# Comparing multinomial, grouped and multivariate control charts for real time monitoring of clean rooms with BioTrak

van Driel, S.A.W.

*Award date:*
2019

# Comparing Multinomial, Grouped and Multivariate Control Charts for Real Time Monitoring of Clean Rooms with BioTrak


EINDHOVEN UNIVERSITY OF TECHNOLOGY


*Author:*

Stephan A.W. van Driel

*TU/e Supervisors:*

Edwin R. van den Heuvel
Zhuozhao Zhan


July 23, 2019

## Abstract

The BioTrak is a new particle count machine to real time monitor control rooms. The BioTrak counts particles and categorises the particles in size bins. It does this for airborne particles and for viable particles. The BioTrak can not count particles smaller than 0.5 $\mu m$. The BioTrak provides the number of particles per size bin every minute . The machine is developed to replace traditional microbiological count methods, which provide daily results only. The BioTrak monitors the clean rooms in real time and so it may detect out of control events earlier. This may lead to better root-cause analysis and indirectly reduce the processing time of batch production. In this thesis we evaluated current statistical process control techniques to monitor multivariate categorical data. We will give a detailed description of multinomial, grouped, and multivariate control charts. The Hotelling $T^2$ chart for Poisson and binomial data, the multivariate np chart, generalized p chart, the multivariate Poisson chart, log likelihood ratio chart, bivariate zero inflated negative binomial chart and bivariate zero inflated Poisson chart. We evaluated the performance of each chart via simulation studies. We generate truncated multinomial data with known underlying log normal distribution for particles sizes. We investigated the average run length of each chart under both in control and out control settings, where the total number of particles is fixed at each time point and where the total number of particles is variable. From the simulation study we conclude that the proposed charts do not perform well when we have to monitor sparse data. But when we have to monitor many particles and we have enough observations in phase I process the performance of the generalized p chart, log likelihood ratio chart and also the multivariate Poisson chart and Hotelling chart is excellent. To monitor the data in practice we would recommend multinomial or multivariate control chart because the grouped data approach is sensitive to the assumption about the underlying particle distribution.

# Contents

## List of symbols

| Symbol | Description |
|---|---|
| $C_i$ | Right bound of interval $i$ |
| $C_{i-1}$ | Left bound of interval $i$ |
| $c_\sigma$ | Constant sigma units from the control limit for the Shewhart type control charts |
| $D_{it}$ | Particle size of measured particle $i$ in period $t$ |
| $Y_{it}$ | Total number of particles in interval $i$ in period $t$ |
| $\boldsymbol{Y_t}$ | Vector $(Y_{2t}, Y_{3t}..., Y_{kt})$ |
| $\overline{Y}_i$ | Mean of total number of particles in interval $i$ |
| $\overline{\boldsymbol{Y}}$ | Vector $(\overline{Y}_2, \overline{Y}_3, ..., \overline{Y}_k)$ |
| $k$ | Total number of intervals |
| $m_N$ | Mean total number of particles |
| $N_t$ | Total number of particles in period $t$ |
| $N_t^*$ | Total number of particles in period $t$ without the first interval |
| $r$ | Number of runs in one simulation study |
| $S$ | Sample covariance matrix |
| $t$ | Time period |
| $T$ | Total time periods phase II |
| $T_{ARL}$ | First time that the process is out of control |
| $T_I$ | Total time period phase I |
| $p_i$ | Probability that a particle is in interval $i$ |
| $\hat{\boldsymbol{p}}$ | Vector $(p_1, p_2, ..., p_k)$ |
| $p_i^*$ | Probability that a particle is in interval $i$ conditional on that the particle is greater than $C_1$ |
| $\boldsymbol{p^*}$ | Vector $(p_2^*, ..., p_k^*)$ |
| $v_N$ | Variance total number of particles |
| $\alpha$ | False alarm rate |
| $\delta_{ij}$ | Covariance between interval $i$ and $j$ |
| $\lambda$ | Parameter (mean) of Poisson distribution |
| $\mu$ | Location parameter of log- normal distribution |
| $\rho_{ij}$ | Correlation coefficient between $Y_i$ and $Y_j$ |
| $\sigma$ | Scale parameter of log- normal distribution |
| $\Sigma$ | Covariance matrix |

# 1 Introduction

Clean rooms are controlled places for specialized industrial production or scientific research. They are used in many industries such as biotechnology, defense industry, microelectronics, pharmaceuticals and nanotechnology. Each industry has its own control criteria for monitoring the quality of the rooms.
In this report we will focus on clean rooms used in the pharmaceutical industry. Their criteria are formulated in terms of maximum allowable numbers of airborne particles and microbial contamination. To keep track of the number of airborne particles real time environmental monitoring machines are used. To count the number of microbial contaminations, they use traditional growth-based methods which have a time to result of 5 to 6 days and report the contamination in colony-forming units.
The delay between time of an excursion event and time of detection can hinder root-cause analysis and cause waste of time and resources in batch production. For that reason, they have developed new monitoring systems. One of these new systems is the BioTrak Real-Time Viable Particle Counter (hereafter *BioTrak*), developed by TSI. The BioTrak is a real time environmental monitor machine which can count real time airborne particles and real time viable particles. A viable particle is a particle that represents a living micro organism. Beside the counting of particles, the machine also measures the size of each particle and classifies particles in fixed size intervals. The machine will reduce the time to get monitoring results of clean rooms and in doing so, lower the time to batch release. In addition, root cause analysis will be more accurate due to real-time excursion data being available.
New criteria are needed for monitoring clean rooms with the BioTrak because the monitoring machine does not report colony-forming units but reports real time the counts of viable particles. A viable particle may not be culturable, so the BioTrak is expected to produce higher counts than traditional microbiological methods. Control charts are useful tools to monitor the number of (viable) particles in real time. Many charts have been developed, also for multivariate count data, and we will study which of them may be useful for the BioTrak. First we explain the case study which was done by the pharmaceutical company MSD. Secondly, we will explain existing control charts that could be used to monitor the data of the BioTrak and we will do a simulation study to check performance of the control techniques to mimic realistic situations.

## 1.1 Case Study

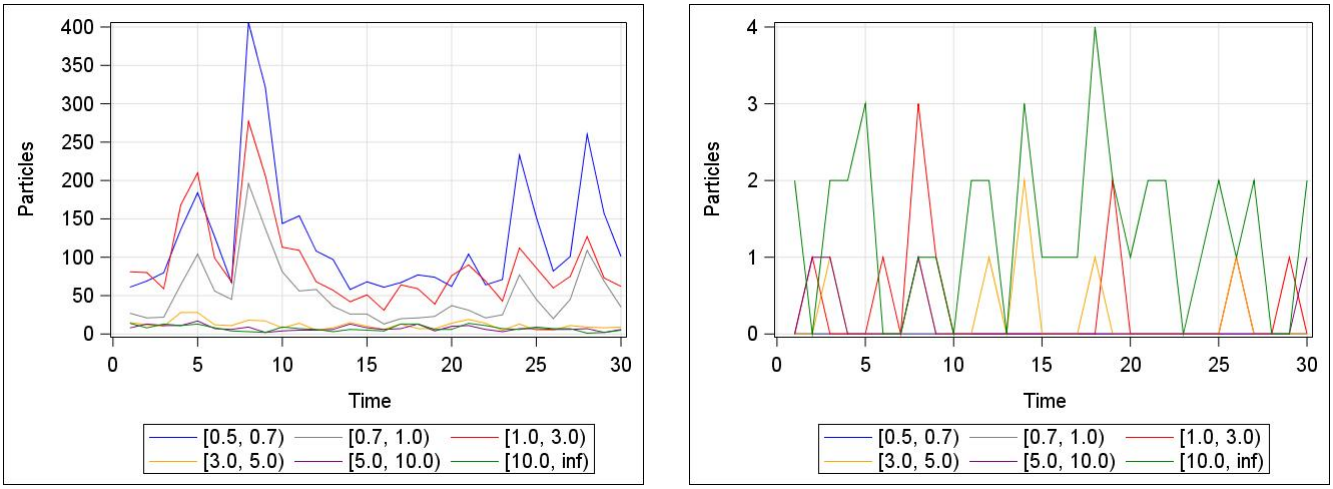
The BioTrak provides particle size, frequency, and auto-fluorescence information. The machine assesses viable particles in real time by exposing the particles to a laser and detecting fluorescence in a process known as laser-induced fluorescence. Viable particles are reported in counts, or auto fluorescing units (AFU). The BioTrak detects all particles with fluorescent characteristics consistent with a viable microorganism irrespective of its metabolic state or culturability. That is the reason that the BioTrak most likely will detect more viable particles in AFU compared to the number of viable particles in colony-forming unit (CFU) detected with traditional growth-based methods.
The BioTrak absorbs air from the clean room. It has a measured flow of 28.3 litre per minute for all particles. 55% of this absorb air, i.e. 15.6 litre per minute will go through a viable particle counter. The other part of absorb air will not go through the viable particles counter and will leave the machine through a HEPA filter. A HEPA filter is a particle absorber. The machine counts (viable) particles and groups the particles into discrete fixed size intervals (see Table 1). The BioTrak counts (viable) particles greater or equal to 0.5 $\mu m$, so very small particles are not counted. In addition to the real time counters, there is also a gelatin filter plate in the BioTrak. The air that comes from the viable particle counter passes this gelatin filter. The filter plate can be removed and incubated to determine the morphology of the colonies present on the filter plate. The gelatin filter could be used to verify the viable particles by a traditional method, but the result of this filter takes still 2 to 5 days [1], the results of the gelatin filter

are reported every three minutes in colony forming unit.

| BioTrak | | | | | |
|---|---|---|---|---|---|
| $[0.5, 0.7)$ | $[0.7, 1.0)$ | $[1.0, 3.0)$ | $[3.0, 5.0)$ | $[5.0, 10.0)$ | $[10.0, \infty)$ |

**Table 1:** Particle size intervals ($\mu m$).

For the proof of concept study, the BioTrak is tested in multiple clean rooms, while the room was at rest and while the room was in operation. At rest means that there were no activities in the room during the monitoring period. In operation means that there were normal work activities in the room. In pharmaceutical industry each room is classified with grade A, B, C or D. Grade A are clean rooms with the strongest requirements, there are almost no particles present, while in clean rooms with grade D much more particles are allowed.
For this report we will use data from a study in a transition airlock, Personnel Gowning/ Degowning clean room (C205) while the room was at rest. That means that there were no activities in the room during the study. This room is in general classified to grade D. The data is collected with the BioTrak. The BioTrak counts particles continuously and reports the results every minute, in total the study was done for 30 minutes. We will use the data as a starting point to set up criteria for the BioTrak with SPC techniques in Grade D clean rooms while the room is at rest. When we have the control limits we can check every minute if the clean room meets all criteria. Trend graphs of the number of particles measured with the BioTrak in room C205 are in Figure 1. It can be seen that the majority of particles were airborne while only limited number of viable particles were present.




**(a)** Airborne particles

**(b)** Viable particles

**Figure 1:** Trend graph per interval, Data from BioTrak in clean room C205 while the room was at rest.

## 1.2 Background

Statistical process control (SPC) techniques are methods of quality control based on statistical methods to monitor and control processes. It is widely used in manufacturing processes from industrial settings, health care, service industries and environments of non-manufacturing quality-improvement efforts. SPC techniques can reduce process and product variation and will therefore lead to a process of higher quality and lower costs. Statistical process control techniques control processes in real time, by checking

the rate at which the process produces incorrect products or is out of control.
Control charts are one of the most used techniques in statistical process control. It will tell if a process is in state of control by a graphical display. Based on rigorous probabilistic calculations it shows a quality characteristic that has been measured or computed from a sample at each time point. The chart contains a centre line which represents the average value, an upper control limit (UCL) and a lower control limit (LCL). These control limits are computed in a way that the false alarm rate $\alpha$ (one new observations fails outside the interval $[LCL, UCL]$) is fixed, in most cases $\alpha = 0.0027$. This means that the number of time points before a false alarm occurs is o average equal to $1/\alpha = 370.4$. A process is assumed in control as long as the sample points fall in between the control limits. A sample point outside of the control limits can be a sign that the process is disturbed, an investigation and corrective action are required to find and eliminate the cause for this occurrence.
There are two phases for the implementation of control charts. In Phase I a set of process data is gathered assuming that the process is in control. Afterwards, control limits are established for monitoring future production process. In Phase II, we use the calculated control limits to monitor the process by comparing the sample statistic to the control limits for each successive sample drawn from the process [2].
A univariate control chart can monitor a process with one quality characteristic. However, in many cases there are more quality characteristics. We could monitor the characteristics separately with univariate charts, but it is better to monitor all variables in one chart, this leads to a multivariate control chart. There are multiple reasons to monitor processes with a multivariate chart and not with multiple univariate charts. One reason is that monitoring with one chart is more practical (requiring less effort) and more economical than several univariate ones. In addition, with a multivariate chart we can take into consideration the relationship between the variables and provide more sensitive control than provided by univariate control charts. With multiple univariate charts the joint effect between variables will induce inflated false alarms and the probability of not detecting the assignable root cause will increase. So, applying univariate control charts to each variable is a possible solution to monitor processes but it is inefficient and can lead to erroneous conclusions [2] [3]. We want to set-up control charts for multivariate count data. Our multivariate count data is related on the multinomial distribution. The multinomial distribution gives the probability of any combination of numbers of successes for the various categories. The multinomial distribution is the generalization of the binomial distribution for more than two categories.

### 1.2.1 Multinomial methods

When a process has two possible outputs, for example the conforming and non-conforming and we can model the data by the binomial distribution, the data can be monitored with the Shewhart type p and np charts. Both charts are very similar, but the p chart focuses on the proportion of nonconforming units and the np chart focuses on the number of nonconforming units. When there are more than two categories multiple p charts can be used but we prefer a single chart that deals with the multiple categories.
One approach is to set up a chi-squared control chart that will serve as a summary statistic chart. This method is well laid out by Duncan (1950) [4] and discussed further by Marcucci (1985) [5] and Nelson (1987) [6]. The method deals with the frequencies of objects in each category and will monitor changes in the parameters of the multinomial distribution. The probabilities are estimated from a phase I study when the process is assumed to be in control or are assumed to be known. In the case that the probabilities are known and the process is in control the chart statistic is simply the Pearson's goodness of fit test statistics and this statistic has approximately a chi-squared distribution with the number of categories minus one degrees of freedom. The chi-squared control chart to the p chart in case of only two categories. We will call this method in the rest of this thesis *generalized p* chart. We will expand the generalized p chart with mathematical formulation in Section 2.

### 1.2.2 Grouped data methods

Recall that when if there exists an underlying continuous scale, but the observations are only classified into categories we refer to this setting as grouped data. In the case study of the BioTrak the data we satisfy this situation since the continuous variable is particle size. One way to monitor the groups is based on midpoint and endpoint approaches and the corresponding method treats the grouped observations as if they are non-grouped. This method will result in bias into for most calculations and requires many categories to be effective. To get more effective control charts, we must consider an underlying distribution for the grouped data. The following methods are based on this underlying distribution and their corresponding parameters.

Tippet (1944) [8] and Stevens (1948) [9] introduce control chart techniques for objects passing a gauge. In this case objects are classified as larger or smaller than the given standard based on whether it does or does not pass the gauge. The proposed methods have been developed only for three categories (smaller, standard or larger). They introduce two Shewhart type control charts for simultaneously monitoring the mean and standard deviation of a normal population. They assume that the distribution is approximately symmetrical, and the two gauges are approximately equidistant above and below the centre of the distribution. Stevens proposes monitoring the number of observations in the first and third group to detect shifts in the process standard deviation and monitoring the difference between the observations in the third group and the first group to detect shifts in the process mean. It is not straightforward to extend the methods to more than three categories. This chart does not fit our case study data where the number of particles in each category is not divided equally, even not when we merge intervals. We will not use this chart in the continuation of this thesis.

A method constructed for more than three categories is proposed by Steiner, Geyer and Wesolowsky (1994) [10] [11]. The appropriate model is multinomial with group probabilities being known functions of the underlying distribution with unknown parameters. The design and implementation methodology are derived assuming either a normal or Weibull underlying continuous scale, but are easily adapted to any other distribution. The methods monitor if location parameter of the underlying distribution is shifted to a specified value. The most uniformly powerful test is based on the likelihood ratio of the multinomial probabilities. They have proposed methods for one-sided and two-sided acceptance sampling plans, acceptance control charts and Shewhart type control charts. Using the proposed method the practitioner will be able to determine the required sample size, control limits for any specific application. We will adapt this chart to our BioTrak data and will explain the mathematical formulation in Section 2.

Tucker, Woodall and Tsui (2002) [12] extend the methods from Stevens [9]. The control chart is based on maximum likelihood estimation (MLE) of the parameters of the underlying distribution of sample category counts. They assume that the order of the observed objects per category can indicate when the distribution changes and they assume that the location parameter increases the probability associated with poorest level of quality. The monitoring statistic is the MLE chart statistic, they calculate the location parameter with MLE and an approximation of the variance of this parameter with the Cramer-Rao lower bound. Tucker shows with a simulation study that the underlying distribution affects the performance (robustness, power and sensitivity) of the MLE statistic. The same authors also indicated the possible use of the multinomial generalized p chart described in Section 1.2.1. Instead, the probabilities of each category are now being estimated. We will not work out the proposed chart of Tucker et al on our case study data because we are not interesting in the effect of the underlying distribution.

### 1.2.3 Multivariate-attribute methods

In industry, most multivariate control charts are for monitoring attribute data. Processes collect multivariate attribute data when several quality characteristics are simultaneously monitored. Multivariate attribute control charts are purely made for multiple binary data: good or bad, yes or no. An example with more variables is a box that can be regarded as non-conforming if its width, length or weight does not meet the specifications. Another example is monitoring various components or performances of an assembly by attributes. Monitoring univariate attribute processes can be done with the np chart in the case of binomial data, or can be done when the data is Poisson distributed with the c or u chart. The multivariate attribute control charts are not always based on the multinomial distribution because an object can have multiple nonconforming quality characteristics. Nevertheless, we will consider only multivariate attribute charts which are constructed for count data. in case of the particle data each interval is considered an attribute and we count the particles per interval. The following methods have been discussed for observations with a discrete distribution like the binomial or Poisson distribution.

Patel (1973) [13] introduced a multivariate attribute chart when observations are from a multivariate binomial or Poisson distribution. In that period most control charts were developed for monitoring multivariate normal data. He suggested to change one of these methods, the Hoteling $T^2$ chart which is a Shewhart type control chart. He uses the property that the distribution of the correlated attributes will approximate the normal distribution when the sample size is sufficiently large. The method is based on a phase I process and an assumption is that we have an homogeneous process variance. A disadvantage of the Hotelling $T^2$ is that it can not handle skewness of the distributions of the attributes and the difficulty to determine which variable has shifted. Another disadvantage is that the $T^2$ chart has also only an upper limit, which cannot be used to detect both process deterioration and process improvement. The method is therefore not widely used in practice, nevertheless we will elaborate this map further in Section 2.

Lu, Xie, Goh and Lai (1998) [14] introduced the multivariate np (mnp) chart. Unlike the Hotelling's $T^2$ chart, the mnp chart is a extension of the univariate np chart. It is used for monitoring the counts of nonconforming units of each quality characteristics. The chart monitors the weighted sum of the counts of nonconforming units of each quality characteristic in a sample. The weights depend on the probability that a particle is in an interval. The probabilities could be estimated based on the proportions or with the underlying distribution. The control limits are constructed with three sigma upper and lower limits. It has been found that the mnp is easy to implement and interpret. It is shown by Lu et al that the chart is more sensitive than the univariate np chart in detecting process shifts. The chart is easy to adapt on our case study data, we will give the mathematical formulation in Section 2.

Jing-Er Chiu and Tsen-I Kuo (2007) [15] constructed a chart for multivariate Poisson count data. We will call this chart the mp chart. The chart is developed to deal with multiple correlated count data. One example of multivariate Poisson data in industry could be the number of defects classified into more than two categories. The paper explains two ways of constructing a chart. The first method assumes that the covariance between the counts in all categories is equal and construct control limits using the total sum of the counts. The distribution of the total sum of multivariate Poisson counts with more then three variables and unequal correlations is less tractable and calculating the corresponding control limit is therefore not trivial. Another disadvantage of this chart is that the constructed control limits are discrete, so the false alarm rate will not be equal to $\alpha$ exactly. The second method of Jing-Er Chiu and Tsen-I Kuo (2007) uses the normal approximation of the Poisson distribution for large samples. Where in the Hotelling $T^2$ chart all categories needs to approximate the normal distribution, in this chart approximates the monitoring statistic, the sum over all intervals, the normal distribution. Chiu et al note

that in case of a sum larger than five the approximation is good. In this chart it is easier to incorporate unequal correlations between variables. In this case the paper shows a Shewart type chart, the control limit is continuous and the monitoring statistic is discrete. To get the right false alarm rate, the number of objects has to be large. The chart monitoring statistic is the sum of all intervals. However, it cannot be used when the total number of objects is fixed, because the control limits are constructed based on the variance of the sum which always will be zero. We will work out the Shewart type mp chart in Section 2.

It is difficult to construct control charts for multivariate processes where some of the variables are often zero and with different correlations between attributes. The joint distribution of the multivariate process is usually difficult to derive directly based on the marginal distributions of each variable and the correlations. Therefore, it has been proposed to use copula functions for the joint distribution instead. Fatahi et al (2012) [16] propose a copula based bivariate zero inflated Poisson (zip) control chart for monitoring rare events. The paper compared the chart with the simultaneous use of two univariate charts and demonstrated that the zip chart has a better performance in terms of average run lengths. They also report a motivation case study where they monitor a count process of particles in a clean room within which two particle count machines simultaneously counts the number of two different particles. The data is similar to our data but we have grouped data. Due to the particle size the monitoring statistic and limit are discrete, and the number of particles may be low. That means it is difficult to construct a control chart with a fixed false alarm rate of $\alpha$ exactly. We will merge intervals of our case study to two intervals and fit the chart on the data in Section 2.

## 2 Methods

The output of the BioTrak is count data (number of particles) spread out in particle size intervals. The BioTrak reports the categorical data periodic, in our case study every minute. We assume that observations from different time points are independent. We also assume that the probability that a particle belongs to a category is fixed and unknown. We also assume that the inflow of particles to the categories, irrespective of whether the particle size exceeds the Biotrak detection limit, occurs random. With these assumptions the counts in intervals conditionally on the total number of particles is assumed to follow a multinomial distribution. The probabilities of the multinomial distribution are then formed by the underlying particle size distribution. A way to monitor the clean rooms is to monitor the parameters of the particle size distribution i.e. using control chart that are designed for our grouped data, but finding the particle size distribution (PSD) is not always possible in clean rooms. It is difficult to determine, because there are many types of particles present in the air. Each particle type behaves differently and may have different particle size distributions, Figure 2 gives an overview of different particles per size [17]. We will apply two grouped methods, the generalized p chart where we assume the underlying particle size distribution is log-normal and use the maximum likelihood chart to monitor the processes. The log normal distribution has been suggested for describing different kinds of particle sizes. In 1879 Galton and McAlister argue based on mathematical arguments that randomness of sizes in nature are log-normal. Their arguments were based on the fact that the log-normal distribution has only positive values and many observations are best described by their geometric rather than arithmetic means [18]. In 1950 Kottler discussed also the usefulness of the log-normal distribution for particle sizes. His arguments were based on the natural growth of particles and the breakage of particles into smaller particles. with each breakage step, more and more particles are produced, resulting in a skewed distribution, so that there are many more smaller particles than larger ones [19].
There are many papers and books which describe models for fitting the log-normal distribution on group data (especially for particles), as in the paper of Raabe [20] and Chapter 4 of the book "Aerosol technology" of Hinds [21]. Hinds mentioned in his book that when the data do not follow a log-normal distribution, it is often better not to assume any distribution function and to use the frequency of particles in the size ranges to describe the distribution, so-called count distribution. A new aersosol book that is more recent is the book "Aerosol measurement" edited by Kulkaini, Baron and Willeke (2011). It also mentions that the particle size distribution is typically log-normal [22]. Because we have different types of particles in the control room we expect that the underlying distribution is based on multiple log normal particle size distributions. Because the sample size (total number of particles) is not fixed we will also use methods which are close to the multinomial methods but which need not a fixed sample size, namely the multivariate methods or multivariate attribute methods. The methods rely on multivariate discrete distributions distributions. In this section we will explain the Hotelling $T^2$ chart, multivariate np chart, multivariate Poisson chart and the Bivariate zero inflated Poisson chart which are all multivariate-attribute charts.
All charts have to take into account that the count data is truncated in the case study. We have seven ($k$) categories but we do not have the information of the first interval. The BioTrak can not detect particles below 0.5 $\mu m$. The control charts have to monitor the process based on the other six ($k-1$) categories. For the multivariate charts this is not relevant but for the grouped data charts the methods need to address this truncation.

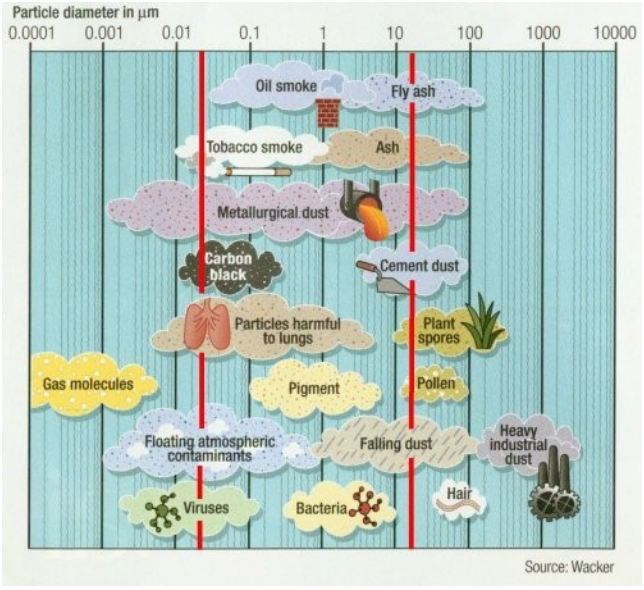



**Figure 2:** Possible particles in cleanrooms per diameter in $\mu m$ [23].

## 2.1 Formulation

In this section, we introduce some notation that will be used throughout the thesis. Let $C_0, C_1, ..., C_k$ denote sizes with $0 = C_0 < C_1 < \cdots < C_k \leq \infty$. The $k$ particle size classes are defined by the intervals $[C_{i-1}, C_i)$ for $i = 1, ..., k$. We count particles at different periods of time. We indicate each period with $t = 1, 2, ..., T$ , where $T$ is the last period. We will count in each period $N_t$ particles. We define a particle size $D_{jt}$ of particle $j$ in period $t$ where $j = 1, 2..., N_t$ and we put each particle based on his size in one of the intervals. We denote the total number of particles presented in interval $i$ with $Y_{it}$ in period $t$ where $i = 1, 2, ..., k$. Each period, the resulting data of the particle counter will be the vector $\boldsymbol{Y_t} = (Y_{1t}, Y_{2t}, ..., Y_{kt})$ which is a vector of all counted particles per interval.

$$Y_{it} = \sum_{j=1}^{N_t} \mathbb{1}_{(D_{jt} \in [C_{i-1}, C_i))}.$$

When $D_{1t}, D_{2t}, ..., D_{nt}$ are i.i.d, $Y_{1t}, Y_{2t}, ..., Y_{kt}$ the vector $\boldsymbol{Y_t}$ follows a multinomial distribution with parameters $N_t$ and $p_i$, where $p_i$ is the probability that an particle is in size interval $i$. Given an underlying particle size density $f_D$, the probability $p_i$ can be calculated based on its cumulative distribution function $F_D$:

$$p_i = \int_{C_{i-1}}^{C_i} f_D(u)du = F_D(C_i) - F_D(C_{i-1}). \tag{1}$$

In case that we do not know the cumulative distribution function we can use the method moment estimation (MME) or the maximum likelihood estimation (MLE) to estimate $p_i$. Both estimation methods MME and MLE estimate the probability with the proportion. Note that the sum over all probabilities is equal to one, $\sum_{i=1}^{k} p_i = 1$.

We define the vector of all probabilities with $\boldsymbol{p} = (p_1, p_2, ..., p_k)$. The corresponding statistics for the

vector $\boldsymbol{Y_t}$ using the multinomial distribution can be calculated as follows.

$$\begin{aligned}&\text{First moment: } \mathbb{E}[Y_{it}] = N_t \cdot p_i.\\&\text{Variance: } Var[Y_{it}] = N_t \cdot p_i \cdot (1-p_i).\\&\text{Covariance: } Cov[Y_{it}, Y_{jt}] = -N_t p_i p_j \quad (\text{for } i \neq j).\end{aligned}$$

Namely, The mean vector ($\overline{\boldsymbol{Y_t}}$) and covariance matrix ($\Sigma$) are

$$\overline{\boldsymbol{Y_t}} = (N_t p_1, N_t p_2, ..., N_t p_k) \tag{2}$$

and

$$\Sigma = \begin{pmatrix} N_t(1-p_1)p_1 & -N_t p_1 p_2 & \cdots & -N_t p_1 p_k \\ -N_t p_2 p_1 & N_t(1-p_2)p_2 & \cdots & -N_t p_2 p_k \\ \vdots & \vdots & \ddots & \vdots \\ -N_t p_k p_1 & -N_t p_k p_2 & \cdots & N_t(1-p_k)p_k \end{pmatrix}. \tag{3}$$

Note that the covariance matrix of the multinomial distribution is always singular. In the case study the intervals do not cover the entire size range, we do not count particles in interval $[C_0, C_1)$. We will define $N_t^*$ which is the total number of particles counted in period $t$, it is the sum over all intervals without the first interval, where

$$N_t^* = N_t - Y_{1t} = \sum_{i=2}^{k} Y_{it}.$$

Similarly, we define the vector $\boldsymbol{Y_t} = (Y_{2t}, Y_{3t}, ..., Y_{kt})$. and $p_i* = p_i/(1-p_1)$ the conditional probability that a counted particle belongs to interval $i$ given that it is greater than $C_1$.

$$\begin{aligned} p_i^* =& \mathbb{P}(X_j \in [C_{i-1}, C_i) \mid X_j > C_1) \\ =& \frac{\mathbb{P}(X_j \in [C_{i-1}, C_i), X_j > C_1)}{\mathbb{P}(X_j > C_1)} \\ =& \frac{\mathbb{P}(X_j \in [C_{i-1}, C_i), X_j > C_1)}{1 - \mathbb{P}(X_j \leq C_1)} \\ =& \begin{cases} \frac{p_i}{1-p_1} & \text{if } i > 0 \\ 0 & \text{if } i = 1 \end{cases} \\ =& \begin{cases} \frac{F_D(C_i) - F_D(C_{i-1})}{1 - F_D(C_1)} & \text{if } i > 0 \\ 0 & \text{if } i = 1 \end{cases} \end{aligned} \tag{4}$$

where, $p_1 = F_D(C_1) - F_D(C_0) = F_D(C_1)$

The distribution of the vector $\boldsymbol{Y_t}$ is not multinomial distributed, but the distribution of the vector conditional on the number of particles in the first interval is multinomial with parameters $N_t^*$ and

$p_2^*, p_3^*, ..., p_k^*$.

$$
\begin{aligned}
&\mathbb{P}(Y_2 = y_2, Y_3 = y_3, ..., Y_k = y_k | Y_1 = y_1) \\
&= \frac{\mathbb{P}(Y_1 = y_1, Y_2 = y_2, ..., Y_k = y_k)}{\mathbb{P}(Y_1 = y_1)} \\
&= \frac{N!}{y_1! \cdot y_2! \cdot ... \cdot y_k!} \cdot p_1^{y_1} \cdot p_2^{y_2} \cdot ... \cdot p_k^{y_k} / \frac{N!}{y_1!(N-y_1)!} p_1^{y_1} (1-p_1)^{N-y_1} \\
&= \frac{(N-y_1)!}{y_2! \cdot y_3! \cdot ... \cdot y_k!} \frac{p_2^{y_2} \cdot ... \cdot p_k^{y_k}}{(1-p_1)^{N-y_1}} \\
&= \frac{N^*!}{y_2! \cdot y_3! \cdot ... \cdot y_k!} \frac{p_2^{y_2} \cdot ... \cdot p_k^{y_k}}{(1-p_1)^{y_2} \cdot (1-p_1)^{y_3} \cdot ... \cdot (1-p_1)^{y_k}} \\
&= \frac{N^*!}{y_2! \cdot y_3! \cdot ... \cdot y_k!} (\frac{p_2}{1-p_1})^{y_2} \cdot (\frac{p_3}{1-p_1})^{y_3} \cdot ... \cdot (\frac{p_k}{1-p_1})^{y_k} \\
&= \frac{N^*!}{y_2! \cdot y_3! \cdot ... \cdot y_k!} (p_2^*)^{y_2} \cdot (p_3^*)^{y_3} \cdot ... \cdot (p_k^*)^{y_k}.
\end{aligned}
$$

When we have a underlying particle size distribution, where $F_D$ is cumulative distribution function we can estimate the parameters with maximum likelihood or with moment estimation with the data from phase I. When we have the parameters we can calculate the probabilities $p_i^*$ with Equation (4). We first have to estimate the parameters of the underlying particle size distribution. We assume that the we have a log normal particle size distribution, so we have to estimate $\mu$ and $\sigma$. We will estimate the parameters with maximum likelihood estimation (MLE). The likelihood function is:

$$L(\mu, \sigma | \boldsymbol{Y}) = c \prod_{j=2}^{k} (p_j^*)^{Y_j}. \tag{5}$$

where $c$ is a constant which we can excluded. $p_j^*$ is a function depending on $\mu$ and $\sigma$, the formula is in Equation (4).
When we do not know the underlying distribution we would estimate the probability that a particle is in interval $i$ given that the particle is greater than $C_1$ with the moment estimator:

$$\hat{p}_i^* = \frac{\overline{Y_i}}{\sum_{i=2}^{k} \overline{Y_i}}. \tag{6}$$

## 2.2 Control charts

As we just indicated in our previous section, the counts observed with the BioTrak would not follow a multinomial distribution, thus it is relevant to investigate which control chart would be best for the case study. Here we list the charts we will study in detail.

### 2.2.1 Hotelling $T^2$ chart (multivariate-attribute chart)

Patel [13] proposed a monitoring chart for the multivariate binomial and Poisson distribution based on an extension of the Hotelling $T^2$ chart. We consider $N_t^*$ particles, where each particle belongs to one of the size intervals excluding the first interval. In this model a particle can have multiple attributes, but of course in our case a particle will only have one attribute. We indicate a particle measured with size

$D_{jt}$ and $p_i$ is the probability that a particle has attribute $i$. The number of particles with attribute $i$ is $Y_i$ follows a Binomial distribution with parameters $N_t$ and $p_i$ for $i = 1, ..., k$.

We assume that $N_t$ and $p_i$ is consistent over time, and also the periods are independent. In phase I of statistical process control, we estimate the mean vector by the sample mean vector ($\overline{\mathbf{Y}} = (\overline{Y}_2, ..., \overline{Y}_k)^\top$) with

$$\overline{Y}_i = \frac{1}{T_I} \sum_{t=1}^{T_I} Y_{it}$$

and the covariance matrix by the sample covariance matrix $S$ given by

$$\boldsymbol{S} = \frac{1}{T_I - 1} \sum_{i=1}^{T_I} (\boldsymbol{Y_t} - \overline{\mathbf{Y}})(\boldsymbol{Y_t} - \overline{\mathbf{Y}})^\top, \tag{7}$$

with $T_I$ the used number of time points used in phase I, $Y_{it}$ the count in interval $i$ at time $t$, and $Y_t = (Y_{1t}, Y_{2t}, ..., Y_{kt})^\top$ in phase I. The means $\overline{Y_t}$ and the covariance matrix $\mathbf{S}$ will be used for future observations to detect out-of-control signals.

In phase II we have on each time point the vector $\boldsymbol{Y_{Ot}} = (Y_{O1t}, Y_{O2t}, ..., Y_{Okt})$, this is a random vector from the counted particles per interval, the monitoring statistic is defined by

$$X_t = (\boldsymbol{Y_{Ot}} - \overline{\boldsymbol{Y}})^\top \boldsymbol{S}^{-1} (\boldsymbol{Y_{Ot}} - \overline{\boldsymbol{Y}}). \tag{8}$$

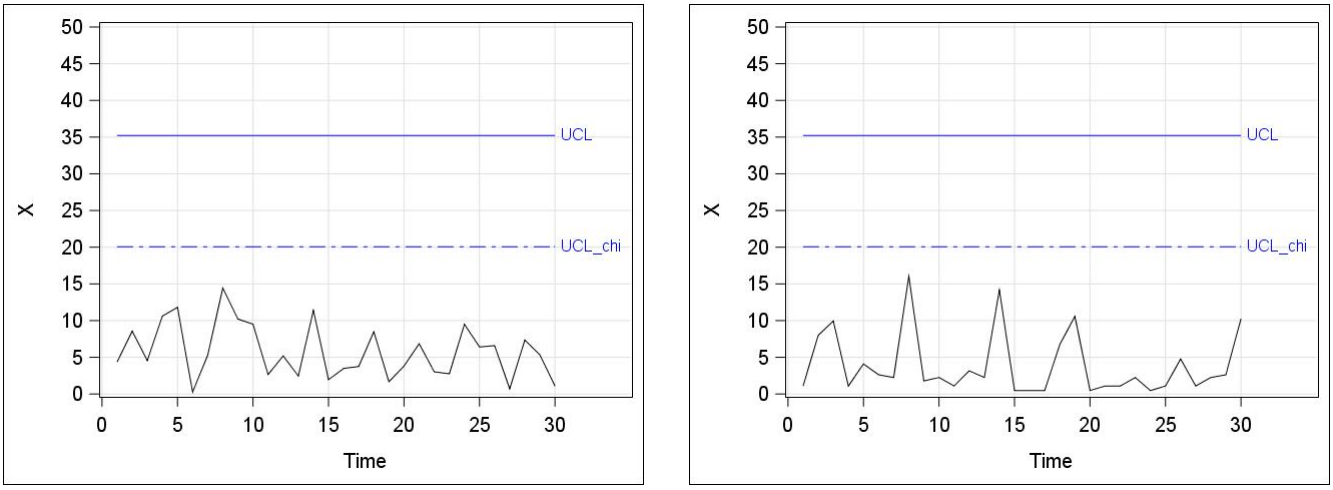
The monitoring statistic only exists when the inverse of the covariance matrix exists. For multinomial data we know that the covariance matrix is always singular and Hotelling's $T^2$ chart can not be used. Research have suggested to remove one interval or attribute [24], which is exactly what we have in our case study. We do not have information of the first interval, so our covariance matrix will be non-singular when the data follow a multinomial distribution.

The monitoring statistic $X_t$ has an approximate $\chi^2$ distribution with $k-1$ degrees of freedom asymptotically [24]. In theory we can use the $\chi^2$ distribution to construct a control limit with $k-1$ degrees of freedom and significance level $\alpha$. For a limited number of observations in phase I, we calculate the control limit based on the F distribution [2][25].

$$UCL = \frac{(k-1)(T_I+1)(T_I-1)}{(T_I-(k-1))T_I} \cdot F(\alpha, (k-1), T_I-(k-1)) \tag{9}$$

where $F(\alpha, k, T_I-(k-1))$ is the $(1-\alpha)^{th}$ quantile of an $F(k, T_I-(k-1))$ distribution . The upper control limit in Equation (9) will convergence to $\chi^2((k-1), \alpha)$ when $T_I$ grows to infinity.

An example of a phase I control chart resulted from applying the Hotelling's $T^2$ method to the dataset of the case study is depicted in Figure 3b when the room was at rest. The upper limit (UCL) is the limit calculated with Equation (9) and the Upper limit ($UCL_X$) is the $\chi^2$ statistics with $k-1$ degrees of freedom and false alarm rate $\alpha = 0.0027$.




**(a)** Airborne particles

**(b)** Viable particles

**Figure 3:** Hotelling's $T^2$ chart applied on multivariate count data generated by the BioTrak in a grade D control room.

### 2.2.2 Multivariate np chart (multivariate-attribute/ grouped method)

The multivariate np chart (mnp chart) could be used for monitoring multivariate attribute processes. It is described by [14] to monitor attribute data. We want to monitor $k$ quality characteristics, we denote the probability that an particle has quality characteristic $i$ is $p_i$. It is possible that an particle has multiple attributes. When $p_i$ is unknown we have to estimate it in phase I of statistical process control. We can estimate $p_i$ with the underlying distribution or when we do not have a underlying distribution with the moment estimator (see Section 2.1).

We will use the estimated $p_i^*$ from phase I in the monitoring statistic. In phase I and phase II there are $N_t^*$ particles counted. The total number of counts for attribute $i$ defined is by $Y_{it}$, is binomial distributed with parameters $N_t$ and $p_i$. In this attribute model the quality characteristics do not have to be independent, to take this into account the method will use the correlation between characteristics. The monitoring statistic is the weighted sum of the nonconforming units of all the quality attributes in a sample. The statistic of interest is.

$$X_t = \sum_{j=2}^{k} \frac{Y_{Ojt}}{\sqrt{\hat{p}_j^*}}, \tag{10}$$

with $\hat{p}^*$ given in Equation (6) or using an alternative method of estimation. When one of the estimated of $\hat{p}_i^*$ is zero we ignore the interval in the calculation.

