paper is to construct some kind of approximate interactive rule between $\hat{u}$ and u_m. That is to find some kind of finite dimensional mapping $\Phi : H_m \to V$ such that $\hat{u}\approx\Phi(u_m)$.

To do so, we introduce an smooth function

$$g(s) \in C^\infty(\mathcal{R}^+)$$

with the following properties

$$0 \le g(s) \le 1,\ |g'(s)| \le 2, \quad \text{and } \forall s \in [0,1],\ g(s) = 1,\ \forall s \in [2,\infty), g(s) = 0.$$

Now let us recall (3.2). Of course, (3.2) is a kind of rule. We may exactly get $\hat{u}$ from u_m. This is, in fact, to solve the Navier-Stokes equations. It is not suitable for our purpose because it is as complex as (2.2). But, enlightened by this equations and noticing $\frac{d\hat{u}}{dt}$ and $B(\hat{u},\hat{u})$ are smaller quantities compared with other terms in (3.2), we introduce the following finite dimensional mapping

$$(4.1) \qquad \begin{cases} \forall \phi \in H_m, \ \text{ find } w \in V \text{ such that} \\ h^{-1}w + \nu A w + g(\frac{2|A^{\frac{1}{2}}\phi|}{M_0})[B(w,\phi) + B(\phi,w)] = g(\frac{2|A^{\frac{1}{2}}\phi|}{M_0})Q_m[f - B(\phi,\phi)]. \end{cases}$$

Here, $h > 0$ is a small constant which will be given soon. Let us introduce bilinear form

$$\mathcal{L}(w,v) := h^{-1}(w,v) + \nu(A^{\frac{1}{2}}w, A^{\frac{1}{2}}v) + g(\frac{2|A^{\frac{1}{2}}\phi|}{M_0})[b(w,\phi,v) + b(\phi,w,v)].$$

It is clear that $\mathcal{L}(\cdot,\cdot)$ is a continuous bilinear form from $V \times V$ to R. Furthermore, we have

Lemma 4.1 *$\mathcal{L}(\cdot,\cdot)$ is continuous and coercive if h is small enough such that*

$$(4.2) \qquad h \le \frac{2\nu}{c_b^2 M_0^2}.$$

Proof Indeed

$$(h^{-1}w + \nu A w + g(\frac{2|A^{\frac{1}{2}}\phi|}{M_0})[B(w,\phi) + B(\phi,w)], w) = \frac{|w|^2}{h} + \nu|A^{\frac{1}{2}}w|^2 + g(\frac{2|A^{\frac{1}{2}}\phi|}{M_0})b(w,\phi,w)$$

$$\ge \frac{|w|^2}{h} + \nu|A^{\frac{1}{2}}w|^2 - c_b|A^{\frac{1}{2}}\phi|\,|w|\,|A^{\frac{1}{2}}w| \ge \frac{|w|^2}{h} + \nu|A^{\frac{1}{2}}w|^2 - c_b M_0|w|\,|A^{\frac{1}{2}}w|$$

$$\ge (\frac{1}{h} - \frac{c_b^2 M_0^2}{2\nu})|w|^2 + \frac{\nu}{2}|A^{\frac{1}{2}}w|^2 \ge \frac{\nu}{2}|A^{\frac{1}{2}}w|^2.$$

Notice we used an implicit condition $|A^{\frac{1}{2}}\phi| < M_0$ in the above inequality. For $|A^{\frac{1}{2}}\phi| \ge M_0$, g will be equal to zero and the above result is obvious. Then we can get the result. □

Consequently, by using Lax-Milgram theorem, we know, for any give $\phi \in H_m$, there exists an unique solution $w = \Phi(\phi)$ of the following variational problem corresponding to (4.1):

$$(4.3) \qquad \begin{cases} \forall \phi \in H_m, \ \text{ find } w \in V \text{ such that} \\ \mathcal{L}(w,v) = g(\frac{2|A^{\frac{1}{2}}\phi|}{M_0})[(Q_m f, v) - b(\phi,\phi,Q_m v)] \quad \forall v \in V \end{cases}$$

Theorem 4.1 *Assume h satisfies (4.2). Then (4.1) can determine a finite dimensional mapping Φ from H_m to V which has the following properties*

i) $\Phi(\phi) = 0$ for $|A^{\frac{1}{2}}\phi| \ge M_0$.

