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Parabolic smoothing for a free boundary problem from porous media flow

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Parabolic smoothing for a free boundary problem from porous media flow

Master's Thesis

Industrial and Applied Mathematics

Eindhoven University of Technology

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Chapter 1

Introduction

The central problem in this report concerns water evaporation in mines, tunnels and other constructions, having contact with natural massifs. The tunnel itself is artificially ventilated, however somewhere above the tunnel is a reservoir with liquid water. Therefore, in the soil between the reservoir and the tunnel, evaporation occurs. The part of the soil above this evaporation front is thus saturated with water, while the part underneath contains a mixture of air and vapour. [14, 15]

The problem described above is an example of a free boundary problem or moving boundary problem. These are partial differential equations where the domain in which the problem has to be solved is unknown and has to be found as part of the solution. The part of the boundary of the domain which is unknown is called the free boundary. In the case described above this is the interface between the two phases. Geometric information is thus part of the solution. Some other examples are the growth of tumors, the winning of oil, and the production of glasses. In these problems the bulk, in which the differential equations are defined, changes shape. These changes are a result of driving forces like nutrition in the cells, pressure differences and surface tension [22]. Two examples which are important in this report, however, are the Stefan problem and the Hele-Shaw problem. The Stefan problem concerns the melting of ice, while the Hele-Shaw problem describes the flow of a fluid between two parallel plates.

One of the standard techniques for these free boundary problems is to transform them to a fixed boundary problem using a time-dependent diffeomorphism. Then techniques from functional analysis can be applied to prove existence, uniqueness and regularity of the solution. For the specific problem described above the first two are already presented in [17]. Proving regularity for the interface is the main objective of this report.

In particular, we will prove that the interface is smooth for all times $t > 0$ regardless of the initial conditions. This is called a smoothing effect and is an important feature of all parabolic differential equations. Take for example the homogeneous heat equation. This equation has the fundamental solution

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & \text{for } x \in \mathbb{R}^n, t > 0, \\ 0 & \text{for } x \in \mathbb{R}^n, t < 0. \end{cases}$$

Now for all $g \in C_0(\mathbb{R}^n)$ the convolution

$$u(x,t) = (\Phi * g)(x,t) = \int_{\mathbb{R}}^{n} \Phi(x-y,t) g(y) dy$$

satisfies

$$\begin{cases} u_t - \Delta u = 0 & \text{for all } x \in \mathbb{R}^n, t > 0, \\ \lim_{\substack{(x,t)\to(x_0,0) \\ t>0}} u(x,t) = g(x_0) & \text{for all } x_0 \in \mathbb{R}^n. \end{cases}$$

Note that this solution u of the initial value problem for the heat equation is smooth for all $t > 0$, since Φ is infinitely many times differentiable.
More general ways to prove smoothness of solutions of linear parabolic differential equations include for example Fourier analysis, energy estimates or semigroup methods [5, 20].
For nonlinear equations, however, these techniques cannot be applied. Therefore, in this report Angenent's parameter trick is applied to the problem. This trick is named after Sigurd Angenent who first introduced it in [4]. In short, it requires a shift in space and time and then applies the Implicit Function Theorem on the resulting system.

This report is organized as follows: In Chapter 2, a mathematical model of the problem is derived. This model is then transformed to a fixed boundary problem and some theorems about existence and uniqueness of solutions from [17] and [6] are stated. In Chapter 3, Angenent's parameter trick is applied to a model problem to demonstrate it. Then, in Chapter 4 and Chapter 5, the same techniques are applied to two different versions of the central problem. Here also the main results are presented in Theorem 4.2 and Theorem 5.2. Then, in Chapter 6, both of these results are compared and discussed. Finally, this report contains two appendixes. Appendix A contains explanations on the Stefan and the Hele-Shaw problem, both mentioned before. Appendix B contains some mathematical definitions and theorems used throughout the report.

Chapter 2

Formulation of the free boundary problem and existence of solutions

In this chapter some previous work on the subject is discussed. First of all, the problem setting is explained and a mathematical model is formulated, which has been done before in [14] and [15]. The resulting system of equations is a free boundary problem. In the remainder of this chapter, we follow [17]. Here, following standard techniques, the problem is first transformed into a fixed boundary problem. As a next step, the resulting system is reduced to the case of homogeneous initial data. Finally, three theorems from [17] and [6] are presented about existence and uniqueness of solutions of the problem itself and the linearisation of the problem.

2.1 Problem setting

The main problem in this report is about water evaporation in tunnels. The tunnel itself is artificially ventilated, however the soil above the tunnel contains liquid water. This water moves downward under the force of gravity and evaporates from the ceiling of the tunnel. Water can enter the tunnel either in liquid or vapour phase. If the surrounding solid has a low permeability, we can reasonably assume that the evaporation occurs in the pores of the soil. The lowest part of the soil thus contains a mixture of vapour and air. We suppose the process of water evaporation is slow compared to heat diffusion. Therefore, the heat absorption is negligible and the temperature T of the surrounding rocks is equal to that of the ventilated air.

The situation is sketched in Figure 1. Here the x_n-axis is directed downwards, so at $x_n = 0$ there is a constant water pressure P_0 and the ceiling of the tunnel is positioned at $x_n = L$. At this boundary the humidity ν_a is constant. We assume there is a sharp interface at which the evaporation occurs. Above this interface the rock is saturated with water and below the rock contains a mixture of air and vapour. Let $h \in (0, L)$ be any fixed reference level and let the interface $\Gamma(t)$ be given by $x_n = h + \eta(x', t)$ at any time $t \in J = [0, T]$, where $\eta \in C(\mathbb{R}^{n-1} \times J)$ such that $h + \eta(x', t) \in (0, L)$ for all $(x', t) \in \mathbb{R}^{n-1} \times J$. At any time $t \in J$ this interface splits the domain $\Omega := \mathbb{T}^{n-1} \times (0, L)$ into $\Omega_-(t)$ (which is saturated with water under a pressure P) and $\Omega_+(t)$ (which is filled with vapour and air with humidity ν), so

$$
\begin{aligned}
\Omega &= \Omega_-(t) \cup \Gamma(t) \cup \Omega_+(t), \\
\Omega_-(t) &:= \{(x', x_n) \mid x' \in \mathbb{T}^{n-1},\ 0 < x_n < h + \eta(x', t)\}, \\
\Gamma(t) &:= \{(x', x_n) \mid x' \in \mathbb{T}^{n-1},\ x_n = h + \eta(x', t)\}, \\
\Omega_+(t) &:= \{(x', x_n) \mid x' \in \mathbb{T}^{n-1},\ h + \eta(x', t) < x_n < L\}.
\end{aligned}
$$

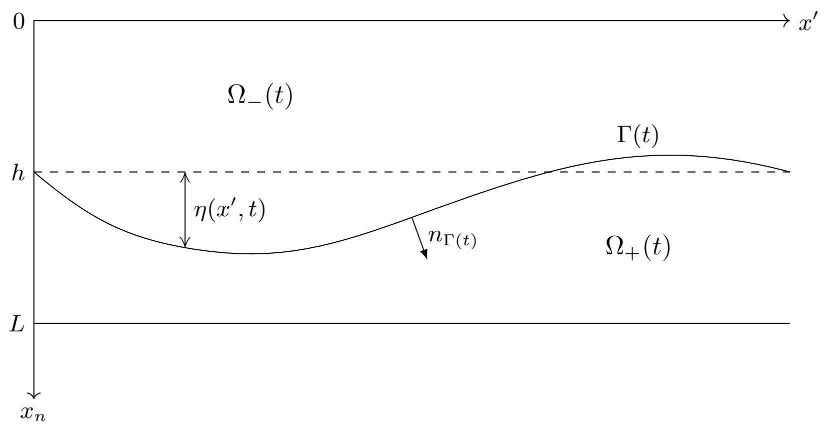

Figure 1: Geometric setting. The upper domain contains liquid water and the lower domain a mixture of vapour and air.

Moreover, define

$$\Sigma_- := \{(x',0) \mid x' \in \mathbb{T}^{n-1}\},$$
$$\Sigma_+ := \{(x',L) \mid x' \in \mathbb{T}^{n-1}\}.$$

If we assume incompressibility, we can derive a system of equations for the model. In this problem the following constants are of importance: [17]

- m : porosity of the medium ($m \in (0,1)$, fraction of free pore space),
- k : its permeability to water,
- μ_w : viscosity of water,
- D : diffusivity of vapor,
- ρ_w : density of water,
- ρ_a : density of air,
- g : gravity,
- P_c : capillary pressure,
- P_0 : hydrodynamic pressure at upper boundary,
- ν_a : humidity at lower boundary.

In the liquid phase we have the continuity equation (conservation of mass) and Darcy's law:

$$\text{div } v_w = 0,$$
$$v_w = \frac{-k}{\mu_w} \text{ grad } (P - \rho_w g x_n),$$

where the superficial velocity v_w and the pressure P are the unknowns. Combining the two equations, they can be reduced to

$$\Delta P = 0 \qquad \text{in } \Omega_-(t).$$

In the vapour phase we have phase we have diffusion

$$\frac{\partial \rho_v}{\partial t} = \text{div } D \text{ grad } \rho_v = D\Delta\rho_v.$$

However, in the remainder of this report, we use the humidity function $\nu = \frac{\rho_v}{\rho_a+\rho_v}$ instead of the vapour density ρ_v. If we assume the density of the vapour in air is low compared to the density of air, we have $\nu \approx \frac{\rho_v}{\rho_a}$ and thus

$$\frac{\partial \nu}{\partial t} = D\Delta\nu \qquad \text{in } \Omega_+(t).$$

The pressure in the air-vapour mixture is equal to the atmospheric pressure. The difference between the pressure in the water domain and the domain filled with air and vapour is equal to the capillary pressure. Therefore, on the interface we have the following boundary condition for the pressure:

$$P = P_+ + P_c = P_a + P_c \qquad \text{on } \Gamma(t).$$

Secondly, because we assume the process to be isothermal and at fixed temperature, evaporation occurs at a fixed humidity

$$\nu = \nu^* \qquad \text{on } \Gamma(t).$$

A third boundary condition on the interface describes mass conservation [14]:

$$\rho_v(\partial_n v_v - V_n) = \rho_w(\partial_n v_w - V_n),$$

where V_n is the normal velocity of the interface $\Gamma(t)$. Combining this with Darcy's law ($v_w = -\frac{k}{\mu_w}$ grad $(P - \rho_w g x_n)$) and Fick's law ($v_v = D$ grad ν) gives that

$$\left(1 - \frac{\rho_v}{\rho_w}\right) V_n = -\frac{k}{m\mu_w}\partial_{n_{\Gamma(t)}}(P - \rho_w g x_n) + D\frac{\rho_a}{\rho_w}\partial_{n_{\Gamma(t)}}\nu \qquad \text{on } \Gamma(t).$$

Note that ρ_v may vary in the vapour phase but is constant on the interface because here $\rho_v = \rho_a\nu^* = \text{const}$.

Finally we have on the upper boundary

$$P = P_0 \qquad \text{on } \Sigma_-$$

and on the lower boundary

$$\nu = \nu_a \qquad \text{on } \Sigma_+.$$

Combining all of the above gives

$$\left.\begin{aligned}
\Delta P &= 0 && \text{in } \Omega_-(t),\\
(\partial_t - D\Delta)\nu &= 0 && \text{in } \Omega_+(t),\\
P &= P_a + P_c && \text{on } \Gamma(t),\\
\nu &= \nu^* && \text{on } \Gamma(t),\\
P &= P_0 && \text{on } \Sigma_-,\\
\nu &= \nu_a && \text{on } \Sigma_+,\\
\left(1 - \frac{\rho_v}{\rho_w}\right) V_n &= -\frac{k}{m\mu_w}\partial_{n_{\Gamma(t)}}(P - \rho_w g x_n) + D\frac{\rho_a}{\rho_w}\partial_{n_{\Gamma(t)}}\nu && \text{on } \Gamma(t).
\end{aligned}\right\} \tag{2.1}$$

Note that this is actually a combination of a Stefan problem (in the vapor phase) and a Hele-Shaw problem (in the liquid phase). These classes of free boundary problems are well-known and discussed in more detail in Appendix A.

2.2 Non-dimensionalization

The problem is now first non-dimensionalized and then normalized. The details are not discussed here but can be found in [17] and [6]. The resulting system is

$$\left.\begin{aligned} \Delta P &= 0 && \text{in } \Omega_-, \\ (\partial_t - \Delta)\nu &= 0 && \text{in } \Omega_+, \\ P &= 1 && \text{on } \Gamma(t), \\ \nu &= 1 && \text{on } \Gamma(t), \\ P &= 0 && \text{on } \Sigma_-, \\ \nu &= 0 && \text{on } \Sigma_+, \\ \gamma V_n &= -\alpha \partial_{n_{\Gamma(t)}} P + n_{\Gamma(t)} \cdot e_{x_n} + \beta \partial_{n_{\Gamma(t)}} \nu && \text{on } \Gamma(t). \end{aligned}\right\} \tag{2.2}$$

with α and β dimensionless numbers, $\gamma = 1 - \frac{\rho_v}{\rho_w}$ and

$$\begin{aligned} \Omega &:= \Omega_-(t) \cup \Gamma(t) \cup \Omega_+(t), \\ \Omega_-(t) &:= \{(x', x_n) \mid x' \in \mathbb{T}^{n-1},\ 0 < x_n < H + \eta(x', t)\}, \\ \Gamma(t) &:= \{(x', x_n) \mid x' \in \mathbb{T}^{n-1},\ x_n = H + \eta(x', t)\}, \\ \Omega_+(t) &:= \{(x', x_n) \mid x' \in \mathbb{T}^{n-1},\ H + \eta(x', t) < x_n < 1\}, \\ \Sigma_- &:= \{(x', 0)\}, \\ \Sigma_+ &:= \{(x', 1)\}, \end{aligned}$$

with $H = \frac{h}{L}$. Note that the scaled versions of the spaces above and η are denoted by the same symbol.

We add initial conditions

$$\begin{aligned} \eta(\cdot, 0) &= \eta_0 && \text{on } \mathbb{T}^{n-1}, \\ \nu(\cdot, 0) &= \nu_0 && \text{in } \Omega_+(0), \end{aligned}$$

with the well-posedness condition [17]

$$\mu \partial_{n_{\Gamma(0)}} [\beta \nu_0 + \alpha P|_{t=0}] \leq -\omega_0 < 0 \quad \text{on } \Gamma(0). \tag{2.3}$$

This is a condition on ν_0 and η_0 only.

Recall that the problem is a combination of a Stefan problem in the vapour phase and a Hele-Shaw problem in the liquid phase. These problems are discussed in Appendix A and they have the following solvability conditions:

$$\partial_{n_{\Gamma_0}} \Theta \leq -\alpha < 0$$

for the Stefan problem (see Equation (A.3)) and

$$\frac{\partial p}{\partial n} < 0$$

for the Hele-Shaw problem (see Equation (A.6)). Note that Equation (2.3) is actually a combination of these two.

2.3 Transformation to fixed domains

We want to transform System (2.2) to a fixed domain. Therefore, we now define

$$\begin{aligned}\Omega_- &:= \{(z', z_n) \mid z' \in \mathbb{T}^{n-1},\ z_n \in (0, H)\},\\ \Omega_+ &:= \{(z', z_n) \mid z' \in \mathbb{T}^{n-1},\ z_n \in (H, 1)\},\\ \Gamma &:= \{(z', H) \mid z' \in \mathbb{T}^{n-1}\}\end{aligned}$$

and consider a function $\hat{\phi} : \Omega \times J \to \mathbb{R}$ such that

- $\hat{\phi}(\cdot, t) = 0$ on $\Sigma_\pm$,
- $\hat{\phi}((z', H), t) = \eta(z', t)$,
- $\hat{\phi}|_{\Omega_\pm}$ and $\hat{\phi}|_\Gamma$ sufficiently smooth and
- $Z_{\hat{\phi}}^{-1} : z \to x = (z', z_n + \hat{\phi}(z, t)) \in$ Diff $(\Omega_\pm, \Omega_\pm(t))$ for all $t \in J$.

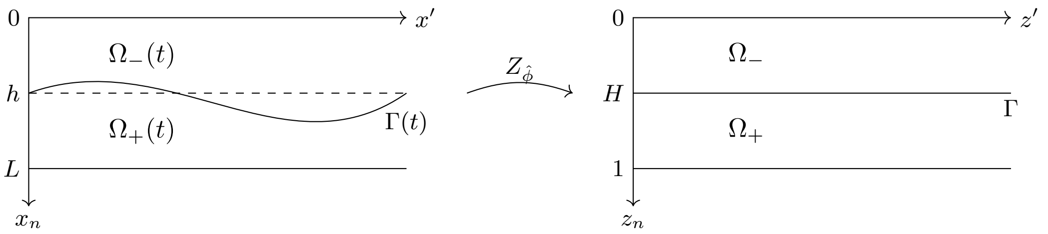

Figure 2: A visualization of the transformation $Z_{\hat{\phi}}$. The functions P and ν are defined on the spaces $\Omega_-(t)$ and $\Omega_+(t)$ in the left figure, while $\hat{P}$ and $\hat{\nu}$ are defined on the spaces Ω_- and Ω_+ in the right figure.

We use the notation $\hat{\phi}^\pm$ for the restrictions of $\hat{\phi}$ to $\Omega_\pm$, respectively, and write $\hat{\phi}$ for the trace at Γ. Using the above, define

$$\begin{aligned}\hat{P}(z, t) &:= P(Z_{\hat{\phi}}^{-1}(z, t), t), && z \in \Omega_-,\\ \hat{\nu}(z, t) &:= \nu(Z_{\hat{\phi}}^{-1}(z, t), t), && z \in \Omega_+.\end{aligned}$$

In the same step, we divide Equation $(2.2)_7$ by $\gamma \neq 0$ and use α and β instead of α/γ and β/γ. The system is transformed to

$$\left.\begin{aligned}\mathcal{A}_{\hat{\phi}^-}\hat{P} - \frac{\hat{P}_{z_n}}{1 + \hat{\phi}^-_{z_n}}\mathcal{A}_{\hat{\phi}^-}\hat{\phi}^- &= 0 && \text{in } \Omega_- \times J,\\ \mathcal{L}_{\hat{\phi}^+}\hat{\nu} - \frac{\hat{\nu}_{z_n}}{1 + \hat{\phi}^+_{z_n}}\mathcal{L}_{\hat{\phi}^+}\hat{\phi}^+ &= 0 && \text{in } \Omega_+ \times J,\\ \hat{P} &= 1 && \text{on } \Gamma \times J,\\ \hat{\nu} &= 1 && \text{on } \Gamma \times J,\\ \hat{\phi}^\pm - \hat{\phi} &= 0 && \text{on } \Gamma \times J,\\ \hat{P} &= 0 && \text{on } \Sigma_- \times J,\\ \hat{\nu} &= 0 && \text{on } \Sigma_+ \times J,\\ \hat{\phi}^\pm &= 0 && \text{on } \Sigma_\pm \times J,\\ \partial_t\hat{\phi} - (1 + |\nabla'\hat{\phi}|^2)\left(\frac{-\alpha\hat{P}_{z_n}}{1 + \hat{\phi}^-_{z_n}} + \frac{\beta\hat{\nu}_{z_n}}{1 + \hat{\phi}^+_{z_n}}\right) &= \frac{1}{\gamma} && \text{on } \Gamma \times J,\\ \hat{\nu}(\cdot, 0) = \hat{\nu}_0 &:= \nu_0 \circ Z_{\hat{\phi}}(\cdot, 0) && \text{in } \Omega_+,\\ \hat{\phi}(\cdot, 0) &= \eta_0 && \text{on } \Gamma,\end{aligned}\right\} \qquad (2.4)$$

where

$$\mathcal{L}_\tau u := \partial_t u - \mathcal{A}_\tau u,$$
$$\mathcal{A}_\tau u := \sum_{i=1}^{n-1} u_{z_i z_i} - \vec{a}(\nabla\tau)\cdot\nabla u_{z_n},$$
$$\vec{a}(\nabla\tau) := \left(\frac{2\nabla'\tau}{1+\tau_{z_n}}, -\frac{1+|\nabla'\tau|^2}{(1+\hat{\phi}_{z_n})^2}\right).$$

In the remainder of this report we assume that

$$p > n + 3/2 + \sqrt{2n+1/4}, \tag{2.5}$$
$$\theta \in \left(\frac{1}{p}, \frac{1}{2}\left(1-\frac{n+1}{p}\right)\right) \tag{2.6}$$

and

$$\eta_0 \in W_p^{2-2/p}(\Gamma), \qquad \eta_0(x') \in (\xi - H, 1-\xi-H) \quad \text{for some } \xi > 0, \tag{2.7}$$
$$\hat{\nu}_0 \in W_p^{2-2/p}(\Omega_+). \tag{2.8}$$

2.4 Transformation to homogeneous initial data

We now set

$$\hat{\phi} = \sigma + \phi,$$
$$\hat{P} = Q + p,$$
$$\hat{\nu} = V + v,$$

where σ is such that

$$\sigma \text{ is smooth for all } t > 0, \tag{2.9}$$
$$\sigma|_{\Gamma\times\{0\}} = \eta_0, \tag{2.10}$$
$$\sigma|_{\Sigma_\pm\times J} = 0, \tag{2.11}$$
$$\sigma_{z_n}(z,t) \geq \varepsilon - 1 > -1 \qquad \text{for all } (z,t)\in\Omega\times J \tag{2.12}$$

and $Q \in L_p(J, W_p^2(\Omega_-)) \cap W_p^\theta(J, W_p^{2-\theta}(\Omega_-))$ and $V \in L_p(J, W_p^2(\Omega_+)) \cap H_p^1(J, L_p(\Omega_+))$ are the unique solutions to the elliptic boundary value problem

$$\left.\begin{aligned} \mathcal{A}_\sigma Q - \frac{Q_{z_n}}{1+\sigma_{z_n}}\mathcal{A}_\sigma\sigma &= 0 && \text{in } \Omega_-\times J,\\ Q &= 1 && \text{on } \Gamma\times J,\\ Q &= 0 && \text{on } \Sigma_-\times J \end{aligned}\right\} \tag{2.13}$$

and the parabolic initial boundary value problem

$$\left.\begin{aligned} \mathcal{L}_\sigma V - \frac{V_{z_n}}{1+\sigma_{z_n}}\mathcal{L}_\sigma\sigma &= 0 && \text{in } \Omega_+\times J,\\ V &= 1 && \text{on } \Gamma\times J,\\ V &= 0 && \text{on } \Sigma_+\times J,\\ V(\cdot,0) &= \hat{\nu}_0 && \text{in } \Omega_+. \end{aligned}\right\} \tag{2.14}$$

The system can now be written as

$$F(\sigma+\phi, Q+p, V+v) = 0,$$

where

$$F(\tau,q,w)=\begin{pmatrix} \mathcal{A}_\tau q-\frac{q_{z_n}}{1+\tau_{z_n}}\mathcal{A}_\tau\tau=0\\ \mathcal{L}_\tau w-\frac{w_{z_n}}{1+\tau_{z_n}}\mathcal{L}_\tau\tau=0\\ q|_\Gamma\\ w|_\Gamma\\ (\tau^\pm-\tau)|_\Gamma\\ q|_{\Sigma_-}\\ w|_{\Sigma_+}\\ \tau^\pm|_{\Sigma_\pm}\\ \left(\partial_t\tau-(1+|\nabla'\tau|^2)\left(\frac{-\alpha q_{z_n}}{1+\tau^-_{z_n}}+\frac{\beta w_{z_n}}{1+\tau^+_{z_n}}\right)\right)\Big|_\Gamma-\frac{1}{\gamma}\\ (\gamma_0 w-\hat{\nu}_0)\,|_{\Omega_+}\\ (\gamma_0\tau-\eta_0)\,|_\Gamma \end{pmatrix}$$

and the time trace operator γ_0 is defined by $\gamma_0 v(x)=v(x,0)$ for all $x\in\Omega_+$. Furthermore, (ϕ,p,v) vanishes at $t=0$ and stays small in suitable norms for small times. Therefore, we can rewrite $F(\sigma+\phi,Q+p,V+v)=0$ using

$$F'(\sigma,Q,V)[\phi,p,v]=-F(\sigma,Q,V)-R,$$

where

$$\begin{aligned} R&=\int_0^1(1-s)\frac{\partial^2}{\partial s^2}[F((\sigma,Q,V+s(\phi,p,v))]ds\\ &=\int_0^1(1-s)F''((\sigma,Q,V+s(\phi,p,v))[(\phi,p,v),(\phi,p,v)]ds\\ &=:(R_1,R_2,0,0,0,0,0,0,R_3,0,0). \end{aligned}$$