Now we can obtain control limits based on the Shewhart control chart principles. The control limits are constructed to be symmetrically placed at a distance of $c_\sigma$-sigma units from the CL, where in general $c_\sigma = 3$. The CL is the mean of the monitoring statistic. The control limits are

$$\begin{aligned} UCL &= \mathbb{E}[X_t] + c_\sigma \cdot \sqrt{Var[X_t]} \\ CL &= \mathbb{E}[X_t] \\ LCL &= \mathbb{E}[X_t] - c_\sigma \cdot \sqrt{Var[X_t]} \end{aligned}$$

where we estimate $\mathbb{E}[X_t]$ with the sample mean

$$\overline{X}_t = \frac{1}{T_I}\sum_{i=1}^{T_I} X_t \tag{11}$$

and $Var[X_t]$ with the sample variance

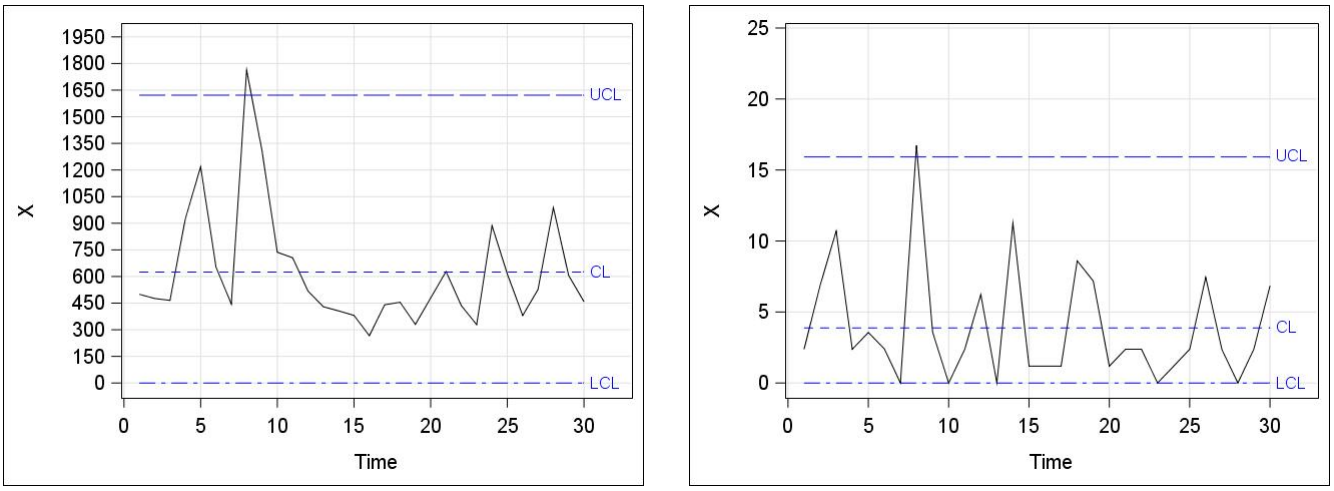
$$v_{X_t} = \frac{1}{T_I - 1}\sum_{i=1}^{T_I} (\overline{X}_t - X_t)^2 \tag{12}$$

with data collected in a phase I study for statistical process control. In phase II when we obtain new data, we calculate $X_t$ in Equation (10) using $Y_{Ot}$ at the observed time points, but we keep the estimates $\hat{p}^*$ from phase I in the calculation (In line with the method used in [14].

Lu et al argue for the use of the weight being the square root of the proportions. First of all the monitoring statistic is a straightforward extension of that of the univariate np chart. Secondly, the expected value of $\mathbb{E}(Y_{jt}/\sqrt{p_j} = N_t\sqrt{p_j})$ (or in our case $\mathbb{E}(Y_{jt}/\sqrt{\hat{p}_j^*} = N_t^*\sqrt{\hat{p}_j^*})$) is also an increasing function of $\hat{p}_i^*$. Another reason is that the choice makes the derivation of the control limits and sample size requirement easy and present a clearly defined mnp chart. An extension to this monitoring statistic could be to classify each characteristic to each severity by including a demerit system. This is useful in the case of attribute data, but in our case, we assume that each interval has the same severity. Lu et al show a numerical example where the mnp chart is more sensitive than the univariate np chart in detecting process shifts of a multivariate attribute process.

To estimate the probabilities $\hat{p}_i^*$ the sample size in phase I has to be large to prevent intervals being discarded. To ensure that we always count a particle in each interval in phase I we could use the cumulative Poisson table to be sure that we count in all intervals a particle with a specified probability (for example 0.95), the sample size condition is therefore derived as $N_t^* \cdot \sum_i \hat{p}_i^* \geq 3(k-1)$ [14].

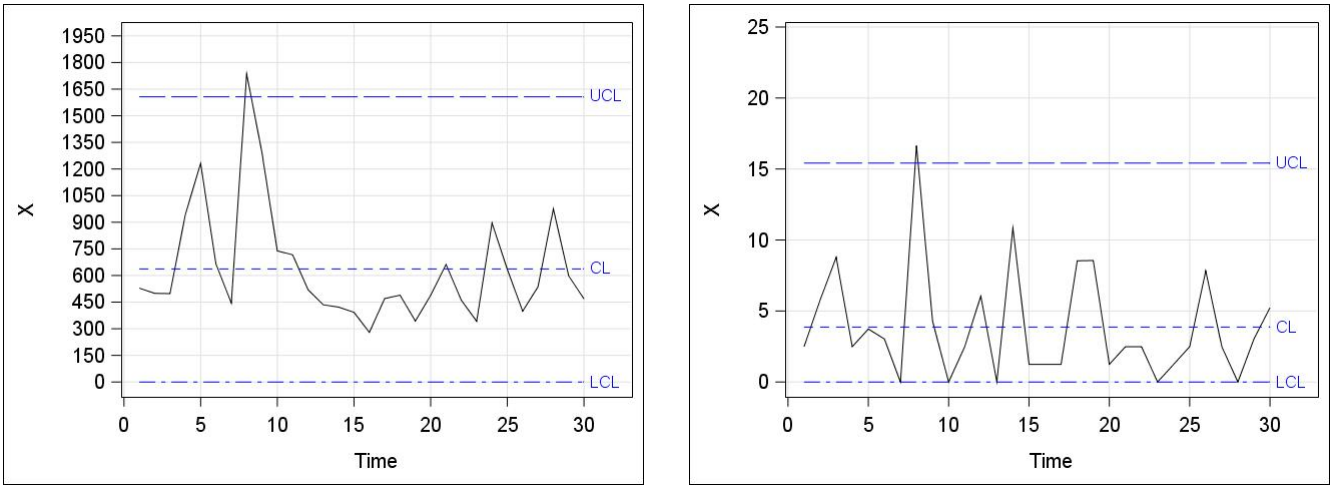
An example of the mnp chart, using data from the case study is depicted in Figures 4 and 5 when the room was at rest. In Figure 4 we estimate the probabilities with the moment estimation and in Figure 5 we estimate the probabilities with the underlying lognormal distribution (5).




**(a)** Airborne particles

**(b)** Viable particles

**Figure 4:** Multivariate np chart applied on multivariate count data generated by the BioTrak in a grade D control room.




**(a)** Airborne particles

**(b)** Viable particles

**Figure 5:** Multivariate np chart applied on multivariate count data generated by the BioTrak in a grade D control room.

### 2.2.3 Generalized p chart (multinomial/grouped method)

The generalized p chart is a chi-squared control chart and recommended for monitoring multinomial data [4][5] [6]. It is a summary control chart and should detect any change in the proportions of the intervals.

The probability vector $\hat{\boldsymbol{p}}$ is estimated in a phase I study when the process is assumed to be in control. We can estimate $\hat{\boldsymbol{p}}$ parametrically (assuming an underlying particle size distribution) or non-parametrically (similar to Equation (6)). We will monitor the vector $\mathbf{Y}_{\mathbf{O}}^{*} = (Y_{O2}, Y_{O3}, ..., Y_{Ok})^{\top}$ in phase II with Pearson goodness of fit statistic.

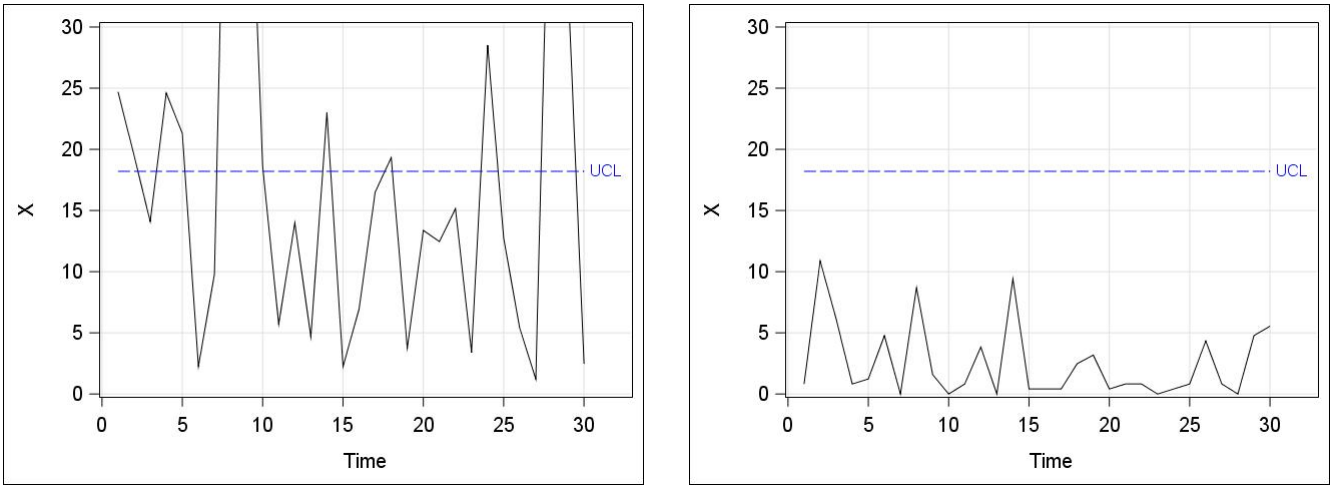
$$X_t = \sum_{j=2}^{k} \frac{(Y_{Otj} - N_{Ot}^{*}\hat{p}_j^{*})^2}{N_{Ot}^{*}\hat{p}_j^{*}}, \tag{13}$$

with $N_{Ot}^{*} = \sum_{j=2}^{k} Y_{Otj}$ and $p_j^{*}$ the estimate for interval $j$ from phase I.

The expectation of the total number of counted particles is $\mathbb{E}(N_t^{*}) = N_t \cdot (1 - p_1)$. Thus $N_t^{*}\hat{p}_j^{*} \approx N_t\hat{p}_j$. The statistic has an upper control limit that is an appropriate percentile of the chi-squared distribution with $k-2$ degrees of freedom ($\chi^2(k-2)$). In the special case of monitoring two categories we get the standardized p chart. We assume that the statistic has an approximate $\chi^2$ distribution. This is only the case when certain conditions are met. Unfortunately only rules of thumb have been proposed in literature.

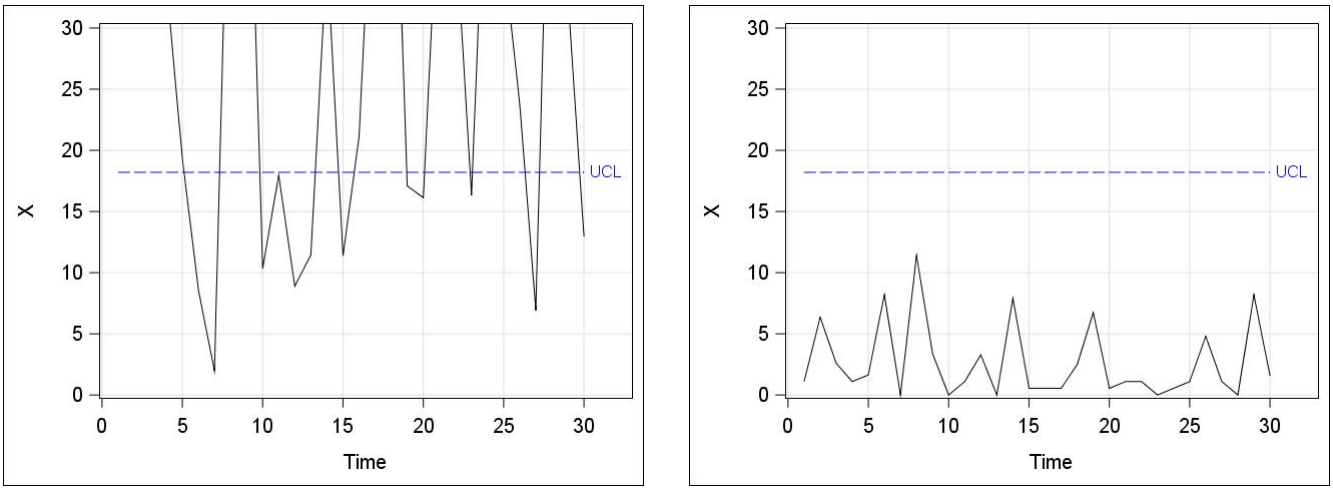
- Cochran (1954) [26], No more than twenty percent of the expected frequencies must be less than five, and none of the expected frequencies should be less than one.
- Yarnold (1970) [27], Let $k-1$ be the number of categories, $r$ be the number of expected frequencies less than five. for $(k-1) \geq 3$, the minimum expectation should at least $5r/(k-1)$.

An example of the generalized p chart to the dataset of the case study is depicted in Figures 6 and 7 when the room was at rest. In Figure 6 we estimate the probabilities with the moment estimation and in Figure 7 we estimate the probabilities with the underlying lognormal distribution (5).




**(a)** Airborne particles **(b)** Viable particles

**Figure 6:** Generalized p chart applied on multivariate count data generated by the BioTrak in a grade D control room.




**(a)** Airborne particles **(b)** Viable particles

**Figure 7:** Generalized p chart applied on multivariate count data generated by the BioTrak in a grade D control room.

### 2.2.4 Multivariate Poisson chart (multivariate-attribute method)

The multivariate Poisson chart is a control chart for multivariate count data [15]. We assume that the vector $\boldsymbol{Y_t} = (Y_{1t}, Y_{2t}, Y_{3t}, ..., Y_{kt})$ follows a jointly k-variate Poisson distribution. $Y_{it}$ follows a Poisson distribution marginally with mean $(\lambda_i)$ and the covariance between all the pairs of random variables $(Y_{it}, Y_{jt})$ is $\delta_0$, $i \neq j$. In our case we know that the covariance between intervals is different for each pair and also negative when $N_t$ is fixed, this will influence the chart.

We monitor again the vector $\mathbf{Y_0 t} = (Y_{2t}, Y_{3t}, ..., Y_{kt})$. The monitoring statistic $X_t$ is the sum of all

$Y_{it}$.

$$X_t = \sum_{i=2}^{k} Y_{Oit}. \tag{14}$$

The control limits are based on the upper and lower $\alpha/2$ percentage point of the exact distribution. For the upper and lower bound we take the largest value of UCL and LCL which satisfy the following properties,

$$\mathbb{P}(X_t > UCL) =$$
$$1 - \sum_{d=0}^{\text{UCL}-1} \exp\left\{-\left(\sum_{j=2}^{k}\lambda_j - (k-2)\delta_0\right)\right\} \sum_{i=0}^{d/k} \frac{(\sum_{j=2}^{k}\lambda_j - (k-1)\delta_0)^{d-(k-1)i}\delta_0^i}{(d-(k-1)i)!i!} \leq \frac{\alpha}{2}$$

and

$$\mathbb{P}(X_t < LCL) =$$
$$\sum_{d=0}^{\text{LCL}} \exp\left\{-\left(\sum_{j=2}^{k}\lambda_j - (k-2)\delta_0\right)\right\} \sum_{i=0}^{d/k} \frac{(\sum_{j=2}^{k}\lambda_j - (k-1)\delta_0)^{d-(k-1)i}\delta_0^i}{(d-(k-1)i)!i!} \leq \frac{\alpha}{2}.$$

In most cases when the mean of the Poisson process is small we take $LCL = 0$, otherwise $\mathbb{P}(X = 0)$ is greater than $\alpha/2$ . In this case we only set up a UCL where the false alarm rate is $\alpha$ (and use $\alpha$ in the formula above instead of $\alpha/2$.

Because $X_t$ is discrete, a unit difference between the UCL will result in a large difference in corresponding probability. In the paper they select the UCL where the corresponding probability is closest to $\alpha$, even if the probability is above $\alpha$.

When the monitoring statistic $X_t$ is greater than five, we may approximate the distribution with a normal one. This means that we can also use Shewhart type control limits obtained from the mean and variance of $X_t$,

$$\mathbb{E}[X_t] = \mathbb{E}[\sum_{j=2}^{k} Y_{jt}] = \sum_{j=2}^{k}\lambda_j \tag{15}$$

and

$$Var(X_t) = Var(\sum_{j=2}^{k} Y_{jt}) = \sum_{j=2}^{k} Var(Y_{jt}) + 2\sum_{j<m}^{k} Cov(Y_{jt}, Y_{mt})$$
$$= \sum_{j=2}^{k}\lambda_j + 2\sum_{j<m}^{k}\rho_{jm}\sqrt{\lambda_j\lambda_m} \tag{16}$$

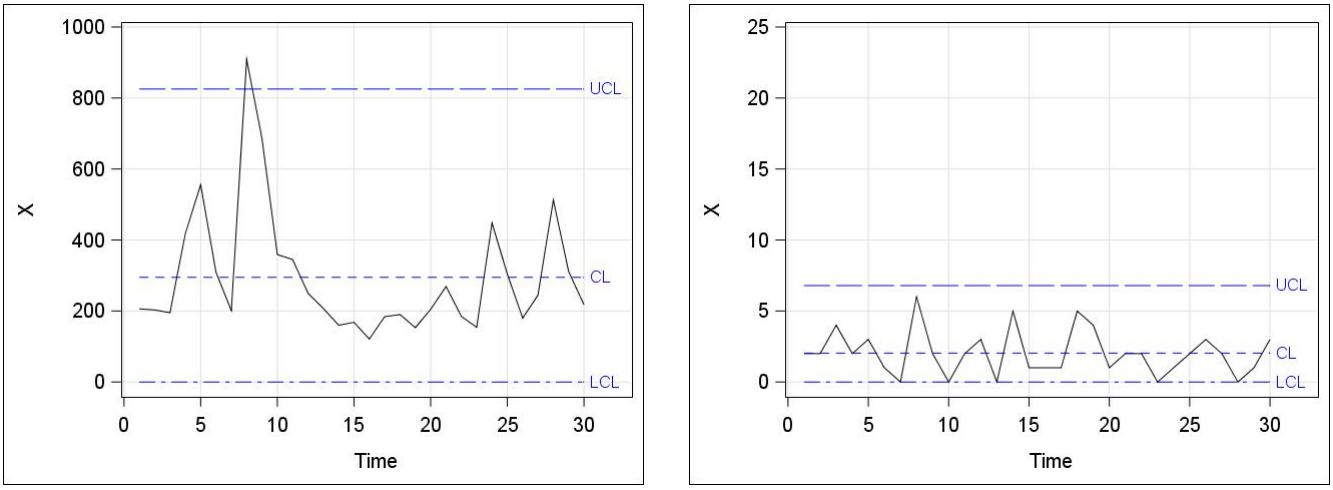
where $\lambda_j$ is the mean of interval $j$ and $\rho_{jm}$ the correlation coefficient between $Y_{jt}$ and $Y_{mt}$.

The Shewhart type control limits with in general $c_\sigma = 3$ are,

$$UCL = \mathbb{E}(X_t) - c_\sigma\sqrt{Var(X_t)}$$
$$CL = \mathbb{E}(X_t)$$
$$LCL = \mathbb{E}(X_t) + c_\sigma\sqrt{Var(X_t)}. \tag{17}$$

To estimate $\mathbb{E}(X_t)$ and $Var(X_t)$ in phase I we may estimate $\lambda_i$ by their sample means and substitute them in equation (15) and (16) but we will estimate the mean and variance of the monitoring statistic with the sample mean and variance determined (not necessarily assuming Poisson) and use these estimates in Equation (17), see also section 2.2.2 Equation (11) and (12).

An example of the multivariate Poisson chart to the dataset of the case study is depicted in Figure 8 when the room was at rest. IN Figure 8 we have estimated the control limits with equation 17.




**(a)** Airborne particles

**(b)** Viable particles

**Figure 8:** Multivariate Poisson chart applied on multivariate count data generated by the BioTrak in a grade D control room.

### 2.2.5 Likelihood ratio chart (grouped method)

Steiner et al. [10] [11] proposes multiple k-step gauge charts. All charts test if the process mean is shifted. We will use the likelihood ratio of two specific hypotheses to create a control chart that can detect one sided shifts in the mean from a underlying distribution from grouped data. The chart constructed in the paper is for an underlying normal distribution, but we will show that it is easy to adjust for other underlying distributions. The expected mean for the underlying distribution is $\mu_0$, the paper test if the mean shifts to $\mu_1$ where $\mu_0 \leq \mu_1$. The paper assumes that we know parameter $\sigma$, and without loss of generality we shall assume that $\sigma$ is unity. So, the paper test every time point the hypotheses $\mu = \mu_0$ versus the alternative $\mu = \mu_1$ with a significance level of $\alpha$ and power $1-\beta$. We will modify this because we will test $\mu = \mu_0$ versus the alternative that $\mu \neq \mu_0$ with a significance level of $\alpha$.

We will use the data $\boldsymbol{Y_t} = (Y_{2t}, Y_{3t}, ..., Y_{kt})$ from a phase I study. First we have to estimate the probabilities $p_i^*$ that a particle belongs to interval $i$ with the cumulative distribution function $F_D$ of the underlying distribution given in Equation (5). In our case we will estimate the parameters of the underlying log normal distribution given by $\hat{\mu}_I$ and $\hat{\sigma}_I$ with maximum likelihood estimation. Then the likelihood function assuming that the data is multinomial is given by

$$L(\mu|\boldsymbol{Y}_t) = c \cdot \prod_{j=2}^{k} (p_j^*)^{Y_{jt}} \tag{18}$$

where the constant of proportionality $c$, is not essential for the calculation below.

In the thesis of Steiner et al. the likelihood ratio for their hypotheses is given by

$$LR(\mu|\boldsymbol{Y}_t) = \frac{L(\mu_1|\boldsymbol{Y}_t)}{L(\hat{\mu}_0|\boldsymbol{Y}_t)} = \prod_{j=2}^{k}\left(\frac{p_j^*(\mu_1)}{p_j^*(\hat{\mu}_0)}\right)^{Y_{it}} \tag{19}$$

In our case study we do not have a pre-specified $\mu_1$. So we have to use the likelihood function $L(\mu_1|\boldsymbol{Y}_t)$ differently. We will estimate the likelihood of one observed data point, but this creates problems with limited observations in each interval for maximum likelihood estimation. For that reason we will estimate the $p_j^*(\mu_1)$ with method of moments ($\hat{p}_{Oj}^*$), given in equation 6. The likelihood ratio that we will use is

$$LR(\mu|\boldsymbol{Y}_t) = \prod_{j=2}^{k}\left(\frac{\hat{p}_{Oj}^*}{p_j^*(\hat{\mu}_0)}\right)^{Y_{Oit}} \tag{20}$$

The log likelihood ratio (LLR) is given by

$$LLR = \sum_{j=2}^{k} Y_{Oit}\ln\left(\frac{\hat{p}_{Oj}^*}{p_j^*(\hat{\mu}_0)}\right) \tag{21}$$

Then the control chart checks if the average likelihood ratio ($X_t$) for a sample is greater then critical value $z$.

$$X_t = \frac{1}{N_t^*}\sum_{j=2}^{k} Y_{Oit}\ln\left(\frac{\hat{p}_{Oj}^*}{p_j^*(\hat{\mu}_0)}\right) \tag{22}$$

When we determine the average log likelihood ratio from Equation (19) and we have a large number of particles we can approximate the statistic with the normal distribution [11]. In that case we can estimate the upper limit $z$ by estimating the normal parameters and then using the $(1-\alpha)$-quantile determined with the inverse normal distribution. Unfortunately our statistic given in Equation (22) is not normal distributed. The distribution is always positive and has a tail to the right (See histogram of simulated data in Appendix A). We will estimate the upper limit by first splitting the phase I data in equal parts. With the first part we will estimate the parameters of the underlying log normal distribution $\hat{\mu}_I$ and $\hat{\sigma}_I$ with MLE (Equation (5)) and with the parameters we can estimate the probability $p_j^*(\hat{\mu}_0)$. With the second part of the phase I data we calculate $X_t$ on each time point. The upper limit is the $1-\alpha$ quantile, we will estimate this with the empirical $1-\alpha$ quantile of the sample values $X_t$ calculated from the second part of the phase I data.

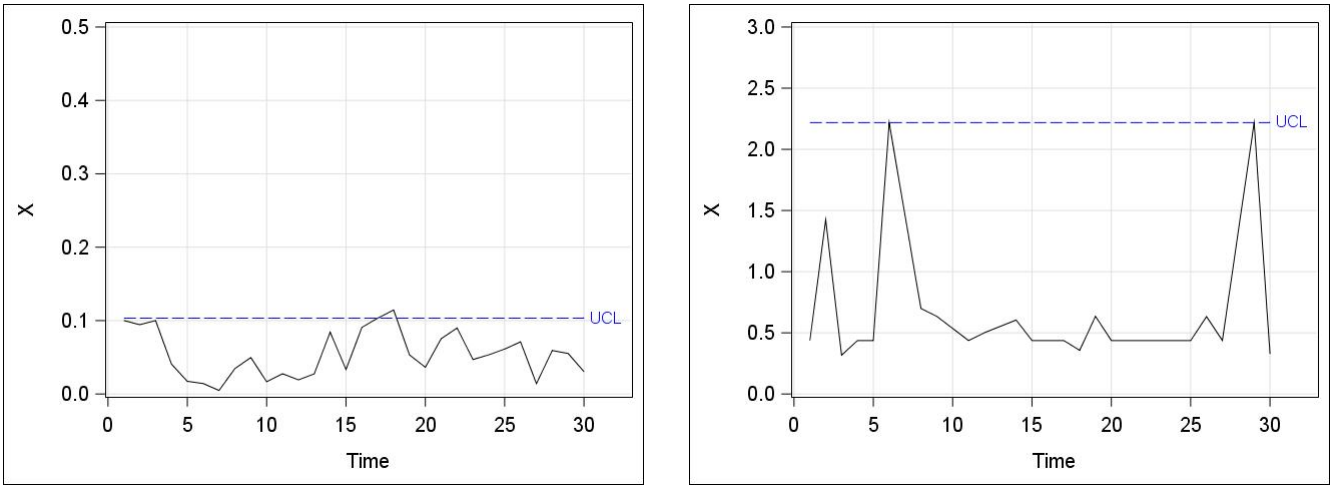
$$Q_{1-\alpha} = max\{x : \mathbb{P}(X_t \leq 1-\alpha)\} \approx \lfloor (T_I/2 \cdot (1-\alpha))\rfloor = q_n \tag{23}$$

We have to use the floor function to get a integer value. The chart will not converge to the predefined average run length (370.4) because the empirical quantile is only an estimate of the distribution quantile. The false alarm rate of this chart will be

$$\alpha_{real} = 1 - \frac{q_n}{T_I/2}. \tag{24}$$

When the phase I process is large $\alpha_{real}$ will be close to $\alpha$.

An example of a phase I control chart resulted from applying the log likelihood ratio method to the dataset of the case study is depicted in Figure 9 when the room was at rest.




**(a)** Airborne particles

**(b)** Viable particles

**Figure 9:** Log likelihood ratio chart applied on multivariate count data generated by the BioTrak in a grade D control room.

### 2.2.6 Zero inflated Poisson chart (bivariate-attribute method)

Fatahi et al. described in their paper a control chart based on Copula functions for monitoring bivariate rare count data with a zero inflated Poisson (zip) distribution [16]. The authors reported a proposed copula but unfortunately does not satisfy the definition of copula (Appendix B, Equation (38)). We will propose a correct copula function. Furthermore, they suggested to use Pearson's correlation coefficient to estimate the association parameter of the copula, but do no provide any argument for this specific choice. We studied their choice with a simulation study on zero inflated Poisson data is in Appendix E. In this section we focus on the modified method that we used for our setting.

Consider the random variables $Y_{1t}$ and $Y_{2t}$, with cumulative density functions $F(y_1)$ and $F(y_2)$, respectively. There exists a copula $C$ function such that, for all $Y_1$ and $Y_2$ in the extended real line, a joint cumulative distribution function $F(y_1, y_2) = C(F(y_1), F(y_2))$ exists (see Sklar's Theorem [28]). In our case $Y_1$ and $Y_2$ may represent counts from two intervals both follow a zero inflated Poisson distribution marginally with respectively parameters $\lambda_1$, $\theta_1$ and $\lambda_2$, $\theta_2$. Where $\lambda_i$ is the mean and $1-\theta_i$ the probability of extra zeros. We propose te following bivariate copula function

$$C(u,v) = \begin{cases} (1-\rho)uv + \rho \min(u,v) & \rho > 0 \\ (1+\rho)uv - \rho \max(u-1+v, 0) & \rho \leq 0 \end{cases} \tag{25}$$

where $\rho$ is the association parameter.

It is straightforward to see that the function in Equation (25) is a copula function according to the theorem that every convex combination of two copulas is a copula too. Since $C^0(u,v) = uv$, $C^-(u,v) = \max(u+v-1, 0)$, and $C^+(u,v) = \min(u,v)$ are all Copula's [29]. where $(u,v) \in [0,1]$ x $[0,1]$ and and we define the bounds as and . We also know the Copula function when the data is completely independent, . The family of copulas that includes $C^0(u,v), C^-(u,v), C^+(u,v)$ is called inclusive or comprehensive. The Fréchet copula is a convex combination of the three copulas described above.

$$C_{\gamma,\beta} = \gamma C^+(u,v) + \beta C^-(u,v) + (1-\gamma-\beta)C^0(u,v) \tag{26}$$

where $\gamma, \beta \geq 0$ and $\gamma + \beta \leq 1$.

Thus, the proposed copula (25) is a special case of the Frechet copula when $\gamma = \rho$ and $\beta = 0$.

Fatahi et al, suggest to use the Pearson's correlation coefficient between the two counts $Y_1$ and $Y_2$ to estimate $\rho$. However, it is unknown whether Pearson's correlation coefficient is superior than other estimators such as Spearman's rank correlation coefficient and Kendall's tau. We know that the Spearman correlation ($\rho_{sp}$) is Pearson correlation coeficient between $U$ and $V$ and is equal to $\gamma - \beta$ for copula (26). The Kendall's $\tau$ between $U$ and $V$ also has an explicit expression for this copula and it is equal to $(\gamma-\beta)(2+\gamma+\beta)/3$ [30]. Pearson's coefficient between $Y_1$ and $Y_2$ may not be equal to $corr(U,V)$ because it is not transformation invariant. On the contrarily, Spearman rank correlation and Kendell's $\tau$ are transformation invariant and can be used in theory to estimate $\rho$ on the zip distributed variables $X_1 = F_1^{-1}(U)$ and $X_2 = F_2^{-1}(V)$ with $F_i$ the $zip(\theta_i, \lambda_i)$ distribution and $F_i^{-1}$ a generalized inverse. However, the transformation should be strictly monotone which makes Spearman and Kendall's tau possibly less favourable too. It is unclear which measure of correlation should be used. To check the performance of the measure of correlations we conducted an ancillary simulation study (Appendix E) on zip distributed data. Results did not favor any of the three methods over all settings considered. Even in some settings none of the proposed measures of correlation should be used due to large biases and we need better alternatives. We will use Spearman's correlation coefficient to estimate $\rho$. We expect that the Spearman's correlation is measuring better in our case where $\lambda_i$ is large and the probabilities of no extra zero ($\theta_i$) is low.

The monitoring statistic discussed in Equation 15 is the sum of the two counts,similar to the multivariate Poisson chart,

$$X_t = Y_{Ot1} + Y_{Ot2}. \tag{27}$$

The control limits have to satisfy the following conditions,

$$\mathbb{P}(X \geq UCL) \leq \alpha/2$$
$$\mathbb{P}(X \leq LCL) \leq \alpha/2.$$

We will only investigate the UCL with a false alarm rate equal to $\alpha$. We now include the copula function and get the formula to determine the UCL.

$$\begin{aligned}
&\mathbb{P}(X \geq UCL) \leq \alpha \\
&\Rightarrow\ \mathbb{P}(X < UCL) \geq 1-\alpha \\
&\Rightarrow\ \mathbb{P}(Y_1 + Y_2 < UCL) \geq 1-\alpha \\
&\Rightarrow\ \sum_{y=1}^{UCL-1} \mathbb{P}(Y_1 + Y_2 = y) \geq 1-\alpha \\
&\Rightarrow\ \sum_{y_1,y_2 \in S_U} \mathbb{P}(Y_1 \leq y_1, Y_2 \leq y_2) - \mathbb{P}(Y_1 \leq y_1, Y_2 \leq y_2 - 1) \\
&- \mathbb{P}(Y_1 \leq y_1 - 1, Y_2 \leq y_2) + \mathbb{P}(Y_1 \leq y_1 - 1, Y_2 \leq y_2 - 1) \geq 1-\alpha \\
&\Rightarrow\ \sum_{y_1,y_2 \in S_U} F(y_1, y_2) - F(y_1, y_2 - 1) - F(y_1 - 1, y_2) + F(y_1 - 1, y_2 - 1) \geq 1-\alpha \\
&\Rightarrow
\end{aligned}$$

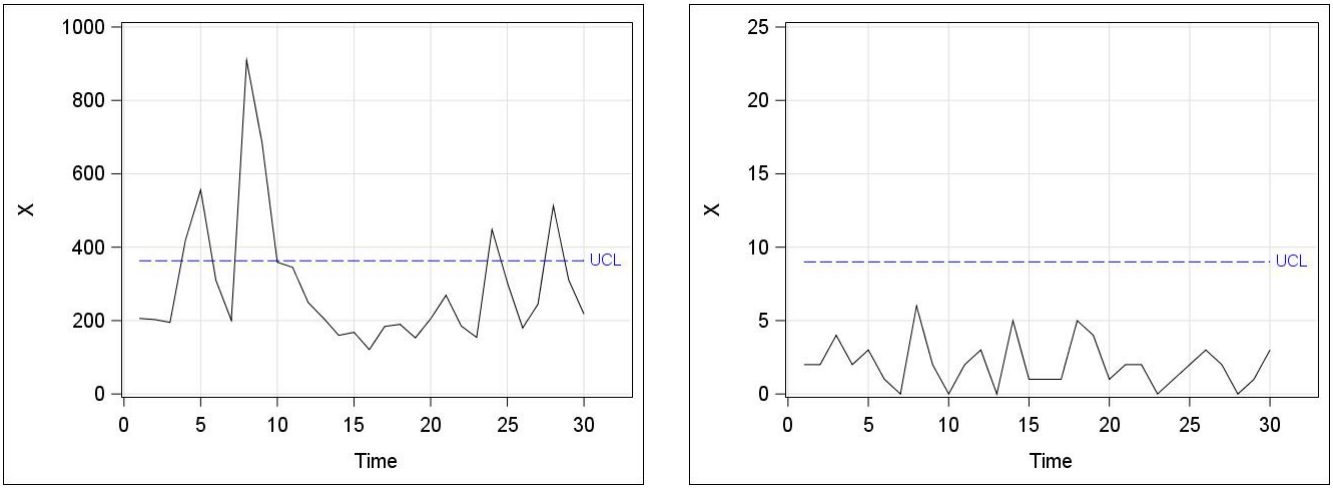
$$\sum_{y_1,y_2 \in S_U} C(F(y_1), F(y_2)) - C(F(y_1-1), F(y_2)) - C(F(y_1), F(y_2-1)) + C(F(y_1-1), F(y_2-1)) \geq 1-\alpha \tag{28}$$

where $S_U = \{(y_1, y_2)|y_1 + y_2 + 1 \leq UCL\}$ and $F(y_i - 1) = 0$ when $y_i = 0$. The smallest UCL satisfy which inequality (28) holds is the upper control limit that we use. The formulation of the UCL in [16] we did not understand, and comparing UCL's from (28) with calculated UCL's in [16] resulted in different UCL's (See appendix B). We will continue with our formulation.

The paper estimates the parameters $\lambda_1$, $\theta_1$ and $\lambda_2$, $\theta_2$ with method of moments estimations (MME) but maximum likelihood estimation (MLE) could be used as well. We know that for finite samples the MLE is superior [31], and considered to use MLE to estimate $\lambda_i$ and $\theta_i$ for $i = 1, 2$ in our study.

One of the benefits of using copula functions to model the joint distribution is that other marginal distributions can also be incorporated in this chart. We can adapt the method easily to other count distributions, for example the zero inflated negative binomial distribution (zinb). When we use zinb we will call the method zero inflated negative binomial method. We use the same copula function and MLE for estimation of the parameters. The chart is for bivariate data only, it would be nice to extend the chart to multivariate data but this is complicated because we then have to consider the correlation between all variables and we have to construct a new copula function.

An example of the zero inflated poisson chart to the dataset of the case study is depicted in Figure 18 when the room was at rest. To make the data bivariate we have combined intervals. The two intervals that we use are for the airborne particles $[0.5, 5.0)$ and $[5.0, \infty)$, for the viable particles $[0.5, 1.0)$ and $[1.0, \infty)$.

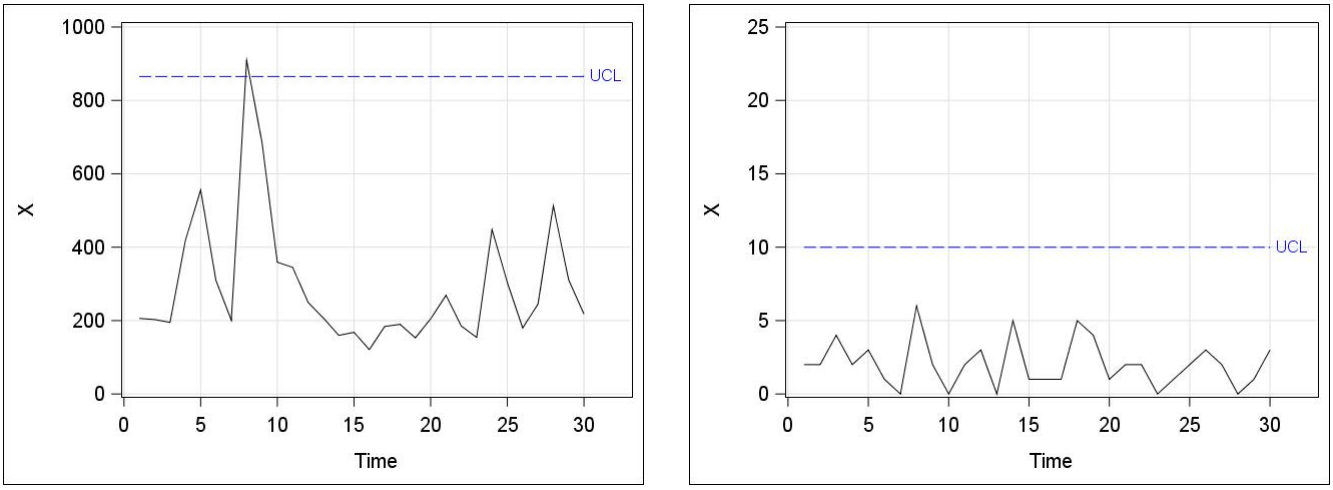



**(a)** Airborne particles

**(b)** Viable particles

**Figure 10:** Zero inflated Poisson chart applied on multivariate count data generated by the BioTrak in a grade D control room.

We also applied the zero inflated negative binomial chart on the same combined intervals and visualized the results in Figure 11.




**(a)** Airborne particles

**(b)** Viable particles

**Figure 11:** Zero inflated negative binomial chart applied on multivariate count data generated by the BioTrak in a grade D control room.

# 3 Simulation

To test the performance of each control chart on our truncated multinomial data, we will do simulation studies where we estimate the average run length (ARL) of each chart. The ARL is a performance measure of a control chart. All control charts will be constructed such that the ARL will be approximately 370.4 when the process is in control. We will calculate the ARL when the process is in control (type-I error), and when the process is out of control (type-II error). We will simulate phase II data with different numbers of particles and time points and evaluate the charts for phase II data. The simulation is programmed in Statistical Analysis Software (SAS), the programs are in Appendix D.

## 3.1 Generating data

Phase I data will be based on an underlying log-normal distribution. The underlying log-normal distribution has parameters $\mu$ and $\sigma$, respectively, the mean and standard deviation of the log particle size distribution. We will not use data from the lowest particle size distribution.Recall that we will use the log normal distribution because literature suggest that this is most appropriate(see Section 2). In control rooms we expect that there will be multiple log normal particle size distributions, to keep it simple we will simulate with only one log normal underlying distribution. The parameters for the phase I study are based on the BioTrak case study, $\mu = 0.5$ and $\sigma = 0.75$ [17]. The corresponding medium particle size is 2.18 with coefficient of variation equal to 86.9%.

We will evaluate processes with different numbers of fixed particles every time point ($N_t = N$). The fixed number of particles is $N = 10$, 100, 1000 and 10000. Note that the total observed particles will not be fixed because we exclude the first interval. We consider two different time periods for phase I $T_I = 250$ and $T_I = 2500$ time points. Secondly, we evaluate the control chart under the more realistic case where the number of particles vary with time points. To simulate this we will generate the total number of particle at each time point $N_t$ according to a negative binomial distribution. The mean of the negative binomial is taken the same values as with the fixed number of particles. So, the mean ($m_N$) will be 10, 100, 1000 and 10000 particles. For the variance we will consider three cases $v_n = m_N/(1-p)$ with $p$ the probability of a success in the negative binomial distribution, we consider $p = 0.25, p = 0.50$ and $p = 0.75$ .