ii) For any $\phi \in H_m$,

$$|A^{\frac{1}{2}}\Phi(\phi)| \le \rho_m = \frac{2}{\nu}(|f| + c_b M_0^2 L_m)\lambda_{m+1}^{-\frac{1}{2}},$$

of course $\rho_m \to 0$ *when* $m \to \infty$.

iii) Φ *is a Lipschitz smooth mapping. That is, there exists some constant* $l_m > 0$ *such that*

$$|A^{\frac{1}{2}}(\Phi(\phi_1) - \Phi(\phi_2))| \le l_m |A^{\frac{1}{2}}(\phi_1 - \phi_2)|.$$

And $l_m \to 0$ *when* $m \to \infty$.

Proof By virtue of lemma 4.1, it asserts that (4.1) can determine a finite dimensional mapping.

i) Let us consider $|A^{\frac{1}{2}}\phi| \ge M_0$. Notice the definition of g, we know $g(\dfrac{2|A^{\frac{1}{2}}\phi|}{M_0}) = 0$ at this time. So (4.1) becomes

$$h^{-1}w + \nu A w = 0.$$

Of course, it only has zero solution. That is, under this circumstance,

$$w = \Phi(\phi) = 0.$$

ii) We only need to consider $|A^{\frac{1}{2}}\phi| < M_0$. Just as being done to prove the uniqueness of the solution, we can get

$$\begin{aligned}\frac{\nu}{2}|A^{\frac{1}{2}}\Phi(\phi)|^2 \le& |(f, Q_m\Phi(\phi)) + b(\phi,\phi,Q_m\Phi(\phi))| \\ \le& (|f| + c_b M_0^2 L_m)\lambda_{m+1}^{-\frac{1}{2}}|A^{\frac{1}{2}}\Phi(\phi)|.\end{aligned}$$

Then we can get the result.

iii) At last, we will show this mapping is also Lipschitz continuous. For any given $\phi_1, \phi_2 \in H_m$, we can get $w_1 = \Phi(\phi_1)$ and $w_2 = \Phi(\phi_2)$ from (4.1). That is

$$h^{-1}w_1 + \nu A w_1 + g(\frac{2|A^{\frac{1}{2}}\phi_1|}{M_0})[B(w_1,\phi_1) + B(\phi_1,w_1)] = g(\frac{2|A^{\frac{1}{2}}\phi_1|}{M_0})Q_m[f - B(\phi_1,\phi_1)],$$

$$h^{-1}w_2 + \nu A w_2 + g(\frac{2|A^{\frac{1}{2}}\phi_2|}{M_0})[B(w_2,\phi_2) + B(\phi_2,w_2)] = g(\frac{2|A^{\frac{1}{2}}\phi_2|}{M_0})Q_m[f - B(\phi_2,\phi_2)].$$

If we denote $\phi_e = \phi_1 - \phi_2$, $w_e = w_1 - w_2$ and $\Delta g = g(\dfrac{2|A^{\frac{1}{2}}\phi_1|}{M_0}) - g(\dfrac{2|A^{\frac{1}{2}}\phi_2|}{M_0})$, we can derive from the above two equations that

$$\begin{aligned}(4.4)\qquad h^{-1}w_e +& \nu A w_e + g(\frac{2|A^{\frac{1}{2}}\phi_1|}{M_0})[B(w_e,\phi_1) + B(w_2,\phi_e)] \\ &+ g(\frac{2|A^{\frac{1}{2}}\phi_1|}{M_0})[B(\phi_e,w_1) + B(\phi_2,w_e)] + \Delta g[B(w_2,\phi_2) + B(\phi_2,w_2)] \\ &= \Delta g Q_m f + g(\frac{2|A^{\frac{1}{2}}\phi_1|}{M_0})Q_m[B(\phi_e,\phi_1) + B(\phi_2,\phi_e)] + \Delta g Q_m B(\phi_2,\phi_2).\end{aligned}$$

For different values of $|A^{\frac{1}{2}}\phi_1|$ and $|A^{\frac{1}{2}}\phi_2|$, we divided our proof into several cases.

Case 1) $|A^{\frac{1}{2}}\phi_1|, |A^{\frac{1}{2}}\phi_2| \ge M_0$.

Noticing the definition of g, (4.4) becomes

$$h^{-1}w_e + \nu A w_e = 0.$$

We can get the Lipschitz continuous result for any $l_m \in \mathcal{R}^+$.