This can be written as

$$\left.\begin{aligned} \tilde{L}^-p-\frac{Q_{z_n}}{1+\sigma_{z_n}}\hat{L}^-\phi^-&=K^-\phi^-+R^-(\phi^-,p) && \text{in } \Omega_-\times J,\\ \tilde{L}^+v-\frac{V_{z_n}}{1+\sigma_{z_n}}\hat{L}^+\phi^+&=K^+\phi^++R^+(\phi^-,v) && \text{in } \Omega_+\times J,\\ p&=0 && \text{on } (\Gamma\cup\Sigma_-)\times J,\\ v&=0 && \text{on } (\Gamma\cup\Sigma_+)\times J,\\ \phi^\pm-\phi&=0 && \text{on } \Gamma\times J,\\ \phi^\pm&=0 && \text{on } \Sigma_\pm\times J,\\ \partial_t\phi-\alpha^-\phi^-_{z_n}-\alpha^+\phi^+_{z_n}&\\ +\zeta\cdot\nabla'\phi-\tilde{\alpha}^-p_{z_n}-\tilde{\alpha}^+v_{z_n}&=g_0+R^B(p,v,\phi^-,\phi^+,\phi) && \text{on } \Gamma\times J,\\ v(\cdot,0)&=0 && \text{in } \Omega_+,\\ \phi(\cdot,0)&=0 && \text{on } \Gamma, \end{aligned}\right\}\qquad(2.15)$$

where

$$
\begin{aligned}
\tilde{L}^{\pm}u &:= \Lambda^{\pm}_{\sigma}u - \frac{u_{z_n}}{1+\sigma_{z_n}}\Lambda^{\pm}_{\sigma}\sigma,\\
\hat{L}^{\pm}\phi^{\pm} &:= \Lambda^{\pm}_{\sigma}\phi^{\pm} \pm A(\nabla\sigma)\nabla\phi^{\pm}\nabla\sigma_{z_n},\\
K^{\pm}\phi^{\pm} &:= A(\nabla\sigma)\nabla\phi^{\pm}\cdot\nabla U^{\pm}_{z_n} - \frac{U^{\pm}_{z_n}}{(1+\sigma_{z_n})^2}\phi^{\pm}_{z_n}\Lambda^{\pm}_{\sigma}\sigma,\\
\Lambda^{+}_{\sigma}u &:= \mathcal{L}_{\sigma}u = \partial_t u - \mathcal{A}_{\sigma}u,\\
\Lambda^{-}_{\sigma}u &:= \mathcal{A}_{\sigma}u = \sum_{i=1}^{n-1} u_{z_i,z_i} - \vec{a}(\nabla\sigma)\cdot\nabla u_{z_n},\\
A(p) &:= D_p\vec{a}(p),\\
g_0 &:= \frac{1}{\mu_w} - \partial_t\sigma + \frac{1+|\nabla'\sigma|^2}{1+\sigma_{z_n}}(-\alpha Q_{z_n} + \beta V_{z_n}),
\end{aligned}
$$

$$
\begin{aligned}
\alpha^{-} &:= \alpha\frac{Q_{z_n}(1+|\nabla'\sigma|^2)}{(1+\sigma_{z_n})^2},\\
\alpha^{+} &:= -\beta\frac{V_{z_n}(1+|\nabla'\sigma|^2)}{(1+\sigma_{z_n})^2},\\
\tilde{\alpha}^{-} &:= -\alpha\frac{1+|\nabla'\sigma|^2}{1+\sigma_{z_n}},\\
\tilde{\alpha}^{+} &:= \beta\frac{1+|\nabla'\sigma|^2}{1+\sigma_{z_n}},\\
\zeta &:= \frac{2(\alpha Q_{z_n} - \beta V_{z_n})}{1+\sigma_{z_n}}\nabla'\sigma,
\end{aligned}
$$

$$
\begin{aligned}
&R^B(u^-,u^+,\phi^-,\phi^+,\phi) :=\\
&(1+|\nabla'\sigma|^2)\Bigg(\alpha\left(\frac{u^-_{z_n}\phi^-_{z_n}}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^-_{z_n})} - \frac{U^-_{z_n}{\phi^-_{z_n}}^2}{(1+\sigma_{z_n})^2(1+\sigma_{z_n}+\phi^-_{z_n})}\right)\\
&\qquad -\beta\left(\frac{u^+_{z_n}\phi^+_{z_n}}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^+_{z_n})} - \frac{U^+_{z_n}{\phi^+_{z_n}}^2}{(1+\sigma_{z_n})^2(1+\sigma_{z_n}+\phi^+_{z_n})}\right)\Bigg)\\
&+2\nabla'\sigma\cdot\nabla'\phi\Bigg(-\alpha\left(\frac{u^-_{z_n}}{1+\sigma_{z_n}+\phi^-_{z_n}} - \frac{U^-_{z_n}\phi^-_{z_n}}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^-_{z_n})}\right)\\
&\qquad +\beta\left(\frac{u^+_{z_n}}{1+\sigma_{z_n}+\phi^+_{z_n}} - \frac{U^+_{z_n}\phi^+_{z_n}}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^+_{z_n})}\right)\Bigg)\\
&+|\nabla'\phi|^2\left(-\alpha\frac{U^-_{z_n}+u^-_{z_n}}{1+\sigma_{z_n}+\phi^-_{z_n}} + \beta\frac{U^+_{z_n}+u^+_{z_n}}{1+\sigma_{z_n}+\phi^+_{z_n}}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&R^{\pm}(u^{\pm},\phi^{\pm}) :=\\
&A(\nabla\sigma)\nabla\phi^{\pm}\nabla u^{\pm}_{z_n} + \vec{b}(\nabla\sigma,\nabla\phi^{\pm})(\nabla U^{\pm}_{z_n} + \nabla u^{\pm}_{z_n})\\
&+\frac{U^{\pm}_{z_n}+u^{\pm}_{z_n}}{1+\sigma_{z_n}+\phi^{\pm}_{z_n}}(A(\nabla\sigma)\nabla\phi^{\pm}\nabla\phi^{\pm}_{z_n} + \vec{b}(\nabla\sigma,\nabla\phi^{\pm})(\nabla\sigma_{z_n}+\nabla\phi^{\pm}_{z_n}))\\
&+\frac{\phi^{\pm}_{z_n}\hat{L}^{\pm}_{\sigma}\sigma}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^{\pm}_{z_n})}\left(u^{\pm}_{z_n} - \frac{U^{\pm}_{z_n}\phi^{\pm}_{z_n}}{1+\sigma_{z_n}}\right),
\end{aligned}
$$

where
$$\vec{b}(\nabla\sigma, \nabla\phi^{\pm}) := \sum_{i=1}^{n} \int_0^1 (1-s)\partial_i A(\nabla\sigma + s\nabla\phi^{\pm})\partial_i\phi^{\pm} \cdot \nabla\phi^{\pm} ds.$$
To close the system and to simplify the equations we set
$$\hat{L}^{\pm}\phi^{\pm} = 0 \qquad \text{in } \Omega_{\pm} \times J, \tag{2.16}$$
$$\phi^{+}(\cdot, 0) = 0 \qquad \text{in } \Omega_{+}. \tag{2.17}$$

2.5 Existence of solutions

In this section three theorems about existence of solutions of certain systems are provided. The proofs of these theorems can be found in [17] and [6].

We define the function spaces
$$\begin{aligned}
X^- &:= L_p(J, W_p^2(\Omega_-)) \cap W_p^{\theta}(J, W_p^{2-\theta}(\Omega_-)),\\
X^+ &:= L_p(J, W_p^2(\Omega_+)) \cap H_p^1(J, L_p(\Omega_+)),\\
X^B &:= L_p(J, W_p^{2-1/p}(\Gamma) \cap H_p^1(J, W_p^{1-1/p}(\Gamma)) \cap W_p^{1+\theta}(J, L_p(\Gamma)),\\
X_0(M) &:= \{f \in W^{2-2/p}(M) : f|_{\Sigma_+} = 0\},\\
Y^- &:= L_p(\Omega_- \times J) \cap W_p^{\theta}(J, W_p^{-\theta}(\Omega_-)),\\
Y^+ &:= L_p(\Omega_+ \times J),\\
Y^B &:= L_p(J, W_p^{1-1/p}(\Gamma)) \cap W_p^{\theta}(J, L_p(\Gamma)),\\
\mathcal{X} &:= (X^- \times X^+)^2 \times X^B,\\
\mathcal{Y} &:= Y^- \times Y^+ \times Y^B,\\
\mathcal{Y}' &:= Y^- \times Y^+ \times Y^B \times X_0(\Omega_+) \times X_0(\bar{\Omega}_+),
\end{aligned}$$
where M can be either the spaces Ω_+ of $\bar{\Omega}_+$.

Theorem 2.1. *There exists a positive time T such that System (2.15) together with (2.16) and (2.17) has a unique solution $(p, v, \phi^-, \phi^+, \phi) \in \mathcal{X}$.*

The linearisation of the system consisting of (2.15), (2.16) and (2.17) is given by
$$\left.\begin{aligned}
\tilde{L}^- p = f^+ &\quad \text{in } \Omega_- \times J,\\
\tilde{L}^+ v = f^- &\quad \text{in } \Omega_+ \times J,\\
\hat{L}^{\pm}\phi^{\pm} = 0 &\quad \text{in } \Omega_{\pm} \times J,\\
p = 0 &\quad \text{on } (\Gamma \cup \Sigma_-) \times J,\\
v = 0 &\quad \text{on } (\Gamma \cup \Sigma_+) \times J,\\
\phi^{\pm} - \phi = 0 &\quad \text{on } \Gamma \times J,\\
\phi^{\pm} = 0 &\quad \text{on } \Sigma_{\pm} \times J,\\
\partial_t\phi - \alpha^-\phi^-_{z_n} - \alpha^+\phi^+_{z_n} &\\
+\zeta \cdot \nabla'\phi - \tilde{\alpha}^- p_{z_n} - \tilde{\alpha}^+ v_{z_n} = g &\quad \text{on } \Gamma \times J,\\
v(\cdot, 0) = 0 &\quad \text{in } \Omega_+,\\
\phi(\cdot, 0) = 0 &\quad \text{on } \bar{\Omega}_+.
\end{aligned}\right\} \tag{2.18}$$

Theorem 2.2. *For each $(f^+, f^-, g) \in \mathcal{Y}$ Problem (2.18) has a unique solution $(p, v, \phi^-, \phi^+, \phi) \in \mathcal{X}$ with*
$$\|(p, v, \phi^-, \phi^+, \phi)\|_{\mathcal{X}} \le C\|(f^+, f^-, g)\|_{\mathcal{Y}}$$
for some constant $C > 0$ independent of T.

Theorem 2.3. *We consider Problem* (2.18) *but with initial conditions*

$$\begin{aligned} v(\cdot, 0) &= \xi_1 \quad \text{in } \Omega_+, \\ \phi(\cdot, 0) &= \xi_2 \quad \text{in } \bar{\Omega}_+ \end{aligned} \tag{2.19}$$

instead of the homogeneous initial conditions. For each $(f^+, f^-, g, \xi_1, \xi_2) \in \mathcal{Y}'$ *the problem above has a unique solution* $(p, v, \phi^-, \phi^+, \phi) \in \mathcal{X}$. *with*

$$\|(p, v, \phi^-, \phi^+, \phi)\|_{\mathcal{X}} \leq C\|(f^+, f^-, g, \xi_1, \xi_2)\|_{\mathcal{Y}'}$$

for some constant $C > 0$ *independent of* T.

Chapter 3

A model problem

In this chapter a model problem is discussed to introduce Angenent's parameter trick and other techniques used in this report. Therefore, we deviate from notation used in the remainder of the report.

3.1 Problem definition

Let $p > 2 + n$, $J = [0, T]$, $\Omega = \mathbb{T}^{n-1}$ and $u \in X := L_p(J, W_p^2(\Omega)) \cap H_p^1(J, L_p(\Omega))$ be a solution of the quasilinear parabolic initial value problem

$$\left.\begin{aligned} \partial_t u &= a_{ij}(\nabla u)\partial_{ij} u && \text{in } \Omega \times J, \\ u &= u_0 && \text{on } \Omega \times \{0\}, \end{aligned}\right\} \tag{3.1}$$

where $u_0 \in X_{tr} := W_p^{2-2/p}(\Omega)$ and $a_{ij} \in C_b^{m+1}(\mathbb{R}^n)$, $m \in \mathbb{N} \cup \{\infty\}$, are coefficient functions that satisfy the usual ellipticity demand

$$\exists c > 0 : \quad \forall \xi \in \mathbb{R}^n : \quad \forall \zeta \in \mathbb{R}^n : \quad \xi_i a_{ij}(\zeta)\xi_j \geq c|\xi|^2. \tag{3.2}$$

The goal is to apply Angenent's parameter trick on this model problem to investigate smoothness of the solution u. The first step in this trick is to introduce two parameters μ and λ which will introduce a shift in place and time of the unknown function u. The shifted function is called $w_{\mu,\lambda}$ and System (3.1) is rewritten in the form $F((\mu, \lambda), w_{\mu,\lambda}) = 0$. This can all be found in Section 3.2. The aim is to apply the Implicit Function Theorem to this equation. However, for this we require that F is differentiable and that G is an isomorphism between appropriate spaces. Here G is defined by $G[h] = D_w F((0, 1), u)[h]$. This is discussed in Section 3.3. Then, the Implicit Function Theorem is applied in Section 3.4. From this theorem it follows that that $(\mu, \lambda) \mapsto w_{\mu,\lambda}$ is m times differentiable. In Section 3.5 it is then concluded that u must be m times differentiable in both of its arguments.

3.2 Setup parameter trick

For $(\mu, \lambda) \in \mathbb{R}^n \times \mathbb{R}$ close to $(0, 1)$ define $w_{\mu,\lambda} \in X$ (with a slightly shorter time interval) by

$$w_{\mu,\lambda}(x, t) := u(x + \mu t, \lambda t). \tag{3.3}$$

Using this definition, we have

$$\nabla w_{\mu,\lambda}(x, t) = \nabla u(x + \mu t, \lambda t),$$

and

$$\partial_{ij} w_{\mu,\lambda}(x,t) = \partial_{ij} u(x+\mu t, \lambda t).$$

Together with the initial value problem for u this gives

$$\begin{aligned}\partial_t w_{\mu,\lambda}(x,t) &= \mu \cdot \nabla u(x+\mu t, \lambda t) + \lambda \partial_t u(x+\mu t, \lambda t)\\ &= \mu \cdot \nabla u(x+\mu t, \lambda t) + \lambda a_{ij}(\nabla u(x+\mu t, \lambda t))\partial_{ij} u(x+\mu t, \lambda t)\\ &= \mu \cdot \nabla w_{\mu,\lambda}(x,t) \qquad + \lambda a_{ij}(\nabla w_{\mu,\lambda}(x,t))\partial_{ij} w_{\mu,\lambda}(x,t),\end{aligned}$$

and

$$w_{\mu,\lambda}(x,0) = u(x,0) = u_0(x).$$

For convenience and readability we now drop the subscripts μ and λ. So w satisfies $F((\mu,\lambda),w) = 0$ with $F : (\mathbb{R}^n \times \mathbb{R}) \times X \to L_p(\Omega \times J) \times X_{tr}$ given by

$$F((\mu,\lambda),w) = \begin{pmatrix} \partial_t w - \mu \cdot \nabla w - \lambda a_{ij}(\nabla w)\partial_{ij} w \\ w|_{t=0} - u_0 \end{pmatrix}. \tag{3.4}$$

The aim is to apply the Implicit Function Theorem to this equation. Therefore, we first need the derivative of F.

3.3 Differentiability of F

Lemma 3.1. *The function $F : (\mathbb{R}^n \times \mathbb{R}) \times X \to L_p(\Omega \times J) \times X_{tr}$ defined in Equation (3.4) is m times Fréchet differentiable with respect to w and the first derivative in the point $((0,1),u)$ is equal to $G : X \to L_p(\Omega \times J) \times X_{tr}$, where*

$$G[h] := D_w F((0,1),u)[h] = \begin{pmatrix} \partial_t h - a_{ij}(\nabla u)\partial_{ij} h - a'_{ij}(\nabla u)[h]\partial_{ij} u \\ h|_{t=0} \end{pmatrix},$$

and

$$a'_{ij}(\nabla u)[h]\partial_{ij} u = (\partial_k a_{ij})(\nabla u)\partial_{ij} u \partial_k h.$$

Proof. First of all we define

$$\begin{aligned} f_1 &: X \to L_p(\Omega) & f_1(w) &= \partial_t w,\\ f_{2,\mu} &: X \to L_p(\Omega) & f_{2,\mu}(w) &= \mu \cdot \nabla w,\\ f_3 &: X \to L_p(\Omega) & f_3(w) &= a_{ij}(\nabla w)\partial_{ij} w,\\ f_4 &: X \to X_{tr} & f_4(w) &= w|_{t=0}, \quad \text{and}\\ f_5 &: X \to X_{tr} & f_5(w) &= u_0. \end{aligned}$$

Because differentiation is linear, we determine the Fréchet derivatives of each of these components separately and combine the results to prove the lemma.
First of all, we note that f_1, f_2 and f_4 are bounded linear functions in w. By Lemma B.2 these functions are infinitely many times differentiable with $f_1'(w)[h] = \partial_t h$, $f_{2,\mu}'(w)[h] = \mu \cdot \nabla h$ and $f_4'(w)[h] = h|_{t=0}$.
Secondly, since f_5 is independent of w, it is also infinitely many times differentiable and $f_5'(w)[h] = 0$.
Finally, differentiability of the term $f_3(w)$ has to be investigated. For this we note that $f_3(w) = g_{ij}(w) \cdot h_{ij}(w)$, where $g_{ij} = a_{ij}(\nabla(w))$ and $h_{ij}(w) = \partial_{ij} w$, and thus we would like to use Lemma B.3 to determine the derivative of f_3.
Now define $\phi_{ij} : W_p^1(\Omega \times J, \mathbb{R}^n) \to L_\infty(\Omega \times J, \mathbb{R})$ by $\phi_{ij}(z)(x) := a_{ij}(z(x))$. Note that g_{ij} can be

written as $g_{ij}(w) = (\phi_{ij} \circ \nabla)(w)$. Since $a_{ij} \in C_b^{m+1}(\mathbb{R}^n)$, by Lemma B.5, we have that ϕ_{ij} is m times differentiable and that its first derivative is given by $\phi'_{ij}(z)[h] = \partial_k a_{ij}(z)h^k$ for all i, j.
From ∇ being a bounded linear function it follows by Lemma B.4 that $g_{ij} = \phi_{ij} \circ \nabla$ is m times differentiable with $g'_{ij} = (\phi_{ij} \circ \nabla)'(w)[h] = \phi'_{ij}(\nabla w)[\nabla h] = \partial_k a_{ij}(\nabla w)\partial_k h$ for all i, j.
Since ∂_{ij} is a bounded linear function we have that h_{ij} is infinitely many times differentiable and $h'_{ij}(w)[h] = \partial_{ij} h$ for all i, j.
Using all of the above and $L_p(\Omega \times J, \mathbb{R}) \cdot L_\infty(\Omega \times J, \mathbb{R}) \hookrightarrow L_p(\Omega \times J, \mathbb{R})$, by Lemma B.3 also $f_3 = g_{ij} \cdot h_{ij}$ is m times differentiable with $f'_3(w)[h] = (g_{ij} \cdot h_{ij})'(w)[h] = g'_{ij}(w)[h] \cdot h_{ij}(w) + g_{ij}(w) \cdot h'_{ij}(w)[h] = \partial_k a_{ij}(\nabla w)\partial_k h \partial_{ij} w + a_{ij}(\nabla w)\partial_{ij} h$.
Note that

$$F((\mu, \lambda), w) = \begin{pmatrix} \partial_t w - \mu \cdot \nabla w - \lambda a_{ij}(\nabla w)\partial_{ij} w \\ w|_{t=0} - u_0 \end{pmatrix} = \begin{pmatrix} f_1(w) - f_{2,\mu}(w) - \lambda f_3(w) \\ f_4(w) - f_5(w) \end{pmatrix}.$$

Combining the results and substitution of $(\mu, \lambda) = (0, 1)$ and $w = u$ therefore concludes the proof that F is m times differentiable with respect to w and that

$$D_w F((0,1), u)[h] = \begin{pmatrix} \partial_t h - a_{ij}(\nabla u)\partial_{ij} h - a'_{ij}(\nabla u)[h]\partial_{ij} u \\ h|_{t=0} \end{pmatrix},$$

with

$$a'_{ij}(\nabla u)[h]\partial_{ij} u = (\partial_k a_{ij})(\nabla u)\partial_{ij} u \partial_k h$$

is the first derivative of $F((\mu, \lambda), w)$ with respect to w in the point $((0, 1), u)$. ■

Lemma 3.2. *There exist $T' < T$ and $2 + n < p' < p$ such that for any $u \in X'$ we have $G \in \mathcal{L}_{is}(X', L_p(\Omega') \times X'_{tr})$, i.e. G is an isomorphism between X' and $L_p(\Omega') \times X'_{tr}$. Here I, X' and X'_{tr} are defined by*

$$\begin{aligned} I &:= (0, T'), \\ X' &:= L_{p'}(I, W^2_{p'}(\Omega)) \cap H^1_{p'}(I, L_{p'}(\Omega)), \\ X'_{tr} &:= W^{2-2/p'}_{p'}(\Omega). \end{aligned}$$

Proof. We split the operator G in two parts $G = G_1 - G_2$, where

$$G_1[z] = \begin{pmatrix} \partial_t h - a_{ij}(\nabla u)\partial_{ij} h \\ h|_{t=0} \end{pmatrix} \qquad \text{and} \qquad G_2[h] = \begin{pmatrix} a'_{ij}(\nabla u)[h]\partial_{ij} u \\ 0 \end{pmatrix}.$$

We first only consider the operator G_1 and define $A = a_{ij}(\nabla u)\partial_{ij}$. Note that the symbol of A is equal to $a(x, \xi) = -a_{ij}(\nabla u(x))\xi_i \xi_j$. Now let $\mathcal{L}$ be the sector $\overline{\Sigma_{\pi/2}} = \left\{z \in \mathcal{C} \backslash \{0\} : |\arg(z)| \le \frac{\pi}{2}\right\}$. Because of the ellipticity demand (Equation (3.2)) we have $a(x, \xi) = -a_{ij}(\nabla u(x))\xi_i\xi_j \le -c|\xi|^2$ for all $x \in \Omega \times I$, $\xi \in \mathbb{R}^n$ and thus that $a(x, \xi) - \lambda \neq 0$ for all $x \in \Omega, \xi \in \mathbb{R}^n, \lambda \in \mathcal{L}$. This means that the operator A is parameter-elliptic in $\mathcal{L}$ and thus that $\partial_t - A$ is parabolic. Now since $a_{ij}(\nabla u) \in C_b(\Omega \times I)$, it must hold by [8] Lemma 3.10 that A has maximal regularity ([7] Lemma 3.10). Since the time interval $(0, T)$ is finite, by [8] Remark 1.7 this implies that G_1 is an isomorphism.
For an estimate on G_2 Lemma B.9 is used. First, we note that $\|G_1^{-1}\|$ does not depend on the interval I. This follows from [2] Theorem III 4.10.7, but is not discussed in detail in this report. In combination with the lemma, where we use that $\partial_{ij} u \in L_p(\Omega \times J)$, this states that for any $2 + n < p' < p$ there exists a T' such that on any interval I of length T' we have that

$$\|\partial_{ij} u\|_{L_{p'}(\Omega \times I)} \le \frac{1}{\|G_1^{-1}\| \|\partial_k a_{ij}\|_{L_\infty(\mathbb{R}^n)}}$$

for all k, i, and j. Together with Lemma B.7 the above is used to estimate

$$\begin{aligned} \|G_2[h]\|_{L_{p'}(\Omega \times I)} &= \|a'_{ij}(\nabla u)[h]\partial_{ij} u\|_{L_{p'}(\Omega \times I)} = \|(\partial_k a_{ij})(\nabla u)\|_{L_\infty(\Omega \times I)} \|\partial_{ij} u\|_{L_{p'}(\Omega \times I)} \|\partial_k h\|_{L_\infty(\Omega \times I)} \\ &\le \|\partial_k a_{ij}\|_{L_\infty(\mathbb{R}^n)} \frac{1}{\|G_1^{-1}\| \|\partial_k a_{ij}\|_{L_\infty(\mathbb{R}^n)}} \|h\|_{X'} = \frac{1}{\|G_1^{-1}\|} \|h\|_{X'}, \end{aligned}$$

and consequently $\|G_2\| \leq \frac{1}{\|G_1^{-1}\|}$. Now since G_1 is an isomorphism, we have that also $G = G_1 - G_2$ is an isomorphism between spaces on I (instead of J). ■

In the following let T', p', I, X' and X'_{tr} be as defined as in (the proof of) the lemma above.

3.4 Implicit Function Theorem

The goal is to apply the Implicit Function Theorem to the equation $F((\mu, \lambda), w) = 0$. For $m \geq 2$ we have the following:

(i) The mapping $F : \mathcal{U}((0,1), u) \subseteq (\mathbb{R}^n \times \mathbb{R}) \times X' \to L_{p'}(\Omega \times I) \times X'_{tr}$ is defined on a open neighbourhood $\mathcal{U}((0,1,u)$ and $F((0,1), u) = 0$.