To generate data for phase I we at each time point $t$. Then we will draw $N_t$ particle sizea t each time point using the log normal distribution.Finally, we count the number $Y_{it}$ of particles whit their sized in each interval $i$ at each time point $t$. We generate data for 3,4 and 7 intervals. The interval bounds are constructed based on the case study. The three and four intervals are merged intervals in a way that the particles are as good as possible equal divided over the intervals. The probabilities that a particle is in interval $i$ and interval sizes for each interval $i$ are shown in Tables 2, 3 and 4. Again, we ignore the first interval in the simulation for constructing control limits and calculating monitoring statistics. So the applied methods are constructed for two, three and six intervals.

| $i$ | 1 | 2 | 3 |
|---|---|---|---|
| $[C_{i-1}, C_i)$ | $[0, 0.5)$ | $[0.5, 3.0)$ | $[3.0, \infty)$ |
| $p_i$ | 0.055820 | 0.73179 | 0.21239 |

**Table 2:** scenario 1: 3 Particle size intervals and corresponding probabilities where mu = 0.5 and sigma = 0.75.

| $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $[C_{i-1}, C_i)$ | $[0, 0.5)$ | $[0.5, 1.0)$ | $[1.0, 3.0)$ | $[3.0, \infty)$ |
| $p_i$ | 0.055820 | 0.19667 | 0.53512 | 0.21239 |

**Table 3:** scenario 2: 4 Particle size intervals and corresponding probabilities where mu = 0.5 and sigma = 0.75.

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $[C_{i-1}, C_i)$ | $[0, 0.5)$ | $[0.5, 0.7)$ | $[0.7, 1.0)$ | $[1.0, 3.0)$ | $[3.0, 5.0)$ | $[5.0, 10.0)$ | $[10.0, \infty)$ |
| $P_i$ | 0.055820 | 0.070858 | 0.12581 | 0.53512 | 0.14286 | 0.061416 | 0.008120665 |

**Table 4:** scenario 3: 7 Particle size intervals and corresponding probabilities where mu = 0.5 and sigma = 0.75.

Each simulation study consist out of 4000 runs (r), in each run we calculate a phase I process with $T_I$ time points and we will determine the corresponding control limits of each chart. So, we will generate in total $r$ phase I processes and control limits for each chart. For the simulation we want to run a phase II process and check when the process is for the first time out of control. Therefore, we will generated for every run also a phase II process. Unfortunately we can not run the phase II study to infinity and we will stop after 4000 time points (T). Thus the time to an out of control censored at 4000. We will keep track of the number of censored times. For each simulation study we will use always the same seed numbers. We will use a seed number to generate the phase I data and a different seed number for the phase II.

In case all charts provide a average ARL of 370.4 for the in-control-setting the charts can be compared on the ARL for out-of-control settings. To make sure that all charts give 370.4 for in control settings we will change the $\alpha$'s such that the ARL $\in (360, 380)$. We will use the same $\alpha$'s in the out of control ARL studies. When it is not possible to change the ARL to 370.4 we can not do an out of control study. In this simulation, studies will change the parameter $\mu$ in the simulated data of the underlying log normal distribution, the phase I data will stay the same. This will result in a different spread of the particles over the categories. Changing the $\mu$ will change the mean and variance of the categorical data. We will change the $\mu$ to 0.35 ,0.4, 0.45 and 0.475. The new probabilities per interval, means and variances corresponding to this parameters are in the tables below.

| $\mu$ | $\sigma$ | $\mathbb{E}[X_{ij}]$ | $Var[X_{ij}]$ | $p_1$ | $p_2$ | $p_3$ |
|---|---|---|---|---|---|---|
| 0.400 | 0.75 | 1.976 | 2.949 | 0.072 | 0.752 | 0.176 |
| 0.450 | 0.75 | 2.078 | 3.259 | 0.064 | 0.743 | 0.194 |
| 0.475 | 0.75 | 2.130 | 3.426 | 0.060 | 0.737 | 0.203 |
| 0.500 | 0.75 | 2.184 | 3.602 | 0.056 | 0.732 | 0.212 |

**Table 5:** Theoretical mean, variance and probabilities multinomial distribution where the number of categories is three.

| $\mu$ | $\sigma$ | $\mathbb{E}[X_{ij}]$ | $Var[X_{ij}]$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ |
|---|---|---|---|---|---|---|---|
| 0.400 | 0.75 | 1.976 | 2.949 | 0.072 | 0.224 | 0.527 | 0.176 |
| 0.450 | 0.75 | 2.078 | 3.259 | 0.064 | 0.211 | 0.532 | 0.194 |
| 0.475 | 0.75 | 2.130 | 3.426 | 0.060 | 0.204 | 0.534 | 0.203 |
| 0.500 | 0.75 | 2.184 | 3.602 | 0.056 | 0.197 | 0.535 | 0.212 |

**Table 6:** Theoretical mean, variance and probabilities multinomial distribution where the number of categories is four.

| $\mu$ | $\sigma$ | $\mathbb{E}[X_{ij}]$ | $Var[X_{ij}]$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $p_6$ | $p_7$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.400 | 0.75 | 1.976 | 2.949 | 0.072 | 0.084 | 0.140 | 0.527 | 0.122 | 0.048 | 0.006 |
| 0.450 | 0.75 | 2.078 | 3.259 | 0.064 | 0.077 | 0.133 | 0.532 | 0.133 | 0.054 | 0.007 |
| 0.475 | 0.75 | 2.130 | 3.426 | 0.060 | 0.074 | 0.130 | 0.534 | 0.138 | 0.058 | 0.007 |
| 0.500 | 0.75 | 2.184 | 3.602 | 0.056 | 0.071 | 0.126 | 0.535 | 0.143 | 0.061 | 0.008 |

**Table 7:** Theoretical mean, variance and probabilities multinomial distribution where the number of categories is seven.

**Pseudo code**

The code for the phase I and phase II are similar, only the parameters of the underlying distribution will change in the Type-II case and the number of time points in the processes is different. The pseudo code for generating our categorical data is given below.

**Input:**

- Total number of time points phase I ($T_I$).
- Is the total number of particle fixed (Yes/No)?
  - Yes: $N$.
  - No: $N \sim Negbin(m_N, v_N)$ .
- Number of intervals (3/4/7).
- Parameters log normal particle size distribution phase I ($\mu_I$ / $\sigma_I$) .
- Parameters log normal particle size distribution phase II ($\mu$ / $\sigma$) .
- Limit on number of time points in a simulated process ($T = 4000$).
- Number of simulations (r).

**Dataset phase I and phase II:**

1. Calculate the probability that a particle is in interval $i$ ($p_i$) for $i = 1, 2, ..., k$ based on the log normal distribution and parameters (phase I: $\mu_I$ and $\sigma_I$, phase II: $\mu$ and $\sigma$) , Equation (1) .
2. **foreach** *runs (r)* **do**
   **foreach** *Time points (phase I: $T_I$, phase II: $T$)* **do**
   **if** $N_t$ *is fixed* **then**
   $N_t = N$
   **else**
   Simulate the total number of counted particles ($N_t$) based on the negative binomial distribution with parameters $m_N$ and $v_N$.
   **end**
   3. Simulate $N_t$ particle sizes. 4. Calculate the number of particles in each interval with the multinomial distribution $(Y_1, Y_2, ..., Y_k)$.
   5. Exclude the first interval and calculate the total number of counted particles ($N_t^*$).
   **end**

**end**

**Output:**

- Dataset Phase I (for each run and time point):
  - Intervals $(Y_2, Y_3, ..., Y_k)$
  - Total number of counted particles ($N_t^*$)
- Dataset Phase II (for each run and time point):
  - Intervals $(Y_2, Y_3, ..., Y_k)$
  - Total number of counted particles ($N_t^*$)

## 3.2 Parameter estimation

With the parameter estimations from the phase I process we determine the control limits and with the parameter estimations of the simulated data we calculate the monitoring statistics for each chart.

First we estimate the parameters for the control limits with the phase I data. The phase I process consist of $T_I$ periods. The resulting data is in each period the vector $Y_t = (Y_{2t}, Y_{3t}, ..., Y_{kt})$ for $t = 1, 2, ..., T_I$.

When we have multivariate (or multinomial) data we have to use the moment estimation of the probability that a particles is in interval $i$,

$$\hat{p_i} = \frac{\sum_{t=1}^{T_I} Y_{it}}{\sum_{t=1}^{T_I} N_t^*}. \tag{29}$$

When we have grouped data we first have to estimate the parameters of the underlying particle size distribution. Of course we know that the distribution is log normal so we have to estimate $\mu$ and $\sigma$. We will estimate the parameters with log likelihood estimation. The likelihood function is:

$$L(\mu, \sigma | \boldsymbol{Y}) = c \prod_{j=2}^{k} (p_j^*)^{Y_j}. \tag{30}$$

Taking the logarithm we get the log likelihood function

$$LL(\mu, \sigma | \boldsymbol{Y}) = log(c) + \sum_{j=2}^{k} \log(p_j^*) Y_j \tag{31}$$

where $c$ is a constant which we can excluded with the log likelihood estimation estimation of $\mu$ and $\sigma$. $p_j^*$ is a function depending on $\mu$ and $\sigma$, the formula is in Equation (4).

When we know $\mu$ and $\sigma$ we can estimate the probability that a particles is in interval $i$ with the log normal cumulative distribution function.

$$p_i^* = \frac{F_D(C_i) - F_D(C_{i-1})}{1 - F_D(C_1)} \tag{32}$$

where, $p_1 = F_D(C_1)$. The explanation of this formula is in Equation (4).

The total number of particles does not depend on the underlying distribution, so is will be for all methods the same. The expected total number of particles counted is

$$m_{N^*} = \frac{1}{T_I} \sum_{t=1}^{T_I} N_t^* = \frac{1}{T_I} \sum_{t=1}^{T_I} \sum_{i=2}^{k} Y_{it}. \tag{33}$$

The formula to calculate the expected number of particles in interval $i$ is given by

$$\overline{Y_{it}} = \mathbb{E}[N^*] \cdot \hat{p}_{it} = \frac{1}{T_I} \sum_{t=1}^{T_I} Y_{it}. \tag{34}$$

The covariance matrix will be estimated with the sample covariance matrix (S). The correlations are calculated with the sample Spearman correlation coefficients for the bivariate zero inflated Poisson chart.

### 3.3 Reporting measures from simulation

The main output from the simulation is an time to event $T_{ARL}$ per simulation, possibly censored at 4000. We will average these times into ARL and calculate the variance and mean squared error as well.

$$\widehat{ARL} = \frac{1}{r} \sum_{i=1}^{r} T_{ARL}. \tag{35}$$

$$STD_{ARL} = \sqrt{\frac{1}{r-1} \sum_{i=1}^{r} (\widehat{ARL} - T_{ARL})^2}. \tag{36}$$

$$RMSE_{ARL} = \sqrt{\frac{1}{r}\sum_{i=1}^{r}(370.4 - T_{ARL})^2}. \tag{37}$$

In case that the process is in control the ARL will be equal to $\frac{1}{\alpha}$, in our case the type-I error is $\alpha = 0.0027$, so we expect to have $ARL = 370.4$. Alternatively, the ARL can be estimated by fitting an exponential distribution to the $T_{ARL}$ while considering the cases with $T_{ARL} = 4000$ as censored observations. However, since the number of cases of censoring is very small, the average ARL is sufficient.

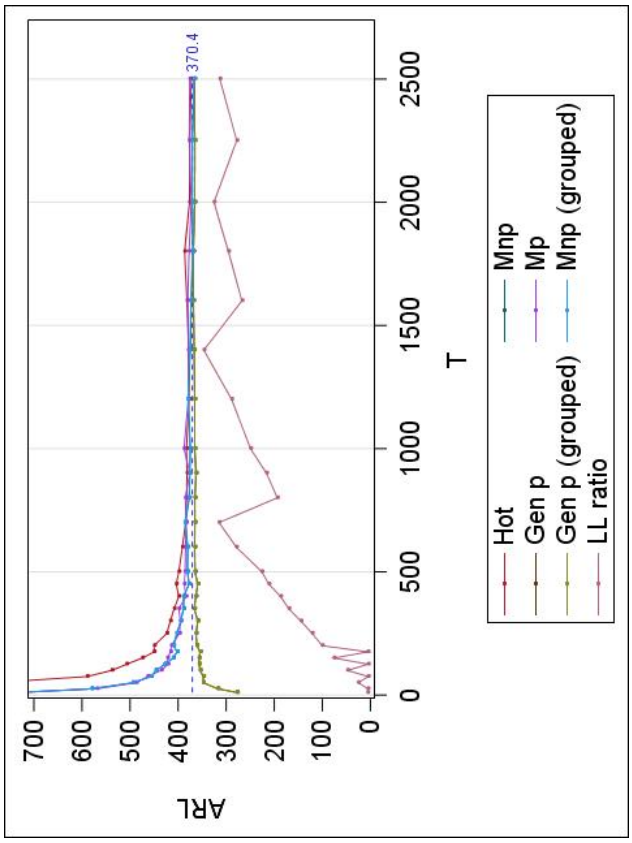
# 4 Results

The results are reported by the number of particles: fixed and varying. Within these settings we report three, four, and seven particle size intervals. We report the ARL's for in control and out-of-control settings for all our control charts. The bivariate ZIP and ZINB charts are onlu studied for three intervals and reported in Section 4.7. the results are reported in tables and visualized in graphs.

## 4.1 Fixed number of particles and three particle size intervals

### In control ARL

We first studied processes that are in control at $\alpha = 0.0027$. The ARl's are vizualized in Figure 12a and 12b. We see that all methods need a sufficiently large number of particles to converges to the expected ARL except for the log likelihood chart which converges to approximately 312.5. From Figure 12b we can conclude that a phase I study requires sufficiently large time points to converges to 370.4. When the number of time points is just 250, only the generalized p chart is close to the nominal ARL of 370.4 when 100 or more particles are observed. The generalized p chart where we estimate the probabilities with the moment estimation or with the estimated parameters of the underlying distribution with maximum likelihood is in many cases the same or close. This also applies to the multivariate np chart, in this method it does not matter if we estimate the parameters with moment estimation or with the parameters of the underlying distribution. When the total phase I time is 2500 all charts have their approximate average run length when the total number of particles is greater than 1000.




**(a)** Total number of particles versus average run length where $T_I = 2500$.




**(b)** Total time points phase I versus average run length where $N_t = 2500$.

**Figure 12:** Three intervals.

| | | Hotelling chart | | | | | Multivariate np chart | | | | | Generalized p chart | | | | | Multivariate poisson chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | N | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 250 | 10 | .0027 | 153.7 | 198.9 | 294.1 | 0 | .0027 | 355.7 | 465.8 | 466.0 | 8 | .0027 | 312.0 | 345.3 | 350.1 | 1 | .0027 | 94.7 | 203.4 | 342.6 | 0 |
| 250 | 100 | .0027 | 320.9 | 359.4 | 362.7 | 1 | .0027 | 387.2 | 448.0 | 448.3 | 5 | .0027 | 354.2 | 359.9 | 359.1 | 0 | .0027 | 275.4 | 341.4 | 354.3 | 0 |
| 250 | 1000 | .0027 | 419.0 | 453.9 | 456.5 | 5 | .0027 | 397.8 | 479.3 | 480.0 | 10 | .0027 | 363.3 | 366.3 | 366.3 | 0 | .0027 | 396.5 | 452.8 | 453.5 | 5 |
| 250 | 10000 | .0027 | 419.3 | 456.9 | 459.4 | 4 | .0027 | 402.6 | 489.9 | 490.9 | 10 | .0027 | 361.5 | 362.0 | 362.1 | 0 | .0027 | 396.0 | 457.2 | 457.9 | 4 |
| 2500 | 10 | .0027 | 114.8 | 129.0 | 286.2 | 0 | .0027 | 279.3 | 310.5 | 323.6 | 0 | .0027 | 308.6 | 311.4 | 317.4 | 0 | .0027 | 64.3 | 61.5 | 312.2 | 0 |
| 2500 | 100 | .0027 | 293.8 | 296.8 | 306.5 | 0 | .0027 | 358.8 | 351.0 | 351.1 | 0 | .0027 | 359.1 | 361.1 | 360.2 | 0 | .0027 | 255.8 | 251.9 | 276.6 | 0 |
| 2500 | 1000 | .0027 | 380.3 | 386.4 | 386.4 | 0 | .0027 | 369.4 | 365.5 | 365.4 | 0 | .0027 | 369.8 | 368.8 | 368.8 | 0 | .0027 | 376.7 | 384.1 | 384.1 | 0 |
| 2500 | 10000 | .0027 | 375.3 | 372.6 | 372.6 | 0 | .0027 | 364.2 | 361.5 | 361.4 | 0 | .0027 | 369.7 | 369.6 | 369.5 | 0 | .0027 | 376.0 | 384.1 | 384.1 | 0 |

**Table 8:** (1/2) Estimated ARL for three intervals with failure rate $\alpha = 0.0027$, fixed $N$ .

| | | multivariate np chart (grouped) | | | | | Generalized p chart (grouped) | | | | | Likelihood ratio control chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | N | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 250 | 10 | .0027 | 355.5 | 466.2 | 466.3 | 8 | .0027 | 312.9 | 346.9 | 351.6 | 1 | .0027 | 110.4 | 177.1 | 314.5 | 0 |
| 250 | 100 | .0027 | 387.4 | 448.2 | 448.5 | 5 | .0027 | 354.5 | 360.1 | 359.4 | 0 | .0027 | 112.8 | 216.3 | 336.3 | 1 |
| 250 | 1000 | .0027 | 397.8 | 479.3 | 480.0 | 10 | .0027 | 363.1 | 365.7 | 365.7 | 0 | .0027 | 120.1 | 234.3 | 342.8 | 3 |
| 250 | 10000 | .0027 | 402.6 | 489.9 | 490.9 | 10 | .0027 | 361.6 | 362.1 | 362.1 | 0 | .0027 | 122.3 | 248.3 | 351.0 | 2 |
| 2500 | 10 | .0027 | 278.9 | 310.1 | 323.3 | 0 | .0027 | 308.6 | 311.4 | 317.4 | 0 | .0027 | 232.6 | 294.7 | 325.3 | 1 |
| 2500 | 100 | .0027 | 358.8 | 351.0 | 351.1 | 0 | .0027 | 359.3 | 361.7 | 360.7 | 0 | .0027 | 301.8 | 368.2 | 374.5 | 1 |
| 2500 | 1000 | .0027 | 369.4 | 365.5 | 365.4 | 0 | .0027 | 369.9 | 368.8 | 368.7 | 0 | .0027 | 311.6 | 399.3 | 403.5 | 4 |
| 2500 | 10000 | .0027 | 364.2 | 361.5 | 361.4 | 0 | .0027 | 369.7 | 369.6 | 369.5 | 0 | .0027 | 308.9 | 387.2 | 392.0 | 0 |

**Table 9:** (2/2) Estimated ARL for three intervals with failure rate $\alpha = 0.0027$, fixed $N$.

### Out of control ARL

In this simulation study we check the performance of the control chart when the phase II process is out of control. To compare all settings we first have changed the out of control significance level $\alpha$ such that the average run length is close to 370.4 for all settings (see Appendix C.1). In this simulation study the phase I monitoring time is 2500 time points. We see that the average run lengths are not smaller than 370.4 in every setting, so all charts observe the out of control setting. In the extreme case that $\mu = 0.35$ all charts have a low average run length, so they detect an out of control point early. The larger the number of particles the faster the control chart detects out of control processes. When the number of particles is low, N=10 , the average run length of the generalized p chart and log likelihood chart is large. When the number of particles is large, n=1000, all charts have an average run length close to 1. When N is greater than 100 we see that the average run length will decrease when parameter $\mu$ shifts from 0.475 to 0.35.

| | | | | Hotelling chart | | | | | Multivariate np chart | | | | | Generalized p chart | | | | | Multivariate poisson chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | N | $\mu$ | $\sigma$ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 10 | 0.350 | 0.75 | .00070 | 139.9 | 140.7 | 270.0 | 0 | .00230 | 146.8 | 183.9 | 289.5 | 0 | .0017 | 1370.5 | 1228.7 | 1583.6 | 351 | .00075 | 95.6 | 133.1 | 305.3 | 0 |
| 2500 | 10 | 0.400 | 0.75 | .00070 | 202.3 | 198.0 | 259.7 | 0 | .00230 | 229.4 | 297.2 | 328.9 | 1 | .0017 | 907.8 | 919.7 | 1064.2 | 99 | .00075 | 146.1 | 204.0 | 303.1 | 0 |
| 2500 | 10 | 0.450 | 0.75 | .00070 | 278.9 | 285.4 | 299.6 | 0 | .00230 | 321.7 | 388.7 | 391.6 | 1 | .0017 | 578.2 | 630.2 | 663.1 | 18 | .00075 | 226.1 | 336.7 | 366.2 | 0 |
| 2500 | 10 | 0.475 | 0.75 | .00070 | 327.0 | 342.9 | 345.6 | 1 | .00230 | 359.2 | 416.2 | 416.3 | 0 | .0017 | 462.2 | 517.7 | 525.7 | 7 | .00075 | 291.0 | 427.7 | 435.0 | 2 |
| 2500 | 100 | 0.350 | 0.75 | .00190 | 17.3 | 16.7 | 353.5 | 0 | .00265 | 12.1 | 12.1 | 358.5 | 0 | .0026 | 59.3 | 58.2 | 316.5 | 0 | .00142 | 18.7 | 20.0 | 352.3 | 0 |
| 2500 | 100 | 0.400 | 0.75 | .00190 | 47.6 | 46.8 | 326.2 | 0 | .00265 | 36.9 | 37.4 | 335.6 | 0 | .0026 | 159.3 | 158.6 | 264.0 | 0 | .00142 | 45.1 | 50.8 | 329.3 | 0 |
| 2500 | 100 | 0.450 | 0.75 | .00190 | 148.4 | 150.8 | 268.3 | 0 | .00265 | 141.2 | 141.5 | 269.4 | 0 | .0026 | 401.6 | 397.6 | 398.7 | 1 | .00142 | 123.1 | 146.4 | 287.4 | 0 |
| 2500 | 100 | 0.475 | 0.75 | .00190 | 250.2 | 256.3 | 283.0 | 0 | .00265 | 275.2 | 280.1 | 295.8 | 0 | .0026 | 473.9 | 478.2 | 489.1 | 3 | .00142 | 210.7 | 269.5 | 313.3 | 0 |
| 2500 | 1000 | 0.350 | 0.75 | .00275 | 1.0 | 0.2 | 369.4 | 0 | .00270 | 1.0 | 0.1 | 369.4 | 0 | .0027 | 1.3 | 0.6 | 369.1 | 0 | .00270 | 1.4 | 0.8 | 369.0 | 0 |
| 2500 | 1000 | 0.400 | 0.75 | .00275 | 1.8 | 1.2 | 368.6 | 0 | .00270 | 1.5 | 0.9 | 368.9 | 0 | .0027 | 3.0 | 2.5 | 367.4 | 0 | .00270 | 3.7 | 3.2 | 366.7 | 0 |
| 2500 | 1000 | 0.450 | 0.75 | .00275 | 15.3 | 15.2 | 355.5 | 0 | .00270 | 10.3 | 10.1 | 360.2 | 0 | .0027 | 23.3 | 22.4 | 347.8 | 0 | .00270 | 28.0 | 28.0 | 343.5 | 0 |
| 2500 | 1000 | 0.475 | 0.75 | .00275 | 89.1 | 90.2 | 295.5 | 0 | .00270 | 64.1 | 64.2 | 312.9 | 0 | .0027 | 109.0 | 108.3 | 282.9 | 0 | .00270 | 115.6 | 117.8 | 280.7 | 0 |
| 2500 | 10000 | 0.350 | 0.75 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .0027 | 1.0 | 0.0 | 369.4 | 0 | .00270 | 1.0 | 0.0 | 369.4 | 0 |
| 2500 | 10000 | 0.400 | 0.75 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .0027 | 1.0 | 0.0 | 369.4 | 0 | .00270 | 1.0 | 0.0 | 369.4 | 0 |
| 2500 | 10000 | 0.450 | 0.75 | .00270 | 1.0 | 0.1 | 369.4 | 0 | .00270 | 1.0 | 0.1 | 369.4 | 0 | .0027 | 1.1 | 0.4 | 369.3 | 0 | .00270 | 1.5 | 0.9 | 368.9 | 0 |
| 2500 | 10000 | 0.475 | 0.75 | .00270 | 3.4 | 3.0 | 367.0 | 0 | .00270 | 2.5 | 2.0 | 367.9 | 0 | .0027 | 5.2 | 4.7 | 365.2 | 0 | .00270 | 9.9 | 9.4 | 360.6 | 0 |

**Table 10:** (1/2) Estimated out of control ARL for three intervals, fixed $N$.

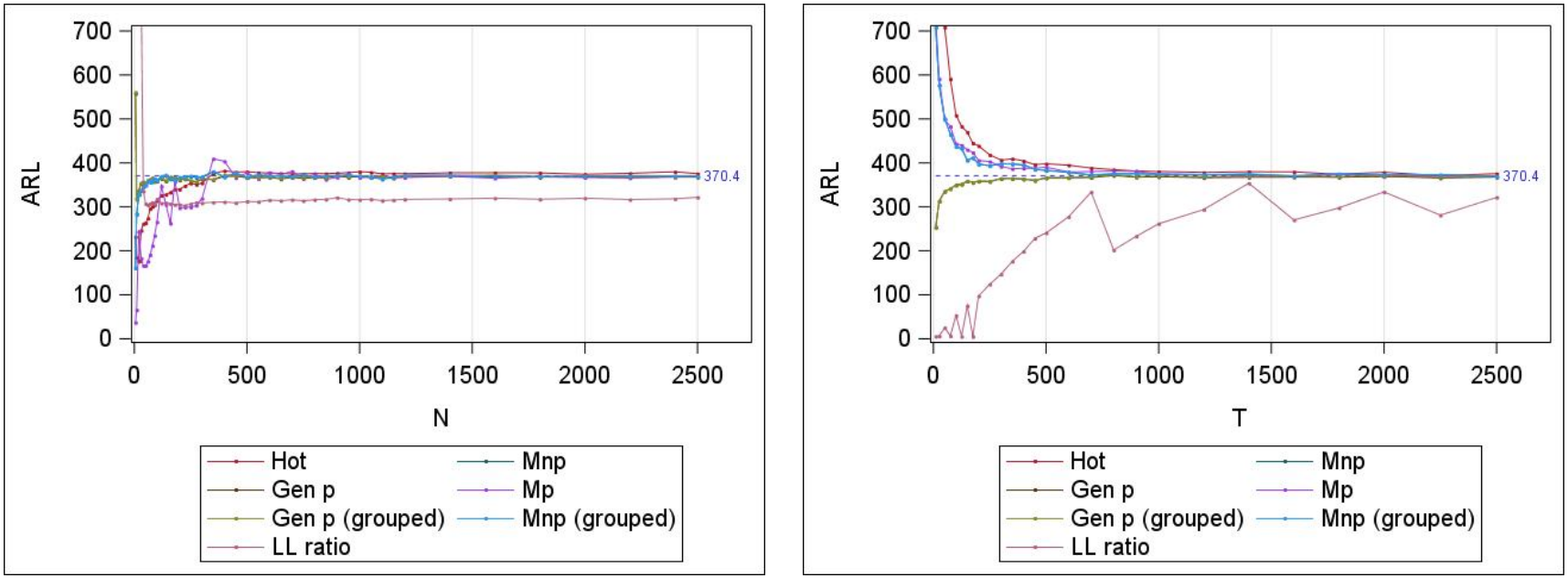
| | | | | multivariate np chart (grouped) | | | | | Generalized p chart (grouped) | | | | | Likelihood ratio control chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | N | $\mu$ | $\sigma$ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 10 | 0.350 | 0.75 | .00230 | 145.9 | 182.5 | 289.3 | 0 | .0017 | 1370.5 | 1227.6 | 1583.3 | 349 | .0014 | 1299.8 | 1269.4 | 1573.1 | 418 |
| 2500 | 10 | 0.400 | 0.75 | .00230 | 228.6 | 296.1 | 328.3 | 1 | .0017 | 906.3 | 918.3 | 1063.2 | 98 | .0014 | 932.2 | 1046.1 | 1187.3 | 188 |
| 2500 | 10 | 0.450 | 0.75 | .00230 | 321.7 | 388.6 | 391.6 | 1 | .0017 | 577.6 | 631.1 | 664.2 | 18 | .0014 | 652.0 | 823.5 | 870.2 | 79 |
| 2500 | 10 | 0.475 | 0.75 | .00230 | 358.7 | 414.0 | 414.1 | 0 | .0017 | 461.7 | 518.1 | 526.0 | 7 | .0014 | 536.9 | 712.4 | 731.6 | 51 |
| 2500 | 100 | 0.350 | 0.75 | .00265 | 12.1 | 12.1 | 358.5 | 0 | .0026 | 59.2 | 58.0 | 316.6 | 0 | .0024 | 27.6 | 32.0 | 344.2 | 0 |
| 2500 | 100 | 0.400 | 0.75 | .00265 | 36.9 | 37.4 | 335.6 | 0 | .0026 | 159.3 | 158.6 | 264.0 | 0 | .0024 | 69.4 | 85.4 | 312.9 | 0 |
| 2500 | 100 | 0.450 | 0.75 | .00265 | 141.2 | 141.5 | 269.4 | 0 | .0026 | 401.7 | 397.6 | 398.7 | 1 | .0024 | 194.1 | 243.4 | 300.5 | 0 |
| 2500 | 100 | 0.475 | 0.75 | .00265 | 274.7 | 278.3 | 294.2 | 0 | .0026 | 474.1 | 478.6 | 489.7 | 3 | .0024 | 311.2 | 382.7 | 387.2 | 3 |
| 2500 | 1000 | 0.350 | 0.75 | .00270 | 1.0 | 0.1 | 369.4 | 0 | .0027 | 1.3 | 0.6 | 369.1 | 0 | .0024 | 1.2 | 0.5 | 369.2 | 0 |
| 2500 | 1000 | 0.400 | 0.75 | .00270 | 1.5 | 0.9 | 368.9 | 0 | .0027 | 3.0 | 2.5 | 367.4 | 0 | .0024 | 2.8 | 2.3 | 367.7 | 0 |
| 2500 | 1000 | 0.450 | 0.75 | .00270 | 10.3 | 10.1 | 360.2 | 0 | .0027 | 23.3 | 22.4 | 347.8 | 0 | .0024 | 21.4 | 23.7 | 349.8 | 0 |
| 2500 | 1000 | 0.475 | 0.75 | .00270 | 64.1 | 64.2 | 312.9 | 0 | .0027 | 109.0 | 108.3 | 282.9 | 0 | .0024 | 99.0 | 124.2 | 298.4 | 0 |
| 2500 | 10000 | 0.350 | 0.75 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .0027 | 1.0 | 0.0 | 369.4 | 0 | .0024 | 1.0 | 0.0 | 369.4 | 0 |
| 2500 | 10000 | 0.400 | 0.75 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .0027 | 1.0 | 0.0 | 369.4 | 0 | .0024 | 1.0 | 0.0 | 369.4 | 0 |
| 2500 | 10000 | 0.450 | 0.75 | .00270 | 1.0 | 0.1 | 369.4 | 0 | .0027 | 1.1 | 0.4 | 369.3 | 0 | .0024 | 1.1 | 0.4 | 369.3 | 0 |
| 2500 | 10000 | 0.475 | 0.75 | .00270 | 2.5 | 2.0 | 367.9 | 0 | .0027 | 5.2 | 4.7 | 365.2 | 0 | .0024 | 5.1 | 5.3 | 365.3 | 0 |

**Table 11:** (2/2) Estimated out of control ARL for three intervals, fixed $N$.

## 4.2 Fixed number of particles and four particle size intervals

### In control ARL

Figure 13a and 13b report the ARL's for in-control processes using $\alpha = 0.0027$ . Figure 13a shows that the multivariate np chart, log likelihood and generalized p chart are always close to the approximate average run length even when $N$ is small. The multivariate Poisson chart and the Hotelling $T^2$ chart need a sufficiently large number of particles to approximate 370.4. For the generalized p chart and multivariate np chart it does not matter if we estimate the probabilities with the moment or with the underlying log normal distribution. Figure 13a and Figure 13b are very similar to the figures described Section 4.1. The log likelihood ratio chart has an expected average run length of approximately 312.5.




**(a)** Total number of particles versus average run length where $T_I = 2500$.

**(b)** Total time points phase I versus average run length where $N_t = 2500$.

**Figure 13:** Four intervals.

| | | Hotelling chart | | | | | Multivariate np chart | | | | | Generalized p chart | | | | | Multivariate poisson chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | N | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 250 | 10 | .0027 | 200.6 | 232.8 | 288.1 | 0 | .0027 | 291.9 | 359.8 | 368.2 | 2 | .0027 | 320.3 | 343.0 | 346.5 | 0 | .0027 | 94.5 | 193.8 | 337.1 | 0 |
| 250 | 100 | .0027 | 349.4 | 389.8 | 390.3 | 0 | .0027 | 388.6 | 471.5 | 471.7 | 5 | .0027 | 357.4 | 352.2 | 352.3 | 0 | .0027 | 290.5 | 360.8 | 369.5 | 3 |
| 250 | 1000 | .0027 | 427.1 | 471.2 | 474.6 | 2 | .0027 | 393.5 | 460.3 | 460.8 | 5 | .0027 | 357.8 | 346.7 | 346.9 | 0 | .0027 | 403.5 | 484.4 | 485.3 | 5 |
| 250 | 10000 | .0027 | 424.7 | 466.7 | 469.8 | 3 | .0027 | 397.8 | 460.4 | 461.1 | 2 | .0027 | 357.5 | 344.8 | 345.0 | 0 | .0027 | 401.1 | 475.5 | 476.2 | 4 |
| 2500 | 10 | .0027 | 185.0 | 189.7 | 264.9 | 0 | .0027 | 283.3 | 287.8 | 300.6 | 0 | .0027 | 319.0 | 325.0 | 327.5 | 0 | .0027 | 65.6 | 65.7 | 311.8 | 0 |
| 2500 | 100 | .0027 | 314.4 | 319.3 | 324.1 | 0 | .0027 | 358.4 | 363.9 | 364.0 | 0 | .0027 | 360.8 | 355.4 | 355.5 | 0 | .0027 | 265.4 | 257.7 | 278.2 | 0 |
| 2500 | 1000 | .0027 | 379.9 | 383.6 | 383.7 | 0 | .0027 | 368.6 | 384.3 | 384.2 | 0 | .0027 | 369.9 | 362.9 | 362.8 | 0 | .0027 | 366.6 | 361.3 | 361.0 | 0 |
| 2500 | 10000 | .0027 | 375.4 | 372.4 | 372.3 | 0 | .0027 | 369.4 | 381.1 | 381.0 | 0 | .0027 | 371.2 | 360.8 | 360.7 | 0 | .0027 | 366.7 | 361.1 | 360.8 | 0 |

**Table 12:** (1/2) Estimated ARL for four intervals with failure rate $\alpha = 0.0027$, fixed $N$.

| | | multivariate np chart (grouped) | | | | | Generalized p chart (grouped) | | | | | Likelihood ratio control chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | N | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 250 | 10 | .0027 | 291.3 | 358.7 | 367.2 | 2 | .0027 | 320.7 | 342.1 | 345.6 | 0 | .0027 | 515.8 | 869.2 | 881.2 | 117 |
| 250 | 100 | .0027 | 388.8 | 472.8 | 473.1 | 5 | .0027 | 358.1 | 352.9 | 353.0 | 0 | .0027 | 122.5 | 248.5 | 351.0 | 2 |
| 250 | 1000 | .0027 | 393.5 | 460.3 | 460.8 | 5 | .0027 | 358.2 | 347.1 | 347.2 | 0 | .0027 | 124.2 | 259.3 | 357.5 | 4 |
| 250 | 10000 | .0027 | 397.8 | 460.4 | 461.1 | 2 | .0027 | 357.4 | 344.8 | 345.0 | 0 | .0027 | 127.5 | 278.3 | 369.4 | 5 |
| 2500 | 10 | .0027 | 283.1 | 287.7 | 300.5 | 0 | .0027 | 318.9 | 323.8 | 326.3 | 0 | .0027 | 1245.7 | 1153.4 | 1447.8 | 258 |
| 2500 | 100 | .0027 | 358.4 | 363.9 | 364.0 | 0 | .0027 | 360.8 | 355.4 | 355.5 | 0 | .0027 | 318.1 | 403.9 | 407.3 | 5 |
| 2500 | 1000 | .0027 | 368.6 | 384.3 | 384.2 | 0 | .0027 | 369.9 | 362.9 | 362.8 | 0 | .0027 | 316.5 | 392.2 | 395.8 | 4 |
| 2500 | 10000 | .0027 | 369.4 | 381.1 | 381.0 | 0 | .0027 | 371.2 | 360.8 | 360.7 | 0 | .0027 | 320.2 | 397.0 | 400.1 | 3 |

**Table 13:** (2/2) Estimated ARL for four intervals with failure rate $\alpha = 0.0027$, fixed $N$.

### Out of control ARL

The average run lengths of all charts in all settings are less than the average run length of 370.4. The approximate average run lengths when the number of particles is $N = 1000$ and $N = 10000$ are very small. When the number of particles is equal to 10 the generalized p chart and multivariate np chart also have smaller ARL's than 370.4. Just like the in-control studies the average run lengths of the generalized p chart where we estimate the probabilities with moments is nearly equal to the grouped generalized p chart. The rest of the results are similar to the simulation studies with three intervals.

| | | | | Hotelling chart | | | | | Multivariate np chart | | | | | Generalized p chart | | | | | Multivariate poisson chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | N | $\mu$ | $\sigma$ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 10 | 0.350 | 0.75 | .0010 | 105.2 | 105.6 | 285.4 | 0 | .0015 | 128.5 | 126.5 | 272.9 | 0 | .0022 | 230.9 | 239.7 | 277.3 | 0 | .00075 | 97.8 | 138.9 | 305.9 | 0 |
| 2500 | 10 | 0.400 | 0.75 | .0010 | 161.4 | 163.5 | 265.4 | 0 | .0015 | 181.1 | 182.8 | 263.2 | 0 | .0022 | 308.8 | 310.7 | 316.7 | 0 | .00075 | 144.9 | 206.2 | 305.6 | 0 |
| 2500 | 10 | 0.450 | 0.75 | .0010 | 246.7 | 249.6 | 278.5 | 0 | .0015 | 256.9 | 259.7 | 283.4 | 0 | .0022 | 361.1 | 361.6 | 361.7 | 0 | .00075 | 235.6 | 346.0 | 371.3 | 0 |
| 2500 | 10 | 0.475 | 0.75 | .0010 | 316.9 | 316.5 | 320.4 | 0 | .0015 | 312.2 | 311.4 | 316.7 | 0 | .0022 | 377.0 | 391.4 | 391.4 | 0 | .00075 | 305.9 | 452.8 | 457.3 | 0 |
| 2500 | 100 | 0.350 | 0.75 | .0022 | 12.9 | 12.6 | 357.8 | 0 | .00260 | 46.2 | 44.7 | 327.3 | 0 | .0026 | 24.4 | 23.8 | 346.9 | 0 | .00145 | 18.2 | 19.7 | 352.8 | 0 |
| 2500 | 100 | 0.400 | 0.75 | .0022 | 40.5 | 40.7 | 332.4 | 0 | .00260 | 95.0 | 95.1 | 291.4 | 0 | .0026 | 68.8 | 67.5 | 309.1 | 0 | .00145 | 43.7 | 49.7 | 330.5 | 0 |
| 2500 | 100 | 0.450 | 0.75 | .0022 | 138.5 | 138.6 | 270.1 | 0 | .00260 | 195.3 | 194.0 | 261.3 | 0 | .0026 | 204.1 | 200.9 | 260.7 | 0 | .00145 | 119.3 | 141.3 | 288.1 | 0 |
| 2500 | 100 | 0.475 | 0.75 | .0022 | 248.2 | 249.5 | 277.7 | 0 | .00260 | 280.4 | 284.6 | 298.4 | 0 | .0026 | 317.2 | 312.7 | 317.2 | 0 | .00145 | 204.2 | 242.8 | 294.1 | 0 |
| 2500 | 1000 | 0.350 | 0.75 | .0027 | 1.0 | 0.1 | 369.4 | 0 | .00270 | 3.8 | 3.3 | 366.6 | 0 | .0027 | 1.1 | 0.3 | 369.3 | 0 | .00270 | 1.4 | 0.8 | 369.0 | 0 |
| 2500 | 1000 | 0.400 | 0.75 | .0027 | 1.5 | 0.9 | 368.9 | 0 | .00270 | 12.1 | 12.0 | 358.5 | 0 | .0027 | 2.1 | 1.5 | 368.3 | 0 | .00270 | 3.8 | 3.2 | 366.6 | 0 |
| 2500 | 1000 | 0.450 | 0.75 | .0027 | 13.1 | 12.8 | 357.5 | 0 | .00270 | 69.7 | 73.4 | 309.6 | 0 | .0027 | 18.9 | 18.2 | 351.9 | 0 | .00270 | 27.6 | 26.8 | 343.8 | 0 |
| 2500 | 1000 | 0.475 | 0.75 | .0027 | 88.7 | 87.3 | 294.9 | 0 | .00270 | 189.5 | 196.0 | 266.6 | 0 | .0027 | 103.8 | 103.1 | 285.9 | 0 | .00270 | 116.4 | 115.6 | 279.1 | 0 |
| 2500 | 10000 | 0.350 | 0.75 | .0027 | 1.0 | 0.0 | 369.4 | 0 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .0027 | 1.0 | 0.0 | 369.4 | 0 | .00270 | 1.0 | 0.0 | 369.4 | 0 |
| 2500 | 10000 | 0.400 | 0.75 | .0027 | 1.0 | 0.0 | 369.4 | 0 | .00270 | 1.1 | 0.2 | 369.3 | 0 | .0027 | 1.0 | 0.0 | 369.4 | 0 | .00270 | 1.0 | 0.0 | 369.4 | 0 |
| 2500 | 10000 | 0.450 | 0.75 | .0027 | 1.0 | 0.1 | 369.4 | 0 | .00270 | 3.7 | 3.2 | 366.8 | 0 | .0027 | 1.0 | 0.2 | 369.4 | 0 | .00270 | 1.5 | 0.9 | 368.9 | 0 |
| 2500 | 10000 | 0.475 | 0.75 | .0027 | 2.9 | 2.4 | 367.5 | 0 | .00270 | 27.3 | 27.1 | 344.1 | 0 | .0027 | 4.1 | 3.6 | 366.3 | 0 | .00270 | 10.1 | 9.7 | 360.4 | 0 |

