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OPTIMAL PARAMETERS IN LAGUERRE AND KAUTZ SERIES

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Abstract

In the context of parsimonious signal and systems representation Laguerre and Kautz series are considered. Arbitrary causal signals having finite energy can be represented in a Laguerre or Kautz series. The Laguerre and Kautz series depend on a single free parameter. In the Laguerre series the parameter represents a pole of the transfer function. The usual Laguerre series evolves when this pole is real-valued. A good parameter choice in the sense of a compaction of the energy in the first terms of the series can then be made on the basis of a few simple signal measurements. It is shown that this also holds for a Laguerre series having a complex-valued pole. This result is subsequently used to estimate the optimal poles for a Kautz series having a repeated complex-conjugated pole pair.

1 Introduction

Orthogonal series expansions have been used in quite some areas. For system identification, we are mainly interested in orthogonal series expansions on a semi-infinite axis. Restricting ourselves to rational transfer functions, we have orthogonalized exponentials [1] as the basis blocks. This leads to what have been called Kautz functions [2, 3], with as specific case the well-known Laguerre functions [4].

In practical situations we work with a finite expansion, i.e., we truncate the series expansion after a number of terms. The restrictions on the optimal parameters in Laguerre and Kautz series have been derived for a fixed number of terms and using a quadratic optimization criterion [5, 6]. Finding the optimal parameters requires to solve a difficult nonlinear minimization problem [7].

Therefore, we are interested in less demanding alternatives yielding a (sub)optimal choice for the parameter but which can be easily established, i.e., without a refined, costly search procedure.

Such a method exists for the Laguerre series [8, 9] and, in fact, more generally for orthogonal functions related to the classical orthogonal polynomials [10]. In that case, an alternative optimization problem is defined, namely the minimization of an enforced convergence criterion or, equivalently, the minimization of an upper bound for the quadratic error using a truncation of the expansion. From this criterion, an explicit expression for the pole of the Laguerre series evolves.

Having the optimal pole, the Laguerre series compactly describes a function that is to be approximated, if it has a low-pass nature. However, on qualitative grounds one must expect that a Kautz series can achieve a more compact representation in the case of approximating a band-pass filter.

Therefore, we first apply the compaction criterion to a Laguerre series described by a complex pole. In practice, this is not a very interesting case: only for complex signals we will find an (optimal) complex pole; in the (usual) case of a real signal or system that is to be approximated the procedure gives a real-valued parameter.

However, the results can be applied as follows. Suppose we have a signal having a pronounced resonant character. The desired approximation is therefore a Kautz series. We restrict to a Kautz series having a repeated complex-conjugated pole pair. From the signal we construct the analytic signal by the Hilbert transformation. Next, we use this complex signal to find the optimal parameter in a complex Laguerre series. Subsequently, we take this pole (and its complex conjugate) as an estimate for the optimal parameters in the Kautz series.

By an example, we show that such procedure can give results close to the optimal parameterization defined by a quadratic error criterion. If, however, the results of the proposed procedure are not satisfactory, the obtained estimated pole can always be used as the initial value in some refined optimization procedure.

2 Kautz and Laguerre series

Consider the set of orthonormal functions $\phi_n(k)$ with $n, k \in \mathbb{N}$ described by their z-transform $\Phi_n(z)$ as

$$\Phi_n(z) = \sqrt{1 - p_n p_n^*} \frac{z}{z - p_n} \prod_{l=0}^{n-1} \frac{1 - z p_l^*}{z - p_l} \tag{1}$$

with $|p_n| < 1$. This set is complete in $\ell^2(\mathbb{N})$ if $\sum_n (1 - |p_n|)$ diverges [1].

If all poles are identical, $p_n = \rho$, and real-valued, we have the Laguerre basis [4, 3]. The functions ϕ_n are now real-valued. If all poles are identical $p_n = \rho$ with $\Im\{\rho\} \neq 0$, we will call it the complex Laguerre basis. In the case that the poles occur as a repeated complex-conjugated pair, i.e., $p_{2n+1}^* = p_{2n} = p$ for $n \in \mathbb{N}$, we will call it a Kautz basis. The more general basis (1) is also referred to as a Kautz basis. Since we will consider the case of a repeated complex-conjugated pole pair only, there will be no confusion in terminology.