(ii) Lemma 3.1 states that the partial derivative $D_w F$ exists on $\mathcal{U}((0,1), u)$ and that it is equal to G in $((0,1), u)$. From Lemma 3.2 we can conclude that $G : X' \to L_{p'}(\Omega \times I) \times X'_{tr}$ is an isomorphism.

(iii) Since all coefficient function a_{ij} are continuously differentiable, also F and G are continuous at $((0,1), u)$.

(iv) The function $F((\mu, \lambda), w)$ is linear in both μ and λ and thus we have that F is smooth in its first argument. By Lemma 3.1 we have that from $a_{ij} \in C^{m+1}$ follows that F is m times Fréchet differentiable with respect to w.

From the Implicit Function Theorem (Theorem B.6) we can now conclude:

(a) Existence and uniqueness: For all (μ, λ) close to $(0, 1)$ there exists a unique $y : \mathbb{R}^n \times \mathbb{R} \to X'$ close to u such that $F((\mu, \lambda), y(\mu, \lambda)) = 0$.

(b) Continuous differentiability. Since F is m times differentiable on a neighbourhood of $((0,1), u)$, also y is on a neighbourhood of $(0, 1)$.

Since we already had $F((\mu, \lambda), w) = 0$, it must hold that $y(\mu, \lambda) = w_{\mu,\lambda}$ and thus that $(\mu, \lambda) \mapsto w_{\mu,\lambda}$ is m times differentiable. Note that for $m = \infty$ we have that F is smooth and consequently that y is smooth in a neighbourhood of $(0, 1)$.

3.5 Conclusion

Theorem 3.3. *If $a_{ij} \in C_b^{m+1}(\mathbb{R}^n)$ satisfies the ellipticity demand* (3.2) *and $u_0 \in X_{tr} := W_p^{2-2/p}(\Omega)$, then the solution $u \in X$ of System* (3.1) *is m times differentiable on the interval* $(0, T)$.

Proof. Let $(x_0, t_0) \in \mathbb{T}^{n-1} \times (0, T')$ be fixed. We define the functional $E_{x_0,t_0} : X' \to \mathbb{R}$ by $E_{x_0,t_0}(u) = u(x_0, t_0)$ for all $u \in X'$. This gives $E_{x_0,t_0} w_{\mu,\lambda} = w_{\mu,\lambda}(x_0, t_0) = u(x_0 + \mu t_0, \lambda t_0)$. Note that since $X' \hookrightarrow C(I, C^1(\Omega))$, we have $|w_{\mu,\lambda}(x_0, t_0)| \leq \|u_{\mu,\lambda}\|_{L^\infty} \leq \|w_{\mu,\lambda}\|_{X'}$ and thus that E_{x_0,t_0} is a bounded linear functional. By Lemma B.2 this gives that E_{x_0,t_0} is infinitely many times differentiable. In combination with $(\mu, \lambda) \mapsto w_{\mu,\lambda}$ being m times differentiable in a neighbourhood of $(0, 1)$ we can conclude that $(\mu, \lambda) \mapsto E_{x_0,t_0} w_{\mu,\lambda} = u(x_0 + \mu t_0, \lambda t_0)$ is m times differentiable in a neighbourhood of $(0, 1)$. From this we can conclude that also $u(x, t)$ must be m times differentiable in both of its arguments around the point $(x, t) = (x_0, t_0)$.
Since T' depends only on p, p', $\|u\|_X$ and $\|\partial_k a_{ij}\|_{L_\infty}$ for $k = 1, ..., n$, the whole argument can be repeated on overlapping open intervals until m times differentiability on the whole interval $(0, T)$ is guaranteed. ■

Chapter 4

Parabolic smoothing I: additional quasilinear condition

4.1 Problem statement

We start from the system of equations (2.4). Because this system is not yet closed and to make the equations easier, we add the equations

$$\begin{aligned} \mathcal{A}_{\hat{\phi}^-}\hat{\phi}^- &= 0 && \text{in } \Omega_-, \\ \mathcal{L}_{\hat{\phi}^+}\hat{\phi}^+ &= 0 && \text{in } \Omega_+, \\ \hat{\phi}^+(\cdot,0) &= \sigma(\cdot,0) && \text{in } \Omega_+. \end{aligned}$$

For convenience we define

$$\begin{aligned} \Lambda_\tau^- &:= \mathcal{A}_\tau, & u^- &:= p, \\ \Lambda_\tau^+ &:= \mathcal{L}_\tau, & u^+ &:= v \end{aligned}$$

and use τ and $\tau^\pm$ instead of $\hat{\phi}$ and $\hat{\phi}^\pm$. The problem can then be written as

$$\left.\begin{aligned} \Lambda_{\tau^\pm}^\pm u^\pm &= 0 && \text{in } \Omega_\pm \times J, \\ \Lambda_{\tau^\pm}^\pm \tau^\pm &= 0 && \text{in } \Omega_\pm \times J, \\ u^\pm &= 1 && \text{on } \Gamma \times J, \\ \tau^\pm - \tau &= 0 && \text{on } \Gamma \times J, \\ u^\pm &= 0 && \text{on } \Sigma_\pm \times J, \\ \tau^\pm &= 0 && \text{on } \Sigma_\pm \times J, \\ \partial_t\tau - (1+|\nabla'\tau|^2)\left(\frac{-\alpha u^-_{z_n}}{1+\tau^-_{z_n}} + \frac{\beta u^+_{z_n}}{1+\tau^+_{z_n}}\right) - \frac{1}{\gamma} &= 0 && \text{on } \Gamma \times J, \\ u^+(\cdot,0) &= \hat{\nu}_0 && \text{in } \Omega_+, \\ \tau^+(\cdot,0) &= \sigma(\cdot,0) && \text{in } \bar{\Omega}_+. \end{aligned}\right\} \tag{4.1}$$

Suppose

$$(u^-,u^+,\tau^-,\tau^+,\tau) \in \mathcal{X} = (X^- \times X^+)^2 \times X^B$$

is a solution of this problem. The goal of this chapter is to apply Angenent's parameter trick to prove smoothness of the solution. Therefore, we apply the same strategy as on the model problem. So first, we introduce shifted versions $w^\pm_{\mu,\lambda}$, $\psi^\pm_{\mu,\lambda}$ and $\psi_{\mu,\lambda}$ of the unknowns $u^\pm$, $\tau^\pm$ and τ, respectively. This is done in Section 4.2.1. Secondly, in Section 4.2.2 the system above is rewritten to an equation of the form $F((\mu,\lambda),(w^-_{\mu,\lambda},w^+_{\mu,\lambda},\psi^-_{\mu,\lambda},\psi^-_{\mu,\lambda},\psi_{\mu,\lambda})) = 0$. Finally, we apply the Implicit Function Theorem to this equation in Section 4.5 to reach a conclusion in Section 4.6. However, for this first differentiability of F needs to be checked and we need that G is an isomorphism between the appropriate spaces. Here G is defined by

$$G[h^-, h^+, \delta^-, \delta^+, \delta] = F'((0,1),(u^-, u^+, \tau^-, \tau^+, \tau))[h^-, h^+, \delta^-, \delta^+, \delta].$$

These topics are addressed in Section 4.3 and 4.4.

Next to the assumption that Problem (4.1) has a unique solution we need a few extra assumptions in this chapter. These are

$$\tau \in H^2_p(\Omega \times J) \tag{4.2}$$

and

$$\partial_i u_{z_n} \in W^\theta_p(J, W^{-\theta''}_p(\Omega_-)) \text{ for some } \theta'' < \theta. \tag{4.3}$$

4.2 Setup parameter trick

4.2.1 Definitions

Let $\lambda \in \mathbb{R}$, $\mu' \in \mathbb{R}^{n-1}$ and $\mu = (\mu', 0) \in \mathbb{R}^n$. Then define

$$w^\pm_{\mu,\lambda}(z,t) := u^\pm(z + \mu t, \lambda t), \tag{4.4a}$$
$$\psi^\pm_{\mu,\lambda}(z,t) := \tau^\pm(z + \mu t, \lambda t), \tag{4.4b}$$
$$\psi_{\mu,\lambda}(z,t) := \tau(z + \mu t, \lambda t), \tag{4.4c}$$

Note that for $z \in \Omega_-$ also $z + \mu t = (z' + \mu' t, z_n) \in \Omega_-$ for all $t \in \mathbb{R}$. The same holds for Ω_+, Γ, Σ_- and Σ_+.

4.2.2 Shifted equations

In the following we use that substitutions gives

$$\left(\mathcal{A}_{\psi^\pm_{\mu,\lambda}} w^\pm_{\mu,\lambda}\right)(z,t) = \left(\mathcal{A}_{\tau^\pm} u^\pm\right)(z + \mu t, \lambda t), \tag{4.5a}$$
$$\left(\mathcal{A}_{\psi^\pm_{\mu,\lambda}} \psi^\pm_{\mu,\lambda}\right)(z,t) = \left(\mathcal{A}_{\tau^\pm} \tau^\pm\right)(z + \mu t, \lambda t). \tag{4.5b}$$

Note that Equation $(4.1)_1$ for the lower (vapour) phase can be written as

$$\partial_t u^+ - \mathcal{A}_{\tau^+} u^+ = 0.$$

Together with Definition (4.4a) and Equation (4.5a) this gives for $w_{\mu,\lambda}$ that

$$\begin{aligned}
\partial_t w^+_{\mu,\lambda}(z,t) &= \mu \cdot \nabla u^+(z + \mu t, \lambda t) + \lambda \partial_t u^+(z + \mu t, \lambda t) \\
&= \mu \cdot \nabla u^+(z + \mu t, \lambda t) + \lambda \left(\mathcal{A}_{\psi^+_{\mu,\lambda}} u^+\right)(z + \mu t, \lambda t) \\
&= \mu \cdot \nabla w^+_{\mu,\lambda}(z,t) \qquad + \lambda \left(\mathcal{A}_{\psi^+_{\mu,\lambda}} w^+_{\mu,\lambda}\right)(z,t)
\end{aligned}$$

and thus

$$\partial_t w^+_{\mu,\lambda} - \mu \cdot \nabla w^+_{\mu,\lambda} - \lambda \mathcal{A}_{\psi^+_{\mu,\lambda}} w^+_{\mu,\lambda} = 0.$$

Analogously, using Equation $(4.1)_2$ in the vapour phase, we get

$$\partial_t \psi^+_{\mu,\lambda} - \mu \cdot \nabla \psi^+_{\mu,\lambda} - \lambda \mathcal{A}_{\psi^+_{\mu,\lambda}} \psi^+_{\mu,\lambda} = 0.$$

Finally, Equation $(4.1)_7$ can be written as

$$\partial_t\tau = (1+|\nabla'\tau|^2)\left(\frac{-\alpha u^-_{z_n}}{1+\tau^-_{z_n}}+\frac{\beta u^+_{z_n}}{1+\tau^+_{z_n}}\right)+\frac{1}{\gamma}$$

and thus

$$\begin{aligned}\partial_t\psi_{\mu,\lambda}(z,t) &= (\mu\cdot\nabla\tau+\lambda\partial_t\tau)\,(z+\mu t,\lambda t)\\ &= \left(\mu\cdot\nabla\tau+\lambda(1+|\nabla'\tau|^2)\left(\frac{-\alpha u^-_{z_n}}{1+\tau^-_{z_n}}+\frac{\beta u^+_{z_n}}{1+\tau^+_{z_n}}\right)+\frac{\lambda}{\gamma}\right)(z+\mu t,\lambda t)\\ &= \left(\mu\cdot\nabla\psi_{\mu,\lambda}+\lambda(1+|\nabla'\psi_{\mu,\lambda}|^2)\left(\frac{-\alpha w^-_{\mu,\lambda,z_n}}{1+\psi^-_{\mu,\lambda,z_n}}+\frac{\beta w^+_{\mu,\lambda,z_n}}{1+\psi^+_{\mu,\lambda,z_n}}\right)+\frac{\lambda}{\gamma}\right)(z,t).\end{aligned}$$

Substitution gives all other results and the entire system can be rewritten as

$$\left.\begin{aligned}\mathcal{A}_{\psi^-_{\mu,\lambda}}w^-_{\mu,\lambda} &= 0 && \text{in } \Omega_-\times J,\\ \partial_t w^+_{\mu,\lambda}-\mu\cdot\nabla w^+_{\mu,\lambda}-\lambda\mathcal{A}_{\psi^+_{\mu,\lambda}}w^+_{\mu,\lambda} &= 0 && \text{in } \Omega_+\times J,\\ \mathcal{A}_{\psi^-_{\mu,\lambda}}\psi^-_{\mu,\lambda} &= 0 && \text{in } \Omega_-\times J,\\ \partial_t \psi^+_{\mu,\lambda}-\mu\cdot\nabla \psi^+_{\mu,\lambda}-\lambda\mathcal{A}_{\psi^+_{\mu,\lambda}}\psi^+_{\mu,\lambda} &= 0 && \text{in } \Omega_+\times J,\\ w^\pm_{\mu,\lambda} &= 1 && \text{on } \Gamma\times J,\\ \psi^\pm_{\mu,\lambda}-\psi_{\mu,\lambda} &= 0 && \text{on } \Gamma\times J,\\ w^\pm_{\mu,\lambda} &= 0 && \text{on } \Sigma_\pm\times J,\\ \psi^\pm_{\mu,\lambda} &= 0 && \text{on } \Sigma_\pm\times J,\\ \partial_t\psi_{\mu,\lambda}-\mu\cdot\nabla\psi_{\mu,\lambda}&\\ -\lambda(1+|\nabla'\psi_{\mu,\lambda}|^2)\left(\frac{-\alpha w^-_{\mu,\lambda,z_n}}{1+\psi^-_{\mu,\lambda,z_n}}+\frac{\beta w^+_{\mu,\lambda,z_n}}{1+\psi^+_{\mu,\lambda,z_n}}\right)-\frac{\lambda}{\gamma} &= 0 && \text{on } \Gamma\times J,\\ w^+_{\mu,\lambda}(\cdot,0) &= \hat{\nu}_0 && \text{in } \Omega_+,\\ \psi^+_{\mu,\lambda}(\cdot,0) &= \sigma && \text{on } \bar{\Omega}_+.\end{aligned}\right\}\tag{4.6}$$

For convenience, in the following sections we drop the subscripts μ and λ. Next to this, we define the spaces

$$\begin{aligned}\tilde{\mathcal{X}} &:= \{(w^-,w^+,\psi^-,\psi^+,\psi)\in\mathcal{X}\mid w^\pm=0 \text{ on } \Sigma_\pm,\ \ \psi^\pm=0 \text{ on } \Sigma_\pm,\ \ \psi^\pm=\psi \text{ on } \Gamma\},\\ \tilde{\mathcal{Y}} &:= Y^-\times Y^+\times Y^-\times Y^+\times Y^B\times Y^B\times X_0(\Omega_+)\times X_0(\bar{\Omega}_+).\end{aligned}$$

Combining all of the above we can conclude that $(w^-,w^+,\psi^-,\psi^+,\psi)$ satisfies

$$F((\mu,\lambda),(w^-,w^+,\psi^-,\psi^+,\psi))=0,$$

with

$$F:(\mathbb{R}^n\times\mathbb{R})\times\tilde{\mathcal{X}}\to\tilde{\mathcal{Y}}$$

given by

$$F((\mu,\lambda),(w^-,w^+,\psi^-,\psi^+,\psi)) := \begin{pmatrix}\mathcal{A}_{\psi^-}w^-\\ \partial_t w^+-\mu\cdot\nabla w^+-\lambda\mathcal{A}_{\psi^+}w^+\\ \mathcal{A}_{\psi^-}\psi^-\\ \partial_t\psi^+-\mu\cdot\nabla\psi^+-\lambda\mathcal{A}_{\psi^+}\psi^+\\ w^\pm-1\\ \partial_t\psi-\mu\cdot\nabla\psi-\lambda(1+|\nabla'\psi|^2)\left(\frac{-\alpha w^-_{z_n}}{1+\psi^-_{z_n}}+\frac{\beta w^+_{z_n}}{1+\psi^+_{z_n}}\right)-\frac{\lambda}{\gamma}\\ \gamma_0 w^+-\hat{\nu}_0\\ \gamma_0\psi^+-\gamma_0\sigma\end{pmatrix}.$$

4.3 Differentiability of F

4.3.1 Partial derivatives of F

As mentioned before, the next step is to investigate Fréchet differentiability of F. Therefore, all partial derivatives of F are calculated in this section.

First of all, note that all terms in F either do not depend on w^- or are linear in w^-. In the first case the derivative of the term with respect to w^- is just zero. In the second case Lemma B.2 can be used to determine the derivative. Since differentiation is a linear operator, the results can easily be combined, which gives that F is infinitely many times differentiable with respect to w^- and that the first derivative is equal to

$$D_{w^-}F((\mu,\lambda),(u^-,u^+,\tau^-,\tau^+,\tau))[h^-] = \begin{pmatrix} \mathcal{A}_{\tau^-}h^- \\ 0 \\ 0 \\ 0 \\ h^- \\ \lambda(1+|\nabla'\tau|^2)\frac{\alpha h^-_{z_n}}{1+\tau^-_{z_n}} \\ 0 \\ 0 \end{pmatrix}.$$

Secondly, all terms are also independent on w^+ or linear in w^+. Using the same reasoning as above, we can conclude that F is infinitely many times differentiable with respect to w^+ and that the first derivative is equal to

$$D_{w^+}F((\mu,\lambda),(u^-,u^+,\tau^-,\tau^+,\tau))[h^+] = \begin{pmatrix} 0 \\ \partial_t h^+ - \mu\cdot\nabla h^+ - \lambda\mathcal{A}_{\tau^+}h^+ \\ 0 \\ 0 \\ h^+ \\ -\lambda(1+|\nabla'\tau|^2)\frac{\beta h^+_{z_n}}{1+\tau^+_{z_n}} \\ \gamma_0 h^+ \\ 0 \end{pmatrix}.$$

For the derivative of F with respect to ψ^- the examples in Section B.1.1 are used. From this we can conclude that F is infinitely many times differentiable with respect to ψ^- and that the first derivative is equal to

$$D_{\psi^-}F((\mu,\lambda),(u^-,u^+,\tau^-,\tau^+,\tau))[\delta^-] = \begin{pmatrix} -A(\nabla\tau^-)\nabla\delta^-\cdot\nabla u^-_{z_n} \\ 0 \\ \sum_{i=1}^{n-1}\delta^-_{z_iz_i} - A(\nabla\tau^-)\nabla\delta^-\cdot\nabla\tau^-_{z_n} - \vec{a}(\nabla\tau^-)\nabla\delta^-_{z_n} \\ 0 \\ 0 \\ -\lambda(1+|\nabla'\tau|^2)\frac{\alpha u^-_{z_n}\delta^-_{z_n}}{(1+\tau^-_{z_n})^2} \\ 0 \\ 0 \end{pmatrix}.$$

Using again Lemma B.2 and the examples in Section B.1.1, we get that F is infinitely many times differentiable with respect to ψ^+ and that the first derivative is equal to

$$
D_{\psi^+}F((\mu,\lambda),(u^-,u^+,\tau^-,\tau^+,\tau))[\delta^+] = \begin{pmatrix} 0 \\ \lambda A(\nabla\tau^+)\nabla\delta^+\cdot\nabla u^+_{z_n} \\ 0 \\ \partial_t\delta^+ - \mu\cdot\nabla\delta^+ - \lambda\left(\sum_{i=1}^{n-1}\delta^+_{z_iz_i} - A(\nabla\tau^+)\nabla\delta^+\cdot\nabla\tau^+_{z_n} - \vec{a}(\nabla\tau^+)\nabla\delta^+_{z_n}\right) \\ 0 \\ \lambda(1+|\nabla'\tau|^2)\frac{\beta u^+_{z_n}\delta^+_{z_n}}{(1+\tau^+_{z_n})^2} \\ 0 \\ \gamma_0\delta^+ \end{pmatrix}.
$$

Finally, using Lemma B.2 and the last example in Section B.1.1, we get that F is infinitely many times differentiable with respect to ψ and that the first derivative is equal to

$$
D_\psi F((\mu,\lambda),(u^-,u^+,\tau^-,\tau^+,\tau))[\delta] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \partial_t\delta - \mu\cdot\nabla\delta - 2\lambda\nabla'\psi\cdot\nabla'\delta\left(\frac{-\alpha u^-_{z_n}}{1+\tau^-_{z_n}} + \frac{\beta u^+_{z_n}}{1+\tau^+_{z_n}}\right) \\ 0 \\ 0 \end{pmatrix}.
$$

4.3.2 Total derivative

Using the results above, we get that F is infinitely many times differentiable and after substitution of $(\mu,\lambda) = (0,1)$ we get

$$
\begin{aligned}
&G[u^+,u^-,\delta^+,\delta^-,\delta] := F'((0,1),(u^-,u^+,\tau^-,\tau^+,\tau))[h^-,h^+,\delta^-,\delta^+,\delta] \\
&= \begin{pmatrix} \mathcal{A}_{\tau^-}h^- - A(\nabla\tau^-)\nabla\delta^-\cdot\nabla u^-_{z_n} \\ \partial_t h^+ - \mathcal{A}_{\tau^+}h^+ + A(\nabla\tau^+)\nabla\delta^+\cdot\nabla u^+_{z_n} \\ \sum_{i=1}^{n-1}\delta^-_{z_iz_i} - A(\nabla\tau^-)\nabla\delta^-\cdot\nabla\tau^-_{z_n} - \vec{a}(\nabla\tau^-)\nabla\delta^-_{z_n} \\ \partial_t\delta^+ - \sum_{i=1}^{n-1}\delta^+_{z_iz_i} + A(\nabla\tau^+)\nabla\delta^+\cdot\nabla\tau^+_{z_n} + \vec{a}(\nabla\tau^+)\nabla\delta^+_{z_n} \\ h^\pm \\ (1+|\nabla'\tau|^2)\left(\frac{\alpha h^-_{z_n}}{1+\tau^-_{z_n}} - \frac{\beta h^+_{z_n}}{1+\tau^+_{z_n}} - \frac{\alpha u^-_{z_n}\delta^-_{z_n}}{(1+\tau^-_{z_n})^2} + \frac{\beta u^+_{z_n}\delta^+_{z_n}}{(1+\tau^+_{z_n})^2}\right) + \partial_t\delta - 2\nabla'\psi\cdot\nabla'\delta\left(\frac{-\alpha u^-_{z_n}}{1+\tau^-_{z_n}} + \frac{\beta u^+_{z_n}}{1+\tau^+_{z_n}}\right) \\ \gamma_0 h^+ \\ \gamma_0\delta^+ \end{pmatrix} \\
&= \begin{pmatrix} \Lambda^-_{\tau^-}h^- - A(\nabla\tau^-)\nabla\delta^-\cdot\nabla u^-_{z_n} \\ \Lambda_{\tau^+}h^+ + A(\nabla\tau^+)\nabla\delta^+\cdot\nabla u^+_{z_n} \\ \Lambda^-_{\tau^-}\delta^- - A(\nabla\tau^-)\nabla\delta^-\cdot\nabla\tau^-_{z_n} \\ \Lambda^+_{\tau^+}\delta^+ + A(\nabla\tau^+)\nabla\delta^+\cdot\nabla\tau^+_{z_n} \\ h^\pm \\ \partial_t\delta - \tilde{\alpha}_1 h^-_{z_n} - \tilde{\beta}_1 h^+_{z_n} - \tilde{\alpha}_2\delta^-_{z_n} - \tilde{\beta}_2\delta^+_{z_n} + \zeta\cdot\nabla'\delta \\ \gamma_0 h^+ \\ \gamma_0\delta^+ \end{pmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\alpha}_1 &:= -(1+|\nabla'\tau|^2)\frac{\alpha}{1+\tau^-_{z_n}}, \\
\tilde{\alpha}_2 &:= (1+|\nabla'\tau|^2)\frac{\alpha u^-_{z_n}}{(1+\tau^-_{z_n})^2}, \\
\tilde{\beta}_1 &:= (1+|\nabla'\tau|^2)\frac{\beta}{1+\tau^+_{z_n}}, \\
\tilde{\beta}_2 &:= -(1+|\nabla'\tau|^2)\frac{\beta u^+_{z_n}}{(1+\tau^+_{z_n})^2}, \\
\zeta &:= -2\left(\frac{-\alpha u^-_{z_n}}{1+\tau^-_{z_n}} + \frac{\beta u^+_{z_n}}{1+\tau^+_{z_n}}\right)\nabla'\psi.
\end{aligned}
$$

4.4 Isomorphism property of F'

In this section the main goal is to prove that G is an isomorphism between the appropriate spaces. In order to do so, we use that $G = G_1 + G_2$ with

$$
G_1[h^-,h^+,\delta^-,\delta^+,\delta] := \begin{pmatrix} \Lambda^-_{\tau^-} h^- \\ \Lambda_{\tau^+} h^+ \\ \Lambda^-_{\tau^-}\delta^- - A(\nabla\tau^-)\nabla\delta^- \cdot \nabla\tau^-_{z_n} \\ \Lambda^+_{\tau^+}\delta^+ + A(\nabla\tau^+)\nabla\delta^+ \cdot \nabla\tau^+_{z_n} \\ h^\pm \\ \partial_t\delta - \tilde{\alpha}_1 h^-_{z_n} - \tilde{\beta}_1 h^+_{z_n} - \tilde{\alpha}_2\delta^-_{z_n} - \tilde{\beta}_2\delta^+_{z_n} + \zeta\cdot\nabla'\delta \\ \gamma_0 h^+ \\ \gamma_0\delta^+ \end{pmatrix}
$$

and

$$
G_2[h^-,h^+,\delta^-,\delta^+,\delta] := \begin{pmatrix} -A(\nabla\tau^-)\nabla\delta^- \cdot \nabla u^-_{z_n} \\ A(\nabla\tau^+)\nabla\delta^+ \cdot \nabla u^+_{z_n} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
$$

The approach we use is to first prove that G_1 is an isomorphism between $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$ and then treat G_2 as a perturbation term.