**Table 14:** (1/2) Estimated out of control ARL for four intervals, fixed $N$.

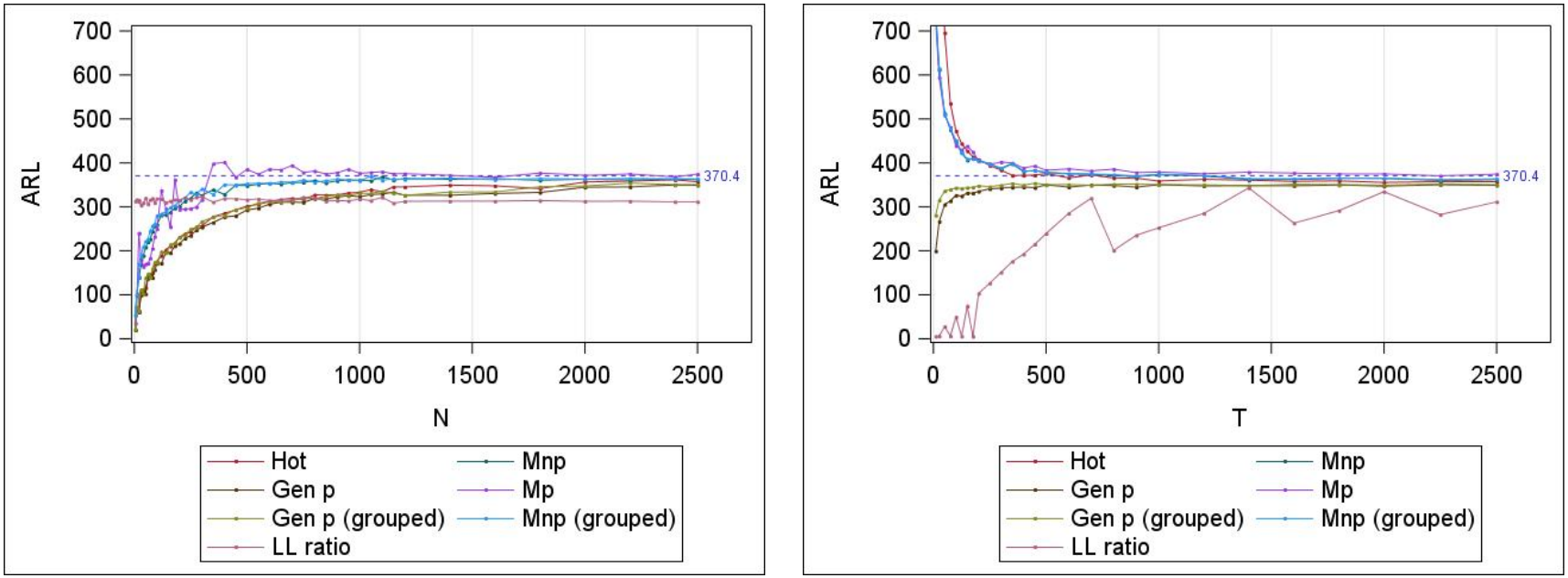
| | | | | multivariate np chart (grouped) | | | | | Generalized p chart (grouped) | | | | | Likelihood ratio control chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | N | $\mu$ | $\sigma$ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 10 | 0.350 | 0.75 | .0015 | 128.5 | 126.5 | 272.9 | 0 | .0022 | 230.5 | 239.0 | 276.9 | 0 | 0.0114 | 241.4 | 319.5 | 344.5 | 0 |
| 2500 | 10 | 0.400 | 0.75 | .0015 | 181.1 | 182.8 | 263.2 | 0 | .0022 | 309.2 | 310.3 | 316.2 | 0 | 0.0114 | 293.9 | 371.9 | 379.6 | 3 |
| 2500 | 10 | 0.450 | 0.75 | .0015 | 256.9 | 259.7 | 283.4 | 0 | .0022 | 362.2 | 362.5 | 362.5 | 0 | 0.0114 | 341.3 | 452.7 | 453.6 | 8 |
| 2500 | 10 | 0.475 | 0.75 | .0015 | 312.2 | 311.4 | 316.7 | 0 | .0022 | 378.4 | 393.4 | 393.4 | 0 | 0.0114 | 342.1 | 435.0 | 435.9 | 4 |
| 2500 | 100 | 0.350 | 0.75 | .0026 | 46.2 | 44.7 | 327.3 | 0 | .0026 | 24.4 | 23.8 | 346.9 | 0 | 0.0024 | 22.8 | 25.2 | 348.5 | 0 |
| 2500 | 100 | 0.400 | 0.75 | .0026 | 95.0 | 95.1 | 291.4 | 0 | .0026 | 68.8 | 67.5 | 309.1 | 0 | 0.0024 | 65.4 | 78.1 | 314.8 | 0 |
| 2500 | 100 | 0.450 | 0.75 | .0026 | 195.3 | 194.0 | 261.3 | 0 | .0026 | 204.0 | 200.8 | 260.8 | 0 | 0.0024 | 204.4 | 256.5 | 305.5 | 1 |
| 2500 | 100 | 0.475 | 0.75 | .0026 | 280.4 | 284.7 | 298.5 | 0 | .0026 | 316.9 | 312.5 | 317.0 | 0 | 0.0024 | 334.1 | 433.8 | 435.3 | 6 |
| 2500 | 1000 | 0.350 | 0.75 | .0027 | 3.8 | 3.3 | 366.6 | 0 | .0027 | 1.1 | 0.3 | 369.3 | 0 | 0.0024 | 1.1 | 0.3 | 369.3 | 0 |
| 2500 | 1000 | 0.400 | 0.75 | .0027 | 12.1 | 12.0 | 358.5 | 0 | .0027 | 2.1 | 1.5 | 368.3 | 0 | 0.0024 | 2.1 | 1.5 | 368.3 | 0 |
| 2500 | 1000 | 0.450 | 0.75 | .0027 | 69.7 | 73.4 | 309.6 | 0 | .0027 | 18.9 | 18.2 | 351.9 | 0 | 0.0024 | 19.5 | 21.3 | 351.6 | 0 |
| 2500 | 1000 | 0.475 | 0.75 | .0027 | 189.5 | 196.0 | 266.6 | 0 | .0027 | 103.7 | 103.0 | 285.9 | 0 | 0.0024 | 113.5 | 141.1 | 293.1 | 0 |
| 2500 | 10000 | 0.350 | 0.75 | .0027 | 1.0 | 0.0 | 369.4 | 0 | .0027 | 1.0 | 0.0 | 369.4 | 0 | 0.0024 | 1.0 | 0.0 | 369.4 | 0 |
| 2500 | 10000 | 0.400 | 0.75 | .0027 | 1.1 | 0.2 | 369.3 | 0 | .0027 | 1.0 | 0.0 | 369.4 | 0 | 0.0024 | 1.0 | 0.0 | 369.4 | 0 |
| 2500 | 10000 | 0.450 | 0.75 | .0027 | 3.7 | 3.2 | 366.8 | 0 | .0027 | 1.0 | 0.2 | 369.4 | 0 | 0.0024 | 1.0 | 0.2 | 369.4 | 0 |
| 2500 | 10000 | 0.475 | 0.75 | .0027 | 27.3 | 27.1 | 344.1 | 0 | .0027 | 4.1 | 3.6 | 366.3 | 0 | 0.0024 | 4.1 | 3.8 | 366.3 | 0 |

**Table 15:** (2/2) Estimated out of control ARL for four intervals, fixed $N$.

## 4.3 Fixed number of particles and seven particle size intervals

### In control ARL

In this case with seven intervals we have diversity in the probabilities $p_i^*$. Now we have more uniform have dominant intervals and intervals with small portions. With three and four intervals the probabilities $p_i^*$ were more uniform. All charts, except the LLR chart, converge to 370.4. The log likelihood ratio chart has an ARL of approximately 312.5 for all $N$. The multivariate Poisson chart behaves similar for studies with three, four and seven intervals. The log likelihood ratio chart is performing better than in the studies with three and four intervals.




**(a)** Total number of particles versus average run length where $T_I = 2500$.

**(b)** Total time points phase I versus average run length where $N_t = 2500$.

**Figure 14:** Seven intervals.

| | | Hotelling chart | | | | | Multivariate np chart | | | | | Generalized p chart | | | | | Multivariate poisson chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | N | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 250 | 10 | .0027 | 74.4 | 83.4 | 307.5 | 0 | .0027 | 103.9 | 128.9 | 296.0 | 0 | .0027 | 65.6 | 74.5 | 313.8 | 0 | .0027 | 96.4 | 213.1 | 347.0 | 0 |
| 250 | 100 | .0027 | 194.1 | 214.9 | 277.7 | 0 | .0027 | 269.3 | 314.5 | 330.3 | 1 | .0027 | 169.7 | 175.0 | 266.2 | 0 | .0027 | 275.0 | 347.4 | 360.2 | 1 |
| 250 | 1000 | .0027 | 365.7 | 403.4 | 403.3 | 1 | .0027 | 381.7 | 452.7 | 452.7 | 3 | .0027 | 316.2 | 314.7 | 319.3 | 0 | .0027 | 409.5 | 502.1 | 503.3 | 7 |
| 250 | 10000 | .0027 | 413.1 | 458.6 | 460.5 | 2 | .0027 | 393.7 | 456.4 | 456.9 | 5 | .0027 | 356.2 | 359.7 | 359.9 | 0 | .0027 | 395.1 | 480.5 | 481.0 | 6 |
| 2500 | 10 | .0027 | 72.2 | 71.7 | 306.6 | 0 | .0027 | 98.1 | 99.3 | 289.9 | 0 | .0027 | 70.9 | 72.6 | 308.2 | 0 | .0027 | 64.2 | 64.0 | 312.8 | 0 |
| 2500 | 100 | .0027 | 175.7 | 178.6 | 264.2 | 0 | .0027 | 259.4 | 260.1 | 282.8 | 0 | .0027 | 171.2 | 172.2 | 263.3 | 0 | .0027 | 249.9 | 252.8 | 280.0 | 0 |
| 2500 | 1000 | .0027 | 333.0 | 339.6 | 341.6 | 0 | .0027 | 360.7 | 367.4 | 367.4 | 0 | .0027 | 322.9 | 324.9 | 328.3 | 0 | .0027 | 376.6 | 388.9 | 388.7 | 0 |
| 2500 | 10000 | .0027 | 372.5 | 370.8 | 370.7 | 0 | .0027 | 377.8 | 376.2 | 376.2 | 0 | .0027 | 365.1 | 366.6 | 366.6 | 0 | .0027 | 368.3 | 373.2 | 373.0 | 0 |

**Table 16:** (1/2) Estimated ARL for seven intervals with failure rate $\alpha = 0.0027$, fixed $N$.

| | | multivariate np chart (grouped) | | | | | Generalized p chart (grouped) | | | | | Likelihood ratio control chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | N | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 250 | 10 | .0027 | 104.2 | 120.2 | 292.1 | 0 | .0027 | 70.8 | 76.3 | 309.2 | 0 | .0027 | 116.5 | 225.7 | 339.7 | 3 |
| 250 | 100 | .0027 | 270.4 | 309.9 | 325.6 | 0 | .0027 | 175.4 | 180.1 | 265.4 | 0 | .0027 | 118.6 | 262.0 | 363.4 | 5 |
| 250 | 1000 | .0027 | 380.5 | 452.4 | 452.5 | 3 | .0027 | 321.6 | 320.5 | 324.2 | 0 | .0027 | 125.1 | 276.1 | 369.3 | 8 |
| 250 | 10000 | .0027 | 393.3 | 455.8 | 456.4 | 5 | .0027 | 364.1 | 364.9 | 364.9 | 0 | .0027 | 125.3 | 266.6 | 362.2 | 5 |
| 2500 | 10 | .0027 | 98.6 | 98.6 | 289.1 | 0 | .0027 | 71.0 | 71.9 | 307.9 | 0 | .0027 | 316.7 | 380.7 | 384.4 | 1 |
| 2500 | 100 | .0027 | 259.7 | 260.6 | 283.1 | 0 | .0027 | 172.9 | 173.7 | 262.9 | 0 | .0027 | 309.6 | 388.4 | 393.1 | 5 |
| 2500 | 1000 | .0027 | 360.0 | 365.5 | 365.6 | 0 | .0027 | 325.3 | 325.5 | 328.6 | 0 | .0027 | 313.7 | 393.1 | 397.1 | 4 |
| 2500 | 10000 | .0027 | 377.0 | 374.3 | 374.3 | 0 | .0027 | 365.2 | 366.5 | 366.4 | 0 | .0027 | 316.4 | 388.1 | 391.8 | 2 |

**Table 17:** (2/2) Estimated ARL for seven intervals with failure rate $\alpha = 0.0027$, fixed $N$.

### Out of control ARL

After adjusting the $\alpha$ levels to obtain an ARL of 370.4 in the in-control setting, we see that the generalized p chart and multivariate np chart do not perform very well when the number of particles is low. The average run length is in this case large. When the number of particles is large, the generalized p chart, log likelihood and multivariate np chart are performing very well. The average run lengths are very small. In the extreme case of $\mu = 0.35$ the average run length is equal to one. That means that the average run length in all runs are one and the chart detects immediately a out of control time point.

| | | | | Hotelling chart | | | | | Multivariate np chart | | | | | Generalized p chart | | | | | Multivariate poisson chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | N | $\mu$ | $\sigma$ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 10 | 0.350 | 0.75 | .000002 | 579.6 | 565.7 | 603.1 | 4 | .00013 | 1039.1 | 992.3 | 1196.5 | 111 | .000001 | 879.3 | 864.0 | 1002.6 | 61 | .00075 | 96.3 | 134.8 | 305.5 | 0 |
| 2500 | 10 | 0.400 | 0.75 | .000002 | 537.1 | 524.5 | 550.3 | 2 | .00013 | 748.1 | 738.4 | 829.3 | 25 | .000001 | 650.8 | 637.0 | 695.9 | 10 | .00075 | 147.4 | 213.0 | 308.3 | 0 |
| 2500 | 10 | 0.450 | 0.75 | .000002 | 437.3 | 439.8 | 444.8 | 0 | .00013 | 535.3 | 554.5 | 578.4 | 4 | .000001 | 481.8 | 483.2 | 495.8 | 0 | .00075 | 229.7 | 334.5 | 362.8 | 0 |
| 2500 | 10 | 0.475 | 0.75 | .000002 | 375.6 | 383.3 | 383.3 | 0 | .00013 | 432.5 | 438.3 | 442.6 | 0 | .000001 | 397.3 | 408.0 | 408.9 | 0 | .00075 | 302.8 | 447.2 | 452.2 | 0 |
| 2500 | 100 | 0.350 | 0.75 | .000800 | 38.6 | 37.6 | 333.9 | 0 | .00170 | 458.1 | 470.9 | 478.9 | 0 | .000700 | 100.7 | 99.3 | 287.4 | 0 | .00140 | 19.3 | 20.4 | 351.7 | 0 |
| 2500 | 100 | 0.400 | 0.75 | .000800 | 126.5 | 126.5 | 274.7 | 0 | .00170 | 692.1 | 696.1 | 766.8 | 18 | .000700 | 252.3 | 249.2 | 275.5 | 0 | .00140 | 45.5 | 53.9 | 329.4 | 0 |
| 2500 | 100 | 0.450 | 0.75 | .000800 | 327.5 | 324.7 | 327.2 | 0 | .00170 | 666.4 | 667.1 | 729.7 | 13 | .000700 | 466.3 | 463.0 | 472.5 | 0 | .00140 | 124.5 | 146.0 | 285.9 | 0 |
| 2500 | 100 | 0.475 | 0.75 | .000800 | 390.3 | 394.2 | 394.5 | 0 | .00170 | 525.6 | 529.1 | 551.3 | 1 | .000700 | 444.0 | 443.1 | 449.1 | 0 | .00140 | 214.9 | 263.8 | 306.1 | 0 |
| 2500 | 1000 | 0.350 | 0.75 | .002400 | 1.0 | 0.2 | 369.4 | 0 | .00260 | 7.7 | 7.2 | 362.8 | 0 | .002400 | 1.2 | 0.5 | 369.2 | 0 | .00270 | 1.4 | 0.8 | 369.0 | 0 |
| 2500 | 1000 | 0.400 | 0.75 | .002400 | 1.9 | 1.3 | 368.5 | 0 | .00260 | 25.1 | 24.3 | 346.1 | 0 | .002400 | 3.4 | 2.9 | 367.0 | 0 | .00270 | 3.8 | 3.2 | 366.6 | 0 |
| 2500 | 1000 | 0.450 | 0.75 | .002400 | 25.5 | 26.0 | 345.8 | 0 | .00260 | 124.1 | 127.1 | 277.1 | 0 | .002400 | 42.7 | 42.0 | 330.4 | 0 | .00270 | 27.4 | 26.9 | 344.0 | 0 |
| 2500 | 1000 | 0.475 | 0.75 | .002400 | 161.7 | 162.5 | 264.5 | 0 | .00260 | 301.7 | 303.9 | 311.5 | 0 | .002400 | 209.3 | 208.3 | 263.3 | 0 | .00270 | 113.1 | 116.0 | 282.3 | 0 |
| 2500 | 10000 | 0.350 | 0.75 | .002700 | 1.0 | 0.0 | 369.4 | 0 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .002700 | 1.0 | 0.0 | 369.4 | 0 | .00270 | 1.0 | 0.0 | 369.4 | 0 |
| 2500 | 10000 | 0.400 | 0.75 | .002700 | 1.0 | 0.0 | 369.4 | 0 | .00270 | 1.1 | 0.3 | 369.3 | 0 | .002700 | 1.0 | 0.0 | 369.4 | 0 | .00270 | 1.0 | 0.0 | 369.4 | 0 |
| 2500 | 10000 | 0.450 | 0.75 | .002700 | 1.0 | 0.1 | 369.4 | 0 | .00270 | 4.3 | 3.8 | 366.2 | 0 | .002700 | 1.1 | 0.3 | 369.3 | 0 | .00270 | 1.5 | 0.9 | 368.9 | 0 |
| 2500 | 10000 | 0.475 | 0.75 | .002700 | 4.0 | 3.5 | 366.4 | 0 | .00270 | 31.3 | 31.4 | 340.6 | 0 | .002700 | 6.5 | 5.9 | 363.9 | 0 | .00270 | 10.0 | 9.7 | 360.5 | 0 |

**Table 18:** (1/2) Estimated out of control ARL for seven intervals, fixed $N$.

| | | | | multivariate np chart (grouped) | | | | | Generalized p chart (grouped) | | | | | Likelihood ratio control chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | N | $\mu$ | $\sigma$ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 10 | 0.350 | 0.75 | .00013 | 1041.0 | 986.1 | 1192.4 | 104 | .000001 | 878.4 | 855.5 | 994.9 | 57 | .0024 | 293.0 | 376.8 | 384.6 | 6 |
| 2500 | 10 | 0.400 | 0.75 | .00013 | 747.7 | 735.5 | 826.6 | 20 | .000001 | 653.5 | 639.8 | 699.6 | 11 | .0024 | 347.6 | 436.3 | 436.8 | 3 |
| 2500 | 10 | 0.450 | 0.75 | .00013 | 526.1 | 537.5 | 559.5 | 4 | .000001 | 483.5 | 485.5 | 498.4 | 0 | .0024 | 409.3 | 522.1 | 523.5 | 13 |
| 2500 | 10 | 0.475 | 0.75 | .00013 | 429.1 | 432.0 | 435.9 | 0 | .000001 | 396.5 | 405.5 | 406.3 | 0 | .0024 | 408.7 | 502.5 | 503.9 | 8 |
| 2500 | 100 | 0.350 | 0.75 | .00170 | 459.1 | 471.3 | 479.5 | 0 | .000700 | 101.1 | 99.3 | 287.0 | 0 | .0024 | 42.0 | 50.5 | 332.2 | 0 |
| 2500 | 100 | 0.400 | 0.75 | .00170 | 693.9 | 701.6 | 772.5 | 22 | .000700 | 252.6 | 249.6 | 276.0 | 0 | .0024 | 110.8 | 132.1 | 291.3 | 0 |
| 2500 | 100 | 0.450 | 0.75 | .00170 | 669.0 | 672.8 | 736.0 | 14 | .000700 | 466.8 | 463.1 | 473.0 | 0 | .0024 | 278.1 | 361.8 | 373.4 | 4 |
| 2500 | 100 | 0.475 | 0.75 | .00170 | 523.1 | 523.6 | 545.3 | 1 | .000700 | 444.8 | 444.9 | 451.0 | 0 | .0024 | 375.7 | 490.3 | 490.3 | 6 |
| 2500 | 1000 | 0.350 | 0.75 | .00260 | 7.7 | 7.2 | 362.8 | 0 | .002400 | 1.2 | 0.5 | 369.2 | 0 | .0024 | 1.1 | 0.4 | 369.3 | 0 |
| 2500 | 1000 | 0.400 | 0.75 | .00260 | 25.1 | 24.2 | 346.2 | 0 | .002400 | 3.3 | 2.8 | 367.1 | 0 | .0024 | 3.0 | 2.6 | 367.4 | 0 |
| 2500 | 1000 | 0.450 | 0.75 | .00260 | 124.0 | 126.9 | 277.2 | 0 | .002400 | 42.9 | 42.3 | 330.2 | 0 | .0024 | 37.4 | 43.5 | 335.8 | 0 |
| 2500 | 1000 | 0.475 | 0.75 | .00260 | 302.7 | 306.6 | 314.0 | 0 | .002400 | 210.0 | 209.2 | 263.6 | 0 | .0024 | 182.3 | 235.6 | 301.5 | 1 |
| 2500 | 10000 | 0.350 | 0.75 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .002700 | 1.0 | 0.0 | 369.4 | 0 | .0024 | 1.0 | 0.0 | 369.4 | 0 |
| 2500 | 10000 | 0.400 | 0.75 | .00270 | 1.1 | 0.3 | 369.3 | 0 | .002700 | 1.0 | 0.0 | 369.4 | 0 | .0024 | 1.0 | 0.0 | 369.4 | 0 |
| 2500 | 10000 | 0.450 | 0.75 | .00270 | 4.3 | 3.8 | 366.2 | 0 | .002700 | 1.1 | 0.3 | 369.3 | 0 | .0024 | 1.1 | 0.3 | 369.3 | 0 |
| 2500 | 10000 | 0.475 | 0.75 | .00270 | 31.3 | 31.4 | 340.6 | 0 | .002700 | 6.5 | 5.9 | 363.9 | 0 | .0024 | 6.5 | 6.5 | 363.9 | 0 |

**Table 19:** (2/2) Estimated out of control ARL for seven intervals, fixed $N$.

## 4.4 Varying numbers of particles and three particle size intervals

### In control ARL

We now have a varying number of total particles. From figure 15 we can conclude that also in this setting all charts converge to the expected average run length of 370.4, except the log likelihood ratio chart will converge to 312.5, when we increase the number of phase I time points. From Table Table 20 and 21 we see that increasing the expected number of particles will result that all chart converge to 370.4, except the log likelihood ratio chart to 312.5 . The generalized p chart performs well in the setting of 250 phase I time points. When the total phase I time is low the Hotelling $T^2$, multivariate np chart, multivariate Poisson chart their average run length are close to 400 and not 370.4, when the base time is $T_I = 2500$ the average run length is in this setting closer to 370.4. The generalized p chart is nearly equal to the generalized p chart (grouped), also the multivariate np chart is nearly equal to multivariate np (grouped) chart. Increasing the variance has effect on the average run lengths in all charts with the exception of the generalized p chart. When the phase I time and number of particles is large, all chart perform well.

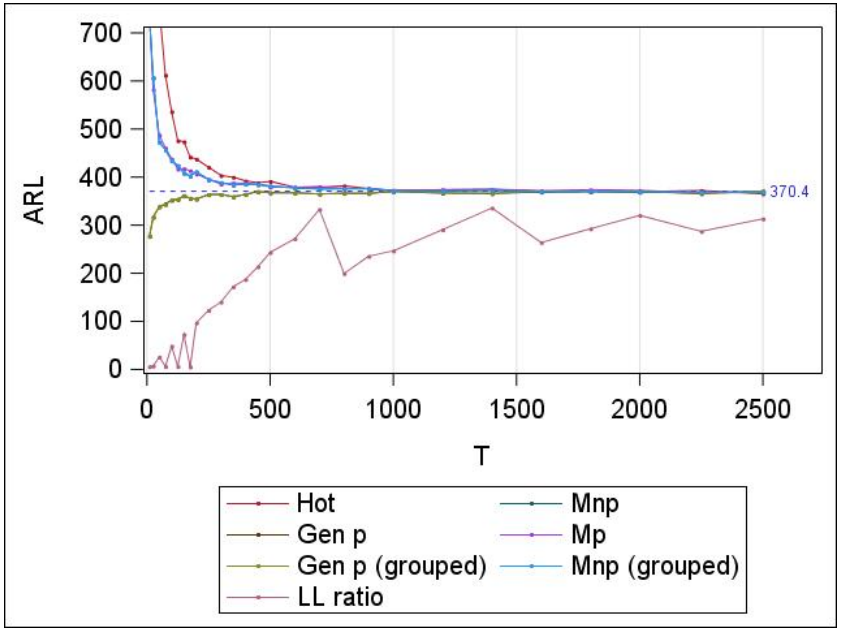



**Figure 15:** Total time points phase I versus average run length where $m_N = 2500$ and $v_N = 3333.33$.

| | | | Hotelling chart | | | | | Multivariate np chart | | | | | Generalized p chart | | | | | Multivariate poisson chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | Mean | Variance | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 250 | 10 | 13.3 | .0027 | 144.5 | 159.0 | 276.2 | 0 | .0027 | 198.1 | 218.6 | 278.3 | 0 | .0027 | 252.9 | 292.0 | 314.7 | 0 | .0027 | 215.7 | 246.1 | 290.7 | 0 |
| 250 | 10 | 20.0 | .0027 | 103.2 | 114.2 | 290.6 | 0 | .0027 | 146.7 | 165.4 | 278.2 | 0 | .0027 | 261.6 | 294.7 | 314.0 | 0 | .0027 | 147.7 | 164.2 | 276.7 | 0 |
| 250 | 10 | 40.0 | .0027 | 63.7 | 66.4 | 313.8 | 0 | .0027 | 91.8 | 102.6 | 296.9 | 0 | .0027 | 263.4 | 304.8 | 322.9 | 1 | .0027 | 95.0 | 106.3 | 295.2 | 0 |
| 250 | 100 | 133.3 | .0027 | 363.7 | 408.7 | 408.5 | 3 | .0027 | 380.8 | 430.1 | 429.6 | 5 | .0027 | 365.1 | 370.7 | 370.6 | 0 | .0027 | 390.5 | 429.8 | 430.2 | 2 |
| 250 | 100 | 200.0 | .0027 | 315.8 | 352.2 | 356.2 | 1 | .0027 | 363.4 | 424.3 | 423.8 | 3 | .0027 | 363.7 | 366.8 | 366.7 | 0 | .0027 | 358.8 | 422.3 | 421.9 | 4 |
| 250 | 100 | 400.0 | .0027 | 224.7 | 248.8 | 288.1 | 0 | .0027 | 264.7 | 300.7 | 318.7 | 0 | .0027 | 365.8 | 367.9 | 367.7 | 0 | .0027 | 272.1 | 319.4 | 334.1 | 0 |
| 250 | 1000 | 1333.3 | .0027 | 430.4 | 495.2 | 498.8 | 0 | .0027 | 406.5 | 511.4 | 512.6 | 9 | .0027 | 363.8 | 363.2 | 363.2 | 0 | .0027 | 395.8 | 479.7 | 480.3 | 7 |
| 250 | 1000 | 2000.0 | .0027 | 418.7 | 491.3 | 493.6 | 5 | .0027 | 404.1 | 479.5 | 480.6 | 4 | .0027 | 361.3 | 370.4 | 370.5 | 0 | .0027 | 408.0 | 474.1 | 475.5 | 5 |
| 250 | 1000 | 4000.0 | .0027 | 395.9 | 457.5 | 458.2 | 2 | .0027 | 376.2 | 436.9 | 436.9 | 1 | .0027 | 360.2 | 374.6 | 374.7 | 1 | .0027 | 383.0 | 453.0 | 453.0 | 3 |
| 250 | 10000 | 13333.3 | .0027 | 423.4 | 483.8 | 486.6 | 2 | .0027 | 398.4 | 491.5 | 492.3 | 10 | .0027 | 359.4 | 366.6 | 366.7 | 0 | .0027 | 400.0 | 478.6 | 479.5 | 6 |
| 250 | 10000 | 20000.0 | .0027 | 416.6 | 470.3 | 472.5 | 2 | .0027 | 393.9 | 472.0 | 472.5 | 6 | .0027 | 364.1 | 372.9 | 372.9 | 0 | .0027 | 402.4 | 481.9 | 482.8 | 7 |
| 250 | 10000 | 40000.0 | .0027 | 440.2 | 512.1 | 516.7 | 5 | .0027 | 402.9 | 486.0 | 487.0 | 6 | .0027 | 361.7 | 367.3 | 367.3 | 0 | .0027 | 406.3 | 482.0 | 483.2 | 7 |
| 2500 | 10 | 13.3 | .0027 | 140.0 | 143.7 | 271.5 | 0 | .0027 | 193.1 | 199.3 | 266.7 | 0 | .0027 | 224.0 | 225.2 | 268.5 | 0 | .0027 | 211.1 | 230.3 | 280.0 | 0 |
| 2500 | 10 | 20.0 | .0027 | 95.8 | 98.9 | 291.9 | 0 | .0027 | 139.8 | 141.2 | 270.4 | 0 | .0027 | 232.2 | 236.9 | 274.2 | 0 | .0027 | 143.9 | 144.6 | 268.7 | 0 |
| 2500 | 10 | 40.0 | .0027 | 60.4 | 59.0 | 315.6 | 0 | .0027 | 86.9 | 87.0 | 296.5 | 0 | .0027 | 230.5 | 228.3 | 267.7 | 0 | .0027 | 88.8 | 90.6 | 295.8 | 0 |
| 2500 | 100 | 133.3 | .0027 | 327.2 | 327.4 | 330.2 | 0 | .0027 | 359.0 | 352.5 | 352.6 | 0 | .0027 | 371.0 | 374.0 | 373.8 | 0 | .0027 | 369.6 | 358.8 | 358.8 | 0 |
| 2500 | 100 | 200.0 | .0027 | 281.2 | 291.4 | 304.7 | 0 | .0027 | 343.4 | 353.3 | 353.9 | 0 | .0027 | 365.2 | 369.1 | 369.0 | 0 | .0027 | 337.3 | 345.3 | 346.8 | 0 |
| 2500 | 100 | 400.0 | .0027 | 202.9 | 201.0 | 261.7 | 0 | .0027 | 249.7 | 253.9 | 281.1 | 0 | .0027 | 371.4 | 379.7 | 379.5 | 0 | .0027 | 252.5 | 255.4 | 281.2 | 0 |
| 2500 | 1000 | 1333.3 | .0027 | 368.0 | 383.9 | 383.8 | 0 | .0027 | 365.8 | 371.3 | 371.2 | 0 | .0027 | 373.1 | 378.5 | 378.5 | 0 | .0027 | 374.5 | 394.4 | 394.4 | 1 |
| 2500 | 1000 | 2000.0 | .0027 | 361.1 | 364.9 | 364.9 | 0 | .0027 | 367.9 | 374.9 | 374.8 | 1 | .0027 | 366.3 | 371.6 | 371.5 | 0 | .0027 | 380.8 | 379.0 | 379.0 | 0 |
| 2500 | 1000 | 4000.0 | .0027 | 343.8 | 349.9 | 350.8 | 0 | .0027 | 359.4 | 367.3 | 367.4 | 0 | .0027 | 366.4 | 379.2 | 379.1 | 1 | .0027 | 356.0 | 360.2 | 360.4 | 0 |
| 2500 | 10000 | 13333.3 | .0027 | 368.0 | 377.8 | 377.7 | 0 | .0027 | 369.1 | 377.1 | 377.1 | 0 | .0027 | 370.0 | 378.4 | 378.3 | 0 | .0027 | 371.1 | 376.4 | 376.3 | 1 |
| 2500 | 10000 | 20000.0 | .0027 | 373.5 | 386.1 | 386.1 | 0 | .0027 | 370.3 | 385.4 | 385.3 | 1 | .0027 | 373.9 | 381.3 | 381.2 | 0 | .0027 | 379.4 | 390.7 | 390.7 | 0 |
| 2500 | 10000 | 40000.0 | .0027 | 380.3 | 388.5 | 388.6 | 0 | .0027 | 370.5 | 369.2 | 369.1 | 1 | .0027 | 372.7 | 376.2 | 376.2 | 0 | .0027 | 371.7 | 372.2 | 372.1 | 0 |

**Table 20:** (1/2) Estimated ARL for three intervals with failure rate $\alpha = 0.0027$, variable $N$.

| | | | multivariate np chart (grouped) | | | | | Generalized p chart (grouped) | | | | | Likelihood ratio control chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | Mean | Variance | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 250 | 10 | 13.3 | .0027 | 198.1 | 218.9 | 278.6 | 0 | .0027 | 245.8 | 269.4 | 296.8 | 0 | .0027 | 161.2 | 481.7 | 525.1 | 52 |
| 250 | 10 | 20.0 | .0027 | 146.2 | 165.1 | 278.4 | 0 | .0027 | 253.1 | 282.4 | 305.8 | 0 | .0027 | 333.9 | 891.2 | 891.9 | 202 |
| 250 | 10 | 40.0 | .0027 | 91.4 | 102.2 | 297.1 | 0 | .0027 | 249.6 | 270.8 | 296.5 | 0 | .0027 | 1460.2 | 1833.6 | 2132.8 | 1347 |
| 250 | 100 | 133.3 | .0027 | 380.7 | 430.2 | 430.2 | 5 | .0027 | 365.0 | 370.8 | 370.8 | 0 | .0027 | 117.6 | 233.0 | 343.8 | 2 |
| 250 | 100 | 200.0 | .0027 | 363.4 | 424.3 | 424.3 | 3 | .0027 | 365.1 | 371.0 | 371.0 | 0 | .0027 | 120.8 | 263.2 | 362.7 | 5 |
| 250 | 100 | 400.0 | .0027 | 264.2 | 298.0 | 316.4 | 0 | .0027 | 368.3 | 370.6 | 370.6 | 0 | .0027 | 122.6 | 246.0 | 349.1 | 2 |
| 250 | 1000 | 1333.3 | .0027 | 406.5 | 511.4 | 512.6 | 9 | .0027 | 363.0 | 362.6 | 362.6 | 0 | .0027 | 123.9 | 251.4 | 352.1 | 3 |
| 250 | 1000 | 2000.0 | .0027 | 404.1 | 479.5 | 480.6 | 4 | .0027 | 360.7 | 369.7 | 369.7 | 0 | .0027 | 125.5 | 277.7 | 370.2 | 6 |
| 250 | 1000 | 4000.0 | .0027 | 376.2 | 436.9 | 436.9 | 1 | .0027 | 361.8 | 376.5 | 376.6 | 1 | .0027 | 125.5 | 275.1 | 368.3 | 5 |
| 250 | 10000 | 13333.3 | .0027 | 398.4 | 491.5 | 492.3 | 10 | .0027 | 359.9 | 367.5 | 367.6 | 0 | .0027 | 124.1 | 263.9 | 360.9 | 3 |
| 250 | 10000 | 20000.0 | .0027 | 393.9 | 472.0 | 472.5 | 6 | .0027 | 364.2 | 372.9 | 372.9 | 0 | .0027 | 127.4 | 269.5 | 362.9 | 2 |
| 250 | 10000 | 40000.0 | .0027 | 402.9 | 486.0 | 487.0 | 6 | .0027 | 361.4 | 366.7 | 366.8 | 0 | .0027 | 123.1 | 266.5 | 363.6 | 5 |
| 2500 | 10 | 13.3 | .0027 | 192.9 | 197.4 | 265.4 | 0 | .0027 | 224.9 | 226.4 | 269.1 | 0 | .0027 | 455.2 | 757.9 | 762.6 | 101 |
| 2500 | 10 | 20.0 | .0027 | 139.6 | 140.9 | 270.4 | 0 | .0027 | 232.1 | 235.9 | 273.4 | 0 | .0027 | 1620.7 | 1706.1 | 2115.0 | 1251 |
| 2500 | 10 | 40.0 | .0027 | 86.9 | 86.9 | 296.5 | 0 | .0027 | 231.5 | 229.1 | 267.9 | 0 | .0027 | 3970.1 | 323.8 | 3614.3 | 3964 |
| 2500 | 100 | 133.3 | .0027 | 358.9 | 352.5 | 352.6 | 0 | .0027 | 371.5 | 374.6 | 374.6 | 0 | .0027 | 307.4 | 371.1 | 376.4 | 2 |
| 2500 | 100 | 200.0 | .0027 | 343.4 | 353.3 | 354.3 | 0 | .0027 | 364.4 | 368.4 | 368.4 | 0 | .0027 | 307.6 | 395.0 | 400.0 | 4 |
| 2500 | 100 | 400.0 | .0027 | 249.7 | 253.9 | 281.1 | 0 | .0027 | 372.3 | 381.6 | 381.6 | 0 | .0027 | 313.9 | 391.3 | 395.3 | 1 |
| 2500 | 1000 | 1333.3 | .0027 | 365.8 | 371.3 | 371.2 | 0 | .0027 | 373.1 | 378.8 | 378.7 | 0 | .0027 | 316.1 | 396.9 | 400.5 | 4 |
| 2500 | 1000 | 2000.0 | .0027 | 367.9 | 374.9 | 374.9 | 1 | .0027 | 366.6 | 371.8 | 371.8 | 0 | .0027 | 309.1 | 375.9 | 380.8 | 0 |
| 2500 | 1000 | 4000.0 | .0027 | 359.4 | 367.3 | 367.4 | 0 | .0027 | 366.4 | 379.3 | 379.2 | 1 | .0027 | 311.0 | 389.5 | 394.0 | 2 |

**Table 21:** (2/2) Estimated ARL for three intervals with failure rate $\alpha = 0.0027$, variable $N$.

### Out of control ARL

For every chart we have changed the out of control rate such that the average run length is close to 370.4 when we monitor a incontrol process (see Appendix C.1). When the number of particles is low, $M_N = 10$ and $m_N = 100$, in all charts the average run length is larger than 370.4 . However, when the number of particles is larger the average run length is much lower than 370.4. In the setting with a average number of particles $N_M = 1000$ The Hotelling $T^2$ chart en generalized p chart monitor early out of control points in comparison with other charts. Increasing the variance has effect on the average run length in all chart. How larger the variance how longer the average run length, when the mean number of particles is large the variance has less effect on the Hotelling $T^2$ chart and generalized p chart. When the average number of particles is large , $M_N = 10000$ and $v_N = 40000$, the multivariate p chart has still a average run length close to 370.4.

