

Robust controller design and performance for polytopic models

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ROBUST CONTROLLER DESIGN AND PERFORMANCE FOR POLYTOPIC MODELS

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Abstract

Polytopic models cover a large class of nonlinear dynamic systems. An algorithm is proposed that partitions the state-space into a number of disjoint clusters on which a gain-scheduling controller is defined. Then, an iterative synthesis algorithm based on LMIs is developed that guarantees globally robust stability of the closed loop system. Finally, an analysis method is presented that makes it possible to associate with certain regions of the state-space (a measure of) performance.

1 Introduction

Promising approaches for the modelling and control of nonlinear systems have emerged over the last few years [Mur97]. The crux of these approaches is to represent a nonlinear dynamic system by a ‘global’ model which is the result of taking convex combinations of locally valid affine models. These ‘global’ models occur frequently in literature and although they all have an equivalent mathematical structure they are given different names i.e. Fuzzy Models [Sug88][Wan92], Multi-Models [Mur97] or Local Model Networks [Joh93]. This modelstructure has several desirable attributes. The model is interpretable on the basis of a regime decomposition [Sug88][Joh93], and the model class is rich since a large class of nonlinear systems can be approximated arbitrarily close with the proposed modelstructure [Wan92][Joh93]. Also the model is of a form lending itself to system analysis and controller synthesis methods on the basis of linear matrix inequalities (LMIs) [Boy94]. This is the subject of this work.

The important property of the proposed modelstructure as exploited within this work is that the model defines a polytope in the model-space. Therefore, from now on these models will be called polytopic models. The objective is to compute a stabilizing gain-scheduling controller,

which is robust against parametric uncertainty. The discrete time equivalent robust stabilization problem is discussed in [Slu98], formulated as a bilinear matrix inequality feasibility problem and solved locally using LMI algorithms. However, contrary to [Slu98] we also demand that some performance level of the closed loop system is achieved in certain regions of the state-space. Related work can be found in [Has98][Ran97] though restricted to piecewise-affine systems. We also formulate the synthesis problem as a matrix inequality feasibility problem. But as mentioned in [Has98] and in conformance with [Slu98] for the case of affine state feedback this leads to nonlinear matrix inequalities. An iterative algorithm involving LMIs will be proposed to solve the synthesis problem. If the LMIs are feasible then the robust controller synthesis problem is solved. In addition, performance of the closed loop in the state-space can be analyzed without doing simulations.

The outline of this paper is as follows. In Section 2 the robust stabilization problem with the objective to meet performance specifications will be stated formally. In Section 3 an algorithm will be proposed that constructs a state-space partitioning. In conformance with this partitioning a gain-scheduling controller will be defined. Then, in Section 4, relevant performance issues will be addressed. An analysis tool is presented that associates with certain regions of the state-space, (a measure of) performance of the closed loop. Section 5 addresses the robust optimal performance problem in the context of LMI conditions. This leads to a two step controller synthesis algorithm. The synthesis algorithm and the analysis tool are illustrated with an example in Section 6. Finally, in Section 7, we come to some conclusions.

2 Problem statement

Polytopic models cover the class of nonlinear systems:

$$\Sigma : \dot{x} = f(x, u, t), \quad x(0) = x_0 \quad (1)$$

with $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ such that the system Σ , subject to stabilization, in some subset $\mathcal{X} \subseteq \mathbb{R}^n$, $\mathcal{U} \subseteq \mathbb{R}^m$

and $\mathcal{T} \subseteq \mathbb{R}^+$ can be described sufficiently accurate by a polytopical model

$$\Pi : \dot{x} = \sum_{j \in I_{Nm}} w_j(x, u, t)(A_j x + B_j u + c_j), \quad x(0) = x_0 \quad (2)$$

The polytopical model consists of (locally valid) models parametrized by the triples (A_j, B_j, c_j) . These models define a polytope in the model-space. Nm denotes the number of models and $I_{Nm} := \{1, \dots, Nm\}$ is the associated indexset. Here $A_j \in \mathbb{R}^{n \times n}$ is the system matrix, $B_j \in \mathbb{R}^{n \times m}$ the input matrix and $c_j \in \mathbb{R}^n$ is a constant vectorfield. Furthermore $w_j : \mathcal{X} \times \mathcal{U} \times \mathcal{T} \rightarrow [0, 1], \forall j \in I_{Nm}$ and also $\sum_{j \in I_{Nm}} w_j(x, u, t) = 1, \forall (x, u, t) \in \mathcal{X} \times \mathcal{U} \times \mathcal{T}$. The w_j 's schedule the triples in the operating space. Within the fuzzy modelling and local modelling framework these functions are called fuzzy membership functions or normalized validation functions [Wan92][Joh94].