4.4.1 Main part: G_1

To prove that G_1 is an isomorphism we need a variation on Theorem 2.3.

Theorem 4.1. *Let*

$$\left.\begin{aligned}
\tilde{L}^{\pm}u^{\pm} &= f_1^{\pm} && \textit{in } \Omega_{\pm}\times J,\\
\hat{L}^{\pm}\psi^{\pm} &= f_2^{\pm} && \textit{in } \Omega_{\pm}\times J,\\
u^{\pm} &= g_1 && \textit{on } \Gamma\times J,\\
\psi^{\pm}-\psi &= 0 && \textit{in } \Gamma\times J,\\
u^{\pm} &= 0 && \textit{on } \Sigma_{\pm}\times J,\\
\psi^{\pm} &= 0 && \textit{on } \Sigma_{\pm}\times J,\\
\partial_t\psi-\hat{\alpha}_2\psi^{-}_{z_n}-\hat{\beta}_2\psi^{+}_{z_n}+\hat{\zeta}\cdot\nabla'\psi-\hat{\alpha}_1u^{-}_{z_n}-\hat{\beta}_1u^{+}_{z_n} &= g_2 && \textit{on } \Gamma\times J,\\
u^{+}(\cdot,0) &= \xi_1 && \textit{in } \Omega_{+},\\
\psi^{+}(\cdot,0) &= \xi_2 && \textit{on } \Gamma,
\end{aligned}\right\} \tag{4.7}$$

where

$$\begin{aligned}
\tilde{L}^{\pm}u^{\pm} &:= \Lambda^{\pm}_{\sigma}u^{\pm}-\frac{u^{\pm}_{z_n}}{1+\sigma_{z_n}}\Lambda^{\pm}_{\sigma}\sigma,\\
\hat{L}^{\pm}\psi^{\pm} &:= \Lambda^{\pm}_{\sigma}\psi^{\pm}\pm A(\nabla\sigma)\nabla\psi^{\pm}\cdot\nabla\sigma_{z_n},\\
\hat{\alpha}_1 &:= -(1+|\nabla'\sigma|^2)\frac{\alpha}{1+\sigma_{z_n}},\\
\hat{\alpha}_2 &:= (1+|\nabla'\sigma|^2)\frac{\alpha U^{-}_{z_n}}{(1+\sigma_{z_n})^2},\\
\hat{\beta}_1 &:= (1+|\nabla'\sigma|^2)\frac{\beta}{1+\sigma_{z_n}},\\
\hat{\beta}_2 &:= -(1+|\nabla'\sigma|^2)\frac{\beta U^{+}_{z_n}}{(1+\sigma_{z_n})^2},
\end{aligned}$$

σ satisfies

$$\begin{aligned}
\sigma &\in H^2_p(\Omega\times J),\\
\sigma|_{\Gamma\times\{0\}} &= \eta,\\
\sigma|_{\Sigma_{\pm}\times J} &= 0,
\end{aligned}$$

U^- is the unique solution of

$$\left.\begin{aligned}
\Lambda^{-}_{\sigma}U^{-}-\frac{U^{-}_{z_n}}{1+\sigma_{z_n}}\Lambda^{-}_{\sigma}\sigma &= 0 && \textit{in } \Omega_{-}\times J,\\
U^{-} &= 1 && \textit{on } \Gamma\times J,\\
U^{-} &= 0 && \textit{on } \Sigma_{-}\times J,
\end{aligned}\right\} \tag{4.8}$$

and U^+ is the unique solution of

$$\left.\begin{aligned}
\Lambda^{+}_{\sigma}U^{+}-\frac{U^{+}_{z_n}}{1+\sigma_{z_n}}\Lambda^{+}_{\sigma}\sigma &= 0 && \textit{in } \Omega_{+}\times J,\\
U^{+} &= 1 && \textit{on } \Gamma\times J,\\
U^{+} &= 0 && \textit{on } \Sigma_{+}\times J,\\
U^{+}(\cdot,0) &= \hat{\nu}_0 && \textit{in } \Omega_{+}.
\end{aligned}\right\} \tag{4.9}$$

For each $(f_1^-, f_1^+, f_2^-, f_2^+, g_1^{\pm}, g_2, \xi_1, \xi_2) \in \tilde{\mathcal{Y}}$ the system above has a unique solution

$$(u^-, u^+, \psi^-, \psi^+, \psi) \in \mathcal{X}$$

and the estimate

$$\|(u^-, u^+, \psi^-, \psi^+, \psi)\|_{\mathcal{X}} \leq C\|(f_1^-, f_1^+, f_2^-, f_2^+, g_1^\pm, g_2, \xi_1, \xi_2)\|_{\tilde{\mathcal{Y}}}$$

is valid with a constant $C > 0$ independent of T.

Proof. Let $y = (f_1^-, f_1^+, f_2^-, f_2^+, g_1^\pm, g_2, \xi_1, \xi_2) \in \tilde{\mathcal{Y}}$ be fixed.
Firstly, by [17] Lemma A.4 and standard parabolic theory we have that the system

$$\left.\begin{aligned} \hat{L}^\pm \phi_2^\pm &= f_2^\pm && \text{in } \Omega_\pm \times J \\ \phi_2^\pm &= 0 && \text{on } (\Sigma_\pm \cup \Gamma) \times J \\ \phi_2^+(\cdot, 0) &= 0 && \text{on } \Gamma \end{aligned}\right\}$$

has a unique solution $\|(\phi_2^-, \phi_2^+)\| \in X^- \times X^+$ with estimate

$$\|(\phi_2^-, \phi_2^+)\|_{X^- \times X^+} \leq C_2\|(f_2^-, f_2^+)\|_{Y^- \times Y^+} \leq C_2\|y\|_{\tilde{\mathcal{Y}}}$$

where C_2 is a constant independent of T.
Secondly, since $(\hat{\alpha}_2\phi_{2,z_n}^- + \hat{\beta}_2\phi_{2,z_n}^+) \in Y^B$, by Theorem 2.3 the system

$$\left.\begin{aligned} \tilde{L}^\pm u^\pm &= f_1^\pm && \text{in } \Omega_\pm \times J, \\ \hat{L}^\pm \phi_1^\pm &= 0 && \text{in } \Omega_\pm \times J, \\ u^\pm &= g_1 && \text{on } \Gamma \times J, \\ \phi_1^\pm - \phi &= 0 && \text{in } \Omega_\pm \times J, \\ u^\pm &= 0 && \text{on } \Sigma_\pm \times J, \\ \phi_1^\pm &= 0 && \text{on } \Sigma_\pm \times J, \\ \partial_t\phi - \hat{\alpha}_2\phi_{1,z_n}^- - \hat{\beta}_2\phi_{1,z_n}^+ + \hat{\zeta}\cdot\nabla'\phi - \hat{\alpha}_1 u_{z_n}^- - \hat{\beta}_1 u_{z_n}^+ &= g_2 + \hat{\alpha}_2\phi_{2,z_n}^- + \hat{\beta}_2\phi_{2,z_n}^+ && \text{on } \Gamma \times J, \\ u^+(\cdot, 0) &= \xi_1 && \text{in } \Omega_+, \\ \phi_1^+(\cdot, 0) &= \xi_2 && \text{on } \Gamma. \end{aligned}\right\}$$

has a unique solution

$$(u^-, u^+, \phi_1^-, \phi_1^+, \phi) \in \mathcal{X}.$$

Note that this already implies uniqueness of any solution to (4.7) and the estimate

$$\|(u^-, u^+, \phi_1^-, \phi_1^+, \phi)\|_{\mathcal{X}} \leq C_1\|y\|_{\tilde{\mathcal{Y}}}$$

is valid with a constant $C_1 > 0$ independent of T.
Together this gives that $(u^-, u^+, \phi^-, \phi^+, \phi)$, with $\phi^\pm = \phi_1^\pm + \phi_2^\pm$, is the unique solution of (4.7) and that the estimate

$$\|(u^-, u^+, \phi^-, \phi^+, \phi)\|_{\mathcal{X}} \leq \|(u^-, u^+, \phi_1^-, \phi_1^+, \phi)\|_{\mathcal{X}} + \|(0, 0, \phi_1^-, \phi_1^+, 0)\|_{\mathcal{X}} \leq (C_1 + C_2)\|y\|_{\tilde{\mathcal{Y}}} = C\|y\|_{\tilde{\mathcal{Y}}}$$

holds with C a constant independent of T ■

Let $y = (f_1^-, f_1^+, f_2^-, f_2^+, g_1^\pm, g_2, \xi_1, \xi_2) \in \tilde{\mathcal{Y}}$. Note that $G_1[h^-, h^+, \delta^-, \delta^+, \delta] = y$ is equivalent to

$$\left.\begin{aligned} \Lambda_{\tau^\pm}^\pm h^\pm &= f_1^\pm && \text{in } \Omega_\pm \times J, \\ \Lambda_{\tau^\pm}^\pm \delta^\pm \pm A(\nabla\tau^\pm)\nabla\delta^\pm \cdot \nabla\tau_{z_n}^\pm &= f_2^\pm && \text{in } \Omega_\pm \times J, \\ h^\pm &= g_1 && \text{on } (\Gamma \cup \Sigma_\pm) \times J, \\ \partial_t\delta - \tilde{\alpha}_1 h_{z_n}^- - \tilde{\beta}_1 h_{z_n}^+ - \tilde{\alpha}_2\delta_{z_n}^- - \tilde{\beta}_2\delta_{z_n}^+ + \zeta\cdot\nabla'\delta &= g_2 && \text{on } \Gamma \times J, \\ h^+(\cdot, 0) &= \xi_1 && \text{in } \Omega_+, \\ \delta^+(\cdot, 0) &= \xi_2 && \text{on } \Gamma. \end{aligned}\right\}$$

We already assumed $\tau \in H_p^2(\Omega \times J)$ (see Equation (4.2)) and by definition of τ (see Equations $(4.1)_4$ and $(4.1)_9$) we have both $\tau|_{\Gamma\times\{0\}} = \eta$ and $\tau|_{\Sigma_\pm \times J} = 0$.

Note that $\tau^\pm$ satisfies $\Lambda_{\tau^\pm}\tau^\pm = 0$, and thus that by definition (see Equations $(4.1)_1$, $(4.1)_5$, $(4.1)_6$ and $(4.1)_7$) u^- and u^+ satisfy Equations (4.8) and (4.9) respectively.

Therefore, we can apply Theorem 4.1 which states that for each $y \in \tilde{\mathcal{Y}}$ the system has a unique solution $(u^-, u^+, \psi^-, \psi^+, \psi) \in \tilde{\mathcal{X}}$. So we can conclude that the function G_1 is an isomorphism. Note that from the final estimate in the theorem it follows that there exists a C independent of J such that $\|G_1^{-1}\| \le C$.

4.4.2 Perturbation: G_2

For an estimate on G_2 Lemma B.9 is used. In the following let any space indicated with a subscript I be the original space restricted to the interval I instead of J.
First, recall $\|G_1^{-1}\| \le C$, where C does not depend on the interval J. In combination with Lemma B.9 this states that for any $2 + n < p' < p$ there exists a T_1' such that on any interval $I \subset (0, T_1')$ we have

$$\|\partial_i u_{z_n}\|_{L_{p'}(\Omega_-\times I)} \le \frac{1}{n^2\|G_1^{-1}\|\|A\|_{L_\infty}}.$$

The above is now used to estimate

$$\begin{aligned}
\|G_2^2[h^\pm, \delta^\pm, \delta]\|_{Y^+} &= \|A(\nabla\tau^+)\nabla\delta^+ \cdot \nabla u_{z_n}^+\|_{L_{p'}(\Omega_+\times I)} \\
&= \sum_{i,j} \|A(\nabla\tau^+)\|_{L_\infty(\Omega_+\times I)}\|\partial_i\delta^+\|_{L_\infty(\Omega_+\times I)}\|\partial_j u_{z_n}^+\|_{L_{p'}(\Omega_+\times I)} \\
&\le \|A\|_{L_\infty}\frac{1}{\|G_1^{-1}\|\|A\|_{L_\infty}}\|\delta\|_{X_I^+} \\
&= \frac{1}{\|G_1^{-1}\|}\|(h^\pm, \delta^\pm, \delta)\|_{\mathcal{X}_I}.
\end{aligned}$$

Analogously, there exists a T_2' such that on any interval $I \subset (0, T_2')$ we estimate

$$\|G_2^1[h^\pm, \delta^\pm, \delta]\|_{L_{p'}(\Omega_-\times I)} \le \frac{1}{\|G_1^{-1}\|}\|(h^\pm, \delta^\pm, \delta)\|_{\mathcal{X}_I}.$$

Next to this, from the assumption $\partial_i u_{z_n} \in W_p^\theta(J, W_p^{-\theta''}(\Omega_-))$ in Equation (4.3) by Lemma B.9 it follows that there exists a T_3' such that on any interval $I \subset (0, T_3')$ it holds that

$$\|\partial_i u_{z_n}\|_{W_{p'}^{\theta'}(I, W_{p'}^{-\theta'}(\Omega_-))} \le \frac{1}{n^2\|G_1^{-1}\|\|A\|_{L_\infty}}.$$

and thus

$$\begin{aligned}
&\|G_2^1[h^\pm, \delta^\pm, \delta]\|_{W_{p'}^{\theta'}(I, W_{p'}^{-\theta'}(\Omega_-))} \\
&= \|-A(\nabla\tau^-)\nabla\delta^- \cdot \nabla u_{z_n}^-\|_{W_{p'}^{\theta'}(I, W_{p'}^{-\theta'}(\Omega_-))} \\
&= \sum_{i,j}\|A(\nabla\tau^-)\|_{L_\infty(\Omega_-\times I)}\|\partial_i\delta^-\|_{L_\infty(\Omega_-\times I)}\|\partial_j u_{z_n}\|_{W_{p'}^{\theta'}(I, W_{p'}^{-\theta'}(\Omega_-))} \\
&\le \|A\|_{L_\infty}\frac{1}{\|G_1^{-1}\|\|A\|_{L_\infty}}\|\delta\|_{X_I^-} \\
&= \frac{1}{\|G_1^{-1}\|}\|(h^\pm, \delta^\pm, \delta)\|_{\mathcal{X}_I}.
\end{aligned}$$

Now define $T' = \min(T_1', T_2', T_3')$ and let $I = (0, T')$. On this interval we can combine the estimates and get $\|G_2\| \le \frac{1}{\|G_1^{-1}\|}$. Now since G_1 is an isomorphism, we have that also $G = G_1 + G_2$ is an isomorphism between spaces on I (instead of J).

4.5 Implicit Function Theorem

Finally, we can apply the Implicit Function Theorem to the equation $F((\mu,\lambda),\tilde{x}) = 0$, where $\tilde{x} = (w^-, w^+, \psi^-, \psi^+, \psi)$.
Since

(i) the mapping $F : \mathcal{U}((0,1),\tilde{x}) \subseteq (\mathbb{R}^n \times \mathbb{R}) \times \tilde{\mathcal{X}}_I \to \mathcal{Y}_I$ is defined on a open neighbourhood $\mathcal{U}((0,1,\tilde{x})$ and $F((0,1),\tilde{x}) = 0$;

(ii) $G = F_{\tilde{x}}$ exists on $\mathcal{U}((0,1),\tilde{x})$ and G is an isomorphism between $(\mathbb{R}^n \times \mathbb{R}) \times \tilde{\mathcal{X}}_I$ and $\mathcal{Y}_I$;

(iii) both F and G are continuous at $((0,1),\tilde{x})$;

(iv) all terms in $F((\mu,\lambda),\tilde{x})$ are either non dependent of or linear in both μ and λ and thus we have that F is smooth in its first argument; also F is infinitely many times Fréchet differentiable with respect to $\tilde{x}$;

we can conclude the following by the Implicit Function Theorem (see Theorem B.6):

(a) Existence and uniqueness: For all (μ,λ) close to $(0,1)$ there exists a unique $\hat{x} : \mathbb{R}^n \times \mathbb{R} \to \tilde{\mathcal{X}}_I$ close to $\tilde{x}$ such that $F((\mu,\lambda),\hat{x}(\mu,\lambda)) = 0$.

(b) Continuous differentiability. Since F is infinitely many times differentiable on a neighbourhood of $((0,1),\tilde{x})$, also $\hat{x}$ is on a neighbourhood of $(0,1)$.

Since we already had $F((\mu,\lambda),\tilde{x}) = 0$, it must hold that $\hat{x}(\mu,\lambda) = \tilde{x}_{\mu,\lambda}$ and thus that $(\mu,\lambda) \mapsto \tilde{x}_{\mu,\lambda}$ is infinitely many times differentiable.

4.6 Conclusion

Theorem 4.2. *Any solution $(u^-, u^+, \tau^-, \tau^+, \tau) \in \mathcal{X}$ of System (4.1) satisfying the assumptions (4.2) and (4.3) are smooth in z_1, ..., z_{n-1} and t on $\Omega_\pm \times (0,T')$. In particular, the interface between the two phases in the original problem is smooth.*

Proof. Let $(z_0,t_0) \in \Omega_- \times (0,T')$ be fixed. We define the function $E_{z_0,t_0} : X_I^+ \to \mathbb{R}$ by $E_{z_0,t_0}(x) = x(z_0,t_0)$ for all $x \in X_I^+$. This gives $E_{z_0,t_0} w^+_{\mu,\lambda} = w^+_{\mu,\lambda}(z_0,t_0) = u^+(z_0 + \mu t_0, \lambda t_0)$. Note that since $X_I^+ \hookrightarrow C((0,T'), C^1(\mathbb{T}^{n-1}))$, we have $|w_{\mu,\lambda}(z_0,t_0)| \leq \|w_{\mu,\lambda}\|_{L^\infty} \leq \|w_{\mu,\lambda}\|_{X_I^+}$ and thus that E_{z_0,t_0} is a bounded linear functional. By Lemma B.2 this gives that E_{x_0,t_0} is infinitely many times differentiable. In combination with $(\mu,\lambda) \mapsto w^+_{\mu,\lambda}$ being infinitely many times differentiable in a neighbourhood of $(0,1)$ we can conclude that $(\mu,\lambda) \mapsto E_{z_0,t_0} w^+_{\mu,\lambda} = u^+(z_0 + \mu t_0, \lambda t_0)$ is infinitely many times differentiable with respect to both μ and λ in a neighbourhood of $(0,1)$. From this we can conclude that also $u^+(x,t)$ must be infinitely many times differentiable in z_1, ..., z_{n-1} and in t around the point $(z,t) = (z_0,t_0)$.
Since T' depends only on p, p', n, $\|u^\pm\|_{X^\pm}$ and A, the whole argument can be repeated on overlapping open intervals until smoothness on the whole interval $(0,T)$ is guaranteed.

Analogously, it can be proven that all of u^-, τ^+, τ^- and τ are smooth in z_1, ..., z_{n-1} and t on $\Omega_\pm \times (0,T')$. Note that this implies that the transformation introduced in Section 2.3 is smooth and thus that the interface between the two phases is smooth. ■

Chapter 5

Parabolic smoothing II: additional linear condition

5.1 Problem statement

In this section let

$$(u^-, u^+, \tau^-, \tau^+, \tau) \in \mathcal{X} = (X^- \times X^+)^2 \times X^B$$

be the unique solution of the problem

$$\left.\begin{aligned}
\tilde{L}^\pm u^\pm &= K^\pm \phi^\pm + R^\pm(u^\pm, \phi^\pm) && \text{in } \Omega_\pm \times J,\\
\hat{L}^\pm \phi^\pm &= 0 && \text{in } \Omega_\pm \times J,\\
u^\pm &= 0 && \text{on } (\Gamma \cup \Sigma_\pm) \times J,\\
\phi^\pm - \phi &= 0 && \text{on } \Gamma \times J,\\
\phi^\pm &= 0 && \text{on } \Sigma_\pm \times J,\\
\partial_t \phi - \alpha^- \phi^-_{z_n} - \alpha^+ \phi^+_{z_n} & &&\\
+\zeta \cdot \nabla' \phi - \tilde{\alpha}^- u^-_{z_n} - \tilde{\alpha}^+ u^+_{z_n} &= g_0 + R^B(u^-, u^+, \phi^-, \phi^+, \phi) && \text{on } \Gamma \times J,\\
u^+(\cdot, 0) &= 0 && \text{in } \Omega_+,\\
\phi^+(\cdot, 0) &= 0 && \text{in } \bar{\Omega}_+.
\end{aligned}\right\} \quad (5.1)$$

Note that this is System (2.15) together with Equations (2.16) and (2.17). Theorem 2.1 states that this system indeed has a unique solution. For convenience we denoted p and v by u^- and u^+ and Q and V by U^- and U^+.

First of all, recall that U^- is defined by the elliptic system in Equation (2.13) and that U^+ is defined by the parabolic system in Equation (2.14). Since these systems are linear parabolic or elliptic problems with smooth coefficients, it follows from standard regularity theory that both U^- and U^+ are smooth. It is also possible to apply Angenent's parameter trick to both of the systems to prove this. This proof is very similar to the other proofs in this report and thus no details are discussed here.

As already mentioned, the system is slightly different from the one in the previous chapter. This is discussed in more detail in Chapter 6. However, the same techniques are used to prove smoothness of the solution. So we introduce shifted version of the unknowns $u^\pm$, $\phi^\pm$ and ϕ in Section 5.2 and the system is written in the form $F((\mu, \lambda), (w^-_{\mu,\lambda}, w^+_{\mu,\lambda}, \psi^-_{\mu,\lambda}, \psi^+_{\mu,\lambda}, \psi_{\mu,\lambda})) = 0$. The Fréchet derivative G of F in $((0, 1), (u^-, u^+, \phi^-, \phi^+, \phi))$ are calculated in Section 5.3 and in Section 5.4 we prove that G is an isomorphism between the appropriate spaces. All of the above is used to apply the Implicit Function Theorem to the equation $F((\mu, \lambda), (w^-_{\mu,\lambda}, w^+_{\mu,\lambda}, \psi^-_{\mu,\lambda}, \psi^+_{\mu,\lambda}, \psi_{\mu,\lambda})) = 0$ in Section 5.5. Finally, we reach the conclusion in Section 5.6.

In this chapter we need the assumption

$$v \in W_p^{\theta}(J, W_p^{-\theta''}(\Omega_-)) \text{ for all } v \in \{U^-_{z_n}, \partial_i U^-_{z_n}, u^-_{z_n}, \partial_i u^-_{z_n}, \partial_i \phi^-\} \text{ and } i \in \{1, ..., n\}. \tag{5.2}$$

5.2 Setup parameter trick

5.2.1 Definitions

Let $\lambda \in \mathbb{R}$, $\mu' \in \mathbb{R}^{n-1}$ and $\mu = (\mu', 0) \in \mathbb{R}^n$. Then define

$$\begin{aligned}
w^{\pm}_{\mu,\lambda}(z,t) &:= u^{\pm}(z + \mu t, \lambda t), \\
\psi^{\pm}_{\mu,\lambda}(z,t) &:= \phi^{\pm}(z + \mu t, \lambda t), \\
\psi_{\mu,\lambda}(z,t) &:= \phi(z + \mu t, \lambda t), \\
\sigma_{\mu,\lambda}(z,t) &:= \sigma(z + \mu t, \lambda t), \\
U^{\pm}_{\mu,\lambda}(z,t) &:= U^{\pm}(z + \mu t, \lambda t).
\end{aligned}$$

Note that for $x \in \Omega_-$ also $z + \mu t = (z' + \mu' t, z_n) \in \Omega_-$ for all $t \in \mathbb{R}$. The same holds for Ω_+, Γ, $\Sigma-$ and Σ_+.