Case 2) One of them exceeds M_0.

Without loss of generality, we suppose $|A^{\frac{1}{2}}\phi_1| \geq M_0$ and $|A^{\frac{1}{2}}\phi_2| < M_0$. Then (4.4) reads

$$h^{-1}w_e + \nu A w_e + \Delta g B(w_2,\phi_2) + \Delta g B(\phi_2, w_2) = \Delta g Q_m f + \Delta g Q_m B(\phi_2,\phi_2). \tag{4.5}$$

Notice that $\Delta g = -g(\dfrac{2|A^{\frac{1}{2}}\phi_2|}{M_0})$ at this time. But we pretend that $g(\dfrac{2|A^{\frac{1}{2}}\phi_1|}{M_0})$ is still there. Then by using the property of g, we have

$$|\Delta g| \leq \frac{4}{M_0}(|A^{\frac{1}{2}}\phi_1| - |A^{\frac{1}{2}}\phi_2|) \leq \frac{4}{M_0}|A^{\frac{1}{2}}\phi_e|. \tag{4.6}$$

Multiply (4.5) by w_e and integrate it on Ω, we get

$$\begin{aligned} h^{-1}|w_e|^2 + \nu|A^{\frac{1}{2}}w_e|^2 \leq & |\Delta g b(w_2,\phi_2,w_e)| + |\Delta g b(\phi_2,w_2,w_e)| \\ & + |\Delta g(f, Q_m w_e)| + |\Delta g b(\phi_2,\phi_2,Q_m w_e)|. \end{aligned} \tag{4.7}$$

Noticing (4.6) and the result of ii), we majorize each term on the right hand side of (4.7) as:

$$|\Delta g b(w_2,\phi_2,w_e)| \leq \frac{4c_b}{M_0}|A^{\frac{1}{2}}w_2|\,|A^{\frac{1}{2}}\phi_2|\,|A^{\frac{1}{2}}w_e|\,|A^{\frac{1}{2}}\phi_e| \leq 4c_b\rho_m|A^{\frac{1}{2}}w_e|\,|A^{\frac{1}{2}}\phi_e|,$$

$$|\Delta g b(\phi_2,w_2,w_e)| \leq 4c_b\rho_m|A^{\frac{1}{2}}w_e|\,|A^{\frac{1}{2}}\phi_e|,$$

$$|\Delta g(f,Q_m w_e)| \leq \frac{4|f|}{M_0\lambda_{m+1}^{\frac{1}{2}}}\,|A^{\frac{1}{2}}w_e|\,|A^{\frac{1}{2}}\phi_e|,$$

$$\begin{aligned} |\Delta g b(\phi_2,\phi_2,Q_m w_e)| \leq & \frac{4c_b}{M_0}|\phi_2|_{L^\infty}|A^{\frac{1}{2}}\phi_2|\,|Q_m w_e|\,|A^{\frac{1}{2}}\phi_e| \\ \leq & 4c_b M_0 L_m \lambda_{m+1}^{-\frac{1}{2}}|A^{\frac{1}{2}}w_e|\,|A^{\frac{1}{2}}\phi_e|. \end{aligned}$$

Combining the above inequalities and omitting $h^{-1}|w_e|^2$ on the left hand side of (4.7), it yields

$$\nu|A^{\frac{1}{2}}w_e|^2 \leq (8c_b\rho_m + 4M_0|f|\lambda_{m+1}^{-\frac{1}{2}} + 4c_b M_0 L_m\lambda_{m+1}^{-\frac{1}{2}})|A^{\frac{1}{2}}w_e|\,|A^{\frac{1}{2}}\phi_e|.$$

Denoting $l_m = \nu^{-1}(8c_b\rho_m + 4M_0|f|\lambda_{m+1}^{-\frac{1}{2}} + 4c_b M_0 L_m\lambda_{m+1}^{-\frac{1}{2}})$, we can get the result.

Case 3) $|A^{\frac{1}{2}}\phi_1|, |A^{\frac{1}{2}}\phi_2| < M_0$.