Consider the mapping $f \to a$ with $f, a \in \ell^2(\mathbb{N})$ and

$$a_n = \langle f, \phi_n \rangle = \sum_{k=0}^{\infty} f(k) \phi_n^*(k). \tag{2}$$

The inverse transformation $a \to f$ is given by

$$f(k) = \sum_{n=0}^{\infty} a_n \phi_n(k). \tag{3}$$

Note that the expansion coefficients a_n (the spectrum) is real-valued for the standard Laguerre basis and real f.

In the case of a real function f and a Kautz series, we can redefine the Kautz basis in order to obtain a real spectrum. The most common way is to define the functions $\psi_{i,n}(t)$ with $i = 1, 2$ having z-transforms

$$\Psi_{i,n}(z) = K_i \frac{z(z - q_i)}{(z - p)(z - p^*)} \left(\frac{(1 - zp)(1 - zp^*)}{(z - p)(z - p^*)} \right)^n \tag{4}$$

with

$$\begin{aligned} K_1 &= \sqrt{(1+p)(1+p^*)(1-pp^*)/2}, \\ K_2 &= \sqrt{(1-p)(1-p^*)(1-pp^*)/2}, \\ q_1 &= -1, \\ q_2 &= 1. \end{aligned}$$

3 Optimal Laguerre parameter

Consider the approximation f_N of f with $f(k) \in \mathbb{R}$ according to

$$f(k) \approx f_N(k) = \sum_{n=0}^{N-1} a_n \phi_n(k) \tag{5}$$

where ϕ_n are the (standard) Laguerre functions, the z-transforms are given by (1) with $p_n = \rho \in (-1, 1)$. The relative quadratic error in this approximation is given by

$$E_r = \frac{\langle f - f_N, f - f_N \rangle}{\langle f, f \rangle} = \sum_{n=N}^{\infty} a_n^2 / \langle f, f \rangle. \tag{6}$$

This relative error is a function of the Laguerre pole ρ. Therefore, we can optimize over this free parameter [5]. This does however not lead to a simple explicit expression for ρ.

We can also consider he following upper bound for the error:

$$\frac{\sum_{n=N}^{\infty} a_n^2}{\langle f, f \rangle} \leq \frac{\sum_{n=N}^{\infty} n a_n^2}{N \langle f, f \rangle} \leq \frac{\sum_{n=0}^{\infty} n a_n^2}{N \langle f, f \rangle} = \frac{M_1}{N} \tag{7}$$

where M_1 is the first-order moment of the energy distribution in the transform domain, i.e., M_1 is the center of the energy distribution. The expression M_1 can also be considered as a compaction criterion.

It turns out that this upper bound (or this center of the energy distribution) can be easily minimized due to the properties that the Laguerre functions inherit from the Laguerre polynomials [8, 9] (in fact, from all classical orthogonal polynomials [10]). The Laguerre functions ϕ_n adheres to the second-order difference equation:

$$D_2 \phi_n(k+1) + D_1 \phi_n(k) + D_0 \phi_n(k-1) = 0, \tag{8}$$

where

$$\begin{aligned} D_2 &= \rho k + \rho, \\ D_1 &= n(1 - \rho^2) - \rho^2 - (1 + \rho^2) k, \\ D_0 &= \rho k. \end{aligned}$$

For the simplicity we introduce the operators $\mathcal{T}_a$ and $\mathcal{K}$

$$\begin{aligned} (\mathcal{T}_a f)(k) &= f(k + a), \\ (\mathcal{K} f)(k) &= k f(k). \end{aligned}$$

Using this operators we now define the functional $\mathcal{L}_\rho$ by

$$\mathcal{L}_\rho(f) = \frac{\langle \frac{1}{1-\rho^2} \left(-\rho \mathcal{T}_1 \mathcal{K} + (1 + \rho^2) \mathcal{K} + \rho^2 \mathcal{I} - \rho \mathcal{K} \mathcal{T}_{-1} \right) f, f \rangle}{\langle f, f \rangle}, \tag{9}$$

where $\mathcal{I}$ is the identity operator. From the second-order difference equation of the Laguerre functions (8), we can write

$$\mathcal{L}_\rho(\phi_n) = n, \text{ for all } n \in \mathbb{N}. \tag{10}$$

Further the functional $\mathcal{L}_\rho$ can be simplified to:

$$\mathcal{L}_\rho = \frac{-2\mu\rho + 2m_1 + m_0}{(1 - \rho^2) m_0} - \frac{m_1}{m_0} - 1, \tag{11}$$

where

$$\begin{aligned} m_0(f) &= \langle f, f \rangle, \\ m_1(f) &= \langle f, \mathcal{K} f \rangle, \\ \mu(f) &= \langle f, \mathcal{T}_1 \mathcal{K} f \rangle. \end{aligned}$$