Definition 1 (robust quadratic stabilizability) *A Model Π Eq.(2) is said to be quadratically stabilizable if there exists a quadratic function $V(t) = x^T P x, P > 0$ that decreases along every nonzero trajectory of the closed loop system, i.e. Π interconnected with a feedback $u(x)$. Then V is called a quadratic Lyapunov function for the closed loop system and the origin is a stable equilibrium point of the closed loop system. A model Π is called robust quadratically stabilizable if it is quadratically stabilizable and furthermore only knowledge of the support in the state space of the $w_j(x)$'s is utilized in the controller design to control Π . If subsequently $u(x)$ is chosen such that also the abstract energy $\int_0^\infty v(x, u)^T v(x, u) dt$ with $v(x, u)$ a linear function of x and u is minimized, then the controller $u(x)$ is said to achieve robust optimal performance e.g. performance in the presence of model uncertainties.*

The objective is to robust quadratically stabilize Π and achieve optimal performance via piecewise-affine state feedback. This notion of optimal performance is closely related to the classical linear quadratic regulator (LQR) problem, see e.g. [Boy94], but extended to handle gain-scheduling controllers for polytopical systems.

3 State-space partitioning and feedback law

A partitioning of the state-space, similar to [Slu98], and the associated feedback law will be formalized. The only knowledge that will be exploited to stabilize Π is the region of support in the state space of the w_j 's. So, first these regions are identified and described by the sets \mathcal{X}_j^s , where s stands for support

$$\mathcal{X}_j^s := \bigcup_{(u,t) \in \mathcal{U} \times \mathcal{T}} \text{supp } w_j(\cdot, u, t) \quad \forall j \in I_{Nm} \quad (3)$$

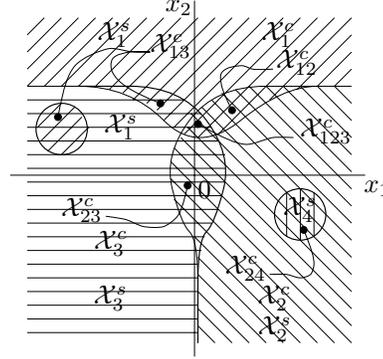


Figure 1: The polytopical model Eq.(2) with $x \in \mathbb{R}^2$ consists of the triples (A_j, B_j, c_j) with $j \in \{1, 2, 3, 4\}$. With the knowledge of the w_j 's the support sets \mathcal{X}_j^s are identified and are given a different hatching. The algorithm, Eq.(5) is initialized, $k = 0$, by taking the intersection of all the support sets \mathcal{X}_j^s which in this case is empty, $\mathcal{X}_{1,2,3,4}^c = \emptyset$. For $k = 1$ the algorithm returns $N_m!(k!(N_m - k)!)^{-1} = 4$ polytopes of size $N_m - k = 3$. Only cluster $\mathcal{X}_{1,2,3,4}^c$ is non-empty. For $k = 2$ there are 6 polytopes of size 2. Again, some of them have non-empty clusters e.g. $\mathcal{X}_{2,3}^c$. Along the same lines all the clusters are defined resulting in a non-overlapping partitioning of the state-space.

where for w_j the support $\text{supp } w_j(\cdot, u, t)$ is defined as

$$\text{supp } w_j(\cdot, u, t) := \{x \in \mathcal{X} \mid w_j(\cdot, u, t) > 0\} \quad (4)$$

Now, a non-overlapping partitioning of the state space in clusters \mathcal{X}_j^c where c stands for cluster can be recursively computed as follows:

Algorithm 2 (state-space partitioning)

for $k = 0$ to $k = N_m - 1$ do

$$\forall J \in V := \{L \mid L \subseteq I_{Nm}, \#L = N_m - k\} \quad (5)$$

$$\mathcal{X}_J^c = \left(\bigcap_{j \in J} \mathcal{X}_j^s \right) \setminus \left(\bigcup_{j \in I_{Nm} \setminus J} \mathcal{X}_{j \cup \{j\}}^c \right)$$

Some of the clusters are possibly empty, and $\bar{\mathcal{X}}_{01}^c = \{\mathcal{X}_J^c \mid \mathcal{X}_J^c \neq \emptyset\}$ is the set containing all non-empty clusters. $\bar{\mathcal{X}}_{01}^c$ can be associated with $\bar{J}_{01} = \{J \mid \mathcal{X}_J^c \in \bar{\mathcal{X}}_{01}^c\}$, the set pointing to all polytopes that have non-empty clusters. Furthermore it will be helpful to have another non-overlapping partitioning of the state-space, i.e. the cluster \mathcal{X}_0^c that contains the origin will be called $\bar{\mathcal{X}}_0^c$ and the set of clusters $\bar{\mathcal{X}}_{01}^c \setminus \bar{\mathcal{X}}_0^c$ will be called $\bar{\mathcal{X}}_1^c$. In resemblance with this partitioning, \bar{J}_0 is the set pointing to models associated with region $\bar{\mathcal{X}}_0^c$. In the same way, \bar{J}_1 can be associated with $\bar{\mathcal{X}}_1^c$, $\bar{J}_1 = \bar{J}_{01} \setminus \bar{J}_0$. An example of the suggested state-space partitioning, Eq.(5), is given in Fig.1.

It is assumed that the models indexed with $j \in \bar{J}_0$ reduce to $(A_j, B_j, 0)$, i.e. $0 \in \bar{\mathcal{X}}_0^c$ is an equilibrium point of

the system Eq.(2). With the partitioning it seems naturally to associate the following piecewise affine state feedback:

$$u = K_J x + k_J \quad \text{iff} \quad x \in \mathcal{X}_J^c \in \bar{\mathcal{X}}_{01}^c \quad (6)$$

This controller can be interpreted as a gain-scheduling controller, where the scheduling of the controller parameters K_J, k_J is defined by the clusters \mathcal{X}_J^c .

4 Bounds on performance

From Def.(1) it is clear that if $\exists P = P^T, M > 0$ such that $\dot{V}(t) \leq -x^T M x$ is satisfied, then the origin of Π is robust quadratically stabilizable. Next, the quality of control will be considered. This means that the energy function $V(t)$ will not only be demanded to decrease, but also to decrease in a prescribed way along every nonzero trajectory of the closed loop system to achieve optimal performance. With respect to optimal performance, two problems are relevant.

Problem 3 (optimal performance) *Given the system Eq.(2) and the pair $(C^T C, D^T D)$ the problem is to determine $P > 0$ from Def.(1) (and $u(x)$ of the structure Eq.(6)) that minimizes $\int_0^\infty (x^T C^T C x + u^T D^T D u) dt$.*

Problem 4 (inverse optimal performance) *Given the system Eq.(2) and a feasible $P > 0$ from Def.(1) the problem is to determine the pair $(C^T C, D^T D)$ and $u(x)$, of the structure Eq.(6) that minimizes $\int_0^\infty (x^T C^T C x + u^T D^T D u) dt$.*

Lowerbounds and Upperbounds analogous to [Ran97], as well as regions for optimal performance will be derived. Assume therefore that a stabilizing feedback is designed such that with $v^T v = x^T C^T C x + u^T D^T D u$, $C^T C$ and $D^T D$ invertible and

$$\dot{V}(t) \leq -v^T v \quad (7)$$

If Eq.(7) is then integrated from 0 to T this equation reads $V(0) \geq \int_0^T v^T v dt + V(T)$. Since $V(t) \geq 0$ for $0 \leq t \leq T$ and by assumption $\lim_{T \rightarrow \infty} V(T) = 0$, an upperbound $V(0)$ is found for the energy $\int_0^\infty v^T v dt$. The best upperbound is the one that minimizes $V(0)$. This means that $\min V(0) = \min x^T(0) P x(0)$. The P that achieves this bound will be denoted \bar{P} .