In the following we use that substitutions gives

$$\begin{aligned}
\left(\mathcal{A}_{\sigma_{\mu,\lambda}} w^{\pm}_{\mu,\lambda}\right)(z,t) &= \left(\mathcal{A}_{\sigma} u^{\pm}\right)(z + \mu t, \lambda t), \\
\left(\mathcal{A}_{\sigma_{\mu,\lambda}} \psi^{\pm}_{\mu,\lambda}\right)(z,t) &= \left(\mathcal{A}_{\sigma} \phi^{\pm}\right)(z + \mu t, \lambda t).
\end{aligned}$$

Therefore, we define $\tilde{L}^-_{\mu,\lambda}$ by

$$\begin{aligned}
\left(\tilde{L}^-_{\mu,\lambda} w^-_{\mu,\lambda}\right)(z,t) &:= \left(\mathcal{A}_{\sigma_{\mu,\lambda}} w_{\mu,\lambda} - \frac{w_{\mu,\lambda,z_n}}{1 + \sigma_{\mu,\lambda,z_n}} \mathcal{A}_{\sigma_{\mu,\lambda}} \sigma_{\mu,\lambda}\right)(z,t) \\
&= \left(\mathcal{A}_{\sigma} w - \frac{w_{z_n}}{1 + \sigma_{z_n}} \mathcal{A}_{\sigma} \sigma\right)(z + \mu t, \lambda t) \\
&= \left(\tilde{L}^- w^-\right)(z + \mu t, \lambda t).
\end{aligned}$$

Analogously to this, $\hat{L}^-_{\mu,\lambda}$, $K^-_{\mu,\lambda}$ $R^{\pm}_{\mu,\lambda}$ and $R^B_{\mu,\lambda}$ are defined. Note that this gives

$$\begin{aligned}
\left(K^-_{\mu,\lambda} \psi^-_{\mu,\lambda}\right)(z,t) &= \left(K^- \phi^-\right)(z + \mu t, \lambda t), \\
\left(R^{\pm}_{\mu,\lambda}(w^{\pm}_{\mu,\lambda}, \psi^{\pm}_{\mu,\lambda})\right)(z,t) &= \left(R^{\pm}(u^{\pm}, \phi^{\pm})\right)(z + \mu t, \lambda t), \\
\left(R^B_{\mu,\lambda}(w^-_{\mu,\lambda}, w^+_{\mu,\lambda}, \psi^-_{\mu,\lambda}, \psi^+_{\mu,\lambda}, \psi_{\mu,\lambda})\right)(z,t) &= \left(R^B(u^-, u^+, \phi^-, \phi^+, \phi)\right)(z + \mu t, \lambda t).
\end{aligned}$$

Finally, since

$$\partial_t \sigma_{\mu,\lambda}(z,t) = \mu \cdot \nabla \sigma_{\mu,\lambda}(z,t) + \lambda \partial_t \sigma(z + \mu t, \lambda t),$$

we have

$$\left(\mathcal{L}_{\sigma} \sigma\right)(z + \mu t, \lambda t) = \left(\partial_t \sigma - \mathcal{A}_{\sigma} \sigma\right)(z + \mu t, \lambda t) \quad = \left(\frac{1}{\lambda}(\partial_t \sigma_{\mu,\lambda} - \mu \cdot \nabla \sigma_{\mu,\lambda}) - \mathcal{A}_{\sigma_{\mu,\lambda}} \sigma_{\mu,\lambda}\right)(z,t),$$

and thus define

$$
\begin{aligned}
\left(K^+_{\mu,\lambda}\psi^+_{\mu,\lambda}\right)(z,t) &= \left(K^+\phi^+\right)(z+\mu t,\lambda t),\\
&= \left(A(\nabla\sigma)\nabla\phi^+\cdot\nabla U^+_{z_n} - \frac{U^+_{z_n}}{(1+\sigma_{z_n})^2}\phi^+_{z_n}\mathcal{L}_\sigma\sigma\right)(z+\mu t,\lambda t)\\
&= \Bigg(A(\nabla\sigma_{\mu,\lambda})\nabla\phi^+_{\mu,\lambda}\cdot\nabla U^+_{\mu,\lambda,z_n}\\
&\qquad -\frac{U^+_{\mu,\lambda,z_n}}{(1+\sigma_{\mu,\lambda,z_n})^2}\phi^+_{\mu,\lambda,z_n}\left(\frac{1}{\lambda}(\partial_t\sigma_{\mu,\lambda}-\mu\cdot\nabla\sigma_{\mu,\lambda}) - \mathcal{A}_{\sigma_{\mu,\lambda}}\sigma_{\mu,\lambda}\right)\Bigg)(z,t)
\end{aligned}
$$

and

$$
\begin{aligned}
g_{0,\mu,\lambda}(z,t) &= g_0(z+\mu t,\lambda t)\\
&= \left(\frac{1}{\gamma} - \partial_t\sigma + \frac{1+|\nabla'\sigma|^2}{1+\sigma_{z_n}}(-\alpha U^-_{z_n} + \beta U^+_{z_n})\right)(z+\mu t,\lambda t)\\
&= \left(\frac{1}{\gamma} - \frac{1}{\lambda}\left(\partial_t\sigma_{\mu,\lambda} - \mu\cdot\nabla\sigma_{\mu,\lambda}\right) + \frac{1+|\nabla'\sigma_{\mu,\lambda}|^2}{1+\sigma_{\mu,\lambda,z_n}}(-\alpha U^-_{\mu,\lambda,z_n} + \beta U^+_{\mu,\lambda,z_n})\right)(z,t).
\end{aligned}
$$

5.2.2 Shifted equations

Note that Equation $(5.1)_1$ for the lower (vapour) phase can be written as

$$
\partial_t u^+ - \mathcal{A}_\sigma u^+ - \frac{u^+_{z_n}}{1+\sigma_{z_n}}\Lambda^+_\sigma\sigma = K^+\phi^+ + R^+(u^+\phi^+)
$$

so for all $(z,t)\in\Omega_+\times J$ we have, using some of the definitions above, that

$$
\begin{aligned}
\partial_t u^+(z+\mu t,\lambda t) &= \left(\mathcal{A}_\sigma u^+ + \frac{u^+_{z_n}}{1+\sigma_{z_n}}\Lambda^+_\sigma\sigma + K^+\phi^+ + R^+(u^+,\phi^+)\right)(z+\mu t,\lambda t)\\
&= \left(\mathcal{A}_{\sigma_{\mu,\lambda}}w^+_{\mu,\lambda} + \frac{w^+_{\mu,\lambda,z_n}}{1+\sigma_{\mu,\lambda,z_n}}\Lambda^+_{\sigma_{\mu,\lambda}}\sigma_{\mu,\lambda} + K^+_{\mu,\lambda}\psi^+_{\mu,\lambda} + R^+_{\mu,\lambda}(w^+_{\mu,\lambda},\psi^+_{\mu,\lambda})\right)(z,t),
\end{aligned}
$$

which gives

$$
\begin{aligned}
\partial_t w^+_{\mu,\lambda}(z,t) &= \mu\cdot\nabla u^+(z+\mu t,\lambda t) + \lambda\partial_t u^+(z+\mu t,\lambda t)\\
&= \mu\cdot\nabla w^+_{\mu,\lambda}(z,t) + \lambda\Bigg(\mathcal{A}_{\sigma_{\mu,\lambda}}w^+_{\mu,\lambda} + \frac{w^+_{\mu,\lambda,z_n}}{1+\sigma_{\mu,\lambda,z_n}}\Lambda^+_{\sigma_{\mu,\lambda}}\sigma_{\mu,\lambda}\\
&\qquad + K^+_{\mu,\lambda}\psi^+_{\mu,\lambda} + R^+_{\mu,\lambda}(w^+_{\mu,\lambda},\psi^+_{\mu,\lambda})\Bigg)(z,t)
\end{aligned}
$$

and thus we can conclude that

$$
\partial_t w^+_{\mu,\lambda} - \mu\cdot\nabla w^+_{\mu,\lambda} - \lambda\mathcal{A}_{\sigma_{\mu,\lambda}}w^+_{\mu,\lambda} - \lambda\frac{w^+_{\mu,\lambda,z_n}}{1+\sigma_{\mu,\lambda,z_n}}\Lambda^+_{\sigma_{\mu,\lambda}}\sigma_{\mu,\lambda} = \lambda K^+_{\mu,\lambda}\psi^+_{\mu,\lambda} + \lambda R^+_{\mu,\lambda}(w^+_{\mu,\lambda},\psi^+_{\mu,\lambda}).
$$

Analogously, using Equation $(5.1)_2$ in the vapour phase, we get

$$
\partial_t\phi^+ - \mathcal{A}_\sigma\phi^+ \pm A(\nabla\sigma)\nabla\phi^+\nabla\sigma_{z_n} = 0,
$$

so for all $(z,t)\in\Omega_+\times J$ we have, again using the definitions in the previous subsection, that

$$
\begin{aligned}
\partial_t\phi^+(z+\mu t,\lambda t) &= \left(\mathcal{A}_{\sigma_{\mu,\sigma}}\phi^+ \pm A(\nabla\sigma)\nabla\phi^+\nabla\sigma_{z_n}\right)(z+\mu t,\lambda t)\\
&= \left(\mathcal{A}_{\sigma_{\mu,\sigma}}\psi^+_{\mu,\lambda} \pm A(\nabla\sigma_{\mu,\lambda})\nabla\psi^+_{\mu,\lambda}\nabla\sigma_{\mu,\lambda,z_n}\right)(z,t),
\end{aligned}
$$

which gives

$$
\begin{aligned}
\partial_t \psi^+_{\mu,\lambda}(z,t) &= \left(\mu \cdot \nabla\phi^+ + \lambda\partial_t\phi^+\right)(z+\mu t, \lambda t) \\
&= \left(\mu \cdot \nabla\psi^+_{\mu,\lambda} + \lambda\left(\mathcal{A}_{\sigma_{\mu,\sigma}}\psi^+_{\mu,\lambda} \pm A(\nabla\sigma_{\mu,\lambda})\nabla\psi^+_{\mu,\lambda}\nabla\sigma_{\mu,\lambda,z_n}\right)\right)(z,t)
\end{aligned}
$$

and thus we conclude here that

$$
\partial_t\psi^+_{\mu,\lambda} - \mu\cdot\nabla\psi^+_{\mu,\lambda} - \lambda\mathcal{A}_{\sigma_{\mu,\lambda}}\psi^+_{\mu,\lambda} \pm \lambda A(\nabla\sigma_{\mu,\lambda})\nabla\psi^+_{\mu,\lambda}\nabla\sigma_{\mu,\lambda,z_n} = 0.
$$

Finally, Equation $(5.1)_6$ gives for all $(z,t) \in \Gamma$ that

$$
\begin{aligned}
&\partial_t\phi(z+\mu t, \lambda t) \\
&= \left(\alpha^-\phi^-_{z_n} + \alpha^+\phi^+_{z_n} - \zeta\cdot\nabla'\phi + \tilde{\alpha}^- u^-_{z_n} + \tilde{\alpha}^+ u^+_{z_n} + g_0 + R^B(u^-, u^+, \phi^-, \phi^+, \phi)\right)(z+\mu, \lambda t) \\
&= \Big(\alpha^-_{\mu,\lambda}\psi^-_{\mu,\lambda,z_n} + \alpha^+_{\mu,\lambda}\psi^+_{\mu,\lambda,z_n} - \zeta_{\mu,\lambda}\cdot\nabla'\psi_{\mu,\lambda} + \tilde{\alpha}^-_{\mu,\lambda} w^-_{\mu,\lambda,z_n} + \tilde{\alpha}^+_{\mu,\lambda} w^+_{\mu,\lambda,z_n} \\
&\qquad\qquad + g_{0,\mu,\lambda} + R^B_{\mu,\lambda}(w^-_{\mu,\lambda}, w^+_{\mu,\lambda}, \psi^-_{\mu,\lambda}, \psi^+_{\mu,\lambda}, \psi_{\mu,\lambda})\Big)(z,t).
\end{aligned}
$$

Together with

$$
\partial_t\psi_{\mu,\lambda}(z,t) = \mu\cdot\nabla\phi(x+\mu t, \lambda t) + \lambda\partial_t\phi(z+\mu t, \lambda t)
$$

we here get

$$
\begin{aligned}
\partial_t\psi_{\mu,\lambda} - \mu\cdot\nabla\psi_{\mu,\lambda} - \lambda\left(\alpha^-_{\mu,\lambda}\psi^-_{\mu,\lambda,z_n} + \alpha^+_{\mu,\lambda}\psi^+_{\mu,\lambda,z_n} - \zeta_{\mu,\lambda}\cdot\nabla'\psi_{\mu,\lambda} + \tilde{\alpha}^-_{\mu,\lambda} w^-_{\mu,\lambda,z_n} + \tilde{\alpha}^+_{\mu,\lambda} w^+_{\mu,\lambda,z_n}\right) \\
= \lambda\left(g_{0,\mu,\lambda} + R^B_{\mu,\lambda}(w^-_{\mu,\lambda}, w^+_{\mu,\lambda}, \psi^-_{\mu,\lambda}, \psi^+_{\mu,\lambda}, \psi_{\mu,\lambda})\right).
\end{aligned}
$$

Substitution gives all other results and the entire system can be rewritten in terms of the shifted variables. For convenience in the following sections we drop the subscripts μ and λ. Together this gives

$$
\left.
\begin{aligned}
\tilde{L}^- w^- &= K^-\psi^- + R^-(\psi^-, w^-) && \text{in } \Omega_- \times J, \\
\partial_t w^+ - \mu\cdot\nabla w^+ - \lambda\mathcal{A}_\sigma w^+ - \lambda\frac{w^+_{z_n}}{1+\sigma_{z_n}}\Lambda^+_\sigma\sigma &= \lambda K^+\psi^+ + \lambda R^+(\psi^+, w^+) && \text{in } \Omega_+ \times J, \\
\hat{L}^-\psi^- &= 0 && \text{in } \Omega_- \times J, \\
\partial_t\psi^+ - \mu\cdot\nabla\psi^+ - \lambda\mathcal{A}_\sigma\psi^+ \pm \lambda A(\nabla\sigma)\nabla\psi^+\nabla\sigma_{z_n} &= 0 && \text{in } \Omega_+ \times J, \\
w^\pm &= 0 && \text{on } (\Gamma\cup\Sigma_\pm)\times J, \\
\psi^\pm - \psi &= 0 && \text{on } \Gamma \times J, \\
\psi^\pm &= 0 && \text{on } \Sigma_\pm \times J, \\
\partial_t\psi - \mu\cdot\nabla\psi - \lambda\big(\alpha^-\psi^-_{z_n} + \alpha^+\psi^+_{z_n} \qquad & && \\
-\zeta\cdot\nabla'\psi + \tilde{\alpha}^- w^-_{z_n} + \tilde{\alpha}^+ w^+_{z_n}\big) &= \lambda\left(g_0 + R^B(\psi^-, \psi^+, \psi, w^-, w^+)\right) && \text{on } \Gamma \times J, \\
w^+(\cdot, 0) &= 0 && \text{in } \Omega_+, \\
\psi^+(\cdot, 0) &= 0 && \text{in } \bar{\Omega}_+.
\end{aligned}
\right\}
$$

Now we define the spaces

$$
\begin{aligned}
\tilde{\mathcal{X}} &:= \{(w^-, w^+, \psi^-, \psi^+, \psi) \in \mathcal{X} \mid w^\pm = 0 \text{ on } \Gamma\cup\Sigma_\pm,\ \ \psi^\pm = \psi \text{ on } \Gamma, \\
&\qquad\qquad \psi^\pm = 0 \text{ on } \Sigma_\pm,\ \ \gamma_0 w^+ = 0,\ \ \gamma_0\psi^+ = 0\}, \\
\tilde{\mathcal{Y}} &= Y^- \times Y^+ \times Y^- \times Y^+ \times Y^B,
\end{aligned}
$$

which gives that $(w^-, w^+, \psi^-, \psi^+, \psi)$ satisfies

$$
F((\mu,\lambda), (w^-, w^+, \psi^-, \psi^+, \psi)) = 0,
$$

with

$$F : (\mathbb{R}^n \times \mathbb{R}) \times \tilde{\mathcal{X}} \to \tilde{\mathcal{Y}}$$

given by

$$F((\mu,\lambda),(w^-,w^+,\psi^-,\psi^+,\psi)) = \begin{pmatrix} \tilde{L}^- w^- - K^-\psi^- - R^-(w^-,\psi^-) \\ \partial_t w^+ - \mu\cdot\nabla w^+ - \lambda\mathcal{A}_\sigma w^+ - \lambda\frac{w^+_{z_n}}{1+\sigma_{z_n}}\Lambda^+_\sigma\sigma - \lambda K^+\psi^+ - \lambda R^+(w^+,\psi^+) \\ \hat{L}^-\psi^- \\ \partial_t\psi^+ - \mu\cdot\nabla\psi^+ - \lambda\mathcal{A}_\sigma\psi^+ \pm \lambda A(\nabla\sigma)\nabla\psi^+\nabla\sigma_{z_n} \\ \partial_t\psi - \mu\cdot\nabla\psi - \lambda\left(\alpha^-\psi^-_{z_n} + \alpha^+\psi^+_{z_n} - \zeta\cdot\nabla'\psi + \tilde{\alpha}^- w^-_{z_n} + \tilde{\alpha}^+ w^+_{z_n}\right) \\ -\lambda\left(g_0 + R^B(w^-,w^+,\psi^-,\psi^+,\psi)\right) \end{pmatrix}.$$

5.3 Differentiability of F

Like in the model problem and the previous chapter, the next step is to investigate Fréchet differentiability of F and to determine an explicit formula for the first derivative. Note that for $R^\pm$ and R^B the derivatives are very elaborate and therefore not fully written out here. They are, however, stated in Section B.1.2 of the appendix for completeness.

5.3.1 Partial derivatives of F

First of all, note that all terms either do not depend on w^- or are linear in w^-. In the first case the derivative of the term with respect to w^- is just zero. In the second case Lemma B.2 can be used to determine the derivative. Since differentiation is a linear operator, the results can easily be combined, which gives that F is infinitely many differential with respect to w^- and that the first derivative is equal to

$$D_{w^-}F((\mu,\lambda),(u^-,u^+,\phi^-,\phi^+,\phi))[h^-] = \begin{pmatrix} \tilde{L}^- h^- - D_{w^-}R^-(u^-,\phi^-)[h^-] \\ 0 \\ 0 \\ 0 \\ -\lambda\tilde{\alpha}^- h^-_{z_n} - \lambda D_{w^-}R^B(u^-,u^+,\phi^-,\phi^+,\phi)[h^-] \end{pmatrix}.$$

Secondly, all terms are also independent on w^+ or linear in w^+. Using the same reasoning as above, we can conclude that F is infinitely many differential with respect to w^+ and that the first derivative is equal to

$$D_{w^+}F((\mu,\lambda),(u^-,u^+,\phi^-,\phi^+,\phi))[h^+] = \begin{pmatrix} 0 \\ \partial_t h^+ - \mu\cdot\nabla h^+ - \lambda\mathcal{A}_\sigma h^+ - \lambda\frac{h^+_{z_n}}{1+\sigma_{z_n}}\Lambda^+_\sigma\sigma - \lambda D_{w^+}R^+(u^+,\phi^+)[h^+] \\ 0 \\ 0 \\ -\lambda\tilde{\alpha}^+ h^+_{z_n} - \lambda D_{w^+}R^B(u^-,u^+,\phi^-,\phi^+,\phi)[h^+] \end{pmatrix}.$$

For the derivative of the first and third equation in F with respect to ψ^- the examples in Section B.1.1 are used. All other terms, with exception of R^- and R^B, do either not depend on ψ^- or are linear in ψ^-. From this we can conclude that F is infinitely many times differentiable with respect to ψ^- and that the first derivative is equal to

$$D_{\psi^-}F((\mu,\lambda),(u^-,u^+,\phi^-,\phi^+,\phi))[\delta^-] = \begin{pmatrix} -K^-\delta^- - D_{\psi^-}R^-(u^-,\phi^-)[\delta^-] \\ 0 \\ \hat{L}^-\delta^- \\ 0 \\ -\lambda\alpha^-\delta^-_{z_n} - \lambda D_{\psi^-}R^B(u^-,u^+,\phi^-,\phi^+,\phi)[\delta^-] \end{pmatrix}.$$

Using the same arguments as for ψ^-, we get that F is infinitely many times differentiable with respect to ψ^+ and that the first derivative is equal to

$$D_{\psi^+}F((\mu,\lambda),(u^-,u^+,\phi^-,\phi^+,\phi))[\delta^+] = \begin{pmatrix} 0 \\ -\lambda K^+\delta^+ - \lambda D_{\psi^+}R^+(u^+,\phi^+)[\delta^+] \\ 0 \\ \partial_t\delta^+ - \mu\cdot\nabla\delta^+ - \lambda\mathcal{A}_\sigma\delta^+ \pm \lambda A(\nabla\sigma)\nabla\delta^+\nabla\sigma_{z_n} \\ -\lambda\alpha^+\delta^+_{z_n} - \lambda D_{\psi^+}R^B(u^-,u^+,\phi^-,\phi^+,\phi)[\delta^+] \end{pmatrix}.$$

Finally, using Lemma B.2 and the last example in Section B.1.1, we get that F is infinitely many times differentiable with respect to ψ and that the first derivative is equal to

$$D_\psi F((\mu,\lambda),(u^-,u^+,\phi^-,\phi^+,\phi))[\delta] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \partial_t\delta - \mu\cdot\nabla\delta + \lambda\zeta\cdot\nabla'\delta - \lambda D_\psi R^B(u^-,u^+,\phi^-,\phi^+,\phi)[\delta] \end{pmatrix}.$$

5.3.2 Total derivative of F with respect to its second argument

Using the results above, we get that F is infinitely many times differentiable and after substitution of $(\mu,\lambda)=(0,1)$ we get

$$\begin{aligned}
G[h^-,h^+,\delta^-,\delta^+,\delta] &= F'((0,1),(u^-,u^+,\phi^-,\phi^+,\phi))[h^-,h^+,\delta^-,\delta^+,\delta] \\
&= \begin{pmatrix} \tilde{L}^-h^- - K^-\delta^- - {R^-}'(u^-,\phi^-)[h^-,\delta^-] \\ \partial_t h^+ - \mathcal{A}_\sigma h^+ - \frac{h^+_{z_n}}{1+\sigma_{z_n}}\Lambda^+_\sigma\sigma - K^+\delta^+ - {R^+}'(u^+,\phi^+)[h^+\delta^+] \\ \hat{L}^-\delta^- \\ \partial_t\delta^+ - \mathcal{A}_\sigma\delta^+ \pm A(\nabla\sigma)\nabla\delta^+\nabla\sigma_{z_n} \\ \partial_t\delta + \tilde{\alpha}^-h^-_{z_n} + \tilde{\alpha}^+w^+_{z_n} - \alpha^-\delta^-_{z_n} - \alpha^+\delta^+_{z_n} + \zeta\cdot\nabla'\delta - {R^B}'(u^-,u^+,\phi^-,\phi^+,\phi)[h^-,h^+,\delta^-,\delta^+,\delta] \end{pmatrix} \\
&= \begin{pmatrix} \tilde{L}^-h^- - K^-\delta^- - {R^-}'(u^-,\phi^-)[h^-,\delta^-] \\ \Lambda^+_\sigma h^+ - \frac{h^+_{z_n}}{1+\sigma_{z_n}}\Lambda^+_\sigma\sigma - K^+\delta^+ - {R^+}'(u^+,\phi^+)[h^+\delta^+] \\ \hat{L}^-\delta^- \\ \Lambda^+_\sigma\delta^+ \pm A(\nabla\sigma)\nabla\delta^+\nabla\sigma_{z_n} \\ \partial_t\delta + \tilde{\alpha}^-h^-_{z_n} + \tilde{\alpha}^+w^+_{z_n} - \alpha^-\delta^-_{z_n} - \alpha^+\delta^+_{z_n} + \zeta\cdot\nabla'\delta - {R^B}'(u^-,u^+,\phi^-,\phi^+,\phi)[h^-,h^+,\delta^-,\delta^+,\delta] \end{pmatrix} \\
&= \begin{pmatrix} \tilde{L}^-h^- - K^-\delta^- - {R^-}'(u^-,\phi^-)[h^-,\delta^-] \\ \tilde{L}^+h^+ - K^+\delta^+ - {R^+}'(u^+,\phi^+)[h^+\delta^+] \\ \hat{L}^-\delta^- \\ \hat{L}^+\delta^+ \\ \partial_t\delta - \tilde{\alpha}^-h^-_{z_n} - \tilde{\alpha}^+w^+_{z_n} - \alpha^-\delta^-_{z_n} - \alpha^+\delta^+_{z_n} + \zeta\cdot\nabla'\delta - {R^B}'(u^-,u^+,\phi^-,\phi^+,\phi)[h^-,h^+,\delta^-,\delta^+,\delta] \end{pmatrix}.
\end{aligned}$$

5.3.3 Derivative of F with respect to its first argument

In order to apply the Implicit Function Theorem, we require some information about differentiability of F with respect to μ and λ. Note that all terms in F can be written as

$$\xi\left(\mu,\lambda,\sigma_{\mu,\lambda},U^\pm_{\mu,\lambda}\right),$$

where ξ is smooth in each of its arguments. Since σ is smooth (see (2.9)) for all $t>0$ and $\sigma_{\mu,\lambda}$ is defined by $\sigma_{\mu,\lambda}(x,t)=\sigma(x+\mu t,\lambda t)$, we have that also the map $(\mu,\lambda)\mapsto\sigma_{\mu,\lambda}$ is smooth for $t>0$. The same holds for the map $(\mu,\lambda)\mapsto U^\pm_{\mu,\lambda}$. Therefore, we have that g is infinitely many times differentiable with respect to μ and λ. Now since this holds for all terms in F, we have that also

F is smooth in its first argument.