| | | | | | Hotelling chart | | | | | Multivariate np chart | | | | | Generalized p chart | | | | | Multivariate poisson chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | $\mu$ | $\sigma$ | Mean | Variance | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 0.400 | 0.75 | 10 | 13.3 | .00040 | 542.1 | 543.8 | 570.2 | 2 | .00090 | 630.1 | 635.5 | 686.4 | 9 | .00170 | 945.0 | 906.4 | 1073.1 | 58 | .00105 | 446.4 | 480.3 | 486.2 | 1 |
| 2500 | 0.450 | 0.75 | 10 | 13.3 | .00040 | 467.4 | 482.0 | 491.6 | 0 | .00090 | 472.6 | 468.0 | 478.9 | 2 | .00170 | 594.9 | 606.5 | 646.7 | 8 | .00105 | 406.4 | 433.2 | 434.6 | 0 |
| 2500 | 0.475 | 0.75 | 10 | 13.3 | .00040 | 418.0 | 436.0 | 438.6 | 1 | .00090 | 415.5 | 409.4 | 411.8 | 0 | .00170 | 472.5 | 489.5 | 500.0 | 3 | .00105 | 391.3 | 418.0 | 418.5 | 0 |
| 2500 | 0.400 | 0.75 | 10 | 20.0 | .00012 | 526.6 | 523.5 | 546.2 | 1 | .00040 | 582.8 | 576.2 | 613.8 | 1 | .00175 | 860.6 | 850.2 | 981.3 | 49 | .00045 | 441.0 | 440.4 | 446.0 | 0 |
| 2500 | 0.450 | 0.75 | 10 | 20.0 | .00012 | 479.8 | 489.6 | 501.6 | 2 | .00040 | 462.4 | 462.3 | 471.0 | 0 | .00175 | 566.0 | 584.9 | 616.7 | 4 | .00045 | 399.0 | 402.6 | 403.6 | 0 |
| 2500 | 0.475 | 0.75 | 10 | 20.0 | .00012 | 431.0 | 437.6 | 441.6 | 1 | .00040 | 412.2 | 411.8 | 413.5 | 0 | .00175 | 465.2 | 486.8 | 495.9 | 3 | .00045 | 386.3 | 390.8 | 391.1 | 0 |
| 2500 | 0.400 | 0.75 | 10 | 40.0 | .00002 | 378.5 | 383.5 | 383.5 | 0 | .00006 | 524.6 | 540.6 | 562.1 | 1 | .00160 | 869.3 | 848.5 | 984.3 | 47 | .00006 | 430.7 | 451.3 | 455.3 | 2 |
| 2500 | 0.450 | 0.75 | 10 | 40.0 | .00002 | 368.9 | 360.8 | 360.7 | 0 | .00006 | 441.0 | 441.4 | 446.9 | 0 | .00160 | 582.5 | 591.9 | 628.7 | 10 | .00006 | 404.5 | 407.7 | 409.1 | 0 |
| 2500 | 0.475 | 0.75 | 10 | 40.0 | .00002 | 350.6 | 352.7 | 353.2 | 0 | .00006 | 403.7 | 411.1 | 412.3 | 0 | .00160 | 460.4 | 476.4 | 484.7 | 4 | .00006 | 387.8 | 399.1 | 399.4 | 0 |
| 2500 | 0.400 | 0.75 | 100 | 133.3 | .00235 | 218.5 | 224.5 | 271.0 | 0 | .00260 | 440.6 | 452.9 | 458.2 | 2 | .00270 | 149.6 | 150.3 | 267.1 | 0 | .00270 | 409.6 | 408.6 | 410.4 | 0 |
| 2500 | 0.450 | 0.75 | 100 | 133.3 | .00235 | 393.1 | 402.8 | 403.4 | 0 | .00260 | 474.2 | 470.9 | 482.1 | 3 | .00270 | 394.6 | 394.9 | 395.6 | 0 | .00270 | 401.3 | 397.7 | 398.8 | 0 |
| 2500 | 0.475 | 0.75 | 100 | 133.3 | .00235 | 431.4 | 436.8 | 441.0 | 1 | .00260 | 435.2 | 425.5 | 430.3 | 1 | .00270 | 479.7 | 488.2 | 500.2 | 1 | .00270 | 386.1 | 382.2 | 382.4 | 0 |
| 2500 | 0.400 | 0.75 | 100 | 200.0 | .00200 | 213.6 | 217.6 | 268.2 | 0 | .00250 | 643.0 | 642.4 | 697.5 | 8 | .00270 | 147.8 | 147.9 | 267.2 | 0 | .00250 | 485.0 | 494.7 | 507.4 | 1 |
| 2500 | 0.450 | 0.75 | 100 | 200.0 | .00200 | 380.0 | 386.1 | 386.2 | 1 | .00250 | 528.0 | 525.6 | 548.3 | 2 | .00270 | 388.3 | 383.7 | 384.1 | 0 | .00250 | 416.7 | 424.5 | 426.5 | 0 |
| 2500 | 0.475 | 0.75 | 100 | 200.0 | .00200 | 423.7 | 419.0 | 422.4 | 0 | .00250 | 450.8 | 453.8 | 460.3 | 0 | .00270 | 464.1 | 460.4 | 469.7 | 0 | .00250 | 388.5 | 393.2 | 393.0 | 0 |
| 2500 | 0.400 | 0.75 | 100 | 400.0 | .00110 | 202.2 | 202.0 | 262.9 | 0 | .00150 | 821.3 | 796.3 | 915.0 | 43 | .00270 | 147.8 | 148.5 | 267.6 | 0 | .00160 | 492.8 | 494.9 | 509.7 | 2 |
| 2500 | 0.450 | 0.75 | 100 | 400.0 | .00110 | 367.2 | 375.3 | 375.3 | 0 | .00150 | 565.4 | 556.5 | 589.6 | 4 | .00270 | 393.1 | 399.5 | 400.0 | 1 | .00160 | 422.2 | 418.3 | 421.5 | 1 |
| 2500 | 0.475 | 0.75 | 100 | 400.0 | .00110 | 409.0 | 418.7 | 420.3 | 1 | .00150 | 458.2 | 448.3 | 456.7 | 0 | .00270 | 460.4 | 446.9 | 455.7 | 0 | .00160 | 389.8 | 384.8 | 385.3 | 0 |
| 2500 | 0.400 | 0.75 | 1000 | 1333.3 | .00270 | 4.3 | 3.8 | 366.1 | 0 | .00270 | 42.9 | 43.5 | 330.4 | 0 | .00270 | 3.0 | 2.5 | 367.4 | 0 | .00270 | 190.9 | 194.4 | 264.6 | 0 |
| 2500 | 0.450 | 0.75 | 1000 | 1333.3 | .00270 | 38.0 | 39.2 | 334.7 | 0 | .00270 | 169.6 | 174.2 | 265.8 | 0 | .00270 | 22.7 | 22.7 | 348.4 | 0 | .00270 | 322.8 | 325.0 | 328.4 | 0 |
| 2500 | 0.475 | 0.75 | 1000 | 1333.3 | .00270 | 155.5 | 158.1 | 266.7 | 0 | .00270 | 311.0 | 326.9 | 332.2 | 0 | .00270 | 107.3 | 108.2 | 284.5 | 0 | .00270 | 369.0 | 372.8 | 372.7 | 0 |
| 2500 | 0.400 | 0.75 | 1000 | 2000.0 | .00260 | 4.4 | 3.9 | 366.0 | 0 | .00270 | 73.5 | 74.5 | 306.1 | 0 | .00270 | 3.0 | 2.5 | 367.4 | 0 | .00275 | 266.5 | 274.9 | 293.8 | 0 |
| 2500 | 0.450 | 0.75 | 1000 | 2000.0 | .00260 | 38.9 | 39.5 | 333.8 | 0 | .00270 | 239.1 | 238.0 | 271.7 | 0 | .00270 | 23.0 | 22.9 | 348.1 | 0 | .00275 | 370.3 | 368.1 | 368.0 | 1 |
| 2500 | 0.475 | 0.75 | 1000 | 2000.0 | .00260 | 163.0 | 164.1 | 264.4 | 0 | .00270 | 361.5 | 359.9 | 359.9 | 0 | .00270 | 109.4 | 108.6 | 282.7 | 0 | .00275 | 386.5 | 389.5 | 389.7 | 1 |
| 2500 | 0.400 | 0.75 | 1000 | 4000.0 | .00250 | 4.6 | 4.0 | 365.8 | 0 | .00260 | 183.0 | 189.1 | 266.1 | 0 | .00270 | 3.1 | 2.5 | 367.3 | 0 | .00260 | 401.0 | 417.7 | 418.8 | 0 |
| 2500 | 0.450 | 0.75 | 1000 | 4000.0 | .00250 | 41.3 | 41.3 | 331.7 | 0 | .00260 | 382.6 | 394.9 | 395.0 | 1 | .00270 | 23.6 | 23.5 | 347.6 | 0 | .00260 | 426.3 | 447.0 | 450.4 | 0 |
| 2500 | 0.475 | 0.75 | 1000 | 4000.0 | .00250 | 158.5 | 161.7 | 266.5 | 0 | .00260 | 428.9 | 439.7 | 443.5 | 0 | .00270 | 107.2 | 106.4 | 283.8 | 0 | .00260 | 403.5 | 423.6 | 424.8 | 0 |
| 2500 | 0.400 | 0.75 | 10000 | 13333.3 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .00270 | 1.4 | 0.8 | 369.0 | 0 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .00270 | 16.0 | 15.5 | 354.7 | 0 |
| 2500 | 0.450 | 0.75 | 10000 | 13333.3 | .00270 | 1.2 | 0.5 | 369.2 | 0 | .00270 | 9.9 | 9.6 | 360.6 | 0 | .00270 | 1.1 | 0.4 | 369.3 | 0 | .00270 | 93.5 | 93.9 | 292.3 | 0 |
| 2500 | 0.475 | 0.75 | 10000 | 13333.3 | .00270 | 7.7 | 7.1 | 362.8 | 0 | .00270 | 61.9 | 62.4 | 314.7 | 0 | .00270 | 5.3 | 4.8 | 365.1 | 0 | .00270 | 235.3 | 237.3 | 273.0 | 0 |
| 2500 | 0.400 | 0.75 | 10000 | 20000.0 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .00270 | 2.1 | 1.6 | 368.3 | 0 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .00270 | 27.1 | 27.0 | 344.3 | 0 |
| 2500 | 0.450 | 0.75 | 10000 | 20000.0 | .00270 | 1.2 | 0.5 | 369.2 | 0 | .00270 | 17.4 | 16.6 | 353.4 | 0 | .00270 | 1.1 | 0.4 | 369.3 | 0 | .00270 | 133.9 | 137.1 | 273.4 | 0 |
| 2500 | 0.475 | 0.75 | 10000 | 20000.0 | .00270 | 7.7 | 7.4 | 362.7 | 0 | .00270 | 91.6 | 93.3 | 294.0 | 0 | .00270 | 5.2 | 4.6 | 365.2 | 0 | .00270 | 284.5 | 292.3 | 304.6 | 0 |
| 2500 | 0.400 | 0.75 | 10000 | 40000.0 | .00280 | 1.0 | 0.0 | 369.4 | 0 | .00270 | 5.8 | 5.2 | 364.6 | 0 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .00270 | 66.4 | 67.4 | 311.4 | 0 |
| 2500 | 0.450 | 0.75 | 10000 | 40000.0 | .00280 | 1.2 | 0.5 | 369.2 | 0 | .00270 | 44.8 | 44.7 | 328.6 | 0 | .00270 | 1.1 | 0.4 | 369.3 | 0 | .00270 | 222.7 | 228.9 | 272.3 | 0 |
| 2500 | 0.475 | 0.75 | 10000 | 40000.0 | .00280 | 7.8 | 7.4 | 362.7 | 0 | .00270 | 169.8 | 168.0 | 261.6 | 0 | .00270 | 5.2 | 4.8 | 365.2 | 0 | .00270 | 342.5 | 351.9 | 353.0 | 0 |

**Table 22:** (1/2) Estimated out of control ARL for three intervals, variable $N$.

| | | | | | multivariate np chart (grouped) | | | | | Generalized p chart (grouped) | | | | | Likelihood ratio control chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | $\mu$ | $\sigma$ | Mean | Variance | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 0.400 | 0.75 | 10 | 13.3 | .00090 | 631.3 | 637.8 | 689.0 | 9 | .00170 | 952.0 | 910.4 | 1080.2 | 60 | 0.0036 | 522.8 | 729.4 | 745.0 | 66 |
| 2500 | 0.450 | 0.75 | 10 | 13.3 | .00090 | 473.0 | 470.1 | 481.1 | 2 | .00170 | 600.7 | 609.0 | 651.0 | 6 | 0.0036 | 407.2 | 609.0 | 610.1 | 49 |
| 2500 | 0.475 | 0.75 | 10 | 13.3 | .00090 | 416.1 | 411.0 | 413.5 | 0 | .00170 | 476.4 | 493.9 | 505.1 | 4 | 0.0036 | 348.5 | 544.6 | 544.9 | 36 |
| 2500 | 0.400 | 0.75 | 10 | 20.0 | .00040 | 582.7 | 575.1 | 612.9 | 1 | .00175 | 875.5 | 858.3 | 995.9 | 53 | 0.0055 | 635.5 | 976.9 | 1012.1 | 209 |
| 2500 | 0.450 | 0.75 | 10 | 20.0 | .00040 | 462.4 | 461.7 | 470.7 | 0 | .00175 | 572.1 | 589.0 | 622.5 | 5 | 0.0055 | 514.8 | 880.7 | 892.3 | 169 |
| 2500 | 0.475 | 0.75 | 10 | 20.0 | .00040 | 412.8 | 411.9 | 414.0 | 0 | .00175 | 468.5 | 487.3 | 497.0 | 3 | 0.0055 | 465.2 | 845.5 | 850.7 | 154 |
| 2500 | 0.400 | 0.75 | 10 | 40.0 | .00006 | 522.9 | 539.7 | 560.8 | 1 | .00160 | 914.0 | 878.4 | 1032.9 | 53 | 0.0138 | 416.3 | 846.4 | 847.5 | 162 |
| 2500 | 0.450 | 0.75 | 10 | 40.0 | .00006 | 441.0 | 441.3 | 446.9 | 0 | .00160 | 617.1 | 614.0 | 661.7 | 11 | 0.0138 | 362.9 | 813.5 | 813.5 | 157 |
| 2500 | 0.475 | 0.75 | 10 | 40.0 | .00006 | 403.6 | 410.9 | 412.2 | 0 | .00160 | 488.1 | 501.1 | 514.7 | 5 | 0.0138 | 336.4 | 794.3 | 794.9 | 148 |
| 2500 | 0.400 | 0.75 | 100 | 133.3 | .00260 | 440.6 | 452.9 | 458.2 | 2 | .00270 | 150.1 | 150.9 | 267.0 | 0 | 0.0024 | 76.9 | 98.3 | 309.5 | 0 |
| 2500 | 0.450 | 0.75 | 100 | 133.3 | .00260 | 474.2 | 470.9 | 482.1 | 3 | .00270 | 394.1 | 393.2 | 393.8 | 0 | 0.0024 | 210.5 | 280.9 | 323.2 | 3 |
| 2500 | 0.475 | 0.75 | 100 | 133.3 | .00260 | 435.3 | 425.6 | 430.5 | 1 | .00270 | 479.9 | 488.9 | 500.9 | 1 | 0.0024 | 332.7 | 421.9 | 423.5 | 7 |
| 2500 | 0.400 | 0.75 | 100 | 200.0 | .00250 | 643.0 | 642.4 | 697.8 | 8 | .00270 | 148.0 | 148.3 | 267.3 | 0 | 0.0024 | 78.9 | 102.6 | 309.0 | 0 |
| 2500 | 0.450 | 0.75 | 100 | 200.0 | .00250 | 528.0 | 525.6 | 548.7 | 2 | .00270 | 389.4 | 384.8 | 385.2 | 0 | 0.0024 | 216.2 | 277.2 | 317.2 | 1 |
| 2500 | 0.475 | 0.75 | 100 | 200.0 | .00250 | 451.0 | 454.2 | 461.3 | 0 | .00270 | 464.2 | 459.0 | 468.4 | 0 | 0.0024 | 330.5 | 447.7 | 449.4 | 4 |
| 2500 | 0.400 | 0.75 | 100 | 400.0 | .00150 | 821.8 | 796.6 | 915.5 | 43 | .00270 | 147.7 | 148.5 | 267.7 | 0 | 0.0024 | 90.9 | 116.0 | 302.6 | 0 |
| 2500 | 0.450 | 0.75 | 100 | 400.0 | .00150 | 565.4 | 556.5 | 589.7 | 4 | .00270 | 392.9 | 401.2 | 401.8 | 1 | 0.0024 | 236.0 | 323.4 | 350.2 | 2 |
| 2500 | 0.475 | 0.75 | 100 | 400.0 | .00150 | 458.1 | 448.3 | 456.7 | 0 | .00270 | 461.5 | 449.0 | 458.1 | 0 | 0.0024 | 346.3 | 455.5 | 456.1 | 9 |
| 2500 | 0.400 | 0.75 | 1000 | 1333.3 | .00270 | 42.9 | 43.5 | 330.4 | 0 | .00270 | 3.0 | 2.5 | 367.4 | 0 | 0.0024 | 2.8 | 2.4 | 367.6 | 0 |
| 2500 | 0.450 | 0.75 | 1000 | 1333.3 | .00270 | 169.6 | 174.2 | 265.8 | 0 | .00270 | 22.8 | 22.7 | 348.4 | 0 | 0.0024 | 20.8 | 23.8 | 350.4 | 0 |
| 2500 | 0.475 | 0.75 | 1000 | 1333.3 | .00270 | 311.0 | 326.9 | 332.2 | 0 | .00270 | 107.4 | 108.2 | 284.4 | 0 | 0.0024 | 99.4 | 121.6 | 297.1 | 0 |
| 2500 | 0.400 | 0.75 | 1000 | 2000.0 | .00270 | 73.5 | 74.5 | 306.1 | 0 | .00270 | 3.0 | 2.5 | 367.4 | 0 | 0.0024 | 2.8 | 2.5 | 367.6 | 0 |
| 2500 | 0.450 | 0.75 | 1000 | 2000.0 | .00270 | 239.1 | 238.0 | 271.8 | 0 | .00270 | 23.0 | 22.9 | 348.1 | 0 | 0.0024 | 21.0 | 24.0 | 350.3 | 0 |
| 2500 | 0.475 | 0.75 | 1000 | 2000.0 | .00270 | 361.5 | 359.9 | 359.9 | 0 | .00270 | 109.2 | 108.3 | 282.7 | 0 | 0.0024 | 100.5 | 124.5 | 297.2 | 0 |
| 2500 | 0.400 | 0.75 | 1000 | 4000.0 | .00260 | 183.0 | 189.1 | 266.2 | 0 | .00270 | 3.1 | 2.5 | 367.3 | 0 | 0.0024 | 2.8 | 2.4 | 367.6 | 0 |
| 2500 | 0.450 | 0.75 | 1000 | 4000.0 | .00260 | 382.6 | 394.9 | 395.0 | 1 | .00270 | 23.6 | 23.5 | 347.6 | 0 | 0.0024 | 21.9 | 25.5 | 349.5 | 0 |
| 2500 | 0.475 | 0.75 | 1000 | 4000.0 | .00260 | 428.7 | 439.6 | 443.4 | 0 | .00270 | 107.2 | 106.4 | 283.9 | 0 | 0.0024 | 101.1 | 131.0 | 299.5 | 0 |
| 2500 | 0.400 | 0.75 | 10000 | 13333.3 | .00270 | 1.4 | 0.8 | 369.0 | 0 | .00270 | 1.0 | 0.0 | 369.4 | 0 | 0.0024 | 1.0 | 0.0 | 369.4 | 0 |
| 2500 | 0.450 | 0.75 | 10000 | 13333.3 | .00270 | 9.9 | 9.6 | 360.6 | 0 | .00270 | 1.1 | 0.4 | 369.3 | 0 | 0.0024 | 1.1 | 0.4 | 369.3 | 0 |
| 2500 | 0.475 | 0.75 | 10000 | 13333.3 | .00270 | 61.9 | 62.4 | 314.7 | 0 | .00270 | 5.3 | 4.8 | 365.1 | 0 | 0.0024 | 5.2 | 5.1 | 365.2 | 0 |
| 2500 | 0.400 | 0.75 | 10000 | 20000.0 | .00270 | 2.1 | 1.6 | 368.3 | 0 | .00270 | 1.0 | 0.0 | 369.4 | 0 | 0.0024 | 1.0 | 0.0 | 369.4 | 0 |
| 2500 | 0.450 | 0.75 | 10000 | 20000.0 | .00270 | 17.4 | 16.6 | 353.4 | 0 | .00270 | 1.1 | 0.4 | 369.3 | 0 | 0.0024 | 1.1 | 0.4 | 369.3 | 0 |
| 2500 | 0.475 | 0.75 | 10000 | 20000.0 | .00270 | 91.6 | 93.3 | 294.0 | 0 | .00270 | 5.2 | 4.6 | 365.2 | 0 | 0.0024 | 5.1 | 4.8 | 365.4 | 0 |
| 2500 | 0.400 | 0.75 | 10000 | 40000.0 | .00270 | 5.8 | 5.2 | 364.6 | 0 | .00270 | 1.0 | 0.0 | 369.4 | 0 | 0.0024 | 1.0 | 0.0 | 369.4 | 0 |
| 2500 | 0.450 | 0.75 | 10000 | 40000.0 | .00270 | 44.8 | 44.7 | 328.6 | 0 | .00270 | 1.1 | 0.4 | 369.3 | 0 | 0.0024 | 1.1 | 0.4 | 369.3 | 0 |
| 2500 | 0.475 | 0.75 | 10000 | 40000.0 | .00270 | 169.8 | 168.0 | 261.7 | 0 | .00270 | 5.2 | 4.8 | 365.2 | 0 | 0.0024 | 5.1 | 4.9 | 365.3 | 0 |

**Table 23:** (2/2) Estimated out of control ARL for three intervals, variable $N$.

## 4.5 Varying numbers of particles and four particle size intervals

### In control ARL

The results are similar as with the three interval simulation study. From figure 16 we see that all charts need a sufficiently large number of phase I time points to converge to 370.4, except the log likelihood chart needs many phase I time points to converge to 312.5. When the number of

particles is low only the generalized p chart has a average run length close to 370.4. When the number of particles is larger than 100 all charts are close to the expected average run length as long as the variance is small. When the number of particles is low, $N = 10$, the log likelihood ratio chart does not observe a out of control. When the number of particles is larger the average run length is close to 312.5.

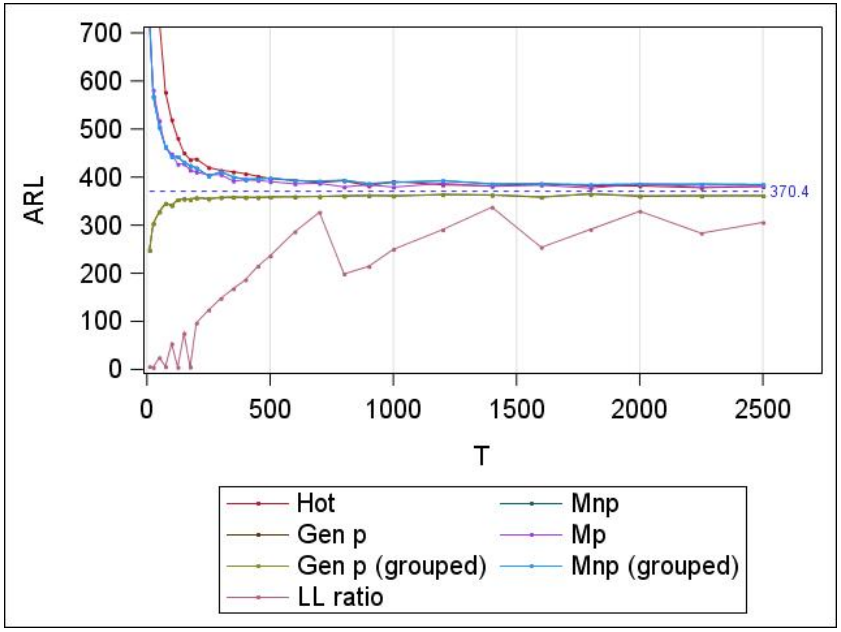



**Figure 16:** Total time points phase I versus average run length where $m_N = 2500$ and $v_N = 3333.33$.

| | | | Hotelling chart | | | | | Multivariate np chart | | | | | Generalized p chart | | | | | Multivariate poisson chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | Mean | Variance | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 250 | 10 | 13.3 | .0027 | 112.9 | 123.0 | 285.3 | 0 | .0027 | 216.7 | 252.6 | 295.6 | 0 | .0027 | 321.0 | 342.3 | 345.8 | 0 | .0027 | 225.5 | 260.5 | 298.0 | 0 |
| 250 | 10 | 20.0 | .0027 | 80.8 | 86.8 | 302.4 | 0 | .0027 | 147.7 | 170.4 | 280.4 | 0 | .0027 | 325.0 | 339.1 | 342.0 | 0 | .0027 | 152.7 | 178.0 | 281.2 | 0 |
| 250 | 10 | 40.0 | .0027 | 50.3 | 51.7 | 324.2 | 0 | .0027 | 89.8 | 97.8 | 297.1 | 0 | .0027 | 346.6 | 377.5 | 378.2 | 1 | .0027 | 90.8 | 96.2 | 295.7 | 0 |
| 250 | 100 | 133.3 | .0027 | 339.3 | 371.7 | 372.9 | 0 | .0027 | 391.3 | 469.7 | 470.1 | 6 | .0027 | 363.9 | 374.1 | 374.1 | 0 | .0027 | 395.7 | 471.1 | 471.7 | 7 |
| 250 | 100 | 200.0 | .0027 | 285.8 | 314.3 | 325.5 | 0 | .0027 | 355.9 | 429.7 | 429.9 | 3 | .0027 | 354.8 | 353.2 | 353.5 | 0 | .0027 | 359.3 | 442.2 | 442.3 | 4 |
| 250 | 100 | 400.0 | .0027 | 193.3 | 205.0 | 270.8 | 0 | .0027 | 264.6 | 302.2 | 320.2 | 0 | .0027 | 361.7 | 363.5 | 363.5 | 0 | .0027 | 265.1 | 302.8 | 320.6 | 0 |
| 250 | 1000 | 1333.3 | .0027 | 419.8 | 458.7 | 461.2 | 2 | .0027 | 399.9 | 465.8 | 466.6 | 5 | .0027 | 358.4 | 358.7 | 358.8 | 0 | .0027 | 409.4 | 473.1 | 474.7 | 1 |
| 250 | 1000 | 2000.0 | .0027 | 406.0 | 455.2 | 456.5 | 1 | .0027 | 399.7 | 477.4 | 478.2 | 6 | .0027 | 362.1 | 354.2 | 354.2 | 0 | .0027 | 405.0 | 496.5 | 497.6 | 9 |
| 250 | 1000 | 4000.0 | .0027 | 370.2 | 412.3 | 412.2 | 3 | .0027 | 379.3 | 454.0 | 454.1 | 3 | .0027 | 359.2 | 355.3 | 355.4 | 0 | .0027 | 379.2 | 455.8 | 455.9 | 6 |
| 250 | 10000 | 13333.3 | .0027 | 423.3 | 460.5 | 463.4 | 3 | .0027 | 402.4 | 465.6 | 466.6 | 5 | .0027 | 359.1 | 360.8 | 360.9 | 0 | .0027 | 400.8 | 465.2 | 466.1 | 4 |
| 250 | 10000 | 20000.0 | .0027 | 424.9 | 471.5 | 474.6 | 1 | .0027 | 403.6 | 467.6 | 468.6 | 3 | .0027 | 356.0 | 355.7 | 355.9 | 0 | .0027 | 409.1 | 484.9 | 486.3 | 7 |
| 250 | 10000 | 40000.0 | .0027 | 414.1 | 469.5 | 471.4 | 0 | .0027 | 407.7 | 478.7 | 480.1 | 3 | .0027 | 352.5 | 355.2 | 355.6 | 0 | .0027 | 402.0 | 472.8 | 473.7 | 3 |
| 2500 | 10 | 13.3 | .0027 | 105.4 | 108.4 | 286.3 | 0 | .0027 | 201.2 | 206.4 | 266.8 | 0 | .0027 | 335.8 | 340.3 | 341.9 | 0 | .0027 | 224.2 | 243.7 | 284.1 | 0 |
| 2500 | 10 | 20.0 | .0027 | 75.8 | 77.4 | 304.5 | 0 | .0027 | 138.9 | 144.5 | 272.9 | 0 | .0027 | 342.8 | 344.2 | 345.3 | 0 | .0027 | 149.3 | 155.0 | 270.0 | 0 |
| 2500 | 10 | 40.0 | .0027 | 47.9 | 46.9 | 325.9 | 0 | .0027 | 88.0 | 87.9 | 295.7 | 0 | .0027 | 368.0 | 369.4 | 369.4 | 0 | .0027 | 89.0 | 87.1 | 294.5 | 0 |
| 2500 | 100 | 133.3 | .0027 | 303.0 | 302.6 | 309.9 | 0 | .0027 | 365.5 | 373.1 | 373.0 | 0 | .0027 | 370.9 | 373.0 | 372.9 | 0 | .0027 | 372.3 | 376.8 | 376.8 | 0 |
| 2500 | 100 | 200.0 | .0027 | 250.8 | 248.8 | 276.0 | 0 | .0027 | 324.7 | 325.8 | 328.9 | 0 | .0027 | 366.5 | 367.7 | 367.6 | 0 | .0027 | 327.5 | 337.2 | 339.9 | 0 |
| 2500 | 100 | 400.0 | .0027 | 175.4 | 175.1 | 262.0 | 0 | .0027 | 251.8 | 263.2 | 288.6 | 0 | .0027 | 367.0 | 360.9 | 360.8 | 0 | .0027 | 251.4 | 259.2 | 285.2 | 0 |
| 2500 | 1000 | 1333.3 | .0027 | 380.4 | 385.5 | 385.5 | 0 | .0027 | 382.1 | 384.3 | 384.4 | 0 | .0027 | 364.2 | 365.6 | 365.6 | 0 | .0027 | 381.4 | 390.2 | 390.3 | 0 |
| 2500 | 1000 | 2000.0 | .0027 | 358.4 | 363.6 | 363.7 | 0 | .0027 | 377.5 | 386.7 | 386.7 | 0 | .0027 | 371.8 | 370.4 | 370.3 | 0 | .0027 | 374.7 | 386.0 | 385.9 | 0 |
| 2500 | 1000 | 4000.0 | .0027 | 332.2 | 330.5 | 332.6 | 0 | .0027 | 350.8 | 365.6 | 366.0 | 0 | .0027 | 372.6 | 370.1 | 370.0 | 0 | .0027 | 350.3 | 370.2 | 370.6 | 0 |
| 2500 | 10000 | 13333.3 | .0027 | 385.0 | 391.9 | 392.1 | 0 | .0027 | 381.1 | 383.8 | 384.0 | 0 | .0027 | 367.7 | 367.5 | 367.5 | 0 | .0027 | 381.3 | 382.4 | 382.5 | 0 |
| 2500 | 10000 | 20000.0 | .0027 | 380.7 | 391.6 | 391.7 | 1 | .0027 | 383.1 | 396.7 | 396.8 | 0 | .0027 | 364.3 | 360.3 | 360.3 | 0 | .0027 | 382.0 | 395.2 | 395.3 | 0 |
| 2500 | 10000 | 40000.0 | .0027 | 372.9 | 370.6 | 370.5 | 0 | .0027 | 376.6 | 371.7 | 371.7 | 0 | .0027 | 359.4 | 361.4 | 361.4 | 0 | .0027 | 383.8 | 389.9 | 390.0 | 0 |

**Table 24:** (1/2) Estimated ARL for four intervals with failure rate $\alpha = 0.0027$, variable $N$.

| | | | multivariate np chart (grouped) | | | | | Generalized p chart (grouped) | | | | | Likelihood ratio control chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | Mean | Variance | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 250 | 10 | 13.3 | .0027 | 215.1 | 248.4 | 293.0 | 0 | .0027 | 321.2 | 346.7 | 350.2 | 0 | .0027 | 2251.8 | 1644.2 | 2498.5 | 1646 |
| 250 | 10 | 20.0 | .0027 | 146.9 | 168.5 | 279.9 | 0 | .0027 | 326.0 | 336.9 | 339.8 | 0 | .0027 | 2995.9 | 1481.1 | 3014.3 | 2565 |
| 250 | 10 | 40.0 | .0027 | 89.6 | 97.6 | 297.3 | 0 | .0027 | 350.2 | 371.8 | 372.3 | 0 | .0027 | 3867.7 | 626.0 | 3552.9 | 3797 |
| 250 | 100 | 133.3 | .0027 | 390.7 | 468.8 | 469.2 | 6 | .0027 | 364.3 | 375.0 | 375.0 | 0 | .0027 | 120.9 | 243.0 | 348.2 | 2 |
| 250 | 100 | 200.0 | .0027 | 355.5 | 429.2 | 429.4 | 3 | .0027 | 353.7 | 352.7 | 353.1 | 0 | .0027 | 126.4 | 280.4 | 371.7 | 5 |
| 250 | 100 | 400.0 | .0027 | 264.7 | 302.5 | 320.4 | 0 | .0027 | 357.3 | 359.6 | 359.7 | 0 | .0027 | 117.7 | 254.9 | 358.9 | 4 |
| 250 | 1000 | 1333.3 | .0027 | 399.9 | 465.8 | 466.6 | 5 | .0027 | 358.8 | 358.8 | 359.0 | 0 | .0027 | 127.3 | 254.7 | 352.0 | 4 |
| 250 | 1000 | 2000.0 | .0027 | 399.7 | 477.4 | 478.3 | 6 | .0027 | 362.7 | 354.9 | 354.9 | 0 | .0027 | 123.6 | 241.5 | 345.2 | 2 |
| 250 | 1000 | 4000.0 | .0027 | 379.3 | 454.0 | 454.1 | 3 | .0027 | 360.4 | 358.0 | 358.1 | 0 | .0027 | 121.2 | 249.8 | 352.8 | 4 |
| 250 | 10000 | 13333.3 | .0027 | 402.4 | 465.6 | 466.6 | 5 | .0027 | 359.2 | 360.6 | 360.7 | 0 | .0027 | 123.2 | 234.4 | 340.6 | 2 |
| 250 | 10000 | 20000.0 | .0027 | 403.6 | 467.6 | 468.7 | 3 | .0027 | 356.0 | 355.8 | 356.0 | 0 | .0027 | 126.4 | 259.1 | 355.9 | 3 |
| 250 | 10000 | 40000.0 | .0027 | 407.7 | 478.7 | 480.1 | 3 | .0027 | 353.2 | 356.3 | 356.7 | 0 | .0027 | 125.4 | 259.3 | 356.7 | 2 |
| 2500 | 10 | 13.3 | .0027 | 201.5 | 206.8 | 267.0 | 0 | .0027 | 336.4 | 340.5 | 342.2 | 0 | .0027 | 3981.5 | 227.7 | 3618.3 | 3969 |
| 2500 | 10 | 20.0 | .0027 | 139.0 | 144.5 | 272.8 | 0 | .0027 | 342.3 | 342.1 | 343.2 | 0 | .0027 | 4000.0 | 0.0 | 3629.6 | 4000 |
| 2500 | 10 | 40.0 | .0027 | 88.0 | 87.9 | 295.7 | 0 | .0027 | 371.4 | 373.0 | 373.0 | 0 | .0027 | 4000.0 | 0.0 | 3629.6 | 4000 |
| 2500 | 100 | 133.3 | .0027 | 365.3 | 373.1 | 373.0 | 0 | .0027 | 371.1 | 372.9 | 372.8 | 0 | .0027 | 312.5 | 389.2 | 393.4 | 5 |
| 2500 | 100 | 200.0 | .0027 | 324.7 | 325.8 | 329.0 | 0 | .0027 | 366.6 | 367.1 | 367.1 | 0 | .0027 | 308.1 | 389.8 | 394.7 | 5 |
| 2500 | 100 | 400.0 | .0027 | 251.8 | 263.2 | 288.6 | 0 | .0027 | 367.1 | 360.6 | 360.6 | 0 | .0027 | 317.4 | 385.4 | 389.0 | 3 |
| 2500 | 1000 | 1333.3 | .0027 | 382.1 | 384.3 | 384.4 | 0 | .0027 | 363.8 | 365.4 | 365.4 | 0 | .0027 | 310.5 | 397.9 | 402.3 | 2 |
| 2500 | 1000 | 2000.0 | .0027 | 377.5 | 386.7 | 386.7 | 0 | .0027 | 372.1 | 370.4 | 370.4 | 0 | .0027 | 314.5 | 400.0 | 403.8 | 5 |
| 2500 | 1000 | 4000.0 | .0027 | 350.8 | 365.6 | 366.0 | 0 | .0027 | 372.5 | 369.9 | 369.9 | 0 | .0027 | 312.7 | 400.1 | 404.2 | 4 |
| 2500 | 10000 | 13333.3 | .0027 | 381.1 | 383.8 | 384.0 | 0 | .0027 | 367.6 | 367.3 | 367.3 | 0 | .0027 | 304.1 | 386.2 | 391.8 | 2 |
| 2500 | 10000 | 20000.0 | .0027 | 383.1 | 396.7 | 396.9 | 0 | .0027 | 364.3 | 360.2 | 360.2 | 0 | .0027 | 309.9 | 397.6 | 402.1 | 3 |
| 2500 | 10000 | 40000.0 | .0027 | 376.6 | 371.7 | 371.7 | 0 | .0027 | 359.5 | 361.6 | 361.8 | 0 | .0027 | 304.9 | 390.8 | 396.2 | 2 |

**Table 25:** (2/2) Estimated ARL for four intervals with failure rate $\alpha = 0.0027$, variable $N$.

### Out of control ARL

When the average number of particles is small, $m_N = 10$, all charts have still a average run length close to 370.4. From this study we see that the Hotelling $T^2$, log likelihood chart and the generalized p chart observe a out of control point early in the process when the number of particles is sufficiently large.

| | | | | | Hotelling chart | | | | | Multivariate np chart | | | | | Generalized p chart | | | | | Multivariate poisson chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | $\mu$ | $\sigma$ | Mean | Variance | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 0.400 | 0.75 | 10 | 13.3 | .00027 | 307.1 | 314.3 | 320.6 | 0 | .00105 | 416.4 | 417.2 | 419.2 | 0 | .0023 | 273.1 | 282.3 | 298.5 | 0 | .00110 | 430.1 | 463.5 | 467.2 | 1 |
| 2500 | 0.450 | 0.75 | 10 | 13.3 | .00027 | 350.3 | 363.7 | 364.2 | 0 | .00105 | 383.9 | 391.3 | 391.4 | 0 | .0023 | 342.0 | 343.5 | 344.5 | 0 | .00110 | 395.4 | 438.8 | 439.4 | 2 |
| 2500 | 0.475 | 0.75 | 10 | 13.3 | .00027 | 363.4 | 371.2 | 371.2 | 0 | .00105 | 374.4 | 380.5 | 380.4 | 0 | .0023 | 360.2 | 369.2 | 369.2 | 0 | .00110 | 383.4 | 415.7 | 415.9 | 1 |
| 2500 | 0.400 | 0.75 | 10 | 20.0 | .00008 | 306.9 | 306.7 | 312.9 | 0 | .00040 | 440.4 | 446.5 | 451.8 | 1 | .0023 | 284.3 | 293.2 | 305.5 | 0 | .00045 | 434.4 | 446.3 | 450.6 | 1 |
| 2500 | 0.450 | 0.75 | 10 | 20.0 | .00008 | 353.0 | 347.4 | 347.6 | 0 | .00040 | 390.7 | 393.5 | 393.8 | 0 | .0023 | 347.5 | 347.6 | 348.3 | 0 | .00045 | 392.4 | 402.0 | 402.4 | 0 |
| 2500 | 0.475 | 0.75 | 10 | 20.0 | .00008 | 363.6 | 366.1 | 365.9 | 0 | .00040 | 386.0 | 386.2 | 386.3 | 0 | .0023 | 374.8 | 380.9 | 380.8 | 0 | .00045 | 384.1 | 394.1 | 394.1 | 0 |
| 2500 | 0.400 | 0.75 | 10 | 40.0 | .00002 | 202.5 | 203.0 | 263.4 | 0 | .00006 | 415.1 | 409.3 | 411.5 | 0 | .0026 | 268.5 | 269.5 | 288.0 | 0 | .00006 | 416.8 | 410.0 | 412.4 | 0 |
| 2500 | 0.450 | 0.75 | 10 | 40.0 | .00002 | 241.8 | 241.0 | 273.2 | 0 | .00006 | 395.1 | 414.4 | 414.9 | 0 | .0026 | 342.7 | 335.9 | 336.9 | 0 | .00006 | 400.4 | 423.9 | 424.8 | 1 |
| 2500 | 0.475 | 0.75 | 10 | 40.0 | .00002 | 241.6 | 241.0 | 273.0 | 0 | .00006 | 385.4 | 389.5 | 389.6 | 0 | .0026 | 369.6 | 374.2 | 374.2 | 0 | .00006 | 395.2 | 400.9 | 401.4 | 1 |
| 2500 | 0.400 | 0.75 | 100 | 133.3 | .00210 | 99.4 | 103.9 | 290.3 | 0 | .00270 | 418.4 | 418.1 | 420.8 | 0 | .0027 | 66.8 | 66.8 | 310.9 | 0 | .00270 | 413.4 | 420.1 | 422.2 | 0 |
| 2500 | 0.450 | 0.75 | 100 | 133.3 | .00210 | 254.2 | 255.9 | 281.0 | 0 | .00270 | 403.3 | 406.5 | 407.8 | 0 | .0027 | 198.5 | 195.7 | 260.4 | 0 | .00270 | 404.4 | 420.6 | 421.9 | 0 |
| 2500 | 0.475 | 0.75 | 100 | 133.3 | .00210 | 342.4 | 347.0 | 348.1 | 0 | .00270 | 384.6 | 396.1 | 396.3 | 1 | .0027 | 313.7 | 315.0 | 320.0 | 1 | .00270 | 387.5 | 391.1 | 391.4 | 0 |
| 2500 | 0.400 | 0.75 | 100 | 200.0 | .00160 | 103.7 | 104.9 | 286.6 | 0 | .00230 | 493.8 | 508.2 | 522.9 | 1 | .0027 | 63.6 | 61.2 | 312.8 | 0 | .00240 | 478.1 | 505.7 | 517.0 | 3 |
| 2500 | 0.450 | 0.75 | 100 | 200.0 | .00160 | 253.9 | 252.1 | 277.7 | 0 | .00230 | 433.3 | 447.0 | 451.4 | 1 | .0027 | 192.2 | 193.8 | 263.3 | 0 | .00240 | 421.6 | 441.0 | 443.9 | 2 |
| 2500 | 0.475 | 0.75 | 100 | 200.0 | .00160 | 352.7 | 359.2 | 359.5 | 0 | .00230 | 403.7 | 414.0 | 415.2 | 2 | .0027 | 300.8 | 300.9 | 308.8 | 0 | .00240 | 389.9 | 407.3 | 407.7 | 1 |
| 2500 | 0.400 | 0.75 | 100 | 400.0 | .00090 | 108.7 | 109.4 | 283.6 | 0 | .00160 | 477.5 | 489.6 | 501.1 | 3 | .0027 | 65.3 | 65.8 | 312.1 | 0 | .00160 | 485.6 | 497.3 | 510.4 | 0 |
| 2500 | 0.450 | 0.75 | 100 | 400.0 | .00090 | 249.4 | 250.3 | 278.0 | 0 | .00160 | 417.2 | 423.0 | 425.5 | 1 | .0027 | 202.4 | 203.2 | 263.6 | 0 | .00160 | 411.6 | 421.2 | 423.2 | 0 |
| 2500 | 0.475 | 0.75 | 100 | 400.0 | .00090 | 336.2 | 341.2 | 342.9 | 0 | .00160 | 387.9 | 392.4 | 392.8 | 0 | .0027 | 303.2 | 296.3 | 303.8 | 0 | .00160 | 384.2 | 394.4 | 394.6 | 0 |
| 2500 | 0.400 | 0.75 | 1000 | 1333.3 | .00275 | 2.6 | 1.9 | 367.9 | 0 | .00275 | 210.6 | 212.6 | 265.9 | 0 | .0027 | 2.1 | 1.5 | 368.3 | 0 | .00275 | 193.5 | 195.8 | 263.8 | 0 |
| 2500 | 0.450 | 0.75 | 1000 | 1333.3 | .00275 | 25.7 | 25.9 | 345.6 | 0 | .00275 | 326.6 | 327.0 | 329.8 | 0 | .0027 | 18.8 | 18.5 | 352.1 | 0 | .00275 | 323.1 | 325.5 | 328.8 | 0 |
| 2500 | 0.475 | 0.75 | 1000 | 1333.3 | .00275 | 132.0 | 134.2 | 273.6 | 0 | .00275 | 369.7 | 372.0 | 372.0 | 1 | .0027 | 105.3 | 106.8 | 285.8 | 0 | .00275 | 372.7 | 378.5 | 378.5 | 0 |
| 2500 | 0.400 | 0.75 | 1000 | 2000.0 | .00260 | 2.6 | 2.0 | 367.8 | 0 | .00270 | 273.4 | 271.3 | 288.0 | 0 | .0027 | 2.1 | 1.5 | 368.3 | 0 | .00270 | 256.5 | 260.8 | 284.5 | 0 |
| 2500 | 0.450 | 0.75 | 1000 | 2000.0 | .00260 | 26.5 | 26.6 | 344.9 | 0 | .00270 | 373.6 | 379.3 | 379.2 | 0 | .0027 | 18.7 | 18.4 | 352.2 | 0 | .00270 | 362.3 | 374.8 | 374.8 | 0 |
| 2500 | 0.475 | 0.75 | 1000 | 2000.0 | .00260 | 134.8 | 136.3 | 272.2 | 0 | .00270 | 390.5 | 401.3 | 401.8 | 0 | .0027 | 105.1 | 106.1 | 285.7 | 0 | .00270 | 391.6 | 401.6 | 402.1 | 1 |
| 2500 | 0.400 | 0.75 | 1000 | 4000.0 | .00245 | 2.7 | 2.1 | 367.7 | 0 | .00260 | 400.9 | 406.8 | 407.9 | 0 | .0027 | 2.1 | 1.6 | 368.3 | 0 | .00260 | 402.7 | 414.9 | 416.1 | 0 |
| 2500 | 0.450 | 0.75 | 1000 | 4000.0 | .00245 | 27.4 | 28.0 | 344.1 | 0 | .00260 | 411.6 | 415.9 | 417.9 | 0 | .0027 | 18.3 | 18.0 | 352.6 | 0 | .00260 | 409.5 | 418.7 | 420.4 | 0 |
| 2500 | 0.475 | 0.75 | 1000 | 4000.0 | .00245 | 137.3 | 141.1 | 272.5 | 0 | .00260 | 397.1 | 415.6 | 416.4 | 2 | .0027 | 104.3 | 105.0 | 286.0 | 0 | .00260 | 395.8 | 406.6 | 407.4 | 0 |
| 2500 | 0.400 | 0.75 | 10000 | 13333.3 | .00275 | 1.0 | 0.0 | 369.4 | 0 | .00275 | 18.6 | 18.6 | 352.3 | 0 | .0027 | 1.0 | 0.0 | 369.4 | 0 | .00275 | 15.1 | 14.8 | 355.7 | 0 |
| 2500 | 0.450 | 0.75 | 10000 | 13333.3 | .00275 | 1.1 | 0.2 | 369.3 | 0 | .00275 | 97.8 | 99.1 | 290.0 | 0 | .0027 | 1.0 | 0.2 | 369.4 | 0 | .00275 | 93.8 | 95.1 | 292.5 | 0 |
| 2500 | 0.475 | 0.75 | 10000 | 13333.3 | .00275 | 5.0 | 4.6 | 365.4 | 0 | .00275 | 235.5 | 239.0 | 274.4 | 0 | .0027 | 4.0 | 3.5 | 366.4 | 0 | .00275 | 230.1 | 233.9 | 272.7 | 0 |
| 2500 | 0.400 | 0.75 | 10000 | 20000.0 | .00275 | 1.0 | 0.0 | 369.4 | 0 | .00275 | 32.5 | 31.9 | 339.4 | 0 | .0027 | 1.0 | 0.0 | 369.4 | 0 | .00275 | 27.9 | 27.4 | 343.6 | 0 |
| 2500 | 0.450 | 0.75 | 10000 | 20000.0 | .00275 | 1.1 | 0.2 | 369.3 | 0 | .00275 | 136.4 | 136.2 | 270.7 | 0 | .0027 | 1.0 | 0.2 | 369.4 | 0 | .00275 | 133.2 | 133.8 | 272.4 | 0 |
| 2500 | 0.475 | 0.75 | 10000 | 20000.0 | .00275 | 5.2 | 4.5 | 365.3 | 0 | .00275 | 278.9 | 280.8 | 295.3 | 0 | .0027 | 4.1 | 3.7 | 366.3 | 0 | .00275 | 277.6 | 273.0 | 288.3 | 0 |
| 2500 | 0.400 | 0.75 | 10000 | 40000.0 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .00270 | 75.5 | 76.8 | 304.7 | 0 | .0026 | 1.0 | 0.0 | 369.4 | 0 | .00275 | 65.9 | 65.7 | 311.5 | 0 |
| 2500 | 0.450 | 0.75 | 10000 | 40000.0 | .00270 | 1.1 | 0.3 | 369.3 | 0 | .00270 | 221.9 | 217.6 | 263.4 | 0 | .0026 | 1.0 | 0.2 | 369.4 | 0 | .00275 | 214.5 | 210.8 | 262.2 | 0 |
| 2500 | 0.475 | 0.75 | 10000 | 40000.0 | .00270 | 5.2 | 4.7 | 365.3 | 0 | .00270 | 340.1 | 341.2 | 342.5 | 0 | .0026 | 4.1 | 3.6 | 366.3 | 0 | .00275 | 330.9 | 330.3 | 332.6 | 0 |