Multiply (4.4) with w_e and integrate it on Ω, we have

$$\begin{aligned} h^{-1}|w_e|^2 + \nu|A^{\frac{1}{2}}w_e|^2 \leq & |b(w_e,\phi_1,w_e)| + |b(\phi_e,w_1,w_e)| \\ & + |\Delta g b(w_2,\phi_2,w_e)| + |\Delta g b(\phi_2,w_2,w_e)| + |\Delta g(f,Q_m w_e)| + |b(\phi_e,\phi_1,Q_m w_e)| \\ & + |b(\phi_2,\phi_e,Q_m w_e)| + |\Delta g b(\phi_2,\phi_2,Q_m w_e)|. \end{aligned} \tag{4.8}$$

Just like the previous case, we majorize each term on the right hand side of (4.8) as:

$$|b(w_e,\phi_1,w_e)| \leq c_b M_0|A^{\frac{1}{2}}w_e|\,|w_e| \leq \frac{\nu}{2}|A^{\frac{1}{2}}w_e|^2 + \frac{c_b^2 M_0^2}{2\nu}|w_e|^2,$$

$$|b(\phi_e, w_1, w_e)| \le c_b\rho_m |A^{\frac{1}{2}} w_e|\, |A^{\frac{1}{2}}\phi_e|,$$

$$|\Delta g b(w_2, \phi_2, w_e)| \le 4c_b\rho_m |A^{\frac{1}{2}} w_e|\, |A^{\frac{1}{2}}\phi_e|,$$

$$|\Delta g b(\phi_2, w_2, w_e)| \le 4c_b\rho_m |A^{\frac{1}{2}} w_e|\, |A^{\frac{1}{2}}\phi_e|,$$

$$|\Delta g(f, Q_m w_e)| \le \frac{4|f|}{M_0}\lambda_{m+1}^{-\frac{1}{2}} |A^{\frac{1}{2}} w_e||A^{\frac{1}{2}}\phi_e|,$$

$$|b(\phi_e, \phi_1, Q_m w_e)| \le c_b|\phi_e|_{L^\infty}|A^{\frac{1}{2}}\phi_1||Q_m w_e| \le c_b M_0 L_m \lambda_{m+1}^{-\frac{1}{2}} |A^{\frac{1}{2}} w_e||A^{\frac{1}{2}}\phi_e|,$$

$$|b(\phi_2, \phi_e, Q_m w_e)| \le c_b M_0 L_m \lambda_{m+1}^{-\frac{1}{2}} |A^{\frac{1}{2}} w_e||A^{\frac{1}{2}}\phi_e|,$$

$$|\Delta g b(\phi_2, \phi_2, Q_m w_e)| \le 4c_b M_0 L_m \lambda_{m+1}^{-\frac{1}{2}} |A^{\frac{1}{2}} w_e||A^{\frac{1}{2}}\phi_e|.$$

Then, from (4.8), we have

$$\frac{\nu}{2}|A^{\frac{1}{2}} w_e| \le (9c_b\rho_m + 6c_b M_0 L_m \lambda_{m+1}^{-\frac{1}{2}} + 4|f|M_0^{-1}\lambda_{m+1}^{-\frac{1}{2}})|A^{\frac{1}{2}}\phi_e|.$$

Once again, if we denote

$$l_m = \frac{2}{\nu}(9c_b\rho_m + 6c_b M_0 L_m \lambda_{m+1}^{-\frac{1}{2}} + 4|f|M_0^{-1}\lambda_{m+1}^{-\frac{1}{2}}),$$

we can derive the result again. □

5 Post Galerkin Method

In previous section, we construct a finite dimensional mapping $\Phi : H_m \to V$. Now, we will use it to give our post Galerkin scheme. In fact, once we get Φ, the construction of post Galerkin scheme is obvious. We state it as following three steps.
(Step 1) Galerkin approximation

$$\begin{cases} \text{find } u_m(t) \text{ such that} \\ \dfrac{du_m}{dt} + \nu A u_m + P_m B(u_m, u_m) = P_m f, \\ u_m(0) = P_m a. \end{cases}$$

(Step 2) Postprocess $u_m(t)$ at any time $t \in \mathcal{R}^+$

$$\begin{cases} \text{find } \bar{u}(t) = \Phi(u_m(t)) \text{ such that} \\ h^{-1}\bar{u} + \nu A\bar{u} + g(\dfrac{2|A^{\frac{1}{2}}u_m|}{M_0})[B(\bar{u}, u_m) + B(u_m, \bar{u})] = g(\dfrac{2|A^{\frac{1}{2}}u_m|}{M_0})Q_m[f - B(u_m, u_m)]. \end{cases}$$

(Step 3) Post Galerkin approximation

$$u^*(t) = u_m(t) + \bar{u}(t) = u_m(t) + \Phi(u_m(t)).$$

In the rest, we will investigate the accuracy presented by this scheme. First of all, we give some classical result as

Lemma 5.1[3] *Under the conditions of theorem 3.1, the solution $u(t)$ of (2.2) is analytic in time, in a neighborhood of the positive real axis, as a $D(A)-$ valued function.*

Denoting by d the distance between the boundary of analytic region and the positive real axis, we could derive the following estimation of $\frac{d\hat{u}}{dt}$ at any time by Cauchy theorem.