Take fore example the term $\tilde{L}^- h^-$. This can be written $\xi\left(\mu, \lambda, \sigma_{\mu,\lambda}, U^-_{\mu,\lambda}\right)$ where $\xi : \mathbb{R}^n \times \mathbb{R} \times C^\infty(\Omega_-) \times C^\infty(\Omega_-) \to Y^-$ is given by

$$\xi(x, y, f, g) = \Lambda^-_f h^- - \frac{h_{z_n}}{1+f} \Lambda^-_f f$$

Note that ξ is only dependent on the third variable and from the examples in Section B.1.1 it follows that ξ is infinitely many times differentiable with respect to this argument. Since $\sigma_{\mu,\lambda}$ depends smoothly on μ and λ, we can apply a chain rule to prove that $(\mu, \lambda) \mapsto \xi(\mu, \lambda, \sigma_{\mu,\lambda}, U^-_{\mu,\lambda})$ is a smooth function from $\mathbb{R}^n \times \mathbb{R}$ to Y^-. Note that Lemma B.4 cannot be applied directly in this case, since it requires linearity of one of the functions, but the proof can be altered to hold for smooth functions.

5.4 Isomorphism property of F'

In this section the main goal is to prove that G is an isomorphism between appropriate spaces. In order to do so, we use that $G = G_1 + G_2$ with

$$G_1[h^-, h^+, \delta^-, \delta^+, \delta] = \begin{pmatrix} \tilde{L}^- h^- \\ \tilde{L}^+ h^+ \\ \hat{L}^- \delta^- \\ \hat{L}^+ \delta^+ \\ \partial_t \delta + \tilde{\alpha}^- h^-_{z_n} + \tilde{\alpha}^+ w^+_{z_n} - \alpha^- \delta^-_{z_n} - \alpha^+ \delta^+_{z_n} + \zeta \cdot \nabla' \delta \end{pmatrix}$$

and

$$G_2[h^-, h^+, \delta^-, \delta^+, \delta] = \begin{pmatrix} -K^- \delta^- - {R^-}'(u^-, \phi^-)[h^-, \delta^-] \\ -K^+ \delta^+ - {R^+}'(u^+, \phi^+)[h^+ \delta^+] \\ 0 \\ 0 \\ -{R^B}'(u^-, u^+, \phi^-, \phi^+, \phi)[h^-, h^+, \delta^-, \delta^+, \delta] \end{pmatrix}.$$

The approach we use, is to first prove that G_1 is an isomorphism between $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$ and then treat G_2 as a perturbation term.

5.4.1 Main part: G_1

In the previous chapter we introduced a variation on Theorem 2.2 to prove that the main part G_1 of G was an isomorphism. Here, we introduce a similar theorem for the same purpose.

Theorem 5.1. *Let*

$$\left.\begin{aligned} \tilde{L}^\pm u^\pm &= f_1^\pm && \text{in } \Omega_\pm \times J, \\ \hat{L}^\pm \phi^\pm &= f_2^\pm && \text{in } \Omega_\pm \times J, \\ u^\pm &= 0 && \text{on } (\Gamma \cup \Sigma_\pm) \times J, \\ \phi^\pm - \phi &= 0 && \text{in } \Omega_\pm \times J, \\ \phi^\pm &= 0 && \text{on } \Sigma_\pm \times J, \\ \partial_t \phi - \hat{\alpha}_2 \phi^-_{z_n} - \hat{\beta}_2 \phi^+_{z_n} + \hat{\zeta} \cdot \nabla' \phi - \hat{\alpha}_1 u^-_{z_n} - \hat{\beta}_1 u^+_{z_n} &= g && \text{on } \Gamma \times J, \\ u^+(\cdot, 0) &= 0 && \text{in } \Omega_+, \\ \phi^+(\cdot, 0) &= 0 && \text{on } \Gamma. \end{aligned}\right\} \quad (5.4)$$

For each $(f_1^-, f_1^+, f_2^-, f_2^+, g) \in \tilde{\mathcal{Y}}$ *the system above has a unique solution*

$$(u^-, u^+, \phi^-, \phi^+, \phi) \in \mathcal{X}$$

and the estimate

$$\|(u^-, u^+, \phi^-, \phi^+, \phi)\|_{\mathcal{X}} \leq C\|(f_1^-, f_1^+, f_2^-, f_2^+, g)\|_{\tilde{\mathcal{Y}}}$$

is valid with a constant $C > 0$ *independent of* T.

Proof. Let $y = (f_1^-, f_1^+, f_2^-, f_2^+, g) \in \tilde{\mathcal{Y}}$ be fixed.
As in Theorem 4.1 above we have that the system

$$\left.\begin{aligned} \hat{L}^{\pm}\phi_2^{\pm} &= f_2^{\pm} && \text{in } \Omega_{\pm} \times J, \\ \phi_2^{\pm} &= 0 && \text{on } (\Sigma_{\pm} \cup \Gamma) \times J, \\ \phi_2^{+}(\cdot, 0) &= 0 && \text{on } \Gamma \end{aligned}\right\}$$

has a unique solution $\|(\phi_2^-, \phi_2^+)\| \in X^- \times X^+$ with estimate

$$\|(\phi_1^-, \phi_1^+)\|_{X^- \times X^+} \leq C_2\|(f_2^-, f_2^+)\|_{Y^- \times Y^+} \leq C_2\|y\|_{\tilde{\mathcal{Y}}}$$

where C_2 is a constant independent of T.
Secondly, since $(\hat{\alpha}_2\phi_{2,z_n}^- + \hat{\beta}_2\phi_{2,z_n}^+)|_\Gamma \in Y^B$, by Theorem 2.2 the system

$$\left.\begin{aligned} \tilde{L}^{\pm}u^{\pm} &= f_1^{\pm} && \text{in } \Omega_{\pm} \times J, \\ \hat{L}^{\pm}\phi_1^{\pm} &= 0 && \text{in } \Omega_{\pm} \times J, \\ u^{\pm} &= 0 && \text{on } (\Gamma \cup \Sigma_{\pm}) \times J, \\ \phi_1^{\pm} - \phi &= 0 && \text{in } \Omega_{\pm} \times J, \\ \phi_1^{\pm} &= 0 && \text{on } \Sigma_{\pm} \times J, \\ \partial_t\phi - \hat{\alpha}_2\phi_{1,z_n}^- - \hat{\beta}_2\phi_{1,z_n}^+ & && \\ +\hat{\zeta} \cdot \nabla'\phi - \hat{\alpha}_1 u_{z_n}^- - \hat{\beta}_1 u_{z_n}^+ &= \phi_2 + \hat{\alpha}_2\phi_{2,z_n}^- + \hat{\beta}_2\phi_{2,z_n}^+ && \text{on } \Gamma \times J, \\ u^+(\cdot, 0) &= 0 && \text{in } \Omega_+, \\ \phi_1^+(\cdot, 0) &= 0 && \text{on } \Gamma. \end{aligned}\right\}$$

has a unique solution

$$(u^-, u^+, \phi_1^-, \phi_1^+, \phi) \in \mathcal{X}.$$

Note that this already implies uniqueness of any solution to (5.4) and that the estimate

$$\|(u^-, u^+, \phi_1^-, \phi_1^+, \phi)\|_{\mathcal{X}} \leq C_1\|y\|_{\tilde{\mathcal{Y}}}$$

is valid with a constant $C_1 > 0$ independent of T.
Together this gives that $(u^-, u^+, \phi^-, \phi^+, \phi)$, with $\phi^{\pm} = \phi_1^{\pm} + \phi_2^{\pm}$, is the unique solution of (5.4) and that the estimate

$$\|(u^-, u^+, \phi^-, \phi^+, \phi)\|_{\mathcal{X}} \leq \|(u^-, u^+, \phi_1^-, \phi_1^+, \phi)\|_{\mathcal{X}} + \|(0, 0, \phi_1^-, \phi_1^+, 0)\|_{\mathcal{X}} \leq (C_1 + C_2)\|y\|_{\tilde{\mathcal{Y}}} = C\|y\|_{\tilde{\mathcal{Y}}}$$

holds with C a constant independent of T ■

From this theorem we can easily conclude that G_1 is an isomorphism between $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$.
Note that from the final estimate in the theorem it follows that there exists a C independent of J such that $\|G_1^{-1}\| \leq C$.

5.4.2 Perturbation: G_2

For an estimate on G_2, defined by

$$G_2[h^-, h^+, \delta^-, \delta^+, \delta] = \begin{pmatrix} -K^-\delta^- - R^{-\prime}(u^-, \phi^-)[h^-, \delta^-] \\ -K^+\delta^+ - R^{+\prime}(u^+, \phi^+)[h^+\delta^+] \\ 0 \\ 0 \\ -R^{B\prime}(u^-, u^+, \phi^-, \phi^+, \phi)[h^-, h^+, \delta^-, \delta^+, \delta] \end{pmatrix},$$

Lemma B.9 is used. First, recall $\|G_1^{-1}\| \leq C$, where C does not depend on the interval J. In the following let any space indicated with a subscript I be the original space restricted to the interval I instead of J. We fix any p' such that $2 + n < p' < p$.

We only elaborate on a few terms. For the remainder of the terms the proof is analogous to the ones below. Note that here we use the demand that for $v \in \{U_{z_n}^-, \partial_i U_{z_n}^-, u_{z_n}^-, \partial_i u_{z_n}^-, \partial_i \phi^-\}$ and all $i = 1, ..., n$ that $v \in W_p^\theta(J, W_p^{-\theta''}(\Omega_-))$ as introduced in Equation (5.2).

- Firstly, we investigate the term $K^+\delta^+$. From Equation (2.12) it follows that there exists a constant c such that $\frac{1}{(1+\sigma_{z_n})^2} \leq c$ on $\Omega \times J$. Lemma B.9 then states that there exists a T_1' such that on any interval $I \subset (0, T_1')$ we have

$$\|\partial_i U_{z_n}^+\|_{L_{p'}(\Omega_+ \times I)} \leq \frac{1}{2n^2 \|G_1^{-1}\| \|A\|_{L_\infty}}$$

  and

$$\|U_{z_n}^+\|_{L_{p'}(\Omega_+ \times I)} \leq \frac{1}{2c\|\mathcal{A}_\sigma \sigma\|_{L_\infty(\Omega_+ \times I)} \|G_1^{-1}\|}.$$

  This gives

$$\begin{aligned} &\|K^+\delta^+\|_{L_{p'}(\Omega_+ \times I)} \\ &= \left\| -A(\nabla\sigma)\nabla\delta^+ \cdot \nabla U_{z_n}^+ - \frac{U_{z_n}^+}{(1+\sigma_{z_n})^2} \delta_{z_n}^+ \mathcal{A}_\sigma \sigma \right\|_{L_{p'}(\Omega_+ \times I)} \\ &\leq \sum_{i,j} \|A(\nabla\sigma^+)\|_{L_\infty(\Omega_+ \times I_1)} \|\partial_i \delta^+\|_{L_\infty(\Omega_+ \times I)} \|\partial_j u_{z_n}^+\|_{L_{p'}(\Omega_+ \times I)} \\ &\quad + \left\| \frac{1}{(1+\sigma_{z_n})^2} \right\|_{L_\infty(\Omega_+ \times I_1)} \|U_{z_n}^+\|_{L_{p'}(\Omega_+ \times I)} \|\delta_{z_n}^+\|_{L_\infty(\Omega_+ \times I)} \|\mathcal{A}_\sigma \sigma\|_{L_\infty(\Omega_+ \times I)} \\ &\leq \left(\|A\|_{L_\infty} \frac{1}{2\|G_1^{-1}\| \|A\|_{L_\infty}} + c\|\mathcal{A}_\sigma \sigma\|_{L_\infty(\Omega_+ \times I)} \frac{1}{2c\|\mathcal{A}_\sigma \sigma\|_{L_\infty(\Omega_+ \times I)} \|G_1^{-1}\|} \right) \|\delta\|_{X_I^+} \\ &\leq \frac{1}{\|G_1^{-1}\|} \|(h^\pm, \delta^\pm, \delta)\|_{\mathcal{X}_I}. \end{aligned}$$

- Secondly, we look at the term $K^-\delta^-$. Analogously to the estimation above, we have that there exists a T_2' such that on any interval $I \subset (0, T_2')$ it holds that

$$\|K^-\delta^-\|_{L_{p'}(\Omega_- \times I)} \leq \frac{1}{\|G_1^{-1}\|} \|(h^\pm, \delta^\pm, \delta)\|_{\mathcal{X}_I}.$$

  Next to this, from $\partial_i U_{z_n}^- \in W_p^\theta(J, W_p^{-\theta''}(\Omega_-))$ and $U_{z_n}^- \in W_p^\theta(J, W_p^{-\theta''}(\Omega_-))$ it follows by Lemma B.9 that there exists a T_3' such that on any interval $I \subset (0, T_3')$ we have that

$$\|\partial_i U_{z_n}^-\|_{W_{p'}^{\theta'}(I, W_{p'}^{-\theta'}(\Omega_-))} \leq \frac{1}{n^2 \|G_1^{-1}\| \|A\|_{L_\infty}}$$

and

$$\|U_{z_n}\|_{W^{\theta'}_{p'}(I,W^{-\theta'}_{p'}(\Omega_-))} \le \frac{1}{2c\|\mathcal{A}_\sigma\sigma\|_{L_\infty(\Omega_-\times I)}\|G_1^{-1}\|}$$

and thus

$$\begin{aligned}
&\|K^-\delta^-\|_{W^{\theta'}_{p'}(I,W^{-\theta'}_{p'}(\Omega_-))}\\
&= \left\| -A(\nabla\sigma)\nabla\delta^-\cdot\nabla U^-_{z_n} - \frac{U^-_{z_n}}{(1+\sigma_{z_n})^2}\delta^-_{z_n}\mathcal{A}_\sigma\sigma \right\|_{W^{\theta'}_{p'}(I,W^{-\theta'}_{p'}(\Omega_-))}\\
&\le \sum_{i,j}\|A(\nabla\sigma^-)\|_{L_\infty(\Omega_-\times I)}\|\partial_i\delta^-\|_{L_\infty(\Omega_-\times I)}\|\partial_j U_{z_n}\|_{W^{\theta'}_{p'}(I,W^{-\theta'}_{p'}(\Omega_-))}\\
&\quad + \left\|\frac{1}{(1+\sigma_{z_n})^2}\right\|_{L_\infty(\Omega_-\times I)}\|U^-_{z_n}\|_{W^{\theta'}_{p'}(I,W^{-\theta'}_{p'}(\Omega_-))}\|\delta^-_{z_n}\|_{L_\infty}\|\mathcal{A}_\sigma\sigma\|_{L_\infty(\Omega_-\times I)}\\
&\le \left(n^2\|A\|_{L_\infty}\frac{1}{n^2\|G_1^{-1}\|\|A\|_{L_\infty}} + c\|\mathcal{A}_\sigma\sigma\|_{L_\infty(\Omega_-\times I)}\frac{1}{2c\|\mathcal{A}_\sigma\sigma\|_{L_\infty(\Omega_-\times I_3)}\|G_1^{-1}\|}\right)\|\delta\|_{X_I^-}\\
&\le \frac{1}{\|G_1^{-1}\|}\|(h^\pm,\delta^\pm,\delta)\|_{\mathcal{X}_I}.
\end{aligned}$$

- Next, we investigate the term $\alpha(1+|\nabla'\sigma|^2)\frac{h^-_{z_n}\phi^-_{z_n}}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^-_{z_n})}$ from $R^{B\prime}(u^-,u^+,\phi^-,\phi^+,\phi)[h^-,h^+,\delta^-,\delta^+,\delta]$. Here we use Lemma B.9 to estimate for some T_4' that

$$\|\phi^-_{z_n}\|_{L_{p'}(I,W^{1-1/p'}_{p'}(\Gamma))} \le \frac{1}{c(1+\|\sigma\|_{X^B})\|G_1^{-1}\|}$$

and

$$\|\phi^-_{z_n}\|_{W^{\theta'}_{p'}(I,L_{p'}(\Gamma))} \le \frac{1}{\alpha c(1+\|\sigma\|_{X^B})\|G_1^{-1}\|}$$

on any interval $I\subset(0,T_4')$. Here c is such that $\frac{1}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^-_{z_n})}\le c$ on $\Gamma\times J$. From the above it follows that

$$\begin{aligned}
&\left\|\alpha(1+|\nabla'\sigma|^2)\frac{h^-_{z_n}\phi^-_{z_n}}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^-_{z_n})}\right\|_{L_{p'}(I,W^{1-1/p'}_{p'}(\Gamma))}\\
&\le \alpha\left\|\frac{1}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^-_{z_n})}\right\|_{L_\infty(\Gamma\times I)}\big(1+\||\nabla'\sigma|^2\|_{L_\infty(\Gamma\times I)}\big)\cdot\\
&\qquad\qquad \|h^-_{z_n}\|_{L_\infty(\Gamma\times I_4)}\|\phi^-_{z_n}\|_{L_{p'}(I,W^{1-1/p'}_{p'}(\Gamma))}\\
&\le \alpha c(1+\|\sigma\|_{X^B})\|h^-\|_{X_I^B}\frac{1}{\alpha c(1+\|\sigma\|_{X^B})\|G_1^{-1}\|}\\
&= \frac{1}{\|G_1^{-1}\|}\|(h^\pm,\delta^\pm,\delta)\|_{\mathcal{X}_I}
\end{aligned}$$

and analogously

$$\left\|\alpha(1+|\nabla'\sigma|^2)\frac{h^-_{z_n}\phi^-_{z_n}}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^-_{z_n})}\right\|_{W^{\theta'}_{p'}(I,L_{p'}(\Gamma))} \le \frac{1}{\|G_1^{-1}\|}\|(h^\pm,\delta^\pm,\delta)\|_{\mathcal{X}_I}.$$

- Finally, we consider the term $\nabla U^+_{z_n}\vec{b}'(\nabla\sigma,\nabla\phi^+)[\delta^+]$. Therefore, first of all we recall

$$\vec{b}'(\nabla\sigma,\nabla\phi^+)[\delta^+] = \Bigg(\sum_{i,j=1}^n \int_0^1 (1-s)\partial_{ij}A(\nabla\sigma + s\nabla\phi^+)\partial_i\phi^+\partial_j\delta^\pm \cdot \nabla\phi^+ ds$$
$$+ \sum_{i=1}^n \int_0^1 (1-s)\partial_i A(\nabla\sigma + s\nabla\phi^+)\partial_i\delta^+ \cdot \nabla\phi^+ ds$$
$$+ \sum_{i=1}^n \int_0^1 (1-s)\partial_i A(\nabla\sigma + s\nabla\phi^+)\partial_i\phi^+ \cdot \nabla\delta^+ ds\Bigg).$$

Secondly, Lemma B.9 states that for every constant $\varepsilon > 0$ there exists a T_5' such that on the interval $I \subset (0, T_5')$ we have

$$\|\partial_i U^+_{z_n}\|_{L_{p'}(\Omega_+\times I)} \leq \frac{1}{\varepsilon\|G_1^{-1}\|}.$$

For the first part this gives

$$\left\|\sum_{i,j=1}^n \int_0^1 (1-s)\nabla U^+_{z_n}\partial_{ij}A(\nabla\sigma + s\nabla\phi^+)\partial_i\phi^+\partial_j\delta^\pm \cdot \nabla\phi^+ ds\right\|_{L_{p'}(\Omega_+\times I)}$$
$$\leq \left\|\sum_{i,j=1}^n \int_0^1 \left|(1-s)\nabla U^+_{z_n}\partial_{ij}A(\nabla\sigma + s\nabla\phi^+)\partial_i\phi^+\partial_j\delta^\pm \cdot \nabla\phi^+\right| ds\right\|_{L_{p'}(\Omega_+\times I)}$$
$$\leq \left\|\sum_{i,j=1}^n \int_0^1 \sum_{k=1}^n |\partial_k U^+_{z_n}|\|\partial_{ij}A\|_{L_\infty}\|\partial_i\phi^+\|_{L_\infty(\Omega_+\times I)}\|\partial_j\delta^\pm\|_{L_\infty(\Omega_+\times I)}\|\partial_k\phi^+\|_{L_\infty(\Omega_+\times I)} ds\right\|_{L_{p'}(\Omega_+\times I)}$$
$$\leq \left\|n^2\sum_{k=1}^n |\partial_k U^+_{z_n}|\|A\|_{W^2_\infty}\|\phi^+\|_{X^+}\|\partial_j\delta^\pm\|_{X^+}\|\phi^+\|_{X^+}\right\|_{L_{p'}(\Omega_+\times I)}$$
$$\leq n^2\sum_{k=1}^n \|\partial_k U^+_{z_n}\|_{L_{p'}(\Omega_+\times I)}\|A\|_{W^2_\infty}\|\phi^+\|_{X^+}\|\partial_j\delta^\pm\|_{X_I^+}\|\phi^+\|_{X_I^+}$$
$$\leq \frac{1}{\|G_1^{-1}\|}\|(h^\pm,\delta^\pm,\delta)\|_{\mathcal{X}_I}.$$

The other two parts can be estimated analogously.

Now define $T' = \min(T_1', T_2', T_3', ...)$ and let $I = (0, T')$. On this interval we can combine the estimates and get $\|G_2\| \leq \frac{1}{\|G_1^{-1}\|}$. Now since G_1 is an isomorphism, we have that also $G = G_1 + G_2$ is an isomorphism between spaces on I (instead of J).

5.5 Implicit Function Theorem

Finally, we can apply the Implicit Function Theorem to the equation $F((\mu,\lambda),\tilde{x}) = 0$, where $\tilde{x} = (w^-, w^+, \psi^-, \psi^+, \psi)$.
Since

(i) the mapping $F : \mathcal{U}((0,1),\tilde{x}) \subseteq (\mathbb{R}^n \times \mathbb{R}) \times \tilde{\mathcal{X}}_I \to \mathcal{Y}_I$ is defined on a open neighbourhood $\mathcal{U}((0,1,\tilde{x})$ and $F((0,1),\tilde{x}) = 0$;
(ii) $G = F_{\tilde{x}}$ exists on $\mathcal{U}((0,1),\tilde{x})$ and G is an isomorphism between $(\mathbb{R}^n \times \mathbb{R}) \times \tilde{\mathcal{X}}_I$ and $\mathcal{Y}_I$;
(iii) both F and G are continuous at $((0,1),\tilde{x})$;

(iv) F is infinitely many times Fréchet differentiable (see Sections 5.3.2 and 5.3.3);

we can conclude the following by the Implicit Function Theorem (see Theorem B.6):

(a) Existence and uniqueness: For all (μ, λ) close to $(0, 1)$ there exists a unique $\hat{x} : \mathbb{R}^n \times \mathbb{R} \to \tilde{\mathcal{X}}_I$ close to $\tilde{x}$ such that $F((\mu, \lambda), \hat{x}(\mu, \lambda)) = 0$.

(b) Continuous differentiability. Since F is infinitely many times differentiable on a neighbourhood of $((0, 1), \tilde{x})$, also $\hat{x}$ is on a neighbourhood of $(0, 1)$.

Since we already had $F((\mu, \lambda), \tilde{x}) = 0$, it must hold that $\hat{x}(\mu, \lambda) = \tilde{x}_{\mu,\lambda}$ and thus that $(\mu, \lambda) \mapsto \tilde{x}_{\mu,\lambda}$ is infinitely many times differentiable.

5.6 Conclusion

Theorem 5.2. *The solution $(u^-, u^+, \tau^-, \tau^+, \tau) \in \mathcal{X}$ of System (5.1) satisfying the assumption in Equation (5.2) is smooth in z_1, ..., z_{n-1} and t on $\Omega_\pm \times (0, T')$. In particular, the interface between the two phases in the original problem is smooth.*

Proof. Let $(z_0, t_0) \in \Omega_- \times (0, T')$ be fixed. We define the function $E_{z_0,t_0} : X_I^+ \to \mathbb{R}$ by $E_{z_0,t_0}(x) = x(z_0, t_0)$ for all $x \in X_I^+$. This gives $E_{z_0,t_0} w^+_{\mu,\lambda} = w^+_{\mu,\lambda}(z_0, t_0) = u^+(z_0 + \mu t_0, \lambda t_0)$. Note that since $X_I^+ \hookrightarrow C((0, T'), C^1(\mathbb{T}^{n-1}))$, we have $|w_{\mu,\lambda}(z_0, t_0)| \leq \|w_{\mu,\lambda}\|_{L^\infty} \leq \|w_{\mu,\lambda}\|_{X_I^+}$ and thus that E_{z_0,t_0} is a bounded linear functional. By Lemma B.2 this gives that E_{x_0,t_0} is infinitely many times differentiable. In combination with $(\mu, \lambda) \mapsto w^+_{\mu,\lambda}$ being infinitely many times differentiable in a neighbourhood of $(0, 1)$ we can conclude that $(\mu, \lambda) \mapsto E_{z_0,t_0} w^+_{\mu,\lambda} = u^+(z_0 + \mu t_0, \lambda t_0)$ is infinitely many times differentiable in a neighbourhood of $(0, 1)$. From this we can conclude that also $u^+(x, t)$ must be infinitely many times differentiable in z_1, ..., z_{n-1} and in t around the point $(z, t) = (z_0, t_0)$.