The corresponding inverse problem can be solved along the same line. This means that for a fixed and feasible $P > 0$ a lowerbound $\int_0^\infty v^T v dt$ is found for the energy $V(0)$. The best lowerbound is the one that maximizes $\int_0^\infty v^T v dt$. This means $\max \int_0^\infty v^T v dt$. The pair $(C^T C, D^T D)$ that achieves this bound will be denoted $(\underline{C}^T \underline{C}, \underline{D}^T \underline{D})$. Then the fixed $P > 0$ becomes the best upperbound \bar{P} for the maximized energy $\int_0^\infty (x^T \underline{C}^T \underline{C} x + u^T \underline{D}^T \underline{D} u) dt$.

Also a lowerbound can be calculated for the energy $\int_0^T v^T v dt$. For this purpose, assume

$$\dot{V}(t) \geq -v^T v \quad (8)$$

If Eq.(8) is then integrated from 0 to T this equation reads $V(0) \leq \int_0^T v^T v dt + V(T)$. Since $V(T) \geq 0$ for $0 \leq t \leq T$ and by assumption $\lim_{T \rightarrow \infty} V(T) = 0$ a lowerbound $V(0)$ is found for the energy $\int_0^T v^T v dt$. The best lowerbound can now be computed that maximizes $V(0)$. This means $\max V(0) = \max x^T(0) P x(0)$. The P that achieves this bound will be denoted \underline{P} .

If now $\bar{P} = \underline{P}$ then $V(0) = \int_0^\infty (x^T C^T C x + u^T D^T D u) dt$ is minimized for every initial condition $x(0)$ and performance is achieved everywhere in the state-space. As a measure of accuracy of the upper- and lowerbound it is therefore natural to look at the following ratio of quadratic terms:

$$R(x(0)) = \frac{x^T(0) \underline{P} x(0)}{x^T(0) \bar{P} x(0)} \quad (9)$$

Of course $0 \leq R(x(0)) \leq 1$. If $R(x(0)) \approx 1$ then the bounds are very accurate. $R(x(0))$, and therefore performance of the controller, depends on the initial condition $x(0)$. Hence there will be regions in the state space with maximum and minimum performance. With the factorization $\bar{P} = \bar{F}^T \bar{F}$ and the nonsingular transformation $z(0) = \bar{F} x(0)$ Eq.(9) becomes

$$R(z(0)) = \frac{z^T(0) \bar{F}^{-T} \underline{P} \bar{F} z(0)}{z^T(0) z(0)} \quad (10)$$

The minimum and maximum of the Rayleigh quotient, Eq.(10), can be obtained from the eigenvector-eigenvalue decomposition $\bar{F}^{-T} \underline{P} \bar{F} = Q_z \Lambda Q_z^T$ with Q_z a matrix containing the orthonormal eigenvectors, and Λ a diagonal matrix containing the eigenvalues λ_i arranged in order of magnitude. Now the $\max_{z \neq 0} R(z(0)) = \lambda_{\max}$ and occurs in the direction of the corresponding eigenvector z_{\max} which is the corresponding column of Q_z . Equally, $\min_{z \neq 0} R(z(0)) = \lambda_{\min}$ with eigenvector z_{\min} . With the transformation $Q_x = \bar{F}^{-1} Q_z$ the direction of minimum and maximum performance (accuracy) in the state-space can be recovered. The regions for constant performance accuracy ($R(x(0)) = c$) are described by lines. This analysis tool makes it possible to associate with certain regions of the state-space, (a measure of) performance.

5 Controller synthesis

Robust optimal performance will be investigated using the piecewise-affine state feedback of Eq.(6). An iterative synthesis algorithm will be proposed that guarantees robust stabilizability via piecewise affine state feedback. The conditions involved can be written as LMIs. Feasibility of these conditions can therefore be checked efficiently

by means of convex optimization routines. If these conditions are feasible then also the corresponding feedback can be computed. Furthermore the iterative synthesis algorithm provides the analysis tool with the upper- and lowerbounds to investigate optimality of the closed loop.