**Table 26:** (1/2) Estimated out of control ARL for four intervals, variable $N$.

| | | | | | multivariate np chart (grouped) | | | | | Generalized p chart (grouped) | | | | | Likelihood ratio control chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | $\mu$ | $\sigma$ | Mean | Variance | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 0.400 | 0.75 | 10 | 13.3 | .00105 | 416.9 | 417.9 | 420.4 | 0 | .0023 | 272.5 | 279.6 | 296.2 | 0 | 0.0440 | 307.5 | 411.3 | 416.1 | 4 |
| 2500 | 0.450 | 0.75 | 10 | 13.3 | .00105 | 383.0 | 390.1 | 390.2 | 0 | .0023 | 339.7 | 340.0 | 341.3 | 0 | 0.0440 | 344.8 | 492.2 | 492.8 | 8 |
| 2500 | 0.475 | 0.75 | 10 | 13.3 | .00105 | 374.4 | 380.6 | 380.6 | 0 | .0023 | 357.9 | 367.3 | 367.4 | 0 | 0.0440 | 366.3 | 523.0 | 522.9 | 12 |
| 2500 | 0.400 | 0.75 | 10 | 20.0 | .00040 | 440.5 | 446.7 | 452.1 | 1 | .0023 | 286.6 | 295.7 | 307.3 | 0 | 0.0630 | 353.0 | 514.7 | 514.9 | 9 |
| 2500 | 0.450 | 0.75 | 10 | 20.0 | .00040 | 390.8 | 393.2 | 393.7 | 0 | .0023 | 347.5 | 346.8 | 347.5 | 0 | 0.0630 | 368.8 | 540.4 | 540.3 | 13 |
| 2500 | 0.475 | 0.75 | 10 | 20.0 | .00040 | 386.2 | 388.1 | 388.4 | 0 | .0023 | 375.5 | 378.3 | 378.3 | 0 | 0.0630 | 385.2 | 569.5 | 569.6 | 17 |
| 2500 | 0.400 | 0.75 | 10 | 40.0 | .00006 | 414.9 | 409.4 | 411.8 | 0 | .0023 | 301.1 | 298.2 | 306.1 | 0 | 0.1160 | 298.4 | 488.1 | 493.3 | 17 |
| 2500 | 0.450 | 0.75 | 10 | 40.0 | .00006 | 394.0 | 409.2 | 409.8 | 0 | .0026 | 362.0 | 352.6 | 352.6 | 0 | 0.1160 | 317.3 | 521.5 | 524.1 | 18 |
| 2500 | 0.475 | 0.75 | 10 | 40.0 | .00006 | 385.1 | 389.9 | 390.1 | 0 | .0026 | 386.5 | 388.0 | 388.3 | 0 | 0.1160 | 334.8 | 573.6 | 574.6 | 26 |
| 2500 | 0.400 | 0.75 | 100 | 133.3 | .00270 | 418.4 | 418.0 | 420.7 | 0 | .0027 | 66.9 | 66.8 | 310.8 | 0 | 0.0024 | 74.1 | 92.2 | 310.3 | 0 |
| 2500 | 0.450 | 0.75 | 100 | 133.3 | .00270 | 403.1 | 406.3 | 407.5 | 0 | .0027 | 198.2 | 195.8 | 260.7 | 0 | 0.0024 | 224.7 | 297.7 | 331.4 | 1 |
| 2500 | 0.475 | 0.75 | 100 | 133.3 | .00270 | 384.2 | 395.8 | 395.9 | 1 | .0027 | 313.7 | 312.2 | 317.3 | 0 | 0.0024 | 341.6 | 443.4 | 444.2 | 8 |
| 2500 | 0.400 | 0.75 | 100 | 200.0 | .00230 | 493.8 | 508.2 | 523.0 | 1 | .0027 | 63.8 | 61.5 | 312.7 | 0 | 0.0024 | 79.7 | 102.1 | 308.1 | 0 |
| 2500 | 0.450 | 0.75 | 100 | 200.0 | .00230 | 432.7 | 444.9 | 449.2 | 1 | .0027 | 192.7 | 194.2 | 263.2 | 0 | 0.0024 | 227.9 | 317.2 | 347.7 | 3 |
| 2500 | 0.475 | 0.75 | 100 | 200.0 | .00230 | 403.7 | 414.0 | 415.2 | 2 | .0027 | 301.5 | 300.9 | 308.7 | 0 | 0.0024 | 346.1 | 453.3 | 453.9 | 8 |
| 2500 | 0.400 | 0.75 | 100 | 400.0 | .00160 | 477.4 | 489.5 | 501.0 | 3 | .0027 | 65.2 | 65.8 | 312.2 | 0 | 0.0024 | 95.8 | 122.5 | 300.6 | 0 |
| 2500 | 0.450 | 0.75 | 100 | 400.0 | .00160 | 416.9 | 422.4 | 424.9 | 1 | .0027 | 201.7 | 202.8 | 263.8 | 0 | 0.0024 | 256.3 | 348.6 | 366.8 | 5 |
| 2500 | 0.475 | 0.75 | 100 | 400.0 | .00160 | 388.0 | 392.4 | 392.8 | 0 | .0027 | 303.4 | 297.1 | 304.6 | 0 | 0.0024 | 372.4 | 500.8 | 500.7 | 12 |
| 2500 | 0.400 | 0.75 | 1000 | 1333.3 | .00275 | 210.6 | 212.6 | 266.0 | 0 | .0027 | 2.1 | 1.5 | 368.3 | 0 | 0.0024 | 2.1 | 1.5 | 368.3 | 0 |
| 2500 | 0.450 | 0.75 | 1000 | 1333.3 | .00275 | 326.6 | 327.0 | 329.8 | 0 | .0027 | 18.8 | 18.6 | 352.1 | 0 | 0.0024 | 19.4 | 21.9 | 351.6 | 0 |
| 2500 | 0.475 | 0.75 | 1000 | 1333.3 | .00275 | 369.7 | 372.0 | 372.0 | 1 | .0027 | 105.2 | 106.8 | 285.9 | 0 | 0.0024 | 115.2 | 149.7 | 295.9 | 0 |
| 2500 | 0.400 | 0.75 | 1000 | 2000.0 | .00270 | 273.4 | 271.3 | 288.0 | 0 | .0027 | 2.1 | 1.5 | 368.3 | 0 | 0.0024 | 2.1 | 1.5 | 368.3 | 0 |
| 2500 | 0.450 | 0.75 | 1000 | 2000.0 | .00270 | 373.6 | 379.3 | 379.3 | 0 | .0027 | 18.7 | 18.4 | 352.2 | 0 | 0.0024 | 19.0 | 20.9 | 352.0 | 0 |
| 2500 | 0.475 | 0.75 | 1000 | 2000.0 | .00270 | 390.5 | 401.3 | 401.8 | 0 | .0027 | 105.0 | 106.1 | 285.8 | 0 | 0.0024 | 114.4 | 149.3 | 296.3 | 0 |
| 2500 | 0.400 | 0.75 | 1000 | 4000.0 | .00260 | 401.1 | 406.9 | 408.0 | 0 | .0027 | 2.1 | 1.6 | 368.3 | 0 | 0.0024 | 2.1 | 1.6 | 368.3 | 0 |
| 2500 | 0.450 | 0.75 | 1000 | 4000.0 | .00260 | 411.6 | 415.9 | 417.9 | 0 | .0027 | 18.3 | 18.0 | 352.6 | 0 | 0.0024 | 20.1 | 23.2 | 351.1 | 0 |
| 2500 | 0.475 | 0.75 | 1000 | 4000.0 | .00260 | 397.1 | 415.6 | 416.4 | 2 | .0027 | 104.3 | 104.9 | 286.0 | 0 | 0.0024 | 114.1 | 144.1 | 294.0 | 0 |
| 2500 | 0.400 | 0.75 | 10000 | 13333.3 | .00275 | 18.6 | 18.6 | 352.3 | 0 | .0027 | 1.0 | 0.0 | 369.4 | 0 | 0.0024 | 1.0 | 0.0 | 369.4 | 0 |
| 2500 | 0.450 | 0.75 | 10000 | 13333.3 | .00275 | 97.8 | 99.1 | 290.0 | 0 | .0027 | 1.0 | 0.2 | 369.4 | 0 | 0.0024 | 1.0 | 0.2 | 369.4 | 0 |
| 2500 | 0.475 | 0.75 | 10000 | 13333.3 | .00275 | 235.5 | 239.0 | 274.4 | 0 | .0027 | 4.0 | 3.5 | 366.4 | 0 | 0.0024 | 4.2 | 4.0 | 366.3 | 0 |
| 2500 | 0.400 | 0.75 | 10000 | 20000.0 | .00275 | 32.5 | 31.9 | 339.4 | 0 | .0027 | 1.0 | 0.0 | 369.4 | 0 | 0.0024 | 1.0 | 0.0 | 369.4 | 0 |
| 2500 | 0.450 | 0.75 | 10000 | 20000.0 | .00275 | 136.4 | 136.2 | 270.8 | 0 | .0027 | 1.0 | 0.2 | 369.4 | 0 | 0.0024 | 1.0 | 0.2 | 369.4 | 0 |
| 2500 | 0.475 | 0.75 | 10000 | 20000.0 | .00275 | 278.9 | 280.8 | 295.3 | 0 | .0027 | 4.1 | 3.7 | 366.3 | 0 | 0.0024 | 4.2 | 3.9 | 366.3 | 0 |
| 2500 | 0.400 | 0.75 | 10000 | 40000.0 | .00270 | 75.5 | 76.8 | 304.7 | 0 | .0026 | 1.0 | 0.0 | 369.4 | 0 | 0.0024 | 1.0 | 0.0 | 369.4 | 0 |
| 2500 | 0.450 | 0.75 | 10000 | 40000.0 | .00270 | 221.9 | 217.6 | 263.4 | 0 | .0026 | 1.0 | 0.2 | 369.4 | 0 | 0.0024 | 1.0 | 0.2 | 369.4 | 0 |
| 2500 | 0.475 | 0.75 | 10000 | 40000.0 | .00270 | 340.1 | 341.2 | 342.5 | 0 | .0026 | 4.1 | 3.6 | 366.3 | 0 | 0.0024 | 4.0 | 3.6 | 366.4 | 0 |

**Table 27:** (2/2) Estimated out of control ARL for four intervals, variable $N$.

## 4.6 Varying numbers of particles and seven particle size intervals

### In control ARL

In Figure 17 we see that all charts need a sufficiently number of phase I points to converse to 370,4, except the log likelihood ratio chart needs 370.4. The generalized p chart, Hotelling $T^2$ chart, multivariate Poisson chart and multivariate np chart need more phase I points to converge to 370.4 in comparison with the four and three interval studies. The log likelihood ratio chart seems to converge faster to 312.5 in comparison with the previous settings with three and four charts. The log likelihood chart performs rely well with low number of particles, even with a average number of particles $N_m$. The generalized p charts needs many particles, on average 10000 particles, to have a average run length close to 370.4.

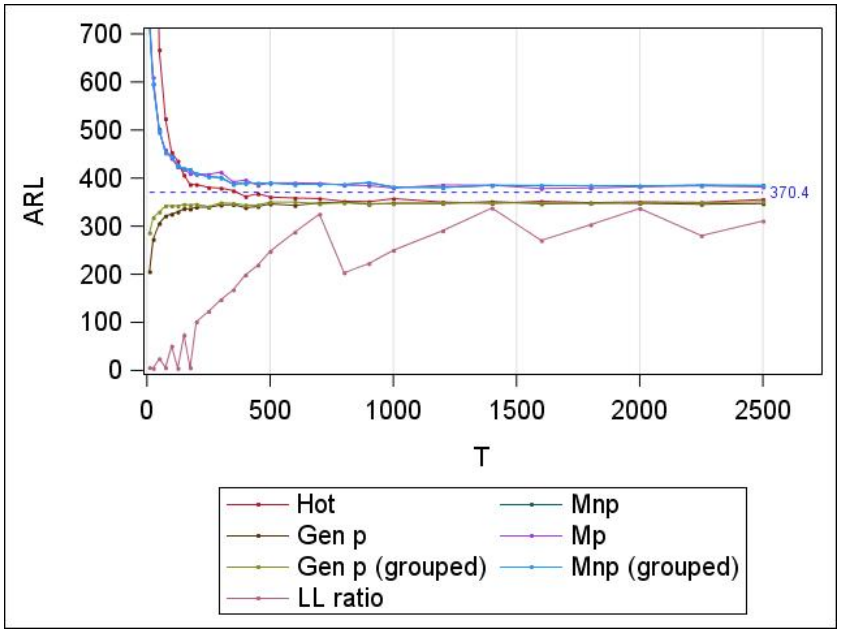



**Figure 17:** Total time points phase I versus average run length where $m_N = 2500$ and $v_N = 3333.33$.

| | | | Hotelling chart | | | | | Multivariate np chart | | | | | Generalized p chart | | | | | Multivariate poisson chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | Mean | Variance | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 250 | 10 | 13.3 | .0027 | 46.0 | 50.9 | 328.4 | 0 | .0027 | 155.5 | 182.3 | 281.8 | 0 | .0027 | 44.6 | 51.6 | 329.9 | 0 | .0027 | 222.7 | 273.6 | 310.9 | 0 |
| 250 | 10 | 20.0 | .0027 | 37.0 | 39.4 | 335.7 | 0 | .0027 | 128.3 | 144.9 | 282.1 | 0 | .0027 | 42.0 | 48.2 | 331.9 | 0 | .0027 | 148.6 | 164.4 | 276.1 | 0 |
| 250 | 10 | 40.0 | .0027 | 26.5 | 28.3 | 345.0 | 0 | .0027 | 92.1 | 101.1 | 296.1 | 0 | .0027 | 41.5 | 48.0 | 332.4 | 0 | .0027 | 93.8 | 101.5 | 294.6 | 0 |
| 250 | 100 | 133.3 | .0027 | 156.7 | 167.3 | 271.4 | 0 | .0027 | 338.5 | 392.0 | 393.2 | 0 | .0027 | 161.3 | 165.7 | 266.8 | 0 | .0027 | 390.9 | 468.0 | 468.4 | 4 |
| 250 | 100 | 200.0 | .0027 | 147.1 | 158.0 | 273.6 | 0 | .0027 | 317.4 | 353.4 | 357.2 | 1 | .0027 | 163.5 | 169.5 | 267.4 | 0 | .0027 | 336.2 | 380.4 | 381.9 | 2 |
| 250 | 100 | 400.0 | .0027 | 112.2 | 115.9 | 283.0 | 0 | .0027 | 269.3 | 306.7 | 322.9 | 0 | .0027 | 163.1 | 168.4 | 267.0 | 0 | .0027 | 278.3 | 311.6 | 324.8 | 0 |
| 250 | 1000 | 1333.3 | .0027 | 355.4 | 389.7 | 389.9 | 0 | .0027 | 403.8 | 489.1 | 490.2 | 6 | .0027 | 316.8 | 318.7 | 323.1 | 0 | .0027 | 401.6 | 495.6 | 496.5 | 8 |
| 250 | 1000 | 2000.0 | .0027 | 346.9 | 374.4 | 375.0 | 0 | .0027 | 395.4 | 466.5 | 467.1 | 2 | .0027 | 310.5 | 324.3 | 329.8 | 0 | .0027 | 399.4 | 478.8 | 479.6 | 3 |
| 250 | 1000 | 4000.0 | .0027 | 317.1 | 354.6 | 358.5 | 0 | .0027 | 387.9 | 457.3 | 457.6 | 4 | .0027 | 318.5 | 321.1 | 325.2 | 0 | .0027 | 397.5 | 468.9 | 469.6 | 5 |
| 250 | 10000 | 13333.3 | .0027 | 404.9 | 433.7 | 435.0 | 0 | .0027 | 404.1 | 492.2 | 493.3 | 11 | .0027 | 364.9 | 360.3 | 360.1 | 0 | .0027 | 402.1 | 480.7 | 481.7 | 0 |
| 250 | 10000 | 20000.0 | .0027 | 401.7 | 427.9 | 428.9 | 2 | .0027 | 393.0 | 457.7 | 458.2 | 5 | .0027 | 362.5 | 357.4 | 357.3 | 0 | .0027 | 399.8 | 477.9 | 478.7 | 5 |
| 250 | 10000 | 40000.0 | .0027 | 395.1 | 432.4 | 433.0 | 0 | .0027 | 392.9 | 460.9 | 461.4 | 4 | .0027 | 363.4 | 363.8 | 363.7 | 0 | .0027 | 398.7 | 474.4 | 475.2 | 5 |
| 2500 | 10 | 13.3 | .0027 | 45.4 | 46.0 | 328.2 | 0 | .0027 | 150.4 | 155.8 | 269.5 | 0 | .0027 | 44.3 | 44.9 | 329.1 | 0 | .0027 | 215.7 | 236.2 | 282.4 | 0 |
| 2500 | 10 | 20.0 | .0027 | 37.3 | 37.1 | 335.1 | 0 | .0027 | 125.1 | 128.4 | 276.9 | 0 | .0027 | 42.2 | 43.9 | 331.1 | 0 | .0027 | 148.8 | 153.6 | 269.6 | 0 |
| 2500 | 10 | 40.0 | .0027 | 26.9 | 27.4 | 344.6 | 0 | .0027 | 87.8 | 87.0 | 295.7 | 0 | .0027 | 40.5 | 39.3 | 332.2 | 0 | .0027 | 90.0 | 87.7 | 293.8 | 0 |
| 2500 | 100 | 133.3 | .0027 | 147.4 | 149.5 | 268.5 | 0 | .0027 | 321.1 | 325.2 | 328.9 | 0 | .0027 | 162.1 | 159.7 | 262.4 | 0 | .0027 | 361.8 | 373.8 | 373.9 | 0 |
| 2500 | 100 | 200.0 | .0027 | 132.5 | 130.8 | 271.5 | 0 | .0027 | 301.3 | 306.5 | 314.1 | 0 | .0027 | 164.0 | 163.3 | 263.1 | 0 | .0027 | 324.0 | 327.7 | 330.9 | 0 |
| 2500 | 100 | 400.0 | .0027 | 102.4 | 102.9 | 287.0 | 0 | .0027 | 253.2 | 261.1 | 286.1 | 0 | .0027 | 160.9 | 161.3 | 264.4 | 0 | .0027 | 258.9 | 260.8 | 283.5 | 0 |
| 2500 | 1000 | 1333.3 | .0027 | 323.2 | 325.8 | 329.1 | 0 | .0027 | 370.4 | 388.4 | 388.3 | 0 | .0027 | 325.4 | 328.3 | 331.3 | 0 | .0027 | 364.9 | 366.3 | 366.2 | 1 |
| 2500 | 1000 | 2000.0 | .0027 | 308.3 | 312.2 | 318.3 | 0 | .0027 | 367.2 | 367.0 | 366.9 | 0 | .0027 | 313.1 | 318.1 | 323.2 | 0 | .0027 | 366.6 | 366.2 | 366.1 | 0 |
| 2500 | 1000 | 4000.0 | .0027 | 286.4 | 280.5 | 292.8 | 0 | .0027 | 355.2 | 346.1 | 346.4 | 0 | .0027 | 319.9 | 320.3 | 324.2 | 0 | .0027 | 363.8 | 370.1 | 370.1 | 0 |
| 2500 | 10000 | 13333.3 | .0027 | 371.3 | 364.5 | 364.4 | 0 | .0027 | 380.7 | 380.7 | 380.8 | 0 | .0027 | 377.7 | 370.9 | 370.8 | 0 | .0027 | 376.9 | 386.5 | 386.4 | 0 |
| 2500 | 10000 | 20000.0 | .0027 | 370.5 | 365.0 | 364.9 | 0 | .0027 | 375.9 | 389.4 | 389.4 | 1 | .0027 | 376.0 | 373.6 | 373.5 | 0 | .0027 | 366.9 | 368.1 | 368.1 | 0 |
| 2500 | 10000 | 40000.0 | .0027 | 368.4 | 378.4 | 378.4 | 0 | .0027 | 372.1 | 374.5 | 374.5 | 0 | .0027 | 375.6 | 368.3 | 368.2 | 0 | .0027 | 368.8 | 375.0 | 375.0 | 0 |

**Table 28:** (1/2) Estimated ARL for seven intervals with failure rate $\alpha = 0.0027$, variable $N$.

| | | | multivariate np chart (grouped) | | | | | Generalized p chart (grouped) | | | | | Likelihood ratio control chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | Mean | Variance | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 250 | 10 | 13.3 | .0027 | 158.6 | 185.2 | 281.3 | 0 | .0027 | 45.4 | 48.6 | 328.6 | 0 | .0027 | 118.5 | 228.6 | 340.2 | 3 |
| 250 | 10 | 20.0 | .0027 | 129.3 | 144.2 | 280.9 | 0 | .0027 | 42.4 | 44.8 | 331.1 | 0 | .0027 | 114.1 | 210.1 | 331.4 | 2 |
| 250 | 10 | 40.0 | .0027 | 92.2 | 101.8 | 296.2 | 0 | .0027 | 42.3 | 43.7 | 331.0 | 0 | .0027 | 97.4 | 185.7 | 330.1 | 1 |
| 250 | 100 | 133.3 | .0027 | 339.9 | 391.5 | 392.7 | 0 | .0027 | 163.0 | 168.5 | 267.2 | 0 | .0027 | 119.7 | 253.5 | 356.5 | 3 |
| 250 | 100 | 200.0 | .0027 | 317.1 | 356.6 | 360.5 | 1 | .0027 | 165.4 | 167.2 | 264.5 | 0 | .0027 | 125.8 | 274.0 | 367.3 | 5 |
| 250 | 100 | 400.0 | .0027 | 267.8 | 305.3 | 322.1 | 0 | .0027 | 164.1 | 169.0 | 266.7 | 0 | .0027 | 120.6 | 240.9 | 347.0 | 3 |
| 250 | 1000 | 1333.3 | .0027 | 403.4 | 488.6 | 489.6 | 6 | .0027 | 320.7 | 320.1 | 324.0 | 0 | .0027 | 123.7 | 246.3 | 348.6 | 2 |
| 250 | 1000 | 2000.0 | .0027 | 395.6 | 465.5 | 466.1 | 3 | .0027 | 313.3 | 326.1 | 331.0 | 0 | .0027 | 121.1 | 258.3 | 358.9 | 4 |
| 250 | 1000 | 4000.0 | .0027 | 387.6 | 456.6 | 456.8 | 4 | .0027 | 321.6 | 321.8 | 325.4 | 0 | .0027 | 130.3 | 284.3 | 372.1 | 4 |
| 250 | 10000 | 13333.3 | .0027 | 404.7 | 493.7 | 494.8 | 11 | .0027 | 370.0 | 362.7 | 362.7 | 0 | .0027 | 131.1 | 282.1 | 369.9 | 5 |
| 250 | 10000 | 20000.0 | .0027 | 393.1 | 457.7 | 458.2 | 5 | .0027 | 369.4 | 366.2 | 366.1 | 0 | .0027 | 126.4 | 268.3 | 362.6 | 3 |
| 250 | 10000 | 40000.0 | .0027 | 392.8 | 460.8 | 461.3 | 4 | .0027 | 371.5 | 366.4 | 366.3 | 0 | .0027 | 125.3 | 260.4 | 357.6 | 2 |
| 2500 | 10 | 13.3 | .0027 | 150.7 | 155.8 | 269.3 | 0 | .0027 | 44.2 | 43.7 | 329.1 | 0 | .0027 | 305.9 | 367.1 | 372.7 | 3 |
| 2500 | 10 | 20.0 | .0027 | 125.3 | 126.7 | 275.9 | 0 | .0027 | 42.2 | 42.6 | 330.9 | 0 | .0027 | 274.1 | 333.0 | 346.6 | 2 |
| 2500 | 10 | 40.0 | .0027 | 87.9 | 86.8 | 295.5 | 0 | .0027 | 40.6 | 39.2 | 332.1 | 0 | .0027 | 223.4 | 268.7 | 306.2 | 0 |
| 2500 | 100 | 133.3 | .0027 | 322.4 | 327.1 | 330.6 | 0 | .0027 | 162.8 | 160.8 | 262.6 | 0 | .0027 | 313.1 | 396.7 | 400.8 | 4 |
| 2500 | 100 | 200.0 | .0027 | 301.1 | 306.2 | 314.0 | 0 | .0027 | 165.1 | 163.8 | 262.7 | 0 | .0027 | 317.2 | 401.4 | 404.9 | 1 |
| 2500 | 100 | 400.0 | .0027 | 253.4 | 261.4 | 286.3 | 0 | .0027 | 161.2 | 161.5 | 264.3 | 0 | .0027 | 321.4 | 403.9 | 406.8 | 5 |
| 2500 | 1000 | 1333.3 | .0027 | 369.1 | 386.5 | 386.4 | 0 | .0027 | 326.1 | 329.3 | 332.2 | 0 | .0027 | 302.0 | 365.0 | 371.3 | 3 |
| 2500 | 1000 | 2000.0 | .0027 | 366.9 | 366.7 | 366.7 | 0 | .0027 | 314.6 | 318.4 | 323.2 | 0 | .0027 | 319.5 | 407.3 | 410.5 | 2 |
| 2500 | 1000 | 4000.0 | .0027 | 355.1 | 345.6 | 345.9 | 0 | .0027 | 320.9 | 320.7 | 324.5 | 0 | .0027 | 318.4 | 414.4 | 417.6 | 6 |
| 2500 | 10000 | 13333.3 | .0027 | 380.8 | 380.7 | 380.8 | 0 | .0027 | 374.3 | 365.9 | 365.9 | 0 | .0027 | 325.5 | 408.2 | 410.6 | 5 |
| 2500 | 10000 | 20000.0 | .0027 | 376.0 | 389.4 | 389.4 | 1 | .0027 | 375.8 | 374.2 | 374.2 | 0 | .0027 | 315.6 | 377.9 | 381.8 | 2 |
| 2500 | 10000 | 40000.0 | .0027 | 372.4 | 374.7 | 374.7 | 0 | .0027 | 375.9 | 367.4 | 367.4 | 0 | .0027 | 316.1 | 385.5 | 389.3 | 3 |

**Table 29:** (2/2) Estimated ARL for seven intervals with failure rate $\alpha = 0.0027$, variable $N$.

### Out of control ARL

In the out of control studies we see that the Hotelling $T^2$ chart, generalized p chart and log likelihood ratio chart observe early out of control points in the out of control processes when the number of particles is large. The average number of particles has to be larger than 1000 to observe early out of control points. The multivariate Poisson chart does not detects that the process is out of control when the number of particles is low. The multivariate np chart does not immediately observe that the process is out of control but it will observe that the process is out of control.

| | | | | | Hotelling chart | | | | | Multivariate np chart | | | | | Generalized p chart | | | | | Multivariate poisson chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | $\mu$ | $\sigma$ | Mean | Variance | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 0.400 | 0.75 | 10 | 13.3 | .00002 | 275.9 | 278.4 | 294.0 | 0 | .000500 | 557.6 | 554.3 | 584.9 | 3 | .00001 | 368.1 | 385.1 | 384.4 | 0 | .00110 | 439.1 | 477.3 | 482.2 | 2 |
| 2500 | 0.450 | 0.75 | 10 | 13.3 | .00002 | 242.2 | 235.7 | 268.3 | 0 | .000500 | 454.0 | 464.4 | 471.8 | 2 | .00001 | 313.0 | 315.3 | 320.4 | 0 | .00110 | 403.3 | 446.0 | 447.1 | 1 |
| 2500 | 0.475 | 0.75 | 10 | 13.3 | .00002 | 209.1 | 209.7 | 264.5 | 0 | .000500 | 409.7 | 405.3 | 407.1 | 0 | .00001 | 267.5 | 283.7 | 301.7 | 0 | .00110 | 388.0 | 442.3 | 442.6 | 2 |
| 2500 | 0.400 | 0.75 | 10 | 20.0 | .00002 | 202.9 | 198.4 | 259.7 | 0 | .000260 | 542.4 | 544.2 | 570.6 | 3 | .00001 | 326.6 | 336.1 | 338.9 | 0 | .00060 | 378.4 | 391.6 | 391.6 | 0 |
| 2500 | 0.450 | 0.75 | 10 | 20.0 | .00002 | 178.3 | 178.1 | 262.0 | 0 | .000260 | 450.3 | 457.6 | 464.4 | 1 | .00001 | 263.7 | 268.7 | 289.1 | 0 | .00060 | 348.0 | 364.6 | 365.2 | 0 |
| 2500 | 0.475 | 0.75 | 10 | 20.0 | .00002 | 165.7 | 162.5 | 261.4 | 0 | .000260 | 418.9 | 416.7 | 419.4 | 0 | .00001 | 244.3 | 249.4 | 279.4 | 0 | .00060 | 343.6 | 353.7 | 354.7 | 1 |
| 2500 | 0.400 | 0.75 | 10 | 40.0 | .00002 | 106.4 | 106.2 | 284.6 | 0 | .000055 | 470.4 | 472.5 | 482.8 | 1 | .00001 | 251.4 | 261.3 | 287.0 | 0 | .00007 | 400.7 | 414.5 | 415.5 | 1 |
| 2500 | 0.450 | 0.75 | 10 | 40.0 | .00002 | 104.0 | 103.1 | 285.6 | 0 | .000055 | 429.5 | 432.5 | 436.4 | 2 | .00001 | 209.7 | 222.2 | 274.2 | 0 | .00007 | 381.4 | 381.6 | 381.7 | 1 |
| 2500 | 0.475 | 0.75 | 10 | 40.0 | .00002 | 93.5 | 94.7 | 292.6 | 0 | .000055 | 386.6 | 395.9 | 396.2 | 1 | .00001 | 185.6 | 196.8 | 269.9 | 0 | .00007 | 369.2 | 379.9 | 379.9 | 0 |
| 2500 | 0.400 | 0.75 | 100 | 133.3 | .00055 | 290.2 | 293.3 | 304.0 | 0 | .002350 | 588.5 | 596.4 | 635.0 | 6 | .00070 | 254.9 | 256.4 | 281.1 | 0 | .00270 | 407.6 | 428.1 | 429.6 | 0 |
| 2500 | 0.450 | 0.75 | 100 | 133.3 | .00055 | 470.1 | 485.0 | 495.0 | 1 | .002350 | 484.9 | 499.0 | 511.9 | 1 | .00070 | 461.9 | 464.3 | 473.2 | 1 | .00270 | 390.8 | 391.5 | 392.0 | 0 |
| 2500 | 0.475 | 0.75 | 100 | 133.3 | .00055 | 447.4 | 449.4 | 455.3 | 1 | .002350 | 440.2 | 450.1 | 455.4 | 0 | .00070 | 453.4 | 451.9 | 459.3 | 0 | .00270 | 375.2 | 399.2 | 399.2 | 2 |
| 2500 | 0.400 | 0.75 | 100 | 200.0 | .00045 | 272.9 | 271.4 | 288.3 | 0 | .002100 | 680.0 | 666.4 | 734.8 | 9 | .00070 | 255.5 | 250.2 | 275.3 | 0 | .00230 | 494.9 | 495.6 | 510.9 | 0 |
| 2500 | 0.450 | 0.75 | 100 | 200.0 | .00045 | 442.0 | 445.1 | 450.7 | 1 | .002100 | 526.1 | 536.5 | 558.5 | 4 | .00070 | 445.9 | 441.1 | 447.4 | 0 | .00230 | 438.0 | 454.5 | 459.4 | 2 |
| 2500 | 0.475 | 0.75 | 100 | 200.0 | .00045 | 437.6 | 443.9 | 448.7 | 0 | .002100 | 455.0 | 453.5 | 461.3 | 1 | .00070 | 445.5 | 449.6 | 455.8 | 0 | .00230 | 417.4 | 429.5 | 432.0 | 0 |
| 2500 | 0.400 | 0.75 | 100 | 400.0 | .00025 | 230.2 | 233.4 | 272.2 | 0 | .001600 | 620.7 | 633.5 | 681.1 | 6 | .00070 | 243.5 | 245.3 | 276.1 | 0 | .00160 | 499.5 | 502.0 | 518.2 | 1 |
| 2500 | 0.450 | 0.75 | 100 | 400.0 | .00025 | 398.6 | 406.3 | 407.2 | 0 | .001600 | 465.2 | 472.8 | 482.2 | 0 | .00070 | 440.7 | 448.6 | 454.0 | 1 | .00160 | 418.5 | 430.1 | 432.6 | 0 |
| 2500 | 0.475 | 0.75 | 100 | 400.0 | .00025 | 423.8 | 431.8 | 435.0 | 1 | .001600 | 406.3 | 418.4 | 419.9 | 0 | .00070 | 448.5 | 440.0 | 446.8 | 0 | .00160 | 389.3 | 402.2 | 402.5 | 1 |
| 2500 | 0.400 | 0.75 | 1000 | 1333.3 | .00230 | 3.9 | 3.5 | 366.5 | 0 | .002700 | 138.4 | 140.6 | 271.2 | 0 | .00230 | 3.3 | 2.8 | 367.1 | 0 | .00270 | 195.2 | 205.6 | 270.1 | 0 |
| 2500 | 0.450 | 0.75 | 1000 | 1333.3 | .00230 | 52.0 | 51.2 | 322.4 | 0 | .002700 | 290.2 | 299.1 | 309.6 | 0 | .00230 | 44.3 | 44.2 | 329.1 | 0 | .00270 | 325.3 | 333.7 | 336.6 | 0 |
| 2500 | 0.475 | 0.75 | 1000 | 1333.3 | .00230 | 238.7 | 240.4 | 274.0 | 0 | .002700 | 369.1 | 376.0 | 376.0 | 0 | .00230 | 218.1 | 217.1 | 265.1 | 0 | .00270 | 365.5 | 376.6 | 376.6 | 2 |
| 2500 | 0.400 | 0.75 | 1000 | 2000.0 | .00220 | 4.0 | 3.4 | 366.4 | 0 | .002700 | 196.7 | 202.4 | 266.7 | 0 | .00230 | 3.3 | 2.7 | 367.1 | 0 | .00270 | 265.8 | 272.8 | 292.1 | 0 |
| 2500 | 0.450 | 0.75 | 1000 | 2000.0 | .00220 | 53.6 | 53.0 | 321.2 | 0 | .002700 | 343.2 | 346.3 | 347.3 | 0 | .00230 | 43.3 | 43.8 | 330.0 | 0 | .00270 | 367.5 | 370.9 | 370.8 | 0 |
| 2500 | 0.475 | 0.75 | 1000 | 2000.0 | .00220 | 242.4 | 250.9 | 281.6 | 0 | .002700 | 391.7 | 394.4 | 394.9 | 0 | .00230 | 214.2 | 214.4 | 265.2 | 0 | .00270 | 387.0 | 391.7 | 391.9 | 0 |
| 2500 | 0.400 | 0.75 | 1000 | 4000.0 | .00200 | 4.2 | 3.7 | 366.2 | 0 | .002650 | 336.6 | 345.1 | 346.7 | 0 | .00230 | 3.3 | 2.8 | 367.1 | 0 | .00270 | 380.6 | 384.9 | 384.9 | 0 |
| 2500 | 0.450 | 0.75 | 1000 | 4000.0 | .00200 | 54.0 | 55.4 | 321.2 | 0 | .002650 | 431.2 | 429.4 | 433.6 | 1 | .00230 | 42.3 | 42.0 | 330.8 | 0 | .00270 | 406.2 | 410.2 | 411.7 | 0 |
| 2500 | 0.475 | 0.75 | 1000 | 4000.0 | .00200 | 238.4 | 240.4 | 274.2 | 0 | .002650 | 423.1 | 424.6 | 427.8 | 1 | .00230 | 215.4 | 214.8 | 264.8 | 0 | .00270 | 389.1 | 391.0 | 391.4 | 0 |
| 2500 | 0.400 | 0.75 | 10000 | 13333.3 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .002750 | 5.6 | 5.2 | 364.8 | 0 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .00270 | 16.0 | 15.8 | 354.8 | 0 |
| 2500 | 0.450 | 0.75 | 10000 | 13333.3 | .00270 | 1.1 | 0.3 | 369.3 | 0 | .002750 | 37.5 | 37.2 | 335.0 | 0 | .00270 | 1.1 | 0.3 | 369.3 | 0 | .00270 | 95.7 | 94.9 | 290.6 | 0 |
| 2500 | 0.475 | 0.75 | 10000 | 13333.3 | .00270 | 7.4 | 7.0 | 363.1 | 0 | .002750 | 144.6 | 144.9 | 268.3 | 0 | .00270 | 6.5 | 6.0 | 363.9 | 0 | .00270 | 241.9 | 238.9 | 271.2 | 0 |
| 2500 | 0.400 | 0.75 | 10000 | 20000.0 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .002700 | 9.3 | 8.8 | 361.2 | 0 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .00270 | 28.1 | 28.2 | 343.5 | 0 |
| 2500 | 0.450 | 0.75 | 10000 | 20000.0 | .00270 | 1.1 | 0.3 | 369.3 | 0 | .002700 | 58.2 | 58.2 | 317.6 | 0 | .00270 | 1.1 | 0.3 | 369.3 | 0 | .00270 | 136.1 | 138.3 | 272.0 | 0 |
| 2500 | 0.475 | 0.75 | 10000 | 20000.0 | .00270 | 7.6 | 7.1 | 362.9 | 0 | .002700 | 186.8 | 187.3 | 262.3 | 0 | .00270 | 6.5 | 5.9 | 364.0 | 0 | .00270 | 279.0 | 281.9 | 296.3 | 0 |
| 2500 | 0.400 | 0.75 | 10000 | 40000.0 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .002700 | 25.6 | 26.3 | 345.8 | 0 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .00270 | 67.7 | 69.2 | 310.5 | 0 |
| 2500 | 0.450 | 0.75 | 10000 | 40000.0 | .00270 | 1.1 | 0.3 | 369.3 | 0 | .002700 | 111.8 | 111.4 | 281.6 | 0 | .00270 | 1.1 | 0.3 | 369.3 | 0 | .00270 | 221.9 | 220.6 | 265.9 | 0 |
| 2500 | 0.475 | 0.75 | 10000 | 40000.0 | .00270 | 7.6 | 7.0 | 362.8 | 0 | .002700 | 255.2 | 259.5 | 283.9 | 0 | .00270 | 6.6 | 6.1 | 363.9 | 0 | .00270 | 331.8 | 336.1 | 338.2 | 0 |