Using the above properties it can be proved that

$$\mathcal{L}_\rho = M_1. \tag{12}$$

The minimization of $\mathcal{L}_\rho$ as a function of ρ results in an explicit expression for the optimal pole $\hat{\rho}$

$$\hat{\rho} = \beta - \operatorname{sgn}(\beta)\sqrt{\beta^2 - 1} \tag{13}$$

with

$$\beta = (m_1 + m_0/2)/\mu. \tag{14}$$

At this minimum we have the lowest possible center of the energy distribution by

$$\mathcal{L}_{\hat{\rho}} = M_1(\hat{\rho}) = \frac{|\mu|}{m_0}\sqrt{\beta^2 - 1} - \frac{1}{2}. \tag{15}$$

With appropriate care the results can be extended to a complex Laguerre basis, $p_n = \rho$, $|\rho| < 1$. As before we consider the approximation $g_N(k)$ of a complex function $g(k) \in \mathbb{C}$ by

$$g(k) \approx g_N(k) = \sum_{n=0}^{N-1} a_n \phi_n(k) \tag{16}$$

where $\phi_n(k)$ are now the complex Laguerre functions. The complex Laguerre functions adheres to the second-order complex difference equation:

$$D_2\phi_n(k+1) + D_1\phi_n(k) + D_0\phi_n(k-1) = 0, \tag{17}$$

where

$$\begin{aligned} D_2 &= \rho^* k + \rho^*, \\ D_1 &= n(1 - \rho\rho^*) - \rho\rho^* - (1 + \rho\rho^*)k, \\ D_0 &= \rho k. \end{aligned}$$

The functional $\mathcal{L}_{\rho_c}$ in the complex case becomes

$$\mathcal{L}_{\rho_c}(f) = \frac{\langle (-\rho^*\mathcal{T}_1\mathcal{K} + (1 + \rho\rho^*)\mathcal{K} + \rho\rho^*\mathcal{I} - \rho\mathcal{K}\mathcal{T}_{-1})\, f, f\rangle}{(1 - \rho\rho *)\langle f, f\rangle}. \tag{18}$$

Taking the same steps as in the real case we arrive at

$$\hat{\rho}_c = \beta_c - \frac{\beta_c}{|\beta_c|}\sqrt{\beta_c\beta_c^* - 1}, \tag{19}$$

with

$$\begin{aligned} \beta_c &= \frac{m_1 + m_0/2}{\mu}, & (20)\\ m_0(f) &= \langle g, g\rangle, \\ m_1(f) &= \langle f, \mathcal{K}f\rangle, \\ \mu(f) &= \langle f, \mathcal{T}_1\mathcal{K}f\rangle. & (21) \end{aligned}$$

Furthermore, we have

$$\mathcal{L}_{\hat{\rho}_c} = M_1(\hat{\rho}_c) = \frac{|\mu|}{m_0}\sqrt{\beta_c\beta_c^* - 1} - \frac{1}{2}. \tag{22}$$

Note that if g is real, then the optimal pole $\hat{\rho}_c$ is real as well, no matter how resonant the character of g is. In that case, M_1 will be large since the resonant character can only be mimicked by many (non-resonant) Laguerre functions.

We now try to give an interpretation to the estimated argument (the angle) of $\hat{\rho}_c$. We have

$$\hat{\rho}_c = \beta_c\left(1 - \sqrt{1 - \frac{1}{\beta_c\beta_c^*}}\right),$$

and in view of (20)-(21) we have

$$\arg\hat{\rho}_c = \arg\beta_c = \arg\{-\mu\}.$$

We rewrite the expression (21) for μ to the frequency domain which yields

$$\arg\hat{\rho}_c = \arg\left\{\int_{\langle 2\pi\rangle} w(\theta)e^{j\theta}d\theta\right\} \tag{23}$$

with

$$w(\theta) = -|H(e^{j\theta})|\left[\frac{\partial}{\partial r}\left\{\frac{1}{\sqrt{r}}H(re^{j\theta})\right\}\right]_{r=1}. \tag{24}$$

We infer that $e^{j\theta}$ is weighted over the unit circle with weight function[1] w and from this the angle is taken.