5.1 Robust quadratic performance

First of all compute the best lowerbound \underline{P} as a function of the desired performance given by the fixed pair $(C^T C, D^T D)$, **step 1: iteration 1**. Compute this best lowerbound for performance without assuming a specific controller structure within a cluster. This can be done by solving the appropriate LMIs based on Eq.(8) together with the maximization of $\text{trace}(P)$.

Secondly, **step 2: iteration 1**, solve the inverse performance problem, this means with $\underline{Q} = \underline{P}^{-1}$ fixed solve the appropriate LMIs based on Eq.(7) together with the minimization of $\text{trace}((C^T C)^{-1}) + \text{trace}((D^T D)^{-1})$. The computed performance matrices will be denoted $\underline{C}^T \underline{C}$ respectively $\underline{D}^T \underline{D}$. By doing so \underline{P} becomes an upperbound \bar{P} for optimal performance, however measured with the obtained performance pair $(\underline{C}^T \underline{C}, \underline{D}^T \underline{D})$. Now one can start again solving **step 1: iteration 2**, with the performance matrices obtained in **step 2: iteration 1** and iterate further to see if some demanded agree of performance accuracy is achieved. At any moment one could stop the iteration and validate performance by comparing the computed upper- and lowerbounds for performance. This means compare \bar{P} obtained at **step 2: iteration k** with \underline{P} obtained at **step 1: iteration k+1** following the analysis presented in section 4. If one is satisfied about the (accuracy of) performance then the controller can be computed.

This leads to the following iterative procedure:

step 1: (find \underline{P})

check if $\exists P, \tau_J \geq 0$ such that for $J \in \bar{J}_{01}, j \in J$

$$\begin{bmatrix} A_j^T P + P A_j + C^T C - \tau_J S_{(J,xx)} & * & * \\ c_j^T P - \tau_J S_{(J,x1)}^T & -\tau_J S_{(J,11)} & * \\ B_j^T P & 0 & D^T D \end{bmatrix} \geq 0 \quad (11)$$

and also maximize $\gamma > 0$

$$\text{trace}(P) > \gamma$$

step 2: (find \bar{P})

and if for $\underline{Q} = \underline{P}^{-1} \exists \tau_J \geq 0, (C^T C)^{-1} > 0, (D^T D)^{-1} > 0, \{Y_J\}, \{y_J\}$ such that for $J \in \bar{J}_{01}, j \in J$

$$\begin{bmatrix} -L_J - \tau_J Q S_{(J,xx)} Q & * & * \\ -M_J^T - \tau_J S_{(J,x1)}^T Q & -\tau_J S_{(J,11)} & * \\ Q & 0 & (C^T C)^{-1} \\ Y_J & y_J & 0 \quad (D^T D)^{-1} \end{bmatrix} \geq 0 \quad (12)$$

with $L_J = Q A_j^T + Y_J^T B_j^T + A_j Q + B_j Y_J, M_J^T = c_j^T + y_J^T B_j^T$ and also minimize $\gamma > 0$

$$\text{trace}(C^T C)^{-1} + \text{trace}(D^T D)^{-1} < \gamma$$

The * elements are induced by symmetry of the matrices.

If the LMIs are feasible, then the controller Eq.(6) with $K_J = Y_J Q^{-1}$ and $k_J = y_J$ robustly stabilizes Π . The algorithm follows from standard Lyapunov arguments and LMI results using the \mathcal{S} -method. The \mathcal{S} -method is introduced to reduce conservatism of the synthesis inequalities [Boy94]. Therefore a quadratic function $S_J(x) = [x \ 1]^T S_J [x \ 1]$ has to be identified that outer approximates \mathcal{X}_J^c i.e. satisfies $\{x \mid S_J(x) \geq 0\} \supseteq \mathcal{X}_J^c$. The elements $S_{(J,\cdot)}$ in the LMIs above follow from a partitioning of S_J according to x and 1. More about how the outer approximation can be done can be found in [Has98].