Since T' depends only on the initial conditions, the whole argument can be repeated on overlapping open intervals until smoothness on the whole interval $(0, T)$ is guaranteed.

Analogously, it can be proven that all of u^-, τ^+, τ^- and τ are infinitely many times differentiable in z_1, ..., z_{n-1} and t. Note that this, together with σ being smooth, implies that the transformation introduced in Section 2.3 is smooth and thus that the interface between the two phases is smooth.

∎

Chapter 6

Discussion

In the previous chapters two analyses are done on solutions of the free boundary problem given in Chapter 2. In this chapter a discussion on the results is given.

There are some differences in the assumptions made in each of the two methods. The assumptions in the first method can be found in Equations (4.2) and (4.3). The assumption made in the second method is stated in Equation (5.2). The most important difference, however, is that the two methods are applied to different systems. They both started from the system of equations (2.4). But since this system is not closed, in both chapters some equations are added. In the second method Angenent's parameter trick is applied to the system as derived in [17]. Here the functions $\hat{P}$, $\hat{\nu}$ and $\hat{\phi}$ were splitted into known functions U^-, U^+ and σ with specific properties and unknown functions u^-, u^+ and ϕ with homogeneous initial data. After this the Equations

$$\begin{aligned} \hat{L}^{\pm}\phi^{\pm} &= 0 \qquad \text{in } \Omega_{\pm} \times J \quad \text{and} \\ \phi^{+}(\cdot, 0) &= 0 \qquad \text{in } \Omega_{+} \end{aligned}$$

were added to close the system. In Chapter 2 some theorems can be found about existence and uniqueness of solutions of the resulting system. In the first method, however, the split mentioned above is not made and other equations are added. These are

$$\begin{aligned} \Lambda_{\tau}^{\pm}\tau^{\pm} &= 0 \qquad \text{in } \Omega_{\pm} \times J \quad \text{and} \\ \tau^{+}(\cdot, 0) &= \sigma(\cdot, 0) \qquad \text{in } \Omega_{+}. \end{aligned}$$

Therefore, as already mentioned in Chapter 4, it is not sure that the resulting system has a solution.

Let us for now consider the upper (liquid) phase. Then the equation from the first method can be written as

$$\sum_{1=1}^{n-1} \tau^{-}_{z_i z_i} - \vec{a}(\nabla \tau^{-}) \cdot \nabla \tau^{-}_{z_n} = 0.$$

We note that this equation is the same as the one in the second method in highest order except that this is the quasilinear version of the linear equation we get when splitting the variables. Existence of solutions of quasilinear equations is often proven with similar techniques as linear equations. We shortly sketch two possible options for proving existence of solutions for this system.

The first option is to linearise the system around some known functions σ, U^- and U^+. Once σ is chosen, define U^- and U^+ analogous to (2.13) and (2.14). Then let $\tau = \sigma + \phi$ and $u^{\pm} = U^{\pm} + h^{\pm}$ and use linearisation to rewrite the system. This system will be similar to the one in Theorem 2.2. Therefore, the structure of the proofs in [17] can be followed. The problem in this option is to choose σ. For the proof we need that $\phi = 0$ at $t = 0$. Note that in the lower phase this is guaranteed by the initial condition. However, in the upper phase this would mean σ has to satisfy

$\Lambda_\sigma^- \sigma = 0$. So in order to fix σ, a quasilinear equation has to be solved and this means no progress is made.

The second option is to approach the problem using an iteration. Suppose $u_n^\pm$, $\tau_n^\pm$ and τ_n are known. Then define $u_{n+1}^\pm$, $\tau_{n+1}^\pm$ and $\tau_{n+1}^\pm$ such that they satisfy (4.1) with

$$\Lambda_{\tau_n^\pm}^\pm \tau_{n+1}^\pm = 0 \qquad \text{in } \Omega_\pm \times J$$

instead of the nonlinear Equation $(4.1)_2$. Now let S be such that $(u_{n+1}^\pm, \tau_{n+1}^\pm, \tau_{n+1}) = S(u_n^\pm, \tau_n^\pm, \tau_n)$. If we now prove that S has a fixed point, using for example the Banach Fixed-Point Theorem, then this is a solution to the nonlinear problem [11].

In conclusion, both of the methods have their advantages and disadvantages. In the first method, in Chapter 4, we first added the two equations to close the system and linearised the resulting system afterwards. As a result the proof that this system has a solution becomes difficult and is not included in this report. On the other hand, the proof that the interface is smooth is easier. In the second method, in Chapter 5, where we first splitted the variables and then added the two equations, proving existence of a solution was easier. However, in proving the solutions are smooth many more terms arose, which all had to be shifted and estimated in the correct norms.

Appendix A

Examples of free boundary problems

In this section two well-known free boundary problems are discussed. Both of these problems show similarities to the main problem in this report. The first one is the Stefan problem, which concerns the melting of ice. The second problem is called the Hele-Shaw problem and describes the flow between two parallel plates (with a small distance) when a sink or source is present. For both of these problems a mathematical model is derived and the solvability conditions are stated.

A.1 The Stefan problem

Let the domain $\Omega = \Omega' \times [0, L]$ with $\Omega' \subset \mathbb{R}^{n-1}$ be divided into two smaller domains by an interface $\Gamma(t)$ at any time $t \geq 0$. Suppose the interface is given by an unknown function $x_n = \xi(x', t)$, where $x' \in \Omega'$. The first domain $\Omega_\ell(t) := \{x \in \Omega \mid x_n < \xi(x', t)\}$ is filled with a liquid (for example water), while the second domain $\Omega_s(t) := \{x \in \Omega \mid x_n > \xi(x', t)\}$ is filled with solid of the same material (for example ice). We define $S = \Omega' \times \{0\}$ and let n be the unit normal of Γ, so $n(x', t) = (-\nabla'(\xi, t), 1)$. The situation is sketched in Figure 3.

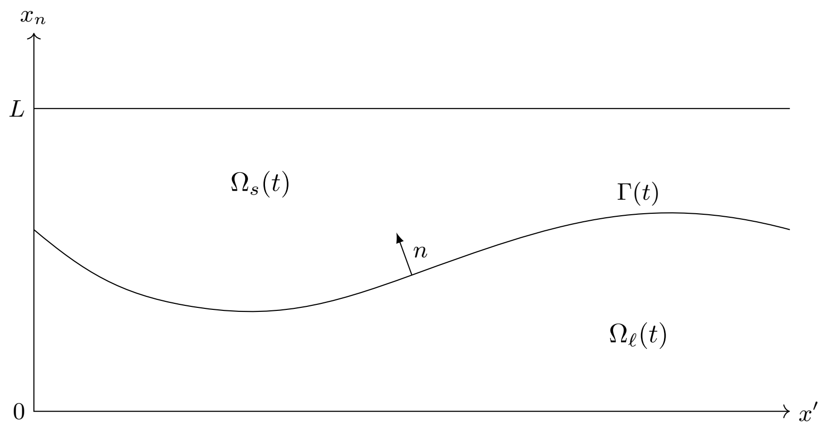

Figure 3: Setting for the Stefan problem. The lower domain is filled with liquid and the upper domain with a solid of the same material.

Let $L > 0$ be the latent heat per unit volume of the material and θ_0 the critical temperature at which the material changes phase. We consider the so-called one-phase Stefan problem. Here the solid phase is entirely at critical temperature and this phase is not visible in the mathematical model, even though it is present in the underlying problem [3]. Furthermore, we assume the temperature is continuous in the entire domain. Combining the above gives

$$\theta = \theta_0 \qquad \text{on } \Gamma(t).$$

We now suppose the ice is melting. Because of energy conservation and the fact that one phase is entirely at critical temperature we have that all energy absorbed by the interface through diffusion is transformed into latent heat [21]. Let us consider a smaller domain $D \times [0, L]$, with $D \subset \Omega'$. First of all, the latent heat between t_0 and t_1 can be found by multiplying the volume of the melted ice by the latent heat per unit volume. This gives

$$Q = L \int_D \xi(x', t_1) - \xi(x', t_0) dx'. \tag{A.1}$$

Secondly, the absorbed heat can also be found using Fouriers law, i.e. $q = -k_l \nabla \theta$, where q is the heat flux and k_l is the heat diffusivity coefficient of the liquid. This gives

$$\begin{aligned} Q &= \int_D \int_{t_0}^{t_1} q((x', \xi(x', t)), t) \cdot n((x', \xi(x', t)), t) dt dx' \\ &= \int_D \int_{t_0}^{t_1} -k_\ell \nabla \theta((x', \xi(x', t)), t) \cdot (-\nabla' \xi((x', \xi(x', t)), t), 1) dt dx' \\ &= \int_D \int_{t_0}^{t_1} k_\ell \nabla' \theta((x', \xi(x', t)), t) \nabla' \xi(x', t) - k_\ell \partial_{x_n} \theta((x', \xi(x', t)), t) dt dx'. \end{aligned} \tag{A.2}$$

As already mentioned, because of energy conservation, we can equate (A.1) and (A.2). Since this holds for all subdomains $D \times [0, L]$ of Ω, it must hold that

$$L(\xi(x', t_1) - \xi(x', t_0)) = \int_{t_0}^{t_1} k_\ell \nabla' \theta((x', \xi(x', t)), t) \nabla' \xi(x', t) - k_\ell \partial_{x_n} \theta((x', \xi(x', t)), t) dt.$$

Now dividing by $t_1 - t_0$ and letting $t_1 \to t_0$ gives

$$L \partial_t \xi(x', t) = k_\ell \nabla' \theta((x', \xi(x', t)), t) \nabla' \xi(x', t) - k_\ell \partial_{x_n} \theta((x', \xi(x', t)), t),$$

so

$$L \partial_t \xi = k_\ell \nabla' \theta \nabla' \xi - k_\ell \partial_{x_n} \theta \qquad \text{on } \Gamma(t).$$

This equation is called the Stefan condition on the free boundary. Note that it also holds in the case of freezing water instead of melting ice. In this case both sides of the equation will be negative.

We now derived two boundary conditions on the free boundary. Inside both the solid and the liquid phases heat conservation should be satisfied. Because there are no heat sources present, this conservation law can be written as

$$\rho c_\ell \partial_t \theta = -\nabla q,$$

where c_ℓ is the specific heat capacity in the liquid phase and ρ the density. Together with Fourier's law this gives

$$\partial_t \theta - \alpha \Delta \theta = 0 \qquad \text{in } \Omega_\ell(t),$$

where $\alpha = \frac{k_\ell}{\rho c_\ell}$ is the thermal diffusivity of the material.

The system can now be completed by initial conditions and a boundary condition at $x_n = 0$. For the boundary conditions at $x_n = 0$ we suppose that the temperature is kept fixed here. This corresponds to applying Dirichlet boundary conditions, so

$$\theta(\cdot, 0) = h \qquad \text{for } x \in S,$$

for some known function $h : \Omega' \to \mathbb{R}$.

For the initial conditions we define

$$\theta(\cdot, 0) = \Theta,$$
$$\xi(\cdot, 0) = \xi_0,$$

where by definition of the domain $\Theta(x, 0) \geq \theta_0$ for all $x \in \Omega_\ell(0)$ and Θ has to satisfy the boundary condition $\Theta|_S = h|_{t=0}$ and $\Theta|_{\Gamma(0)} = \theta_0$.
Combining all of the above gives

$$\left.\begin{array}{rll} \partial_t\theta - \alpha\Delta\theta = 0 & \text{for } x \in \Omega_\ell(t), & t > 0, \\ \theta = h & \text{for } x \in S, & t > 0, \\ \theta = \theta_0 & \text{for } x \in \Gamma(t), & t > 0, \\ L\partial_t\xi - k_\ell\nabla'\theta\nabla'\xi + k_\ell\partial_{x_n}\theta = 0 & \text{for } x \in \Gamma(t), & t > 0, \\ \theta = \Theta & \text{for } x \in \Omega(t), & t = 0, \\ \xi = \xi_0 & \text{for} & t = 0. \end{array}\right\}$$

The solvability condition for this problem is [12]

$$\partial_{n_{\Gamma_0}}\Theta \leq -\alpha < 0. \tag{A.3}$$

A.2 The Hele-Shaw problem

The Hele-Shaw problem describes a flow between two parallel plates. The fluid occupies a bounded domain with a free boundary. Inside this domain a source or sink is placed. This is a small tube trough which injection or suction of fluid occurs. As a result the boundary of the fluid will move. The situation is sketched in Figure 4. We define h as the height between the two plates and consider a three dimensional setting where the plates are positioned at $x_3 = 0$ and $x_3 = h$ (see Figure 5).

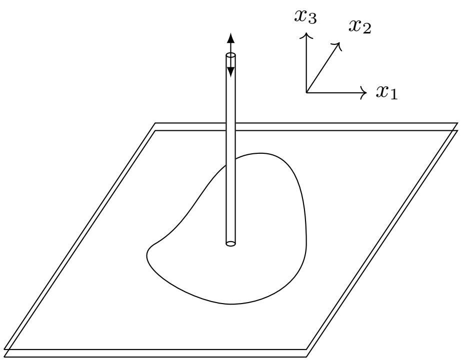

Figure 4: Setting for the Hele-Shaw problem.

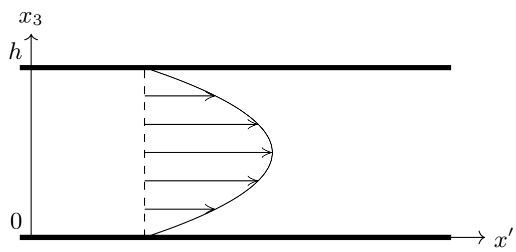

Figure 5: Cross section of the set-up.

A.2.1 Equations in the bulk

For the equations in the bulk we consider a position in the fluid away from the boundary. The situation is sketched in Figure 5. In the fluid both the continuity equation (mass conservation) and the Navier-Stokes equation (momentum conservation) hold. [16] These can be written as

$$\nabla v = 0, \tag{A.4}$$

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = \frac{1}{\rho}(-\nabla p + \mu\Delta v), \tag{A.5}$$

where v the velocity of the fluid, ρ the density, p the pressure and μ the dynamic viscosity. Nest to this we have no-slip boundary conditions

$$v_1(x_1, x_2, 0) = v_1(x_1, x_2, h) = 0,$$
$$v_2(x_1, x_2, 0) = v_2(x_1, x_2, h) = 0$$

and we have

$$v_3(x_1, x_2, 0) = v_3(x_1, x_2, h) = 0$$

because the wall is impenetrable.

Furthermore, we assume that h is small enough to assume that the derivatives of v with respect to t, x_1 and x_2 are negligible compared to the derivatives with respect to x_3. This can be proved using asymptotic analysis, but are not discussed in detail here. From mass conservation (Equation (A.4)) it then follows that

$$\frac{\partial v_3}{\partial x_3} = 0$$

and together with the third boundary condition we get

$$v_3 = 0.$$

Combining this with the Navier-Stokes equation (A.5) the latter reduces to

$$\frac{\partial p}{\partial x_1} = \mu \frac{\partial^2 v_1}{\partial x_3^2},$$
$$\frac{\partial p}{\partial x_2} = \mu \frac{\partial^2 v_2}{\partial x_3^2},$$
$$\frac{\partial p}{\partial x_3} = 0.$$

From the last equation it follows that p is independent of x_3 and combining this with the first two equations gives that both v_1 and v_2 are polynomials in x_3 with degree at most 2. Together with the first two boundary conditions we now get

$$v_1 = -\frac{1}{2\mu} \frac{\partial p}{\partial x_1} x_3 (h - x_3),$$
$$v_2 = -\frac{1}{2\mu} \frac{\partial p}{\partial x_2} x_3 (h - x_3).$$

The flow rates per unit length are then equal to

$$\phi_1 = \int_0^h v_1 dx_3 = -\frac{h^3}{12\mu} \frac{\partial p}{\partial x_1},$$
$$\phi_2 = \int_0^h v_2 dx_3 = -\frac{h^3}{12\mu} \frac{\partial p}{\partial x_2},$$

which means that the average velocity across the gap is equal to

$$\tilde{v}_1 = \frac{1}{h} \phi_1 = -\frac{h^2}{12\mu} \frac{\partial p}{\partial x_1},$$
$$\tilde{v}_2 = \frac{1}{h} \phi_2 = -\frac{h^2}{12\mu} \frac{\partial p}{\partial x_2}.$$

Note that this can be written as

$$\tilde{v} = -\frac{h^2}{12\mu}\nabla p.$$

The equation above is called the Hele-Shaw equation. Because the averaged velocity field also has to satisfy the continuity equation, we get

$$\Delta p = 0$$

in the entire fluid.

A.2.2 Boundary and sources

Now let there be a source or sink of constant strength Q in the origin. In case of a sink (i.e. suction of fluid) we have $Q < 0$ and in case of a source (i.e. injection of fluid) we have $Q < 0$. We define the domain containing the fluid at time t by $\Omega(t)$ and let $\Gamma(t)$ be the boundary of the domain. The pressure p now satisfies

$$\Delta p = Q\delta(x_1, x_2),$$

where δ is the Dirac distribution.
For the boundary of the fluid we have to adapt two boundary conditions [10, 16]. First of all we assume that the effect of surface tension can be ignored and thus that the pressure p at the boundary is equal to the atmospheric pressure, and thus constant along the entire boundary. The second boundary condition states that fluid particles on the interface remain on the interface and is given by

$$V_n = v \cdot n = -\frac{h^2}{12\mu}\frac{\partial p}{\partial n}$$

where V_n is the normal velocity of the interface and n the unit normal vector of the interface $\Gamma(t)$. The Hele-Shaw problem is well defined and has a unique solution for $Q < 0$. This means the fluid contains a source and is expanding. Therefore, this condition can also be stated as

$$V_n > 0 \quad \text{or equivalently} \quad \frac{\partial p}{\partial n} < 0. \tag{A.6}$$

In the case that $Q < 0$ however, the problem is ill-posed and might not have a solution [13].

Appendix B

Mathematical techniques

B.1 Fréchet differentiability

We first introduce some notation. For any function $f : X \to Y$ we define $f(x) = o(\|x\|)$ if and only if $f(x)/\|x\| \to 0$ for $x \to 0$.

Definition B.1. *[23] In Banach spaces a function $f : X \to Y$ is Fréchet differentiable in the point x if there exists a function $f'(x) \in L(X, Y)$ such that*

$$\|f(x+h) - f(x) - f'(x)[h]\|_Y = o(\|h\|_X).$$

In this case $f'(x)$ is called the (Fréchet) derivative of f in the point x.
Higher order Fréchet derivatives are defined successively. So for example the the second order derivative of f is a bilinear function $f^{(2)}(x) \in L^{(2)}(X, Y)$ such that

$$\|f'(x+h_2)[h_1] - f'(x)[h_1] - f^{(2)}(x)[h_1, h_2]\|_Y = o(\|h_2\|_X).$$

Let $f : X_1 \times ... \times X_n \to Y$, where X_1, ..., X_n and Y are Banach spaces. Let $x_2, ..., x_n$ be fixed and let g be such that $g(x_1) = f(x_1, ..., x_n)$. If g is Fréchet differentiable at x_1, then we define the partial Fréchet derivative of f with respect to x_1 by $D_{x_1} f(x_1, ..., x_n) = g'(x_1)$.
If all partial Fréchet derivatives of f exist, then f is Fréchet differentiable and the total derivative is given by $f'(x_1, ..., x_n)$ with

$$f'(x_1, ..., x_n)[h_1, ..., h_n] = D_{x_1} f(x_1, ..., x_n)[h_1] + ... + D_{x_n} f(x_1, ..., x_n)[h_n].$$

Lemma B.2 (Derivative of a bounded linear function)**.** *Let X and Y be two Banach spaces and $A : X \to Y$ a bounded linear function. Then A is infinitely many times differentiable and the first derivative is defined by*

$$A'(x)[h] = A(h).$$

Proof. The statement follows easily from $A(x+h) - A(x) - A(h) = 0$ and the fact that $A'(x)$ is independent of x. ■

Lemma B.3 (Product rule)**.** *Let U, X, Y and Z be four function spaces that map from Ω to $\mathbb{R}$ such that $X \cdot Y \hookrightarrow Z$, i.e. for all $u \in X$, $v \in Y$ it holds that $\|uv\|_Z \leq C\|u\|_X\|v\|_Y$ for some fixed constant C. Here the product is defined by $uv(x) = u(x)v(x)$ for all $x \in \Omega$. Suppose that $f : U \to X$ and $g : U \to Y$ are $K \geq 1$ times differentiable in $u \in U$. Then $fg : U \to Z$ defined by $(fg)(u) = f(u)g(u)$ is k times differentiable in u for all $k \leq K$ and the first derivative is given by*

$$(fg)'(u)[h] = f'(u)[h]g(u) + f(u)g'(u)[h].$$

Proof. We prove the statement using induction on k. Therefore, we first prove it for $k = 1$. Since f and g are at least 1 time differentiable, we have that

$$\begin{aligned}
&\|(fg)(u+h) - (fg)(u) - f'(u)[h]g(u) - f(u)g'(u)[h]\|_Z \\
&= \|f(u+h)g(u+h) - f(u)g(u) - f'(u)[h]g(u) - f(u)g'(u)[h]\|_Z \\
&= \|(f(u+h) - f(u) - f'(u)[h])g(u+h) + f(u)(g(u+h) - g(u) - g'(u)[h]) \\
&\qquad + f'(u)[h](g(u+h) - g(u))\|_Z \\
&\leq C\,(\|f(u+h) - f(u) - f'(u)[h]\|_X \|g(u+h)\|_Y + \|f(u)\|_X \|g(u+h) - g(u) - g'(u)[h]\|_Y \\
&\qquad + \|f'(u)\|_{L(U,X)} \|h\|_U \|g(u+h) - g(u)\|_Y)
\end{aligned}$$

By definition of the derivative of f we have

$$\lim_{h \to 0} \frac{1}{\|h\|_U} \|f(u+h) - f(u) - f'(u)[h]\|_X = 0,$$

and because differentiability implies continuity also

$$\lim_{h \to 0} \|g(u+h) - g(u)\|_Y = 0.$$

So we can conclude

$$\begin{aligned}
\lim_{h \to 0} \Bigg(& \frac{1}{\|h\|_U} \|f(u+h) - f(u) - f'(u)[h]\|_X \|g(u+h)\|_Y \\
& + \|f(u)\|_X \frac{1}{\|h\|_U} \|g(u+h) - g(u) - g'(u)[h]\|_Y \\
& + \|f'(u)\|_{L(U,X)} \|g(u+h) - g(u)\|_Y \Bigg) = 0
\end{aligned}$$

and thus

$$\lim_{h \to 0} \frac{1}{\|h\|_U} \|(fg)(u+h) - (fg)(u) - f'(u)[h]g(u) - f(u)g'(u)[h]\|_Z = 0.$$