The weighting function w is intuitively a good measure. If the function $G(z)$ has poles p_G (inside the unit circle) than if $\theta \simeq \arg p_G$ both $|H(e^{j\theta})|$ and $\left[-\frac{\partial}{\partial r}\left\{\frac{1}{\sqrt{r}}H(re^{j\theta})\right\}\right]_{r=1}$ will be large.

4 Optimal Kautz parameter

The results above depend on the existence of a difference equation for the (complex) Laguerre series [9, 10]. This leads to the existence of a linear operator $\mathcal{A}$ such that

$$\mathcal{A}\phi_n = n\phi_n \tag{25}$$

For the Kautz series, it is as yet not clear whether an operator of this form exists, even though the structure of the Kautz functions is similar to that of the Laguerre functions in the sense that

$$\Psi_{i,n} = \Psi_{i,0}\left\{A(z)\right\}^n \tag{26}$$

where $A(z)$ is the transfer function of an all-pass filter, see (4). Thus, differentiation with respect to z automatically gives rise to the appearance of the desired multiplicative factor n needed in the linear operator in order that the same procedure can be applied to the Kautz functions. However, we have not been able to derive such operator so far.

The intuitive way to estimate the optimal pole in the Kautz series is to look at the amplitude spectrum, search

[1]The weight function is not necessarily positive for all θ.

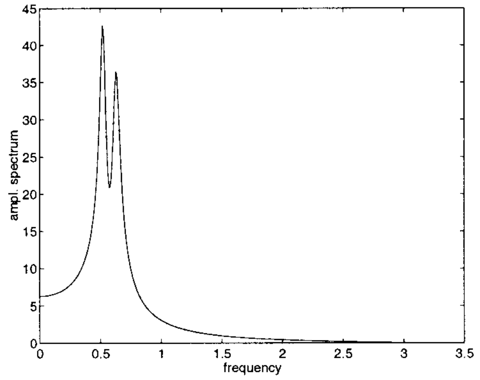

Figure 1: Amplitude spectrum of the function f

for the dominant peak (assuming it exists) or the center of energy distribution as a function of the frequency θ ($0 \leq \theta \leq \pi$ for a real-valued function). This frequency is then identified with the argument of p. Next, one considers the signal as a modulated low-pass signal and identifies $|p|$ with the optimal Laguerre parameter for the low-pass signal.

A slightly different approach is the following. From a real function f having a clear resonant character, we construct the analytic function g as its Hilbert transform[2]. Next, the function g is written as a complex Laguerre series and the optimal complex Laguerre parameter for g is calculated. Since

$$f = \{g + g^*\}/2 = \frac{1}{2}\left\{\sum_{n=0}^{\infty}[a_n\phi_n + a_n^*\phi_n^*]\right\} \qquad (27)$$

and since the set of functions $\{\phi_n, \phi_n^* | n = 0, \cdots N\}$ spans the same space as $\{\psi_{1,n}, \psi_{2,n} | n = 0, \cdots, N\}$ we expect the optimal complex parameter for an approximation of g to be a good candidate for a compact representation of f in a Kautz series.

5 Example

As an example, we took the function $f(k)$ defined by a rational z-transform

$$F(z) = \frac{\prod_{i=1}^{3}(z - q_i)}{\prod_{i=1}^{4}(z - p_i)}$$

with

$$\begin{aligned} q_1 = q_2^* &= 0.96e^{j\pi/5.5}, \\ q_3 &= -1, \\ p_1 = p_2^* &= 0.97e^{j\pi/5}, \\ p_3 = p_4^* &= 0.975e^{j\pi/6}. \end{aligned}$$

[2]The imaginary part of g is in general non-causal. Only the causal part of g is considered.

In view of the complex conjugated pole pairs of the poles and zeros, f is obviously real-valued. In Fig. 1 the amplitude spectrum is given. We observe two closely-spaced resonant peaks. We also note that the function $f(k)$ can never be expressed exactly in a finite Kautz series defined by a repeated complex-conjugated pole pair.

As discussed in the previous section, we consider the causal part of the Hilbert transform of f, and from this function calculated the optimal complex pole for a complex Laguerre series. This was found that for $\hat{\rho}_c$ we have $|\hat{\rho}_c| = 0.922$ and $\arg\{\hat{\rho}_c\} = 0.562$ (rad).

Next, we used this pole in a Kautz series and calculated the relative modeled energy as a function of the number of sections N. By this we mean the number of complex-conjugated pole pairs. Thus, the order of the approximating transfer function equals $2N$.