6 Illustrative example

Assume that the polytopic model Π Eq.(2) consists of three models, parametrized by $(A_j, B_j, c_j), j \in \{1, 2, 3\}$ where

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad c_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad c_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 1 \\ -4 & 2 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad c_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

In this example it is assumed that the clusters are identified such that $\bar{\mathcal{X}}_{01}^c = \{\mathcal{X}_1^c, \mathcal{X}_{12}^c, \mathcal{X}_3^c\}$ and $\bar{\mathcal{X}}_0^c = \mathcal{X}_1^c$. In order to be able to introduce the \mathcal{S} -method the different regions $\mathcal{X}_J^c \in \{\mathcal{X}_1^c, \mathcal{X}_{12}^c, \mathcal{X}_3^c\}$ have to be outer approximated by quadratic function $S_J(x)$. In this typical example $\mathcal{X}_1^c = \mathbb{R}^2 \setminus \bar{\mathcal{X}}_1^c, \mathcal{X}_{12}^c = \{x \mid -((x_1 - 3)^2 + (x_2 - 3)^2) + 1 > 0\}, \mathcal{X}_3^c = \{x \mid -((x_1 + 3)^2 + (x_2 + 3)^2) + 1 > 0\}$ and the associated quadratic outer approximations $S_J(x)$ are equal except $S_1(x) = 0$. Some conservatism is introduced since $S_1 = 0 \forall x \in \mathbb{R}^2$. Without the \mathcal{S} -method, for every polytope defined by $J \in \bar{J}_{01}$, the triples (A_j, B_j, c_j) have to be stabilizable via the feedback Eq.(6). Therefore the synthesis LMIs without the \mathcal{S} -method would fail. In this case however Eq.(11,12) are feasible and the following results are obtained.

step 1: iteration 1 (find \underline{P})

Solve Eq.(11) with the suggested maximization of P for fixed $C^T C = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$ and $D^T D = 1$. The following

result is obtained: $\underline{P} = \begin{bmatrix} 11.8148 & 1.7044 \\ 1.7044 & 2.7034 \end{bmatrix}$.

step 2: iteration 1 (find \bar{P})

Solve Eq.(12) with i.e. $\min(\text{trace}(C^T C)^{-1} + 0.01 \text{trace}(D^T D)^{-1})$, the off-diagonal elements of $C^T C$ are chosen to be structural zeros. The following results are obtained:

$\underline{C}^T \underline{C} = \begin{bmatrix} 7.9147 & 0 \\ 0 & 14.1451 \end{bmatrix}, \underline{D}^T \underline{D} = 0.2547$. For the performance pair $(\underline{C}^T \underline{C}, \underline{D}^T \underline{D})$ the fixed \underline{P} becomes \bar{P} .

	min. accuracy	max. accuracy
$x(0)$	$x_{\min} = [-0.019 \ -4.853]^T$	$x_{\max} = [3.051 \ -1.971]^T$
$R(x(0))$	$\lambda_{\min} = 0.6287$	$\lambda_{\max} = 0.9967$
$V(0)$	64.0072	99.9826
E	61.9262	99.9071
$\underline{V}(0)$	40.2440	99.6574

Table 1: Upperbound, lower bound and achieved performance

The computed controller parameters are

$$\begin{aligned} K_1 &= \begin{bmatrix} -6.6911 & -10.6129 \end{bmatrix} & k_1 &= 0 \\ K_{12} &= \begin{bmatrix} -6.6917 & -10.6084 \end{bmatrix} & k_{12} &= 0.2714 \\ K_3 &= \begin{bmatrix} -6.7007 & -10.5092 \end{bmatrix} & k_3 &= -0.0117 \end{aligned}$$

step 1: iteration 2(find \underline{P})

Solve Eq.(11) with the suggested maximization of P however with the substitution of $(C^T C, D^T D)$ with the pair $(\underline{C}^T \underline{C}, \underline{D}^T \underline{D})$. The outcome of this step is $\underline{P} = \begin{bmatrix} 11.3612 & 1.0562 \\ 1.0562 & 1.6998 \end{bmatrix}$.