For any given $t > 0$, we know from the above lemma and Cauchy theorem

$$\frac{d\hat{u}(t)}{dt} = \frac{1}{2\pi i}\int_{|z-t|=d}\frac{\hat{u}(z)}{(t-z)^2}dz.$$

Thus,

$$|A^{-\frac{1}{2}}\frac{d\hat{u}(t)}{dt}| \leq \frac{1}{2\pi}|\int_{|z-t|=d}\frac{A^{-\frac{1}{2}}\hat{u}(z)}{(t-z)^2}dz| \tag{5.1}$$

$$\leq d^{-1}|||A^{-\frac{1}{2}}\hat{u}|||_t \leq d^{-1}\nu^{-\frac{1}{2}}\lambda_{m+1}^{-\frac{1}{2}}T_1(t)|||\hat{u}|||_t.$$

Generally, d is a small constant related to ν and f. For its concrete express, we refer readers to [3]. Now, we give our main result as

Theorem 5.1 *Under the conditions of theorem 3.1 and (4.2), we have*

$$|A^{\frac{1}{2}}(u(t) - u^*(t))| \leq T_2(t)\lambda_{m+1}^{-\frac{3}{2}},$$

where

$$T_2(t) = 2\nu^{-1}C_1(t)((h^{-1} + d^{-1})\nu^{-\frac{1}{2}}T_1(t) + c_1C_2(t)).$$

Proof Notice that

$$u^*(t) = u_m(t) + \Phi(u_m(t)), \quad u(t) = u_m(t) + \hat{u}(t).$$

Thus, to get the estimation, we only need to concern about

$$\hat{u}(t) - \Phi(u_m(t)) = \hat{u}(t) - \bar{u}(t).$$

Because of $|||A^{\frac{1}{2}}u_m||| \leq \frac{1}{2}M_0$, we have

$$g(\frac{2|A^{\frac{1}{2}}u_m|}{M_0}) = 1.$$

Thus $\bar{u}$ satisfies the following equations at any give time

$$h^{-1}\bar{u} + \nu A\bar{u} + B(\bar{u}, u_m) + B(u_m, \bar{u}) = Q_m[f - B(u_m, u_m)]. \tag{5.2}$$

Subtracting (5.2) from (3.2) and denoting $e = \hat{u} - \bar{u}$, we have

$$h^{-1}e + \nu Ae + B(e, u_m) + B(u_m, e) = h^{-1}\hat{u} - \frac{d\hat{u}}{dt} - B(\hat{u}, \hat{u}).$$

From lemma 4.1, we know

$$||h^{-1}e + \nu Ae + B(e, u_m) + B(u_m, e)||_{\mathcal{L}(V,V^*)} \geq \frac{\nu}{2}|A^{\frac{1}{2}}e|.$$

And from (2.5), theorem 3.1 and (5.1), we have

$$||h^{-1}\hat{u} - \frac{d\hat{u}}{dt} - B(\hat{u}, \hat{u})||_{\mathcal{L}(V,V^*)} \leq (h^{-1} + d^{-1})|||A^{-\frac{1}{2}}\hat{u}|||_t + c_1|\hat{u}||A^{\frac{1}{2}}\hat{u}|$$

$$\leq ((h^{-1} + d^{-1})\nu^{-\frac{1}{2}}T_1(t)\lambda_{m+1}^{-\frac{1}{2}} + c_1|A^{\frac{1}{2}}\hat{u}|)|\hat{u}|. \tag{5.3}$$

By using (2.5) we conclude

$$|A^{\frac{1}{2}}e| \le 2\nu^{-1}C_1(t)((h^{-1}+d^{-1})\nu^{-\frac{1}{2}}T_1(t)+c_1C_2(t))\lambda_{m+1}^{-\frac{3}{2}}.$$