In Fig. 2, the relative energy loss E_r is shown as a function of N. We also calculated this loss for a Kautz series where, for each order N, the pole was optimized in such a way that a maximum of energy was contained in the first N terms (Fig. 2, dashed line). As is to be expected, the latter is always better (in fact, by definition the best in a quadratic sense). The proposed energy compaction criterion yields results which are close to the optimal ones. This is also shown in Table 1.

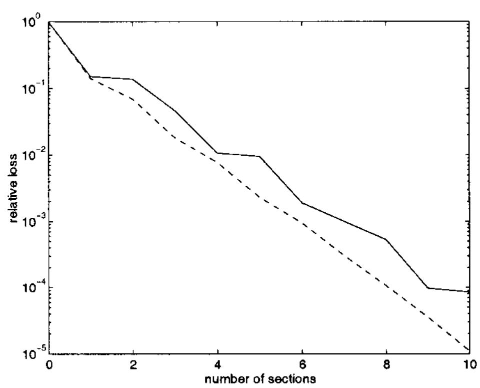

Figure 2: Relative energy loss in the Kautz series with an optimal pole. Optimality of the pole is defined according to the energy compaction criterion (solid line) and the quadratic error (dashed line).

In Figs. 3 and 4, we plotted the optimal pole radius and angle according to a maximum energy criterion as a function of the number of sections in the Kautz series (dashed lines) together with the optimal pole according to the energy compaction criterion. We note that the optimal pole defined by the compaction criterion is not quite in line with those obtained by a quadratic criterion.

Even though the poles estimated by the two procedures do not seem to agree, the behaviour in terms of the relative loss is similar. One additional term is needed to obtain roughly the same relative loss in a series expansion using the optimal pole defined by the compaction criterion compared to the optimal pole in a quadratic error criterion.

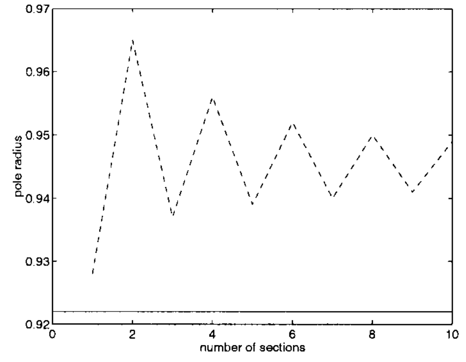

Figure 3: Optimal pole radius as a function of the number of expansion terms. Optimality according to the energy compaction criterion (solid line) and the quadratic error (dashed line).

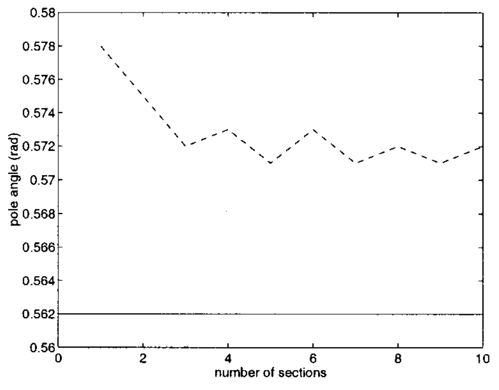

Figure 4: Optimal pole angle as a function of the number of expansion terms. Optimality according to the energy compaction criterion (solid line) and the quadratic error (dashed line).

6 Discussion

We proposed a simple procedure to establish the optimal parameter in a Kautz series expansion on the basis of an energy compaction criterion. This criterion is suboptimal in a quadratic sense, but yields simple explicit expressions for the optimal pole in terms of the Hilbert transform of the given function. In contrast to an optimal pole according to a quadratic error criterion, the compaction criterion

order	radius	angle	E_{r1}	E_{r2}
1	0.928	0.578	0.152	0.141
2	0.965	0.575	0.138	0.069
3	0.937	0.572	0.046	0.018
4	0.956	0.573	0.011	0.008
5	0.939	0.571	0.009	0.002

Table 1: Optimal pole radius and angle as a function of the number of sections, for a quadratic error criterion, together with the relative modeled energy loss E_{r2}. The relative modeled energy loss E_{r1} for the optimal pole according the energy compaction criterion (see text) is given in the last column.

has the convenient property that it defines a pole independent of the number of the terms in the expansion.

By an example, it was shown that such procedure can give results close to the optimal one in a quadratic sense. Finally, we note that the optimal pole defined by the compaction criterion can always be used as the initial estimate in a more refined optimization procedure.

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