The quality of the two step synthesis algorithm can be quantized by looking at the distance between the performance achieved by the controller and the desired performance. The achieved performance of the controller is measured by the loss function $E = \int_0^T (x^T(t) \underline{C}^T \underline{C} x(t) + u^T(t) \underline{D}^T \underline{D} u(t)) dt$. $\bar{V}(0) = x^T(0) \bar{P} x(0)$ is an upperbound for the desired loss function E , see Eq.(7,12). The corresponding lowerbound for the desired performance reads $\underline{V}(0) = x^T(0) \underline{P} x(0)$, see Eq.(8,11). So clearly $\underline{V}(0) \leq E \leq \bar{V}(0)$.

The upperbound ($\bar{V}(0)$), lowerbound ($\underline{V}(0)$), achieved performance (E) and performance accuracy ($R(x(0))$) with the associated regions of minimal performance accuracy (βx_{\min}) and regions of maximal performance accuracy (βx_{\max}) in the state-space is given in Table 1.

In Fig.(2) simulation results are shown for the polytopic system with $w_1 = w_2 = 0.5$ for $x \in \mathcal{X}_{12}^c$. Trajectories are initiated from: $x(0) = [3 \ 4.5]^T$, $x(0) = x_{\min}$ and $x(0) = x_{\max}$. Also the (dotted) lines along which maximal and minimal performance accuracy is obtained are drawn in this figure. From the Lyapunov level curves it can be seen that along the trajectories energy decreases. Since trajectories of the closed loop system with initial values $x(0) = x_{\max}$ and $x(0) = x_{\min}$ stay within cluster \mathcal{X}_1^c one could compare the performance of the controller for these initial conditions with a LQR design for the triple (A_1, B_1, c_1) with performance pair $(\underline{C}^T \underline{C}, \underline{D}^T \underline{D})$. This means solving the algebraic Riccati equation for the unknown P ,

$$P A_1 + A_1^T P - P B_1 (\underline{D}^T \underline{D})^{-1} B_1^T P + \underline{C}^T \underline{C} = 0$$

with optimal controller $K = -(\underline{D}^T \underline{D})^{-1} B_1^T P$. The following solution is obtained: $P = \begin{bmatrix} 11.8036 & 1.6972 \\ 1.6972 & 2.6835 \end{bmatrix}$ and $K = [-6.6635 \ -10.5359]$. This leads to the costs $E_{x_{\max}} = 99.8224$ and $E_{x_{\min}} = 56.6821$ evaluated for initial

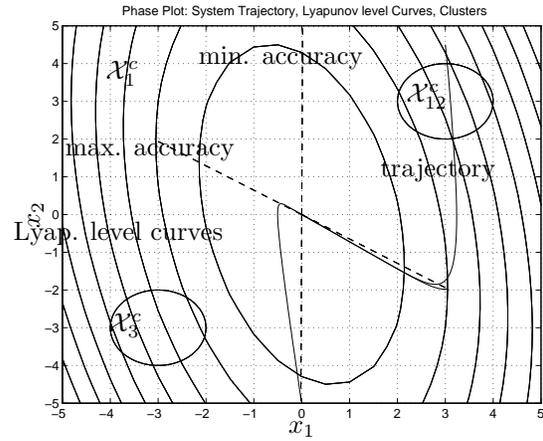


Figure 2: Trajectories in the state-space for different initial conditions of the closed loop polytopic system.

conditions of maximal and minimal performance accuracy as given in Table 1. Comparing these costs with the costs from Table 1 shows that indeed performance is achieved for initial conditions $x(0)$ close to ax_{\max} .

7 Conclusions

Polytopic models cover a large class of nonlinear dynamic systems. An algorithm is proposed that partitions the state-space into a number of disjoint clusters on which a gain-scheduling controller is defined. The objective is to parametrize this controller such that robust stabilization and performance of the closed loop polytopic model is achieved.

An iterative algorithm involving LMIs is proposed to solve the robust stabilization synthesis problem. If the LMIs are feasible then the synthesis problem is solved. Furthermore performance of the closed loop in the state-space can be analyzed with the presented analysis tool that builds naturally on the outcome of the iterative synthesis algorithm presented in this paper.

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