End of the proof. □

Remark. Notice that (Step 2) of the post Galerkin scheme is solved in whole space V. When we consider the realistic implementation of this scheme, we should restrict this step in a larger finite dimensional subspace of V, namely, H_M with $M \gg m$ and to get a finite dimensional approximation $u_M^*(t)$. That is we should modify the (Step 2) and (Step 3) as

(Step 2') Postprocess $u_m(t)$ at any time $t \in \mathcal{R}^+$

$$\begin{cases} \text{find } \ u^M(t) = \Phi_M(u_m(t)) \in H_M \ \text{ such that} \\ h^{-1}u^M + \nu A u^M + g(\frac{2|A^{\frac{1}{2}}u_m|}{M_0})P_M[B(u^M,u_m)+B(u_m,u^M)] \\ \qquad = g(\frac{2|A^{\frac{1}{2}}u_m|}{M_0})P_M Q_m[f - B(u_m,u_m)]. \end{cases}$$

(Step 3') Post Galerkin aprroximation

$$u_M^*(t) = u_m(t) + u^M(t) = u_m(t) - \Phi_M(u_m(t)).$$

It is easy to show that this finite dimensional scheme has the following error estimation

$$|A^{\frac{1}{2}}(u(t) - u_M^*(t))| \le T_2(t)\lambda_{m+1}^{-\frac{3}{2}} + C\lambda_{M+1}^{-\frac{1}{2}}.$$

Of course, to balance the two terms on the right hand side of the above inequality, we should choose $M \sim m^3$. That is the performance of our proposed scheme is just like that of a standard Galerkin scheme with very large computing scale.

On the other hand, the results here are also valid when we consider the periodic boundary conditions case.

6 Numerical experiment

In this section, we will present a simple numerical experiment for our scheme. For the sake of simplicity, we will consider problem (2.2) in a square domain $\Omega = (-\pi,\pi)^2$ with periodical boundary conditions. Under this circumstance, H is

$$H = \{u = \sum_{k\in Z, k\neq 0} u_k e^{ix\cdot k},\ u_k = \overline{u_{-k}},\ \mathrm{div}u = 0 \ \text{ under weak sense },\ \sum_{k\in Z, k\neq 0} |u_k|^2 < +\infty\}.$$

We assume that we have a true solution of (2.2). In fact, we give some function $u(t) \in H$ with

$$|u_k(t)| \sim |k|^{-2}.$$

And for given $m \in \mathcal{N}$, we define P_m is the orthogonal projection from H onto

$$H_m = \{u = \sum_{|k_1|,|k_2|\le m} u_k e^{ix\cdot k},\ u_k = \overline{u_{-k}},\ \mathrm{div}u = 0\},$$

where $k = (k_1,k_2)^T$. Meanwhile, for the periodical case, it is very easy to get the divergence free projection P. Then we use this $u(t)$ to compute $P_m f(t)$ in (2.4) and solve it to get the standard Galerkin approximation.

Concerning about the computing scale, we only give a small scale simulation here. For example, in our numerical implementation, we take $m = 9$, $M = 2m$ and $h = \nu = 1$. Following table indicates the relative error of standard Galerkin method and post Galerkin method defined as

$$\text{SGM} = \frac{||u - u_m||}{||u||},$$

and

$$\text{PGM} = \frac{||u - u_M^*||}{||u||},$$

where SGM and PGM mean the relative error of standard Galerkin method and post Galerkin method respectively.

Time	SGM	PGM
2.0	2.08%	0.400%
4.0	2.86%	1.33%
6.0	4.87%	2.71%
8.0	10.6%	5.98%
10.0	9.06%	4.90%
12.0	4.87%	2.38%
14.0	3.43%	1.47%
16.0	2.83%	1.06%
18.0	2.56%	0.845%
20.0	2.50%	0.757%
22.0	2.56%	0.747%

It seems that $\dfrac{PGM}{SGM} \sim \dfrac{1}{2}$. From the remark at the end of last section, we know this ratio is restricted by the truncation error because we just take $M = 2m$ instead of $M \sim m^3$ when concerning about the large computing scale. So this ratio is reasonable because $\dfrac{\lambda_{M^2+1}^{-\frac{1}{2}}}{\lambda_{m^2+1}^{-\frac{1}{2}}}$ should be close to $\dfrac{M^{-1}}{m^{-1}} = \dfrac{1}{2}$.

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