**Table 30:** (1/2) Estimated out of control ARL for seven intervals, variable $N$.

| | | | | | multivariate np chart (grouped) | | | | | Generalized p chart (grouped) | | | | | Likelihood ratio control chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | $\mu$ | $\sigma$ | Mean | Variance | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 0.400 | 0.75 | 10 | 13.3 | .000500 | 563.3 | 561.4 | 593.5 | 4 | .00001 | 379.7 | 380.4 | 380.5 | 0 | .0024 | 369.2 | 469.7 | 469.6 | 8 |
| 2500 | 0.450 | 0.75 | 10 | 13.3 | .000500 | 459.5 | 464.7 | 473.1 | 0 | .00001 | 318.5 | 317.6 | 321.8 | 0 | .0024 | 377.2 | 474.4 | 474.4 | 6 |
| 2500 | 0.475 | 0.75 | 10 | 13.3 | .000500 | 413.0 | 410.5 | 412.7 | 0 | .00001 | 269.3 | 275.7 | 293.6 | 0 | .0024 | 371.5 | 471.2 | 471.1 | 6 |
| 2500 | 0.400 | 0.75 | 10 | 20.0 | .000260 | 543.6 | 545.3 | 572.1 | 3 | .00001 | 332.8 | 335.2 | 337.3 | 0 | .0024 | 326.9 | 449.5 | 451.6 | 11 |
| 2500 | 0.450 | 0.75 | 10 | 20.0 | .000260 | 451.4 | 455.2 | 462.3 | 0 | .00001 | 267.1 | 263.9 | 283.4 | 0 | .0024 | 339.4 | 452.3 | 453.3 | 7 |
| 2500 | 0.475 | 0.75 | 10 | 20.0 | .000260 | 419.9 | 416.8 | 419.7 | 0 | .00001 | 247.0 | 246.1 | 275.3 | 0 | .0024 | 338.6 | 437.5 | 438.6 | 10 |
| 2500 | 0.400 | 0.75 | 10 | 40.0 | .000055 | 469.1 | 472.8 | 482.9 | 1 | .00001 | 265.7 | 263.7 | 283.7 | 0 | .0024 | 317.3 | 409.6 | 413.0 | 3 |
| 2500 | 0.450 | 0.75 | 10 | 40.0 | .000055 | 430.2 | 434.7 | 438.8 | 2 | .00001 | 221.1 | 225.2 | 270.2 | 0 | .0024 | 293.5 | 361.4 | 369.4 | 1 |
| 2500 | 0.475 | 0.75 | 10 | 40.0 | .000055 | 386.2 | 394.7 | 394.9 | 1 | .00001 | 194.4 | 200.5 | 266.8 | 0 | .0024 | 287.5 | 350.6 | 360.2 | 3 |
| 2500 | 0.400 | 0.75 | 100 | 133.3 | .002350 | 589.2 | 597.7 | 636.4 | 6 | .00070 | 257.2 | 259.9 | 283.5 | 0 | .0024 | 132.2 | 172.7 | 294.2 | 0 |
| 2500 | 0.450 | 0.75 | 100 | 133.3 | .002350 | 485.1 | 497.5 | 510.4 | 1 | .00070 | 463.2 | 467.6 | 476.7 | 1 | .0024 | 299.9 | 410.2 | 416.1 | 7 |
| 2500 | 0.475 | 0.75 | 100 | 133.3 | .002350 | 441.2 | 450.7 | 456.2 | 0 | .00070 | 451.8 | 448.1 | 455.4 | 0 | .0024 | 387.4 | 487.3 | 487.6 | 8 |
| 2500 | 0.400 | 0.75 | 100 | 200.0 | .002100 | 680.4 | 666.4 | 734.9 | 9 | .00070 | 255.6 | 251.1 | 276.0 | 0 | .0024 | 141.5 | 182.2 | 292.5 | 0 |
| 2500 | 0.450 | 0.75 | 100 | 200.0 | .002100 | 527.8 | 541.0 | 563.4 | 5 | .00070 | 450.2 | 444.4 | 451.5 | 0 | .0024 | 312.3 | 403.0 | 407.1 | 5 |
| 2500 | 0.475 | 0.75 | 100 | 200.0 | .002100 | 454.7 | 454.8 | 462.5 | 1 | .00070 | 441.6 | 440.8 | 446.5 | 0 | .0024 | 381.7 | 493.2 | 493.2 | 9 |
| 2500 | 0.400 | 0.75 | 100 | 400.0 | .001600 | 620.9 | 633.6 | 681.3 | 6 | .00070 | 245.2 | 248.8 | 278.5 | 0 | .0024 | 171.8 | 240.9 | 312.2 | 1 |
| 2500 | 0.450 | 0.75 | 100 | 400.0 | .001600 | 464.8 | 472.2 | 481.5 | 0 | .00070 | 437.2 | 441.9 | 446.8 | 1 | .0024 | 316.7 | 423.4 | 426.7 | 5 |
| 2500 | 0.475 | 0.75 | 100 | 400.0 | .001600 | 405.7 | 417.8 | 419.2 | 0 | .00070 | 450.5 | 439.9 | 447.1 | 0 | .0024 | 390.9 | 504.0 | 504.4 | 13 |
| 2500 | 0.400 | 0.75 | 1000 | 1333.3 | .002700 | 138.3 | 140.6 | 271.3 | 0 | .00230 | 3.3 | 2.8 | 367.1 | 0 | .0024 | 3.0 | 2.6 | 367.4 | 0 |
| 2500 | 0.450 | 0.75 | 1000 | 1333.3 | .002700 | 289.9 | 298.3 | 308.9 | 0 | .00230 | 44.4 | 44.4 | 329.0 | 0 | .0024 | 37.3 | 44.4 | 336.1 | 0 |
| 2500 | 0.475 | 0.75 | 1000 | 1333.3 | .002700 | 368.7 | 375.8 | 375.7 | 0 | .00230 | 219.8 | 219.9 | 266.5 | 0 | .0024 | 187.9 | 261.6 | 318.9 | 3 |
| 2500 | 0.400 | 0.75 | 1000 | 2000.0 | .002700 | 196.5 | 201.9 | 266.5 | 0 | .00230 | 3.3 | 2.7 | 367.1 | 0 | .0024 | 3.0 | 2.6 | 367.4 | 0 |
| 2500 | 0.450 | 0.75 | 1000 | 2000.0 | .002700 | 343.0 | 345.9 | 346.9 | 0 | .00230 | 43.5 | 43.9 | 329.8 | 0 | .0024 | 37.3 | 45.7 | 336.2 | 0 |
| 2500 | 0.475 | 0.75 | 1000 | 2000.0 | .002700 | 392.2 | 394.7 | 395.3 | 0 | .00230 | 213.6 | 213.4 | 264.8 | 0 | .0024 | 180.5 | 242.7 | 308.1 | 2 |
| 2500 | 0.400 | 0.75 | 1000 | 4000.0 | .002650 | 336.9 | 345.4 | 346.9 | 0 | .00230 | 3.3 | 2.8 | 367.1 | 0 | .0024 | 3.1 | 2.6 | 367.4 | 0 |
| 2500 | 0.450 | 0.75 | 1000 | 4000.0 | .002650 | 431.0 | 429.2 | 433.4 | 1 | .00230 | 42.3 | 42.0 | 330.7 | 0 | .0024 | 36.9 | 46.4 | 336.7 | 0 |
| 2500 | 0.475 | 0.75 | 1000 | 4000.0 | .002650 | 422.9 | 424.3 | 427.5 | 1 | .00230 | 214.2 | 211.3 | 262.8 | 0 | .0024 | 190.1 | 252.9 | 310.6 | 0 |
| 2500 | 0.400 | 0.75 | 10000 | 13333.3 | .002750 | 5.6 | 5.2 | 364.8 | 0 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .0024 | 1.0 | 0.0 | 369.4 | 0 |
| 2500 | 0.450 | 0.75 | 10000 | 13333.3 | .002750 | 37.5 | 37.2 | 335.0 | 0 | .00270 | 1.1 | 0.3 | 369.3 | 0 | .0024 | 1.1 | 0.3 | 369.3 | 0 |
| 2500 | 0.475 | 0.75 | 10000 | 13333.3 | .002750 | 144.5 | 144.9 | 268.4 | 0 | .00270 | 6.5 | 6.0 | 363.9 | 0 | .0024 | 6.5 | 6.5 | 364.0 | 0 |
| 2500 | 0.400 | 0.75 | 10000 | 20000.0 | .002700 | 9.3 | 8.8 | 361.2 | 0 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .0024 | 1.0 | 0.0 | 369.4 | 0 |
| 2500 | 0.450 | 0.75 | 10000 | 20000.0 | .002700 | 58.1 | 58.1 | 317.6 | 0 | .00270 | 1.1 | 0.3 | 369.3 | 0 | .0024 | 1.1 | 0.3 | 369.3 | 0 |
| 2500 | 0.475 | 0.75 | 10000 | 20000.0 | .002700 | 186.8 | 187.3 | 262.3 | 0 | .00270 | 6.5 | 5.9 | 363.9 | 0 | .0024 | 6.5 | 6.3 | 364.0 | 0 |
| 2500 | 0.400 | 0.75 | 10000 | 40000.0 | .002700 | 25.6 | 26.3 | 345.8 | 0 | .00270 | 1.0 | 0.0 | 369.4 | 0 | .0024 | 1.0 | 0.0 | 369.4 | 0 |
| 2500 | 0.450 | 0.75 | 10000 | 40000.0 | .002700 | 111.8 | 111.5 | 281.5 | 0 | .00270 | 1.1 | 0.3 | 369.3 | 0 | .0024 | 1.1 | 0.3 | 369.3 | 0 |
| 2500 | 0.475 | 0.75 | 10000 | 40000.0 | .002700 | 255.2 | 259.5 | 283.9 | 0 | .00270 | 6.6 | 6.1 | 363.8 | 0 | .0024 | 6.5 | 6.4 | 364.0 | 0 |

**Table 31:** (2/2) Estimated out of control ARL for seven intervals, variable $N$.

## 4.7 Zero inflated methods

### Fixed total number of particles

**In control ARL**

in the case of a fixed number of particles the ZIP and ZINB charts do not detect out of control points. The data does not fit the zero inflated poisson or zero inflated negative binomial distribution when the total number of particles is fixed. The estimated location parameter is way to high, that results in a upper limit that is always greater than the maximum sample value. For that reason we will not study a out of control study when the number of particles is fixed.

| | | Zero inflated poisson chart | | | | | Zero inflated negative binomial chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | N | $\alpha$ | ARL | STD | RMSE | $\otimes$ | $\alpha$ | ARL | STD | RMSE | $\otimes$ |
| 250 | 10 | .0027 | 4000 | 0 | 3629.6 | 4000 | .0027 | 4000 | 0 | 3629.6 | 4000 |
| 2500 | 10 | .0027 | 4000 | 0 | 3629.6 | 4000 | .0027 | 4000 | 0 | 3629.6 | 4000 |

**Table 32:** Estimated ARL with failure rate $\alpha = 0.0027$, fixed $N$.

### Variable total number of particles

**In control ARL**

When the number of particles is variable the ZIP chart has a low average run length the and ZINB chart has a high average run length. In the ZIP chart a high variance means that the charts detects earlier out of control points. In the ZINB chart this is not always the case.

| | | | Zero inflated Poisson chart | | | | | Zero inflated negative binomial chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | Mean | Variance | $\alpha$ | ARL | STD | RMSE | $\otimes$ | $\alpha$ | ARL | STD | RMSE | $\otimes$ |
| 250 | 10 | 13.3 | .0027 | 147.5 | 147.9 | 267.5 | 0 | .0027 | 496.9 | 495.7 | 511.5 | 0 |
| 250 | 10 | 20.0 | .0027 | 101.4 | 101.8 | 287.6 | 0 | .0027 | 901.2 | 850.2 | 1002.2 | 51 |
| 250 | 10 | 40.0 | .0027 | 28.7 | 27.3 | 342.8 | 0 | .0027 | 466.4 | 473.0 | 482.6 | 1 |
| 2500 | 10 | 13.3 | .0027 | 270.4 | 272.6 | 290.4 | 0 | .0027 | 496.9 | 495.7 | 511.5 | 0 |
| 2500 | 10 | 20.0 | .0027 | 101.4 | 101.8 | 287.6 | 0 | .0027 | 574.0 | 571.6 | 606.7 | 4 |
| 2500 | 10 | 40.0 | .0027 | 28.7 | 27.3 | 342.8 | 0 | .0027 | 466.4 | 473.0 | 482.6 | 1 |

**Table 33:** Estimated ARL with failure rate $\alpha = 0.0027$, variable $N$.

**Out of control ARL**

To compare the charts in a out of control study we first had to change the average run length to the same number. We have tried this by changing the rate $\alpha$, unfortunately because the low sample value and upper limit is discrete we where not always able to do this. The charts do not detect a early out of control point, the average run lengths are in all settings still close to 370.4.

| | | | | | Zero inflated Poisson chart | | | | | Zero inflated negative binomial chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | Mu | Sigma | Mean | Variance | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 0.400 | 0.75 | 10 | 13.3 | .002100 | 322.9 | 314.9 | 318.4 | 0 | .00500 | 322.9 | 314.9 | 318.4 | 0 |
| 2500 | 0.450 | 0.75 | 10 | 13.3 | .002100 | 296.0 | 290.3 | 299.6 | 0 | .00500 | 296.0 | 290.3 | 299.6 | 0 |
| 2500 | 0.475 | 0.75 | 10 | 13.3 | .002100 | 284.3 | 280.1 | 293.0 | 0 | .00500 | 284.3 | 280.1 | 293.0 | 0 |
| 2500 | 0.400 | 0.75 | 10 | 20.0 | .000500 | 442.2 | 437.0 | 442.8 | 0 | .00350 | 442.2 | 437.0 | 442.8 | 0 |
| 2500 | 0.450 | 0.75 | 10 | 20.0 | .000500 | 401.2 | 401.5 | 402.6 | 0 | .00350 | 401.2 | 401.5 | 402.6 | 0 |
| 2500 | 0.475 | 0.75 | 10 | 20.0 | .000500 | 387.4 | 387.1 | 387.5 | 0 | .00350 | 387.4 | 387.1 | 387.5 | 0 |
| 2500 | 0.400 | 0.75 | 10 | 40.0 | .000006 | 417.5 | 431.6 | 434.1 | 0 | .00345 | 417.5 | 431.6 | 434.1 | 0 |
| 2500 | 0.450 | 0.75 | 10 | 40.0 | .000006 | 395.6 | 392.5 | 393.3 | 0 | .00345 | 395.6 | 392.5 | 393.3 | 0 |
| 2500 | 0.475 | 0.75 | 10 | 40.0 | .000006 | 378.4 | 382.6 | 382.6 | 0 | .00345 | 378.4 | 382.6 | 382.6 | 0 |

**Table 34:** Estimated out of control ARL, variable $N$.

# 5 Discussion

## Case study

We discuss the control charts fitted on the data from the case study, data generated in a grade D control room at rest. We assume that the data represents a room in control since no work activities were conducted. For these charts we used the data to determine the control limits and we checked if all time points are in control. The control limits are all constructed in a way that the false alarm rate of the charts is in theory $\alpha = 0.0027$, but we had only limited data to work with.

We start with the airborne particles counted by the BioTrak, in this case the total number of particles is large and spread over seven intervals. We also see in the trend graph in Figure 1a that there are four peaks on time periods $5, 8, 24$ and 28, where the peak on time 8 is prominent. At first sight the peaks could be a out of control time point and it is expected for the control chart to recognize these points. In the Hotelling $T^2$ control chart we see that the monitoring statistic is always in control and that the UCL is far above all values. For the multivariate np chart, we have one out-of-control signal at time point 8, that was also one of the peaks in the trend graph. The generalized p chart is not monitoring the data as it should since, there are 12 time periods out of control. The strange results may be cause by empty categories. When we use the estimation based on the log normal distribution the chart got worse, possibly due to a misfit of the log normal distribution. In the mp chart, where the monitoring static is simply the sum of the interval counts only time point 8 is out of control, so similar as the mnp chart. The chart is, like the mnp chart, a Shewhart type control chart. The bivariate zero inflated Poisson chart has six time point out of control. The monitoring statistic of this chart is the same as in the mp chart, but in this chart the upper control limit is much lower. The time points that are out of control are 4, 5, 8, 24 and 28. In the zero inflated negative binomial chart we only have one out of control point at time 8. The log likelihood ratio (LLR) chart has an out of control signal at time point 18. Based on the airborne data we believe that the generalized p chart and zero inflated poisson chart should not be used, but the rest may be appropriate.
The viable particles are a subset of the airborne particles. For the viable particles (Figure 1b) we do not see clear peaks like in the airborne trend graph. We also see that the number of particles is low. Thus the chart that we need for controlling this data has to be able to monitor low number of counts. The Hotelling $T^2$ chart has no out of control signals. The multivariate np chart has one out of control point signal at time point 8. This is the same out of control point as in the airborne particles mnp control chart. The generalized p chart seems to do better for viable particles, with no out of control points. The reason that the generalized p chart works better is because we do not count many particles and the estimated proportions on each time point are often the same. The multivariate Poisson chart has no out of control time point, either. For the zero inflated Poisson chart and zero inflated negative binomial chart we see no out of signal. The log likelihood ratio has also no out of control time points. When there are no particles on a time point counted, the statistic is zero in all charts except in the Hotelling $T^2$ chart, but this is never an out of control point when there are no particles counted. Based on the viable data there does not seem to be inappropriate charts.

## Simulation

### Fixed number of particles

To check the performance of the charts we have conducted a simulation study. We start with the in control simulations when the total number of particles is fixed, where the phase I data and simulated data are generated with the same distributions and corresponding parameters. The total number of

observed particles are still variable because we ignore the particles in the first interval.

First, we have done an in control ARL study, when the data meets all requirements of the charts than we will have an estimated in control ARL close to 370.4. The simulation shows how the control charts performs under different conditions where we do not meet all requirements. We see that not all charts performed well when sample size of the phase I data was to small. Performances also deteriorated when the number of particles decreased, the distribution of counts over the intervals was not uniform and the particles in each category was too small. When we increased the number of particles all charts converges to the average run length of 370.4 except the log likelihood chart. The generalized p chart and multivariate np converges faster to 370.4 when the categories were three and four. It does not matter for the generalized p if we use non-parametrically or parametrically form of the interval probabilities. In practice, we would prefer non-parametric form since we do not have to make assumptions about the underlying distribution. In the case where we spread the data over seven intervals the generalized p and Hotelling $T^2$ performed worse. In this case there are intervals with many particles and other intervals with a small number of particles. So, it is important to choose intervals limits in a way that the data is spread-out over-all intervals. The performances of the mp chart were quite stable among all settings, because the test statistic and the control limits are in all charts the same; just the sum of al counts over all intervals. It seems that the multivariate np chart has the least difficulty with few particles. The test statistic is the weighted sum, that means that a shift in one of the smaller intervals has less effect on the test statistic. In all cases the Hotelling $T^2$ chart needs large number of particles, the reason for that is that we assume that the counts in every interval approximate the normal distribution. So, the primary conclusion from the number of particles is that they all need a certain number of particles to reach the specified false alarm rate. When we compare the charts, we see that the Hotelling $T^2$ needs the most particle to work nicely, the mnp chart needs less particles to perform well. The zero inflated Poisson and zero inflated negative binomial charts are never close to 370.4. The reason for that is the discreteness of the test statistics and control limits. The log likelihood ratio chart will not converge to the predefined average run length because the empirical quantile is only an estimate of the distribution quantile. But when we take this into account we see that the chart performed well when the estimated parameters from phase I are close to the real parameters. This means that the number of time points during the phase I process must be sufficiently large. Biotrak measures at minutes. Thus one week of 8 hours is 2400 time points. In practice we can not collect data for years, so the question is how many time points need the control chart so that the performance is as we expect. What we see in our study is that the ARL of Hotelling $T^2$, mnp and mp chart is always far above 370.4 when the process time of the phase I data is too low. The estimated ARL of generalized p chart is in this case too low. The generalized p chart had fastest convergence rate to 370.4 when we increase the time points in phase I. Again, we did not see big difference between the multivariate and grouped multivariate np and generalized Poisson chart

What we must keep in mind is that we have discrete data, when the number of particles is low also the possible combinations of the particles over the intervals is limited and small. We could combine intervals but still the combination are limited. That means it is not always the case that a proper control limit can be constructed with a predetermined false alarm rate. For that reason,it is preferred to change the $\alpha$'s in a way that all average run lengths are close to 370.4. In the out of control studies we have changed the parameters of the underlying distribution which resulted in a shift in the corresponding categorical data. When the number of particles is low, it seems that the mp chart had timely detection of the out of control. When we have three intervals and the number of particles is large, all chart reports a low average run length but the Hotelling and Multivariate mnp chart have lower ARL values than the generalized p and on the fourth place is taken by the mp chart. The mp chart and generalized p chart perform well when the number of particles is low. When we have more categories such as in the case of four and seven, the mnp chart is performing worse than the generalized p chart and the mp chart.

### Varying number of particles

We have done a similar simulation study in the case that the total number of particles on each time point is varying. We generate the total number of particles with the negative binomial distribution. The simulation studies are testing the average run length with different parameters (mean and variance) of the negative binomial total particles distribution. Increasing the variance of the negative binomial means that it is harder to monitor the processes. Under the in control setting, performances of all charts improved as the mean of the negative binomial was increased similar to the situation with fixed N. Furthermore, when we increased the variance the required number of particles and the sample size of phase I process for the charts to reach the predefined false alarm rate was also increased. We see in the in-control study that the generalized p chart is performing well when we have three and four categories, and in the case of seven intervals. It needed substantially more particles to perform well. Though the generalized p chart relies on the estimation of the p's, the ARL were similar between the moment estimator and the MLE. We also changed the false alarm rate in the studies to be sure that every simulation study will have an in control ARL of 370.4. When the number of particles is small, the ARL's are for all four charts larger than 370.4. When the number of particles becomes larger, the average run length becomes smaller than 370.4. On average, the generalized p chart had the fastest detection time of the out of control. But also, the Hotelling charts and log likelihood ratio chart perform very well. The mnp and mp chart need many particles to become comparable to the other chart

## Overall conclusion

To come to an overall conclusion, we must think about the settings in the case study as well. So, in the case study the number particles are variable and depending on the grade of the room there will be many or few airborne particles in the room. In case of viable particles there will be zero or few particles in the room, in case of all airborne particles there could be many particles in the room. When we have a few particles in the room all charts do not have satisfactory performances, modification of the predefined $\alpha$ is thus required to improve the performances. Based on the simulation results the generalized p chart is performing well when the number of categories is limited bu this one fails completely in the case study. It is important that the interval limits are spread out in a such a way that each interval receives an equal number of particles when the room are in control. This will provide better results. It is also recommended to use different interval limits for the airborne and viable particles. In case of seven intervals the spread is not good, and the phase I time sample must be larger to estimate all parameters. When we have many particles present in the room then the Hotelling $T^2$ or generalized p chart is performing very well. But it would be nice to see this chart in practice in a case study where we have data from one control room and many continuous time periods, when the control is at rest and when the room is in operation. In both situation it is very important that we also report the activities that happens in the room. We must look back in this data to see if it can declare peaks or other out of control points.

# 6 Further research

There is still a lot of work to do to monitor the categorical BioTrak particle data in real time. In this thesis we have seen the performances of possible existing control charts for categorical data. We do not know how the charts perform on real phase I and phase II data, we must do again a case study. In the new case study, we must focus on one control room and we must monitor the process longer. We also must investigate which control limits we have to use in practice, this is difficult because each control room has is own the number of categories is limited. We can also try to further investigate the underlying distribution of the categorical distribution. When we know the underlying distribution, we could monitor the parameters of this distribution instead of all categories. It is also possible to extend the control charts. We could add weights to each category, in that way we have influence on the importance of each category. When we know that a kind of hazardous particle is in interval four, then we want to monitor interval four extra. We also must optimize the monitoring time and interval limits. more time means that we expect to count more particles, and so the control chart performance will be better. To use the charts in practice we must construct control limits based on a phase I process which is considered in control. A larger phase I process will lead to better control limits and parameter estimates. In practice due to limited time constraints we must find a way to construct reasonable control limits and parameter estimation methods based on a small phase I dataset. The control charts do not perform well when we expect a few particles in the categories, so we need other charts or adjust the chart in a way that they will work also for low counts data.

## References


[1] TSI. TSI Biotrak Real-Time Viable Particle Counter, performance qualification studie: abridged drug master file. September 2016 (US).

[2] D.C. Montgomery. *Statistical Quality Control, 7th Edition*. Wiley, 2012.

[3] Elisabeth Topalidou and Stelios Psarakis. Review of multinomial and multiattribute quality control charts. *Quality and Reliability Engineering International*, 25(7):773–804.

[4] A. J. Duncan. Quality control and industrial statistics (fifth edition). *Quality and Reliability Engineering International*, 5(3), 1989.

[5] Mark Marcucci. Monitoring multinomial processes. *Journal of Quality Technology*, 17:86–91, 04 1985.

[6] Lloyd S. Nelson. A chi-square control chart for several proportions. *Journal of Quality Technology*, 19(4):229–231, 1987.

[7] Chia-Ding Hou, Yuehjen E. Shao, and Sheng Huang. A combined mle and generalized p chart approach to estimate the change point of a multinomial process. *Appl. Math.*, 7(4):1487–1493, 2013.

[8] L.H.C. Tippett. The efficient use of gauges in quality control. *Engineer.*, 177:481–483, 1944.

[9] W. L. Stevens. Control by gauging. *Journal of the Royal Statistical Society. Series B (Methodological)*, 10(1):54–108, 1948.

[10] S. H. Steiner. Quality control and improvement based on grouped data. *Managment Science/Systems*, 1994.

[11] S. H. Steiner, P. L. Geyer, and G. O. Wesolowsky. Control charts based on grouped observations. *International Journal of Production Research*, 32(1):75–91, 1994.

[12] Gary R. Tucker, William H. Woodall, and Kwok-Leung Tsui. A control chart method for ordinal data. *American Journal of Mathematical and Management Sciences*, 22(1-2):31–48, 2002.

[13] H. I. Patel. Quality control methods for multivariate binomial and poisson distributions. *Technometrics*, 15(1):103–112, 1973.

[14] X. S. Lu. Control chart for multivariate attribute processes. *International Journal of Production Research*, 36(12):3477–3489, 1998.

[15] Jing-Er Chiu and Tsen-I Kuo. Attribute control chart for multivariate poisson distribution. *Communications in Statistics - Theory and Methods*, 37(1):146–158, 2007.

[16] Amir Afshin Fatahi, Rassoul Noorossana, Pershang Dokouhaki, and Babak Farhang Moghaddam. Copula-based bivariate zip control chart for monitoring rare events. *Communications in Statistics - Theory and Methods*, 41(15):2699–2716, 2012.

[17] Stephan .A.W van Driel. Particle size statistics with grouped and truncated data measured by the biotrak real time viable particle counter in cleanrooms, 2018.

[18] F. Galton and B. A McALISTER. Xii. the geometric mean, in vital and social statistics. *Proceedings of the Royal Society of London*, 29(196-199):365–367, 1879.

[19] F. Kottler. The distribution of particle sizes. *Journal of the Franklin Institute*, 250(4):339–356, 1950.

[20] O. G. Raabe. Particle size analysis utilizing grouped data and the log-normal distribution. *Journal of Aerosol Science*, 2(3):289–303, 1971.

[21] W.C. Hinds. *Aerosol technology: properties, behavior, and measurement of airborne particles*. Wiley-Interscience. Wiley, 1999.

[22] K. Willeke 3rd ed. ed. P Kulkani, P.A. Baron. *Aerosol Measurements, Principles, techniques and applications*, chapter 36, pages 771–784. Wiley-Blackwell, 2011.

[23] Andy Holbrook. Cleanroom contamination. 2015. Source: Wacker.

[24] Warren May and William Johnson. On the singularity of the covariance matrix for estimates of multinomial proportions. *Journal of biopharmaceutical statistics*, 8:329–36, 06 1998.

[25] Youn-Min Chou, Robert L. Mason, and John C. Young. The control chart for individual observations from a multivariate non-normal distribution. *Communications in Statistics - Theory and Methods*, 30(8-9):1937–1949, 2001.

[26] William G. Cochran. The $\chi^2$ test of goodness of fit. *Ann. Math. Statist.*, 23(3):315–345, 09 1952.

[27] James K. Yarnold. The minimum expectation in $\chi^2$ goodness of fit tests and the accuracy of approximations for the null distribution. *Journal of the American Statistical Association*, 65(330):864–886, 1970.

[28] Abe Sklar. Random variables, joint distribution functions, and copulas. *Kybernetika*, 9:449–460, 1973.

[29] Lai Chin-Diew N Balakrishnan. *Continuous Bivariate Distributions*. Springer-Verlag New York, 2009. Originally published by Rumsby Scientific Publishing, Adelaide, South Australia, Australia, 1990.

[30] R.B. Nelsen. Copulas and association in: Dall'aglio g., kotz s., salinetti g. (eds) advances in probability distributions with given marginals.mathematics and its applications. *Springer, Dordrecht*, 67:51–73, 1991.

[31] Sadie Beckett, Joshua Jee, Thapelo Ncube, Sophia Pompilus, Quintel Washington, Anshuman Singh, and Nabendu Pal. Zero-inflated poisson (ZIP) distribution: parameter estimation and applications to model data from natural calamities. *Involve, a Journal of Mathematics*, 7(6):751–767, oct 2014.

# Appendices

## A Log likelihood ratio chart

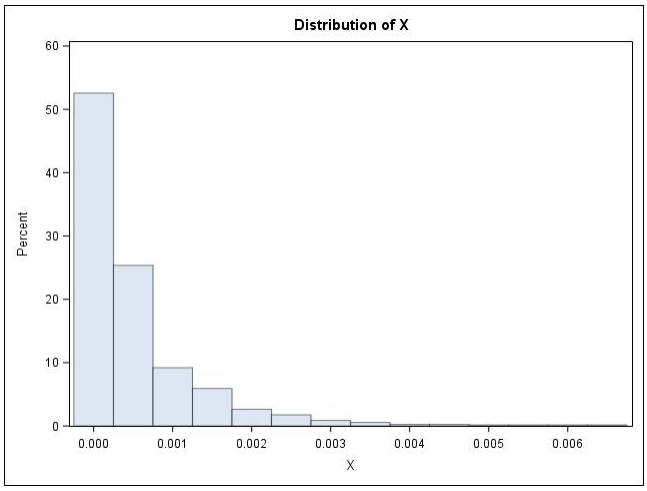



**Figure 18:** Histogram from the monitoring statistic based on a phase II dataset with 3 intervals, 1250 time points and on each time point a fixed number of 1000 particles.

## B Bivariate zero inflated Poisson chart

### B.1 Copula

The copula function in the paper is:

$$C(u,v) = \begin{cases} (1-\rho)uv + \rho \min(u,v) & \rho > 0 \\ (1+\rho)uv + \rho \max(u-1+v,0) & \rho \leq 0 \end{cases} \tag{38}$$

The first part for $\rho > 0$ is a well defined copula function, it is the same as in Equation (25), but the second part for $\rho \geq 0$ is not a copula function. We will proof that Equation (38) is not a copula function for all $\rho \in (0,-1)$.

Let $C(u,v) = (1+\rho)uv + \rho \max(u-1+v,0)$ be a copula function, where $\rho \leq 0$. Take $u = v = 1$ then $C(u,v) = 1+2\rho \geq 1$. When $\rho > 0$ we have $C(u,v) = 1+2\rho > 1$. We have a contradiction because by definition $C(u,v) \in (0,1)$.

### B.2 Reformulation inequality upper control limit

$$
\begin{aligned}
&\mathbb{P}(X \geq UCL) \leq \alpha \\
\Rightarrow\ &1 - \mathbb{P}(X < UCL) \leq \alpha \\
\Rightarrow\ &\mathbb{P}(X < UCL) \geq 1 - \alpha \\
\Rightarrow\ &\mathbb{P}(X \leq UCL - 1) \geq 1 - \alpha \\
\Rightarrow\ &\mathbb{P}(Y_1 \leq y_1, Y_2 \leq y_2 \mid \min(y_1 + y_2 - 1) = UCL - 1) \geq 1 - \alpha \\
\Rightarrow\ &F(y_1, y_2 \mid \min(y_1 + y_2 - 1) = UCL - 1) \geq 1 - \alpha \\
\Rightarrow\ &C(u, v \mid \min(y_1 + y_2 - 1) = UCL) \geq 1 - \alpha
\end{aligned} \tag{39}
$$

## C Extra results

## C.1 Charts with same average run length

For every simulation study the false alarm rate $\alpha$ is changed to get a ARL close to 370.4. We change the alpha by hand to get a ARL $\in (360, 380)$. It is not always possible to change the chart in a way that the chart becomes 370, we indicate this by making the table cell light grey. **Three intervals, fixed N**

| | | Hotelling chart | | | | | Multivariate np chart | | | | | Generalized p chart | | | | | Multivariate poisson chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | N | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 10 | .00070 | 367.3 | 383.5 | 383.5 | 0 | .00230 | 378.2 | 423.9 | 423.9 | 1 | .0017 | 369.7 | 410.6 | 410.6 | 2 | .00075 | 370.7 | 542.9 | 542.8 | 3 |
| 2500 | 100 | .00190 | 365.7 | 366.3 | 366.3 | 0 | .00265 | 367.5 | 361.0 | 361.0 | 0 | .0026 | 374.5 | 377.2 | 376.2 | 0 | .00142 | 370.4 | 470.7 | 470.6 | 4 |
| 2500 | 1000 | .00275 | 373.2 | 380.5 | 380.4 | 0 | .00270 | 369.4 | 365.5 | 365.4 | 0 | .0027 | 369.8 | 368.8 | 368.8 | 0 | .00270 | 376.7 | 384.1 | 384.1 | 0 |
| 2500 | 10000 | .00270 | 375.3 | 372.6 | 372.6 | 0 | .00270 | 364.2 | 361.5 | 361.4 | 0 | .0027 | 369.7 | 369.6 | 369.5 | 0 | .00270 | 376.0 | 384.1 | 384.1 | 0 |

**Table 35:** (1/2) Changed $\alpha$ to get close to ARL = 370.4 for three intervals, fixed $N$.

| | | multivariate np chart (grouped) | | | | | Generalized p chart (grouped) | | | | | Likelihood ratio control chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | N | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 10 | .00230 | 378.2 | 423.5 | 423.5 | 1 | .0017 | 368.7 | 410.6 | 410.6 | 2 | .0014 | 434.6 | 600.8 | 604.1 | 29 |
| 2500 | 100 | .00265 | 367.5 | 361.0 | 361.0 | 0 | .0026 | 375.2 | 378.5 | 378.5 | 0 | .0024 | 399.1 | 506.9 | 507.7 | 15 |
| 2500 | 1000 | .00270 | 369.4 | 365.5 | 365.4 | 0 | .0027 | 369.9 | 368.8 | 368.8 | 0 | .0024 | 412.0 | 538.8 | 540.4 | 16 |
| 2500 | 10000 | .00270 | 364.2 | 361.5 | 361.5 | 0 | .0027 | 369.7 | 369.6 | 369.5 | 0 | .0024 | 409.9 | 524.2 | 525.7 | 11 |

**Table 36:** (2/2) Changed $\alpha$ to get close to ARL = 370.4 for three intervals, fixed $N$.

**Four intervals, fixed N**

| | | Hotelling chart | | | | | Multivariate np chart | | | | | Generalized p chart | | | | | Multivariate poisson chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | N | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 10 | .0010 | 378.5 | 380.2 | 379.8 | 1 | .0015 | 378.2 | 387.4 | 387.4 | 0 | .0022 | 367.2 | 378.4 | 377.1 | 0 | .00075 | 377.3 | 555.8 | 555.8 | 5 |
| 2500 | 100 | .0022 | 378.0 | 378.9 | 377.6 | 0 | .00260 | 371.0 | 375.7 | 375.6 | 0 | .0026 | 371.9 | 365.6 | 365.5 | 0 | .00145 | 361.5 | 442.2 | 442.2 | 4 |
| 2500 | 1000 | .0027 | 379.9 | 383.6 | 383.7 | 0 | .00270 | 368.6 | 384.3 | 384.2 | 0 | .0027 | 369.9 | 362.9 | 362.8 | 0 | .00270 | 366.6 | 361.3 | 361.0 | 0 |
| 2500 | 10000 | .0027 | 375.4 | 372.4 | 372.3 | 0 | .00270 | 369.4 | 381.1 | 381.0 | 0 | .0027 | 371.2 | 360.8 | 360.7 | 0 | .00270 | 366.7 | 361.1 | 360.8 | 0 |

**Table 37:** (1/2) Changed $\alpha$ to get close to ARL = 370.4 for four intervals, fixed $N$.

| | | multivariate np chart (grouped) | | | | | Generalized p chart (grouped) | | | | | Likelihood ratio control chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | N | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 10 | .00145 | 385.6 | 397.4 | 397.6 | 0 | .0022 | 367.0 | 378.3 | 378.3 | 0 | 0.0114 | 365.8 | 487.7 | 487.6 | 11 |
| 2500 | 100 | .00260 | 370.6 | 374.9 | 374.8 | 0 | .0026 | 371.9 | 365.6 | 365.5 | 0 | 0.0024 | 410.8 | 537.3 | 538.7 | 13 |
| 2500 | 1000 | .00270 | 368.6 | 384.3 | 384.2 | 0 | .0027 | 369.9 | 362.9 | 362.8 | 0 | 0.0024 | 410.4 | 519.7 | 521.2 | 12 |
| 2500 | 10000 | .00270 | 369.4 | 381.1 | 381.0 | 0 | .0027 | 371.2 | 360.8 | 360.7 | 0 | 0.0024 | 413.0 | 526.6 | 528.2 | 12 |

**Table 38:** (2/2) Changed $\alpha$ to get close to ARL = 370.4 for four intervals, fixed $N$.

### Seven intervals, fixed N

| | | Hotelling chart | | | | | Multivariate np chart | | | | | Generalized p chart | | | | | Multivariate poisson chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | N | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 10 | .000002 | 333.5 | 330.8 | 332.8 | 0 | .00013 | 379.4 | 386.7 | 386.8 | 0 | .000001 | 346.8 | 348.9 | 349.7 | 0 | .00075 | 371.5 | 540.5 | 540.4 | 2 |
| 2500 | 100 | .000800 | 366.2 | 367.2 | 366.8 | 0 | .00170 | 370.0 | 374.1 | 374.1 | 0 | .000700 | 369.1 | 370.0 | 369.7 | 0 | .00140 | 373.1 | 468.6 | 468.6 | 3 |
| 2500 | 1000 | .002400 | 372.9 | 375.5 | 375.4 | 0 | .00260 | 371.6 | 378.7 | 378.7 | 0 | .002400 | 363.0 | 361.3 | 361.4 | 0 | .00270 | 376.6 | 388.9 | 388.7 | 0 |
| 2500 | 10000 | .002700 | 372.5 | 370.8 | 370.7 | 0 | .00270 | 377.8 | 376.2 | 376.2 | 0 | .002700 | 365.1 | 366.6 | 366.6 | 0 | .00270 | 368.3 | 373.2 | 373.0 | 0 |

**Table 39:** (1/2) Changed $\alpha$ to get close to ARL = 370.4 for seven intervals, fixed $N$.

| | | multivariate np chart (grouped) | | | | | Generalized p chart (grouped) | | | | | Likelihood ratio control chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | N | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 10 | .00013 | 377.4 | 381.9 | 381.9 | 0 | .000001 | 346.8 | 349.0 | 349.7 | 0 | .0024 | 417.3 | 531.5 | 533.5 | 10 |
| 2500 | 100 | .00170 | 371.0 | 374.1 | 374.0 | 0 | .000700 | 367.3 | 365.7 | 365.6 | 0 | .0024 | 409.1 | 531.0 | 532.4 | 13 |
| 2500 | 1000 | .00260 | 370.9 | 377.3 | 377.3 | 0 | .002400 | 364.7 | 365.4 | 365.4 | 0 | .0024 | 408.7 | 519.2 | 520.6 | 12 |
| 2500 | 10000 | .00270 | 377.0 | 374.3 | 374.3 | 0 | .002700 | 365.2 | 366.5 | 366.5 | 0 | .0024 | 425.6 | 563.4 | 566.0 | 15 |

**Table 40:** (2/2) Changed $\alpha$ to get close to ARL = 370.4 for seven intervals, fixed $N$.

### Three intervals, varying N

| | | | Hotelling chart | | | | | Multivariate np chart | | | | | Generalized p chart | | | | | Multivariate poisson chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | Mean | Variance | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 10 | 13.3 | .00040 | 361.0 | 378.7 | 378.7 | 1 | .00090 | 363.8 | 358.8 | 358.9 | 0 | .00170 | 368.1 | 378.6 | 378.6 | 0 | .00105 | 371.0 | 401.5 | 401.4 | 0 |
| 2500 | 10 | 20.0 | .00012 | 375.8 | 377.5 | 377.5 | 0 | .00010 | 716.3 | 717.5 | 796.5 | 17 | .00175 | 370.3 | 388.7 | 388.6 | 0 | .00045 | 372.3 | 376.2 | 376.2 | 0 |
| 2500 | 10 | 40.0 | .00002 | 318.3 | 320.8 | 324.9 | 0 | .00006 | 364.1 | 368.6 | 368.6 | 0 | .00160 | 368.2 | 375.4 | 375.4 | 0 | .00006 | 373.6 | 389.3 | 389.2 | 1 |
| 2500 | 100 | 133.3 | .00235 | 373.4 | 375.0 | 375.0 | 0 | .00260 | 372.7 | 363.4 | 363.3 | 0 | .00270 | 371.0 | 374.0 | 374.0 | 0 | .00270 | 369.6 | 358.8 | 358.8 | 0 |
| 2500 | 100 | 200.0 | .00200 | 364.8 | 375.3 | 375.3 | 0 | .00250 | 366.7 | 374.8 | 374.7 | 0 | .00270 | 365.2 | 369.1 | 369.1 | 0 | .00250 | 363.3 | 369.7 | 369.8 | 0 |
| 2500 | 100 | 400.0 | .00110 | 362.7 | 362.1 | 362.1 | 0 | .00150 | 373.4 | 372.0 | 371.9 | 0 | .00270 | 371.4 | 379.7 | 379.6 | 0 | .00160 | 363.4 | 362.4 | 362.4 | 0 |
| 2500 | 1000 | 1333.3 | .00270 | 368.0 | 383.9 | 383.9 | 0 | .00270 | 365.8 | 371.3 | 371.2 | 0 | .00270 | 373.1 | 378.5 | 378.5 | 0 | .00270 | 374.5 | 394.4 | 394.4 | 1 |
| 2500 | 1000 | 2000.0 | .00260 | 377.5 | 383.5 | 383.5 | 0 | .00270 | 367.9 | 374.9 | 374.9 | 1 | .00270 | 366.3 | 371.6 | 371.6 | 0 | .00275 | 374.0 | 374.4 | 374.4 | 0 |
| 2500 | 1000 | 4000.0 | .00250 | 370.4 | 375.3 | 375.3 | 0 | .00260 | 374.2 | 382.5 | 382.5 | 0 | .00270 | 366.4 | 379.2 | 379.1 | 1 | .00260 | 371.5 | 379.1 | 379.1 | 0 |
| 2500 | 10000 | 13333.3 | .00270 | 368.0 | 377.8 | 377.8 | 0 | .00270 | 369.1 | 377.1 | 377.1 | 0 | .00270 | 370.0 | 378.4 | 378.3 | 0 | .00270 | 371.1 | 376.4 | 376.3 | 1 |
| 2500 | 10000 | 20000.0 | .00270 | 373.5 | 386.1 | 386.1 | 0 | .00270 | 370.3 | 385.4 | 385.4 | 1 | .00270 | 373.9 | 381.3 | 381.2 | 0 | .00270 | 379.4 | 390.7 | 390.7 | 0 |
| 2500 | 10000 | 40000.0 | .00280 | 365.6 | 373.6 | 373.5 | 0 | .00270 | 370.5 | 369.2 | 369.1 | 1 | .00270 | 372.7 | 376.2 | 376.2 | 0 | .00270 | 371.7 | 372.2 | 372.2 | 0 |

**Table 41:** (1/2) Changed $\alpha$ to get close to ARL = 370.4 for three intervals, variable $N$.

| | | | multivariate np chart (grouped) | | | | | Generalized p chart (grouped) | | | | | Likelihood ratio control chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | Mean | Variance | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 10 | 13.3 | .00090 | 364.3 | 358.3 | 358.3 | 0 | .00170 | 370.9 | 379.0 | 379.0 | 0 | 0.0036 | 304.5 | 507.5 | 511.7 | 32 |
| 2500 | 10 | 20.0 | .00040 | 369.6 | 371.5 | 371.4 | 0 | .00175 | 374.0 | 388.2 | 388.2 | 0 | 0.0055 | 422.4 | 809.8 | 811.4 | 138 |
| 2500 | 10 | 40.0 | .00006 | 363.8 | 368.7 | 368.7 | 0 | .00160 | 388.4 | 393.3 | 393.6 | 0 | 0.0138 | 313.9 | 782.4 | 784.3 | 148 |
| 2500 | 100 | 133.3 | .00260 | 372.7 | 363.3 | 363.3 | 0 | .00270 | 371.5 | 374.6 | 374.6 | 0 | 0.0024 | 407.7 | 513.8 | 515.1 | 14 |
| 2500 | 100 | 200.0 | .00250 | 366.7 | 374.8 | 374.8 | 0 | .00270 | 364.4 | 368.4 | 368.4 | 0 | 0.0024 | 402.3 | 518.5 | 519.4 | 11 |
| 2500 | 100 | 400.0 | .00150 | 373.3 | 371.8 | 371.8 | 0 | .00270 | 372.3 | 381.6 | 381.6 | 0 | 0.0024 | 411.6 | 523.7 | 525.2 | 11 |
| 2500 | 1000 | 1333.3 | .00270 | 365.8 | 371.3 | 371.2 | 0 | .00270 | 373.1 | 378.8 | 378.7 | 0 | 0.0024 | 408.7 | 515.1 | 516.4 | 10 |
| 2500 | 1000 | 2000.0 | .00270 | 367.9 | 374.9 | 374.9 | 1 | .00270 | 366.6 | 371.8 | 371.8 | 0 | 0.0024 | 400.8 | 489.7 | 490.6 | 8 |
| 2500 | 1000 | 4000.0 | .00260 | 374.2 | 382.5 | 382.5 | 0 | .00270 | 366.4 | 379.3 | 379.2 | 1 | 0.0024 | 407.4 | 524.9 | 526.1 | 9 |
| 2500 | 10000 | 13333.3 | .00270 | 369.1 | 377.1 | 377.1 | 0 | .00270 | 370.0 | 378.4 | 378.3 | 0 | 0.0024 | 407.7 | 536.0 | 537.3 | 11 |
| 2500 | 10000 | 20000.0 | .00270 | 370.3 | 385.4 | 385.4 | 1 | .00270 | 373.5 | 380.9 | 380.9 | 0 | 0.0024 | 415.4 | 534.3 | 536.1 | 10 |
| 2500 | 10000 | 40000.0 | .00270 | 370.5 | 369.2 | 369.1 | 1 | .00270 | 372.8 | 376.3 | 376.3 | 0 | 0.0024 | 413.4 | 532.5 | 534.1 | 14 |

**Table 42:** (2/2) Changed $\alpha$ to get close to ARL = 370.4 for three intervals, variable $N$.

**Four intervals, varying N**

| | | | Hotelling chart | | | | | Multivariate np chart | | | | | Generalized p chart | | | | | Multivariate poisson chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | Mean | Variance | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 10 | 13.3 | .00027 | 361.1 | 363.2 | 363.3 | 0 | .00105 | 369.2 | 373.8 | 373.7 | 0 | .0023 | 368.3 | 367.3 | 367.3 | 0 | .00110 | 375.0 | 407.9 | 407.9 | 1 |
| 2500 | 10 | 20.0 | .00008 | 369.0 | 366.0 | 366.0 | 0 | .00040 | 374.3 | 375.6 | 375.6 | 0 | .0023 | 372.2 | 375.2 | 375.2 | 0 | .00045 | 371.5 | 375.9 | 375.8 | 0 |
| 2500 | 10 | 40.0 | .00002 | 237.0 | 235.6 | 270.7 | 0 | .00006 | 366.0 | 375.2 | 375.2 | 0 | .0026 | 360.7 | 362.7 | 362.8 | 0 | .00006 | 368.6 | 383.0 | 383.0 | 0 |
| 2500 | 100 | 133.3 | .00210 | 377.4 | 385.9 | 385.9 | 0 | .00270 | 365.5 | 373.1 | 373.1 | 0 | .0027 | 370.9 | 373.0 | 372.9 | 0 | .00270 | 372.3 | 376.8 | 376.8 | 0 |
| 2500 | 100 | 200.0 | .00160 | 376.9 | 367.2 | 367.2 | 0 | .00230 | 375.3 | 382.6 | 382.6 | 0 | .0027 | 366.5 | 367.7 | 367.7 | 0 | .00240 | 366.5 | 383.7 | 383.7 | 0 |
| 2500 | 100 | 400.0 | .00090 | 366.9 | 383.7 | 383.6 | 1 | .00160 | 361.6 | 369.1 | 369.2 | 0 | .0027 | 367.0 | 360.9 | 360.9 | 0 | .00160 | 361.6 | 374.4 | 374.5 | 0 |
| 2500 | 1000 | 1333.3 | .00275 | 374.9 | 380.6 | 380.5 | 0 | .00275 | 372.6 | 374.5 | 374.4 | 0 | .0027 | 364.2 | 365.6 | 365.6 | 0 | .00275 | 373.8 | 380.3 | 380.3 | 0 |
| 2500 | 1000 | 2000.0 | .00260 | 373.0 | 377.2 | 377.1 | 0 | .00270 | 377.5 | 386.7 | 386.7 | 0 | .0027 | 371.8 | 370.4 | 370.4 | 0 | .00270 | 374.7 | 386.0 | 385.9 | 0 |
| 2500 | 1000 | 4000.0 | .00245 | 364.6 | 362.4 | 362.4 | 0 | .00260 | 361.4 | 375.8 | 375.9 | 0 | .0027 | 372.6 | 370.1 | 370.0 | 0 | .00260 | 365.4 | 386.9 | 386.9 | 1 |
| 2500 | 10000 | 13333.3 | .00275 | 375.2 | 381.5 | 381.5 | 0 | .00275 | 373.3 | 376.8 | 376.8 | 0 | .0027 | 367.7 | 367.5 | 367.5 | 0 | .00275 | 372.6 | 373.0 | 373.0 | 0 |
| 2500 | 10000 | 20000.0 | .00275 | 373.2 | 386.1 | 386.1 | 1 | .00275 | 376.3 | 390.0 | 390.0 | 0 | .0027 | 364.3 | 360.3 | 360.3 | 0 | .00275 | 376.4 | 388.5 | 388.5 | 0 |
| 2500 | 10000 | 40000.0 | .00270 | 372.9 | 370.6 | 370.6 | 0 | .00270 | 376.6 | 371.7 | 371.7 | 0 | .0026 | 371.5 | 371.2 | 371.2 | 0 | .00275 | 376.5 | 382.7 | 382.7 | 0 |

**Table 43:** (1/2) Changed $\alpha$ to get close to ARL = 370.4 for four intervals, variable $N$.

| | | | multivariate np chart (grouped) | | | | | Generalized p chart (grouped) | | | | | Likelihood ratio control chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | Mean | Variance | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 10 | 13.3 | .00105 | 369.1 | 374.1 | 374.0 | 0 | .0023 | 367.5 | 370.3 | 370.3 | 0 | 0.0440 | 373.5 | 531.3 | 531.3 | 16 |
| 2500 | 10 | 20.0 | .00040 | 374.2 | 375.6 | 375.6 | 0 | .0023 | 374.3 | 376.1 | 376.1 | 0 | 0.0630 | 393.7 | 578.7 | 579.1 | 21 |
| 2500 | 10 | 40.0 | .00006 | 365.8 | 374.9 | 374.9 | 0 | .0026 | 378.6 | 380.5 | 380.5 | 0 | 0.1160 | 351.7 | 590.2 | 590.4 | 20 |
| 2500 | 100 | 133.3 | .00270 | 365.3 | 373.1 | 373.0 | 0 | .0027 | 371.1 | 372.9 | 372.8 | 0 | 0.0024 | 411.9 | 530.9 | 532.5 | 15 |
| 2500 | 100 | 200.0 | .00230 | 375.3 | 382.6 | 382.6 | 0 | .0027 | 366.6 | 367.1 | 367.1 | 0 | 0.0024 | 406.9 | 518.8 | 520.0 | 15 |
| 2500 | 100 | 400.0 | .00160 | 361.6 | 369.1 | 369.2 | 0 | .0027 | 367.1 | 360.6 | 360.6 | 0 | 0.0024 | 421.0 | 538.7 | 541.0 | 15 |
| 2500 | 1000 | 1333.3 | .00275 | 372.6 | 374.5 | 374.4 | 0 | .0027 | 363.8 | 365.4 | 365.4 | 0 | 0.0024 | 399.5 | 503.7 | 504.5 | 7 |
| 2500 | 1000 | 2000.0 | .00270 | 377.5 | 386.7 | 386.7 | 0 | .0027 | 372.1 | 370.4 | 370.4 | 0 | 0.0024 | 407.6 | 519.2 | 520.5 | 12 |
| 2500 | 1000 | 4000.0 | .00260 | 361.4 | 375.8 | 375.9 | 0 | .0027 | 372.5 | 369.9 | 369.9 | 0 | 0.0024 | 412.8 | 533.5 | 535.1 | 10 |
| 2500 | 10000 | 13333.3 | .00275 | 373.3 | 376.8 | 376.8 | 0 | .0027 | 367.6 | 367.3 | 367.3 | 0 | 0.0024 | 390.2 | 492.6 | 493.0 | 9 |
| 2500 | 10000 | 20000.0 | .00275 | 376.3 | 390.0 | 390.0 | 0 | .0027 | 364.3 | 360.2 | 360.2 | 0 | 0.0024 | 392.2 | 492.1 | 492.5 | 7 |
| 2500 | 10000 | 40000.0 | .00270 | 376.6 | 371.7 | 371.7 | 0 | .0026 | 371.7 | 371.2 | 371.2 | 0 | 0.0024 | 391.8 | 502.7 | 503.1 | 6 |

**Table 44:** (2/2) Changed $\alpha$ to get close to ARL = 370.4 for four intervals, variable $N$.

**Seven intervals, varying N**

| | | | Hotelling chart | | | | | Multivariate np chart | | | | | Generalized p chart | | | | | Multivariate poisson chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | Mean | Variance | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 10 | 13.3 | .00002 | 188.7 | 190.3 | 263.1 | 0 | .000500 | 362.6 | 372.0 | 372.1 | 0 | .00001 | 245.6 | 259.1 | 287.5 | 0 | .00110 | 371.6 | 412.9 | 412.9 | 0 |
| 2500 | 10 | 20.0 | .00002 | 147.3 | 150.6 | 269.2 | 0 | .000260 | 381.7 | 384.0 | 384.1 | 0 | .00001 | 211.3 | 212.7 | 265.6 | 0 | .00045 | 367.7 | 379.3 | 379.3 | 0 |
| 2500 | 10 | 40.0 | .00002 | 90.7 | 90.2 | 293.9 | 0 | .000055 | 379.7 | 390.0 | 390.0 | 0 | .00001 | 164.7 | 167.5 | 265.3 | 0 | .00007 | 362.2 | 360.6 | 360.7 | 0 |
| 2500 | 100 | 133.3 | .00055 | 366.7 | 380.5 | 380.4 | 0 | .002350 | 364.4 | 366.6 | 366.6 | 0 | .00070 | 371.5 | 371.9 | 371.8 | 0 | .00270 | 361.8 | 373.8 | 373.9 | 0 |
| 2500 | 100 | 200.0 | .00045 | 362.3 | 356.2 | 356.3 | 0 | .002100 | 374.7 | 382.1 | 382.1 | 0 | .00070 | 366.7 | 369.9 | 369.9 | 1 | .00230 | 372.3 | 383.0 | 382.9 | 2 |
| 2500 | 100 | 400.0 | .00025 | 368.7 | 377.2 | 377.2 | 0 | .001600 | 362.3 | 365.7 | 365.8 | 0 | .00070 | 367.5 | 374.2 | 374.2 | 0 | .00160 | 370.5 | 378.1 | 378.1 | 0 |
| 2500 | 1000 | 1333.3 | .00230 | 372.3 | 371.7 | 371.7 | 0 | .002700 | 370.4 | 388.4 | 388.3 | 0 | .00230 | 378.8 | 374.6 | 374.6 | 0 | .00270 | 364.9 | 366.3 | 366.3 | 1 |
| 2500 | 1000 | 2000.0 | .00220 | 377.3 | 382.6 | 382.7 | 0 | .002700 | 367.2 | 367.0 | 367.0 | 0 | .00230 | 369.1 | 379.5 | 379.4 | 0 | .00270 | 366.6 | 366.2 | 366.2 | 0 |
| 2500 | 1000 | 4000.0 | .00200 | 374.3 | 370.6 | 370.6 | 0 | .002650 | 361.8 | 354.8 | 354.9 | 0 | .00230 | 372.5 | 375.8 | 375.7 | 0 | .00270 | 363.8 | 370.1 | 370.1 | 0 |
| 2500 | 10000 | 13333.3 | .00270 | 371.3 | 364.5 | 364.4 | 0 | .002750 | 373.2 | 371.8 | 371.8 | 0 | .00270 | 377.7 | 370.9 | 370.9 | 0 | .00270 | 376.9 | 386.5 | 386.5 | 0 |
| 2500 | 10000 | 20000.0 | .00270 | 370.5 | 365.0 | 365.0 | 0 | .002700 | 375.9 | 389.4 | 389.4 | 1 | .00270 | 376.0 | 373.6 | 373.6 | 0 | .00270 | 366.9 | 368.1 | 368.1 | 0 |
| 2500 | 10000 | 40000.0 | .00270 | 368.4 | 378.4 | 378.4 | 0 | .002700 | 372.1 | 374.5 | 374.5 | 0 | .00270 | 375.6 | 368.3 | 368.3 | 0 | .00270 | 368.8 | 375.0 | 375.0 | 0 |

**Table 45:** (1/2) Changed $\alpha$ to get close to ARL = 370.4 for seven intervals, variable $N$.

| | | | multivariate np chart (grouped) | | | | | Generalized p chart (grouped) | | | | | Likelihood ratio control chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | Mean | Variance | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 10 | 13.3 | 0.000500 | 362.1 | 373.3 | 373.3 | 0 | .00001 | 244.2 | 247.9 | 278.1 | 0 | .0024 | 393.9 | 502.4 | 502.9 | 12 |
| 2500 | 10 | 20.0 | .00026 | 383.1 | 385.3 | 385.4 | 0 | .00001 | 215.0 | 215.0 | 265.3 | 0 | .0024 | 351.4 | 448.1 | 448.4 | 8 |
| 2500 | 10 | 40.0 | 0.000055 | 378.9 | 391.4 | 391.5 | 1 | .00001 | 172.4 | 169.8 | 260.8 | 0 | .0024 | 268.3 | 312.0 | 328.3 | 0 |
| 2500 | 100 | 133.3 | 0.002350 | 363.4 | 363.9 | 363.9 | 0 | .00070 | 374.6 | 374.5 | 374.4 | 0 | .0024 | 405.3 | 507.1 | 508.3 | 11 |
| 2500 | 100 | 200.0 | 0.002100 | 375.7 | 383.5 | 383.4 | 0 | .00070 | 369.1 | 372.7 | 372.6 | 1 | .0024 | 416.2 | 545.3 | 547.1 | 12 |
| 2500 | 100 | 400.0 | .00160 | 362.6 | 366.3 | 366.4 | 0 | .00070 | 370.3 | 375.8 | 375.7 | 0 | .0024 | 421.1 | 528.1 | 530.5 | 11 |
| 2500 | 1000 | 1333.3 | 0.002700 | 369.1 | 386.5 | 386.4 | 0 | .00230 | 376.7 | 372.6 | 372.6 | 0 | .0024 | 405.6 | 512.5 | 513.6 | 14 |
| 2500 | 1000 | 2000.0 | 0.002700 | 366.9 | 366.7 | 366.7 | 0 | .00230 | 366.4 | 371.3 | 371.3 | 0 | .0024 | 418.1 | 552.2 | 554.2 | 16 |
| 2500 | 1000 | 4000.0 | 0.002650 | 361.6 | 354.3 | 354.4 | 0 | .00230 | 371.7 | 375.4 | 375.3 | 0 | .0024 | 421.8 | 560.1 | 562.3 | 23 |
| 2500 | 10000 | 13333.3 | 0.002750 | 373.3 | 371.9 | 371.9 | 0 | .00270 | 374.3 | 365.9 | 365.9 | 0 | .0024 | 428.1 | 545.0 | 547.9 | 15 |
| 2500 | 10000 | 20000.0 | 0.002700 | 376.0 | 389.4 | 389.4 | 1 | .00270 | 375.8 | 374.2 | 374.2 | 0 | .0024 | 421.4 | 538.8 | 541.1 | 14 |
| 2500 | 10000 | 40000.0 | 0.002700 | 372.4 | 374.7 | 374.7 | 0 | .00270 | 375.9 | 367.4 | 367.4 | 0 | .0024 | 419.3 | 537.2 | 539.3 | 11 |

**Table 46:** (2/2) Changed $\alpha$ to get close to ARL = 370.4 for seven intervals, variable $N$.

**Zero inflated methods**

**Three intervals, variable N**

| | | | Zero inflated Poisson chart | | | | | Zero inflated negative binomial chart | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{base}$ | Mean | Variance | $\alpha$ | ARL | STD | RMSE | ⊗ | $\alpha$ | ARL | STD | RMSE | ⊗ |
| 2500 | 10 | 13.3 | .002100 | 270.4 | 272.6 | 290.4 | 0 | .00500 | 270.4 | 272.6 | 290.4 | 0 |
| 2500 | 10 | 20.0 | .000500 | 373.4 | 372.3 | 372.2 | 0 | .00350 | 373.4 | 372.3 | 372.2 | 0 |
| 2500 | 10 | 40.0 | .000006 | 360.2 | 365.2 | 365.3 | 0 | .00345 | 360.2 | 365.2 | 365.3 | 0 |

**Table 47:** Changed $\alpha$ to get close to ARL = 370.4, variable $N$.

# D SAS programs

- `Monitor_BioTrak_data.sas`
  (monitoring the BioTrak data and making the control charts, Section 2.2.)
- `Simulation.sas`
  (Simulation studies, Section 3. )
- `Figures_simulation.sas`
  (simulation and making figures from simulation study, Section 3.)

# E Control chart for the bivariate zero-inflated Poisson process, correction of Fatahi et.al (2012)

# Control chart for the bivariate zero-inflated Poisson process

## Corrections of Fatahi et.al (2012)

E. R. van den Heuvel[1], S. A. W. van Driel[1], and Z. Zhan[1]

[1]Department of Mathematics and Computer Science, Eindhoven University of Technology

Fatahi *et al.* (2012), proposed a bivariate zero-inflated Poisson (bi-ZIP) control chart based on copula for the purpose of monitoring correlated rare events. They introduce the following copula for modeling the joint distribution of the bi-ZIP process:

$$C(u,v) = \begin{cases} (1-\rho)uv + \rho \min(u,v) & \rho > 0 \\ (1+\rho)uv + \rho \max(u+v-1,0) & \rho \leq 0 \end{cases}$$

We noticed that when $\rho \leq 0$, the expression $C^-(u,v) = (1+\rho)uv + \rho \max(u+v-1,0)$ is not guaranteed to be a copula. To see this, let $u = 1$ and $v \in [0,1]$, then $C^-(u,v) = (1+2\rho)v$, which can take values below zero. The author of the original paper did not encounter any issues because they didn't study negative correlations. It is a priori unclear how to repair this. The most obvious option would be take a convex combination of the two copulas $\Pi(u,v) = uv$ and $W(u,v) = \max(u+v-1,0)$, such that it is guaranteed to be a copula as well (Balakrishna & Lai, 2009). This choice would then leads to the following copula:

$$C(u,v) = \begin{cases} (1-\rho)uv + \rho \min(u,v) & \rho > 0 \\ (1+\rho)uv - \rho \max(u+v-1,0) & \rho \leq 0. \end{cases} \tag{1}$$

Note that, copula (1) is a special case of the two-parameter comprehensive copula due to Fréchet (1958):

$$C_F(u,v) = \alpha \min(u,v) + \beta \max(u+v-1,0) + (1-\alpha-\beta)uv$$

for $\alpha > 0$, $\beta > 0$ (Nelsen, 1999). When $\rho > 0$, we may take $\alpha = \rho$, $\beta = 0$, and then $C_F(u,v)$ simplifies to $C^+(u,v) = (1-\rho)uv + \rho \min(u,v)$ which remains the same as the original paper. When $\rho \leq 0$, $C_F(u,v)$ becomes equal to $C^-(u,v) = (1+\rho)uv - \rho \max(u+v-1,0)$ when we take $\alpha = 0$ and $\beta = -\rho$. This demonstrate the minus sign in front of the maximum.

To estimate the association parameter $\rho$ in copula (1), Fatahi *et al.* (2012) used Pearson's correlation coefficient between the two ZIP distributed counts (i.e., $r = \text{corr}(X_1, X_2)$, with $X_i \sim \text{ZIP}(\pi_i, \lambda_i)$). However, they did not provide any arguments to support this specific choice. For two uniformly distributed random variables $U$ and $V$ with the Fréchet copula, it is known that Spearman's rank correlation $\rho_s$ is Pearson's correlation coefficient $r = \text{corr}(U,V)$ between $U$ and $V$ and is equal to $\alpha - \beta$. Kendall's $\tau$ of $(U,V)$ is equal to $(\alpha-\beta)(2+\alpha+\beta)/3$ (Nelsen, 1991). Setting $\alpha$ and $\beta$ according to the aforementioned specification in terms of the association parameter $\rho$ yields $\rho_s = \text{corr}(U,V) = \rho$ and $\tau = \rho(\rho+2)/3$ for copula (1). However, Pearson's correlation coefficient $r = \text{corr}(X_1, X_2)$ may not be equal to parameter $\rho = \text{corr}(U,V)$ since Pearson's correlation coefficient is not transformation invariant. Both Spearman's rank correlation and Kendall's $\tau$ is invariant under strictly monotone transformations. Therefore, in theory, both Spearman's rank correlation and Kendall's $\tau$ can be used to estimate the $\rho$ on the bivariate ZIP distributed random variables $X_1 = F_1^{-1}(U)$ and $X_2 = F_2^{-1}(V)$ with $F_i$ the $\text{ZIP}(\pi_i, \lambda_i)$ distribution and $F_i^{-1}$ a generalized inverse. However, the discreteness of $X_1$ and $X_2$ for ZIP may cause issues as well and it is a priori unclear which measure of correlation should be used on $X_1$ and $X_2$. We conducted a simulation study to assess the performances of the three estimators based on parameter settings used in the original paper (simulation code for simulating bivariate ZIP distributed random variables can be found in the Appendix). The results (see Table 1) suggest both Pearson's correlation and Kendall's $\tau$ are competitive with each other under

certain settings without a clear winner for all settings while Spearman's correlation had larger biases. More disturbingly, for some settings, none of the three approaches are satisfactory for practical usage due to large biases and clearly indicates the need for better alternatives.

Furthermore, the marginal ZIP distribution parameters need to be estimated as well. In favor of its simplicities, the authors of the original paper used the method of moment estimators (MME) in place of the maximum likelihood estimators (MLE). Though MLE and MME are asymptotically equivalent for ZIP (Nanjundan & Naika, 2012), MLE has been shown to be superior for finite samples (Beckett *et al.*, 2014). In addition, MME has the disadvantage that the estimate of the probability of extra zeros would be negative when the sample mean is larger than the sample variance. Thus the recommended methods of estimation for $\pi_i$, $\lambda_i$, and $\rho$ are not at all optimal for all settings.

The authors proposed control limits for the sum of the bivariate zero-inflated Poisson distributed random variables (i.e. $X_1 + X_2$, with $X_i \sim \text{ZIP}(\pi_i, \lambda_i)$ and having copula (1)). They focused on the upper control limit (UCL) and they indicated that the UCL should satisfy

$$F(x_1, x_2 | \min(x_1 + x_2 - 1) = \text{UCL}) \geq 1 - \alpha$$

in their formula (9), with $F$ the joint distribution of $(X_1, X_2)$. They also reformulated the inequality in terms of their copula as $C(F_1(x_1), F_2(x_2) | \max(x_1 + x_2 - 1) = \text{UCL}) \geq 1 - \alpha$ in their formula (10), with $C(\cdot, \cdot)$ the copula and $F_i$ the marginal distribution of $X_i$.

First of all, we do not understand the notation and secondly the formulae seem to contradict since a minimum is used in their formula (9) and a maximum is used in their formula (10). When we would construct an upper control limit UCL that satisfies $P(X_1 + X_2 \geq \text{UCL}) \leq \alpha$, we obtain an equivalent specification in terms of joint probabilities $P(X_1 = x_1, X_2 = x_2)$ as

$$\sum_{x_1, x_2 \in S_U} P(X_1 = x_1, X_2 = x_2) \geq 1 - \alpha, \tag{2}$$

where the summation is over the set $S_U = \{(x_1, x_2) | x_1 + x_2 + 1 \leq \text{UCL}\}$. The joint probabilities can be calculated based on the copula as follows:

$$\begin{aligned} P(X_1 = x_1, X_2 = x_2) = {} & C(F_1(x_1), F_2(x_2)) - C(F_1(x_1 - 1), F_2(x_2)) \\ & - C(F_1(x_1), F_2(x_2 - 1)) + C(F_1(x_1 - 1), F_2(x_2 - 1)), \end{aligned}$$

with $F_i(x_i - 1) = 0$ when $x_i = 0$. Note that we use $x_1 + x_2 + 1$ instead of $x_1 + x_2 - 1$ used in the original paper. The inequality in (2) can be obtained numerically when all parameters are known (or estimated), by choosing the smallest UCL that would satisfy this inequality. Implementation of this proposed methods to calculate UCL can be found in the Appendix.

Comparing the upper control limit with our inequality to the upper control limit of the paper we obtain some differences, see our Table 2. Note that this is not caused by the incorrectly formulated copula, since the copula of the paper was correct for positive correlations. The upper control limit for the case study was calculated at 15 in the original paper, but we obtained an upper control limit of 13, using the same estimates. In some settings we calculated the same UCL, but we did not obtain the same theoretical average run length (ARL) (see Table 2). We also simulated the ARL using the UCL calculated according to our proposed method and the UCL provided in the original paper of Fatahi et. al for each of the corresponding settings. It can be seen in Table 2 that the empirical ARL using the UCL provided by the paper did not yield correct ARL while the theoretical ARL of our calculation were very close to the empirical ones. For negative $\rho$'s, this has also been verified and the results can be obtained from the last author (ZZ).

Summarizing our findings, we corrected the copula of Fatahi *et al.* (2012) to a true copula, demonstrated that the proposed parameter estimates are not always the most appropriate ones, and provided a correct calculation of the upper control limit for the sum of the bivariate ZIP distributed random variables. With these corrections, the control chart may be an appropriate control chart for sparse bivariate numbers of events with either positive or negative correlations.

**Table 1:** Comparison of estimators of the association parameter $\rho$ for settings of the paper ($\pi_1 = 0.10$, $\pi_2 = 0.15$)

| $\lambda_1$ | $\lambda_2$ | $\rho = 0$ | | | | | | $\rho = 0.2$ | | | | | | $\rho = 0.5$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Kendall | | Pearson | | Spearman | | Kendall | | Pearson | | Spearman | | Kendall | | Pearson | | Spearman | |
| | | Bias | MSE | Bias | MSE | Bias | MSE | Bias | MSE | Bias | MSE | Bias | MSE | Bias | MSE | Bias | MSE | Bias | MSE |
| 0.5 | 0.5 | -0.0022 | 0.0021 | -0.0011 | 0.0009 | -0.0008 | 0.0010 | 0.0146 | 0.0050 | -0.0259 | 0.0066 | -0.0388 | 0.0047 | -0.0145 | 0.0046 | -0.0618 | 0.0120 | -0.0940 | 0.0132 |
| 0.5 | 1 | -0.0014 | 0.0021 | -0.0007 | 0.0009 | -0.0002 | 0.0010 | -0.0246 | 0.0044 | -0.0262 | 0.0057 | -0.0696 | 0.0073 | -0.0955 | 0.0129 | -0.0608 | 0.0106 | -0.1701 | 0.0323 |
| 0.5 | 2 | -0.0022 | 0.0020 | -0.0008 | 0.0009 | -0.0008 | 0.0010 | -0.0480 | 0.0055 | -0.0414 | 0.0059 | -0.0869 | 0.0096 | -0.1451 | 0.0243 | -0.0994 | 0.0159 | -0.2135 | 0.0483 |
| 1 | 0.5 | -0.0018 | 0.0021 | -0.0009 | 0.0009 | -0.0005 | 0.0010 | 0.0528 | 0.0068 | -0.0087 | 0.0052 | -0.0065 | 0.0029 | 0.0612 | 0.0071 | -0.0212 | 0.0072 | -0.0152 | 0.0038 |
| 1 | 1 | -0.0015 | 0.0022 | -0.0005 | 0.0010 | -0.0003 | 0.0010 | 0.0155 | 0.0036 | -0.0175 | 0.0047 | -0.0366 | 0.0036 | -0.0134 | 0.0030 | -0.0418 | 0.0076 | -0.0897 | 0.0109 |
| 1.5 | 1.5 | -0.0010 | 0.0021 | 0.0001 | 0.0010 | 0.0001 | 0.0010 | 0.0164 | 0.0032 | -0.0127 | 0.0039 | -0.0345 | 0.0032 | -0.0107 | 0.0024 | -0.0298 | 0.0057 | -0.0840 | 0.0094 |
| 2 | 2 | -0.0008 | 0.0021 | -0.0002 | 0.0010 | 0.0002 | 0.0010 | 0.0171 | 0.0031 | -0.0106 | 0.0035 | -0.0330 | 0.0030 | -0.0099 | 0.0022 | -0.0251 | 0.0049 | -0.0814 | 0.0088 |

**Table 2:** Comparison of upper control limits for settings of the paper ($\pi_1 = 0.10$, $\pi_2 = 0.15$, $\alpha = 0.0027$)

| $\lambda_1$ | $\lambda_2$ | $\rho = 0$ | | | | | | $\rho = 0.2$ | | | | | | $\rho = 0.5$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | UCL | | ARL | | ARL sim | | UCL | | ARL | | ARL sim | | UCL | | ARL | | ARL sim | |
| | | Paper | Letter | Paper | Letter | Paper | Letter | Paper | Letter | Paper | Letter | Paper | Letter | Paper | Letter | Paper | Letter | Paper | Letter |
| 0.5 | 0.5 | 4 | 4 | 587.84 | 1492.14 | 1477.38 | 1477.38 | 4 | 4 | 606.55 | 427.31 | 446.02 | 446.02 | 4 | 5 | 636.96 | 888.53 | 217.59 | 879.69 |
| 0.5 | 1 | 5 | 5 | 503.28 | 1270.09 | 1292.89 | 1292.89 | 5 | 5 | 532.67 | 410.85 | 422.67 | 422.67 | 5 | 6 | 583.82 | 667.45 | 214.58 | 665.36 |
| 0.5 | 2 | 7 | 7 | 472.17 | 1212.18 | 1232.07 | 1232.07 | 7 | 7 | 504.53 | 446.52 | 459.39 | 459.39 | 7 | 8 | 562.33 | 741.76 | 239.78 | 729.03 |
| 1 | 0.5 | 5 | 5 | 462.74 | 1631.42 | 1565.03 | 1565.03 | 5 | 5 | 474.25 | 477.01 | 497.59 | 497.59 | 5 | 6 | 492.63 | 878.19 | 237.92 | 892.09 |
| 1 | 1 | 6 | 5 | 408.71 | 626.97 | 158.28 | 665.43 | 6 | 6 | 427.87 | 523.73 | 535.28 | 535.28 | 6 | 7 | 460.24 | 681.54 | 242.22 | 697.37 |
| 1.5 | 1.5 | 8 | 6 | 396.08 | 446.61 | 2819.55 | 453.73 | 8 | 7 | 418.18 | 394.58 | 676.81 | 404.20 | 8 | 9 | 456.38 | 701.87 | 302.97 | 714.47 |
| 2 | 2 | 10 | 7 | 428.21 | 376.29 | 3324.39 | 376.48 | 10 | 9 | 454.65 | 532.55 | 862.02 | 532.36 | 10 | 10 | 501.04 | 370.54 | 378.70 | 378.70 |

## References

Balakrishna, N., & Lai, Chin Diew. 2009. *Construction of Bivariate Distributions*. Springer New York.

Beckett, Sadie, Jee, Joshua, Ncube, Thapelo, Pompilus, Sophia, Washington, Quintel, Singh, Anshuman, & Pal, Nabendu. 2014. Zero-inflated Poisson (ZIP) distribution: parameter estimation and applications to model data from natural calamities. *Involve, a Journal of Mathematics*, **7**(6), 751–767.

Fatahi, Amir Afshin, Noorossana, Rassoul, Dokouhaki, Pershang, & Moghaddam, Babak Farhang. 2012. Copula-Based Bivariate ZIP Control Chart for Monitoring Rare Events. *Communications in Statistics - Theory and Methods*, **41**(15), 2699–2716.

Fréchet, M. 1958. Remarque au sujet de la note précédente. *Comptes Rendus de l'Académiedes Sciences, Série I. Mathématique*, **246**(2719-2720).

Nanjundan, G., & Naika, T. Raveendra. 2012. Asymptotic Comparison of Method of Moments Estimators and Maximum Likelihood Estimators of Parameters in Zero-Inflated Poisson Model. *Applied Mathematics*, **03**(06), 610–616.

Nelsen, Roger B. 1991. Copulas and Association. *Pages 51–74 of: Advances in Probability Distributions with Given Marginals*. Springer Netherlands.

Nelsen, Roger B. 1999. *An Introduction to Copulas*. Springer New York.

# Appendix

## Simulation code (SAS)

```
%MACRO SIMULATE(SIM=, N=, LAMBDA1= , LAMBDA2=, P1=, P2=, RHO=, UCL=, SEED=);

DATA SIMZIP;
  CALL STREAMINIT(&SEED);
  DO SIM = 1 TO &SIM;
    DO ID = 1 TO &N;
      X1 = RAND('BERNOULLI', &P1) * RAND('POISSON', &LAMBDA1);
      F1 = (1-&P1) + &P1 * CDF('POISSON', X1, &LAMBDA1);
      MPROB1 = (1-&P1) * (X1=0) + &P1 * PDF('POISSON', X1, &LAMBDA1);
      IF X1 = 0 THEN F1M = 0;
      ELSE F1M = (1-&P1) + &P1 * CDF('POISSON', X1-1, &LAMBDA1);
      STOP = 0;
      TMP = 0;
      UU = RAND('UNIFORM');
      DO UNTIL(STOP = 1);
        F2 = (1 - &P2) + &P2 * CDF('POISSON', TMP, &LAMBDA2);
        IF (&RHO >= 0) THEN
          COND_F2 = ((1-&RHO)*(F1*F2-F1M*F2) + &RHO*(MIN(F1,F2)-MIN(F1M, F2)))/MPROB1;
        IF (&RHO < 0) THEN
          COND_F2 = ((1+&RHO)*(F1*F2-F1M*F2)-&RHO*(MAX(F1+F2-1,0)-MAX(F1M+F2-1,0)))/MPROB1
              ;
        IF COND_F2 > UU THEN DO;
          STOP = 1;
          X2 = TMP;
        END;
        TMP = TMP + 1;
      END;
    OUTPUT;
    END;
  END;
RUN;

DATA SIMZIP;
  SET SIMZIP;
  UCL = &UCL;
  IF X1 + X2 >= UCL THEN OC=1;
  ELSE OC = 0;
RUN;

%MEND;
```

## UCL calculation (SAS)

```
%MACRO UCL_BIZIP(MEAN1=, MEAN2=, PROB1=, PROB2=, CORR=, UCLMAX=, ALPHA=);

TITLE1 "MU1=&MEAN1, MU2=&MEAN2, P1=&PROB1, P2=&PROB2, RHO=&CORR";
DATA BIZIPCHART;
  RHO          =  &CORR;
  MU1          =  &MEAN1;
  MU2          =  &MEAN2;
  P1           =  &PROB1;
  P2           =  &PROB2;
  DO UCL = 0 TO &UCLMAX;
    DO X1 = 0 TO UCL-1 BY 1;
      DO X2 = 0 TO UCL-1-X1 BY 1;
       F1 = 1-P1 + P1*CDF('POISSON',X1,MU1);
       F2 = 1-P2 + P2*CDF('POISSON',X2,MU2);
       F1M= (X1>0)*(1-P1 + P1*CDF('POISSON',X1-1,MU1));
       F2M= (X2>0)*(1-P2 + P2*CDF('POISSON',X2-1,MU2));
       C  = (RHO>=0)*( ((1-RHO)*F1*F2 + RHO*MIN(F1,F2))
                 - ((1-RHO)*F1M*F2 + RHO*MIN(F1M,F2))
                 - ((1-RHO)*F1*F2M + RHO*MIN(F1,F2M))
                 + ((1-RHO)*F1M*F2M + RHO*MIN(F1M,F2M)) )
         + (RHO<0)*( ((1+RHO)*F1*F2 - RHO*MAX(F1+F2-1,0))
                 - ((1+RHO)*F1M*F2 - RHO*MAX(F1M+F2-1,0))
                 - ((1+RHO)*F1*F2M - RHO*MAX(F1+F2M-1,0))
                 + ((1+RHO)*F1M*F2M - RHO*MAX(F1M+F2M-1,0)) );
       OUTPUT;
       END;
     END;
  END;
RUN;

PROC MEANS DATA=BIZIPCHART NOPRINT;
  VAR C;
  BY UCL;
  OUTPUT OUT=PROB SUM=PROB;
RUN;

DATA PROB;
  SET PROB;
  WHERE PROB>=1-&ALPHA;
RUN;

DATA PROB;
  SET PROB;
  OBS = _N_;
  ARL = 1/(1-PROB);
  KEEP OBS UCL PROB ARL;
RUN;

PROC PRINT DATA=PROB;
  WHERE OBS=1;
RUN;
%MEND;
```