Note that since both $f'(u)[h]$ and $g'(u)[h]$ are linear in h, also $f'(u)[h]g(u) + f(u)g'(u)[h]$ is. Finally, because $h \mapsto f'(u)[h]$ and $h \mapsto g'(u)[h]$ are bounded, also $h \mapsto f'(u)[h]g(u) + f(u)g'(u)[h]$ is, which concludes the proof of the statement for $k = 1$.
We now perform the induction to prove that fg is $k \leq K$ times differentiable with

$$(fg)^{(k)}(u)[h_1, ..., h_k] = \sum_{\sigma} \sum_{i=0}^{k} c_{i,k} f^{(i)}(u)[h_{\sigma_1}, ..., h_{\sigma_i}] \cdot g^{(k-i)}(u)[h_{\sigma_{i+1}}, ..., h_{\sigma_k}], \tag{B.1}$$

where $c_{i,k}$ is some constant depending on k and i. Note that for $k = 1$ the derivative of fg is indeed of this form. So now suppose for some $k < K$ that the above statement holds, i.e. that fg is k times differentiable with $(fg)^{(k)}(u)$ as in Equation (B.1). Since differentiation is a linear operator, it is sufficient to prove that $f^{(i)}(u)[h_{\sigma_1}, ..., h_{\sigma_i}] \cdot g^{(k-i)}(u)[h_{\sigma_{i+1}}, ..., h_{\sigma_k}]$ is differentiable for any fixed $i \leq k$. Note that f and g are both at least $k+1$ times differentiable and thus for $i \leq k$ it must hold that both $f^{(i)}(u)$ and $g^{(k-i)}(u)$ are at least 1 time differentiable. This means we can apply the lemma for $k = 1$ (which we proved above). This gives that $f^{(i)}(u)[h_{\sigma_1}, ..., h_{\sigma_i}] \cdot g^{(k-i)}(u)[h_{\sigma_{i+1}}, ..., h_{\sigma_k}]$ is differentiable and that the derivative is equal to the map $(h_1, h_2, ..., h_{k+1}) \mapsto f^{(i+1)}(u)[h_{\sigma_1}, ..., h_{\sigma_i}, h_{k+1}] \cdot g^{(k-i)}(u)[h_{\sigma_{i+1}}, ..., h_{\sigma_k}] + f^{(i)}(u)[h_{\sigma_1}, ..., h_{\sigma_i}] \cdot g^{(k-i+1)}(u)[h_{\sigma_{i+1}}, ..., h_{\sigma_k}, h_{k+1}]$. Combining all terms gives thus that fg is differentiable and that there exist constants $c_{i,k+1}$ such that $(fg)^{(k+1)}(u)$ is defined by

$$(fg)^{(k+1)}(u)[h_1, ..., h_k, h_{k+1}] = \sum_{\sigma} \sum_{i=0}^{k+1} c_{i,k+1} f^{(i)}(u)[h_{\sigma_1}, ..., h_{\sigma_i}] \cdot g^{(k+1-i)}(u)[h_{\sigma_{i+1}}, ..., h_{\sigma_{k+1}}]$$

which proves the lemma for all $k \leq K$. ■

Lemma B.4 (A chain rule)**.** *Let X, Y and Z be three Banach spaces. Suppose that $A : X \to Y$ is bounded linear and $\phi : Y \to Z$ is $K \geq 1$ times differentiable. Then $\phi \circ A : X \to Z$ is k times differentiable for all $k \leq K$ and the first derivative is given by*

$$(\phi \circ A)'(x)[h] = \phi'(A(x))[Ah].$$

Proof. We use induction to prove that $\phi \circ A : X \to Z$ is k times differentiable with $(\phi \circ A)^{(k)} \in L^{(k)}(X, Z)$ defined by

$$(\phi \circ A)^{(k)}(x)[h_1, ..., h_k] = \phi^{(k)}(A(x))[Ah_1, ..., Ah_k] \tag{B.2}$$

for all $k \leq K$. Note that for $k = 0$ the statement is trivial. In order to perform the induction, suppose for some $k < K$ that the above statement holds, i.e. $\phi \circ A$ is k times differentiable with $(\phi \circ A)^{(k)}(x)$ as in Equation (B.2). Since $k < K$, it must hold that $k + 1 \leq K$ and thus that ϕ is $k + 1$ times differentiable. Using the above together with the fact that $A \in L(X, Y)$, we have

$$\begin{aligned}
&\|(\phi \circ A)^{(k)}(x + h_{k+1})[h_1, ..., h_k] - (\phi \circ A)^{(k)}(x)[h_1, ..., h_k] - \phi^{(k+1)}(A(x))[Ah_1, ..., Ah_{k+1}]\|_Z \\
&= \|\phi^{(k)}(A(x + h_{k+1}))[Ah_1, ..., Ah_k] - \phi^{(k)}(A(x))[Ah_1, ..., Ah_k] - \phi^{(k+1)}(A(x))[Ah_1, ..., Ah_{k+1}]\|_Z \\
&= \|\phi^{(k)}(A(x) + A(h_{k+1}))[Ah_1, ..., Ah_k] - \phi^{(k)}(A(x))[Ah_1, ..., Ah_k] - \phi^{(k+1)}(A(x))[Ah_1, ..., Ah_{k+1}]\|_Z \\
&= o(\|A(h_{k+1})\|_Y) \\
&= o(\|h_{k+1}\|_X).
\end{aligned}$$

Note that since A is a linear map and $\phi^{(k+1)}(A(x))$ is linear in all of its arguments, also $(h_1, ..., h_{k+1}) \mapsto \phi^{(k+1)}(A(x))[Ah_1, ..., Ah_{k+1}]$ is. Secondly, since $A \in L(X, Y)$ and $\phi^{(k+1)} \in L^{(k)}(Y, Z)$, we have

$$\begin{aligned}
&\|\phi^{(k+1)}(A(x))[Ah_1, ..., Ah_{k+1}]\|_Z \\
&\leq \|\phi^{(k+1)}(A(x))\|_{L^{(k)}(Y,Z)} \|Ah_1\|_Y \cdot ... \cdot \|Ah_{k+1}\|_Y \\
&\leq \|\phi^{(k+1)}(A(x))\|_{L^{(k)}(Y,Z)} \|A\|^{k+1}_{L(X,Y)} \|h_1\|_X \cdot ... \cdot \|h_{k+1}\|_X
\end{aligned}$$

and thus that $(h_1, ..., h_{k+1}) \mapsto \phi^{(k+1)}(A(x))[Ah_1, ..., Ah_{k+1}]$ is a bounded linear map. We can now conclude that $\phi \circ A$ is $k + 1$ times differentiable with $(\phi \circ A)^{(k+1)} \in L^{(k+1)}(X, Z)$ defined by

$$(\phi \circ A)^{(k+1)}(x)[h_1, ..., h_{k+1}] = \phi^{(k+1)}(A(x))[Ah_1, ..., Ah_{k+1}]$$

which proves the first part of the lemma. The second part is easily checked by substituting $k = 1$ in Equation (B.2).

■

Lemma B.5. *Let $\phi : Z = W^1_p(\Omega, \mathbb{R}^n) \to L_\infty(\Omega, \mathbb{R})$, with Ω bounded, be defined by*

$$\phi(z)(x) := a(z(x)),$$

where $a \in C^{K+1}_b(\mathbb{R}^n)$ for some $K \geq 1$. Then ϕ is k times differentiable for all $k \leq K$ and the first derivative is equal to

$$\phi'(z)[h] = \partial_i a(z)h^i,$$

where h^i denotes the i^{th} component of h.

Proof. We prove that ϕ is k times differentiable for all $k \leq K$ with $\phi^{(k)}(z) \in L^{(k)}(W^1_p(\Omega, \mathbb{R}^n), L_\infty(\Omega, \mathbb{R}^n))$ defined by

$$\phi^{(k)}(z)[h_1, ..., h_k] = \partial_{i_1,...,i_k} a(z)h^{i_1}_1 ...h^{i_k}_k \tag{B.3}$$

using induction on k. Therefore, we first prove this statement for $k = 1$. Let $h \in Z$. Note that since $W^1_p(\Omega, \mathbb{R}^n) \hookrightarrow C(\Omega, \mathbb{R}^n)$, we have that $h \in L_\infty(\Omega, \mathbb{R}^n)$ and that $\|h^j\|_{L_\infty} \leq \|h\|_Z$. Furthermore, for now let $x \in \Omega$ be fixed and define the function $\varphi : \mathbb{R} \to \mathbb{R}$ by $\varphi(\lambda) = a(z(x) + \lambda h(x))$. We note a is at least twice differentiable and thus that φ is twice differentiable with $\varphi'(\lambda) = \partial_i a(z(x) + \lambda h(x))h^i(x)$ and $\varphi''(\lambda) = \partial_{ij} a(z(x) + \lambda h(x))h^i(x)h^j(x)$. Now we use Taylor series to write $\varphi(1) - \varphi(0) - \varphi'(0) = \frac{1}{2}\int_0^1 \varphi''(\lambda)d\lambda$. It follows that

$$\begin{aligned}
&|a(z(x) + h(x)) - a(z(x)) - \partial_i a(z(x))h^i(x)| \\
&= |\varphi(1) - \varphi(0) - \varphi'(0)| \\
&= \left|\frac{1}{2}\int_0^1 \varphi''(\lambda)d\lambda\right| \\
&= \frac{1}{2}\left|\int_0^1 \partial_{ij} a(z(x) + \lambda h(x))h^i(x)h^j(x)d\lambda\right| \\
&\leq \int_0^1 |\partial_{ij} a(z(x) + \lambda h(x))h^i(x)h^j(x)|d\lambda \\
&\leq \int_0^1 \|\partial_{ij} a(z + \lambda h)h^i h^j\|_{L_\infty} d\lambda \\
&\leq \int_0^1 \|\partial_{ij} a\|_{L_\infty} \|h^i\|_{L_\infty} \|h^j\|_{L_\infty} d\lambda \\
&= \|\partial_{ij} a\|_{L_\infty} \|h^i\|_{L_\infty} \|h^j\|_{L_\infty} \\
&\leq \|\partial_{ij} a\|_{L_\infty} \|h\|_Z^2
\end{aligned}$$

from which we get

$$\begin{aligned}
&\|\phi(z + h) - \phi(z) - \partial_i a(z)h^i\|_{L_\infty} \\
&= \sup_{x\in\Omega} |\phi(z + h)(x) - \phi(z)(x) - (\partial_i a(z)h^i)(x)| \\
&= \sup_{x\in\Omega} |a(z(x) + h(x)) - a(z(x)) - \partial_i a(z(x))h^i(x)| \\
&\leq \|\partial_{ij} a\|_{L_\infty} \|h\|_Z^2 \\
&= o(\|h\|_Z).
\end{aligned}$$

It is easily checked that $h \mapsto \partial_i a(z)h^i$ is linear and that

$$\|\partial_i a(z)h^i\|_{L_\infty} \leq \|\partial_i a(z)\|_{L_\infty} \|h^i\|_{L_\infty} \leq \|\partial_i a(z)\|_{L_\infty} \|h\|_Z.$$

So we have that $h \mapsto \partial_i a(z)h^i$ is a bounded linear map and thus that ϕ is differentiable with $\phi'(z)[h] = \partial_i a(z)h^i$. Now that the statement is proven for $k = 1$ we suppose it holds for any $k < K$, i.e. suppose that ϕ is k times differentiable with $\phi^{(k)}(z)$ as in Equation (B.3). By repetition of the arguments above we have for all fixed $x \in \Omega$ that

$$|\partial_{i_1,\ldots,i_k} a(z(x)+h(x)) - \partial_{i_1,\ldots,i_k} a(z(x)) - \partial_{i_1,\ldots,i_k,i_{k+1}} a(z(x))h_1^{i_1} \ldots h^{i_{k+1}}(x)| \leq \|\partial_{i_1,\ldots,i_k} a\|_{L_\infty} \|h\|_Z^{k+1}$$

and thus that

$$\begin{aligned}
&\|\phi^{(k)}(z + h_{k+1})[h_1, \ldots, h_k] - \phi^{(k)}(z)[h_1, \ldots, h_k] - \partial_{i_1,\ldots,i_k,i_{k+1}} a(z)h_1^{i_1} \ldots h_{k+1}^{i_{k+1}}\|_{L_\infty} \\
&= \sup_{x\in\Omega} \|\phi^{(k)}(z + h_{k+1})(x) - \phi^{(k)}(z)(x) - \partial_{i_1,\ldots,i_k,i_{k+1}} a(z)h_1^{i_1} \ldots h_{k+1}^{i_{k+1}}(x)\| \\
&= \sup_{x\in\Omega} \|a(z(x) + h_{k+1}(x)) - a(z(x)) - \partial_{i_1,\ldots,i_k,i_{k+1}} a(z)h_1^{i_1} \ldots h_{k+1}^{i_{k+1}}(x)\| \\
&\leq \|\partial_{i_1,\ldots,i_k,i_{k+1}} a\|_{L_\infty} \|h\|_Z^{k+1} \\
&= o(\|h\|_Z).
\end{aligned}$$

Now using similar arguments as used above, we can prove that a function defined by $(h_1, \ldots, h_{k+1}) \mapsto \partial_{i_1,\ldots,i_k,i_{k+1}} a(z)h_1^{i_1} \ldots h_k^{i_k} h_{k+1}^{i_{k+1}}$ is bounded and linear in all of its arguments. Therefore, we proved that ϕ is $k+1$ times differentiable with $\phi^{(k)}(z)$ as in Equation (B.3), which concludes the induction proof. ■

B.1.1 Examples

In this section some derivatives of functions from the main problem in this report are discussed. The definitions of the spaces below can, therefore, be found in Chapter 2. The following functions are discussed:

$$\begin{aligned}
f_1 &: X^{\pm} \to L_p(\Omega_{\pm}) && f_1(\phi) = \mathcal{A}_\phi w, \\
f_2 &: X^{\pm} \to L_p(\Omega_{\pm}) && f_2(\phi) = \mathcal{A}_\phi \phi, \\
f_3 &: X^{B} \to L_p(\Gamma) && f_3(\phi) = \frac{1}{1+\phi_{z_n}}, \\
f_4 &: X^{B} \to L_p(\Gamma) && f_4(\phi) = |\nabla' \phi|^2.
\end{aligned}$$

First of all we define

$$\begin{aligned}
g_i &: W_p^1(\Omega_{\pm}, \mathbb{R}^n) \to L_\infty(\Omega_{\pm}, \mathbb{R}) && g_i(z)(x) := \vec{a}_i(z(x)), \\
G &: W_p^2(\Omega_{\pm}, \mathbb{R}) \to W_p^1(\Omega_{\pm}, \mathbb{R}^n) && G(\phi) = \nabla \phi, \\
G_i &: W_p^2(\Omega_{\pm}, \mathbb{R}) \to L_p(\Omega_{\pm}, \mathbb{R}) && G_i(\phi) = \partial_i \phi_{z_n}.
\end{aligned}$$

Note that $f_1(\phi)$ can be written as $f_1(\phi) = \sum_{i=1}^{n-1} w_{z_i z_i} - \sum_{i=1}^{n} (g_i \circ G)(\phi) \cdot \nabla w_{z_n}$ and therefore f_1 is differentiable if $(g_i \circ G)$ is for all i. In that case we have $f_1'(\phi)[\delta] = -\sum_{i=1}^{n} (g_i \circ G)'(\phi)[\delta] \cdot \nabla w_{z_n}$. Since $a_i \in C_b^\infty(\mathbb{R}^n)$, by Lemma B.5, we have that g_i is infinitely many times differentiable and that its first derivative is given by $g_i'(z)[h] = \sum_{k=1}^{n} \partial_k a_i(z) h^k$ for all i.
From G being a bounded linear function it follows by Lemma B.4 that $g_i \circ G$ is infinitely many times differentiable with $(g_i \circ G)'(\phi)[\delta] = g_i'(G(\phi))[G(\delta)] = \sum_{k=1}^{n} \partial_k a_i(\nabla\phi)\partial_k \delta = \nabla a_i(\nabla \phi) \cdot \nabla \delta$ for all i.
So we can conclude from the above that f_1 is infinitely times differentiable and that its first derivative is given by

$$f_1'(\phi)[\delta] = -A(\nabla\phi)\nabla\delta \cdot \nabla w_{z_n},$$

where $A(p) = D_p \vec{a}(p)$.

Now we write $f_2(\phi) = \sum_{i=1}^{n-1} \phi_{z_i z_i} - \sum_{i=1}^{n} ((g_i \circ G) \cdot G_i)(\phi)$.
We already proved that since $a_i \in C_b^\infty(\mathbb{R}^n)$, we have that $(g_i \circ G)$ is infinitely many times differentiable with $(g_i \circ G)'(\phi)[\delta] = \nabla a_i(\nabla\phi) \cdot \nabla\delta$ for all i.
Since G_i is a bounded linear function, it is infinitely many times differentiable and $G_i'(\phi)[\delta] = G_i \delta$ for all i.
Using all of the above and $L_p(\Omega_{\pm}, \mathbb{R}) \cdot L_\infty(\Omega_{\pm}, \mathbb{R}) \hookrightarrow L_p(\Omega, \mathbb{R})$, by Lemma B.3 also $((g_i \circ G) \cdot G_i)$ is infinitely many times differentiable with $((g_i \circ G) \cdot G_i)'(\phi)[\delta] = (g_i \circ G)'(\phi)[\delta] \cdot G_i(\phi) + (g_i \circ G)(\phi) \cdot G_i'(\phi)[\delta] = \sum_{k=1}^{n} \partial_k a_i(\nabla\phi)\partial_k\delta \partial_i \phi_{z_n} + a_i(\nabla\phi)\partial_i \delta_{z_n} = \nabla a_i(\nabla\phi)\nabla\delta \cdot \partial_i \phi_{z_n} + a_i(\nabla\phi)\partial_i\delta_{z_n}$ for all i.
Now combining these results with the fact that $\phi \mapsto \phi_{z_i z_i}$ is linear in ϕ for all i gives that f_2 is infinitely many times differentiable with

$$f_2'(\phi)[\delta] = \sum_{i=1}^{n-1} \delta_{z_i z_i} - A(\nabla\phi)\nabla\delta \cdot \nabla\phi_{z_n} - \vec{a}(\nabla\phi)\nabla\delta_{z_n}.$$

For the third function we use that $\phi_{z_n}(x,t) > \varepsilon - 1$ for all $(x,t) \in \Omega \times J$ (see (2.12)). This gives the estimate

$$
\begin{aligned}
&\left\| f_3(\phi+\delta) - f_3(\phi) - \frac{-\delta_{z_n}}{(1+\phi_{z_n})^2} \right\|_{L_p(\Gamma)} \\
&= \left\| \frac{1}{1+\phi_{z_n}+\delta_{z_n}} - \frac{1}{1+\phi_{z_n}} + \frac{\delta_{z_n}}{(1+\phi_{z_n})^2} \right\|_{L_p(\Gamma)} \\
&= \left\| \frac{(1+\phi_{z_n})^2 - (1+\phi_{z_n})(1+\phi_{z_n}+\delta_{z_n}) + \delta_{z_n}(1+\phi_{z_n}+\delta_{z_n})}{(1+\phi_{z_n}+\delta_{z_n})(1+\phi_{z_n})^2} \right\|_{L_p(\Gamma)} \\
&\leq \left\| \frac{\delta_{z_n}^2}{(1+\phi_{z_n}+\delta_{z_n})(1+\phi_{z_n})^2} \right\|_{L_p(\Gamma)} \\
&\leq \left\| \frac{1}{(1+\phi_{z_n}+\delta_{z_n})(1+\phi_{z_n})^2} \right\|_{L_\infty(\Gamma)} \|\delta_{z_n}\|_{L_\infty(\Gamma)} \|\delta_{z_n}\|_{L_p(\Gamma)} \\
&\leq c\|\delta\|_X^2
\end{aligned}
$$

It is easily checked that the map $\delta \mapsto \frac{-\delta_{z_n}}{(1+\phi_{z_n})^2}$ is linear and since

$$
\left\| \frac{-\delta_{z_n}}{(1+\phi_{z_n})^2} \right\|_{L_p(\Gamma)} \leq \left\| \frac{1}{(1+\phi_{z_n})^2} \right\|_{L_\infty(\Gamma)} \|\delta_{z_n}\|_{L_p(\Gamma)} \leq c\|\delta\|_{X^B},
$$

it is also bounded so

$$
f_3'(\phi)[\delta] = \frac{-\delta_{z_n}}{(1+\phi_{z_n})^2}.
$$

Using induction and arguments analogue to those above, we have that f_3 is infinitely many times differentiable.

For the last function we estimate

$$
\begin{aligned}
&\|f_4(\phi+\delta) - f_4(\phi) - 2\nabla'\phi \cdot \nabla'\delta\|_{L_p(\Gamma)} \\
&= \left\| \sum_{i=1}^{n-1} (\phi_{z_i}+\delta_{z_i})^2 - \sum_{i=1}^{n-1} \phi_{z_i}^2 - 2\sum_{i=1}^{n-1} \phi_{z_i}\delta_{z_i} \right\|_{L_p(\Gamma)} \\
&= \left\| \sum_{i=1}^{n-1} \delta_{z_i}^2 \right\|_{L_p(\Gamma)} \\
&\leq \sum_{i=1}^{n-1} \|\delta_{z_i}\|_{L_p(\Gamma)} \|\delta_{z_i}\|_{L_\infty(\Gamma)} \\
&\leq (n-1)\|\delta\|_X^2.
\end{aligned}
$$

Now since

$$
\|2\nabla'\phi \cdot \nabla'\delta\|_{L_p(\Gamma)} \leq 2\|\nabla'\phi\|_{L_\infty(\Gamma)} \|\nabla'\delta\|_{L_p(\Gamma)} \leq 2\|\phi\|_X \|\delta\|_X,
$$

it is clear that $\delta \mapsto 2\nabla'\phi \cdot \nabla'\delta$ is a bounded linear map and thus that

$$
f_4'(\phi)[\delta] = 2\nabla'\phi \cdot \nabla'\delta.
$$

Note that the $f_4'(\phi)[\delta]$ is linear in ϕ and using Lemma B.2, this gives that f_4 is infinitely many times differentiable.

B.1.2 The derivatives of $R^{\pm}$ and R^B

The derivatives of $R^{\pm}$ and R^B are given by

$$
\begin{aligned}
&R^{B\prime}(u^-,u^+,\phi^-,\phi^+,\phi)[h^-,h^+,\delta^-,\delta^+,\delta] = \\
&(1+|\nabla'\sigma|^2)\left(\alpha\frac{h^-_{z_n}\phi^-_{z_n}}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^-_{z_n})} - \beta\frac{h^+_{z_n}\phi^+_{z_n}}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^+_{z_n})}\right) \\
&+2\nabla'\sigma\cdot\nabla'\phi\left(-\alpha\frac{h^-_{z_n}}{1+\sigma_{z_n}+\phi^-_{z_n}} + \beta\frac{h^+_{z_n}}{1+\sigma_{z_n}+\phi^+_{z_n}}\right) \\
&+|\nabla'\phi|^2\left(-\alpha\frac{h^-_{z_n}}{1+\sigma_{z_n}+\phi^-_{z_n}} + \beta\frac{h^+_{z_n}}{1+\sigma_{z_n}+\phi^+_{z_n}}\right), \\
&+(1+|\nabla'\sigma|^2)\Bigg(\ \alpha\left(\frac{u^-_{z_n}\delta^-_{z_n}}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^-_{z_n})} - \frac{u^-_{z_n}\phi^-_{z_n}\delta^-_{z_n}}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^-_{z_n})^2}\right. \\
&\qquad\qquad \left. - \frac{U^-_{z_n}\phi^-_{z_n}\delta^-_{z_n}}{(1+\sigma_{z_n})^2(1+\sigma_{z_n}+\phi^-_{z_n})} + \frac{U^-_{z_n}{\phi^-_{z_n}}^2\delta^-_{z_n}}{(1+\sigma_{z_n})^2(1+\sigma_{z_n}+\phi^-_{z_n})^2}\right) \\
&\qquad -\beta\left(\frac{u^+_{z_n}\delta^+_{z_n}}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^+_{z_n})} - \frac{u^+_{z_n}\phi^+_{z_n}\delta^+_{z_n}}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^+_{z_n})^2}\right. \\
&\qquad\qquad \left. - \frac{U^+_{z_n}\phi^+_{z_n}\delta^+_{z_n}}{(1+\sigma_{z_n})^2(1+\sigma_{z_n}+\phi^+_{z_n})} + \frac{U^+_{z_n}{\phi^+_{z_n}}^2\delta^+_{z_n}}{(1+\sigma_{z_n})^2(1+\sigma_{z_n}+\phi^+_{z_n})^2}\right) \\
&+2\nabla'\sigma\cdot\nabla'\phi\left(-\alpha\left(\frac{u^-_{z_n}\delta^-_{z_n}}{(1+\sigma_{z_n}+\phi^-_{z_n})^2} - \frac{U^-_{z_n}\delta^-_{z_n}}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^-_{z_n})} + \frac{U^-_{z_n}\phi^-_{z_n}\delta^-_{z_n}}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^-_{z_n})^2}\right)\right. \\
&\qquad \left. +\beta\left(\frac{u^+_{z_n}\delta^+_{z_n}}{(1+\sigma_{z_n}+\phi^+_{z_n})^2} - \frac{U^+_{z_n}\delta^+_{z_n}}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^+_{z_n})} + \frac{U^+_{z_n}\phi^+_{z_n}\delta^+_{z_n}}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^+_{z_n})^2}\right)\right) \\
&+|\nabla'\phi|^2\left(-\alpha\frac{(U^-_{z_n}+u^-_{z_n})\delta^-_{z_n}}{(1+\sigma_{z_n}+\phi^-_{z_n})^2} + \beta\frac{(U^+_{z_n}+u^+_{z_n})\delta^+_{z_n}}{(1+\sigma_{z_n}+\phi^+_{z_n})^2}\right), \\
&+2\nabla'\sigma\cdot\nabla'\delta\left(-\alpha\left(\frac{u^-_{z_n}}{1+\sigma_{z_n}+\phi^-_{z_n}} - \frac{U^-_{z_n}\phi^-_{z_n}}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^-_{z_n})}\right)\right. \\
&\qquad \left. +\beta\left(\frac{u^+_{z_n}}{1+\sigma_{z_n}+\phi^+_{z_n}} - \frac{U^+_{z_n}\phi^+_{z_n}}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^+_{z_n})}\right)\right) \\
&+2\nabla'\phi\nabla'\delta\left(-\alpha\frac{U^-_{z_n}+u^-_{z_n}}{1+\sigma_{z_n}+\phi^-_{z_n}} + \beta\frac{U^+_{z_n}+u^+_{z_n}}{1+\sigma_{z_n}+\phi^+_{z_n}}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&R^{\pm\prime}(u^{\pm},\phi^{\pm})[h^{\pm},\delta^{\pm}] = \\
&A(\nabla\sigma)\nabla\phi^{\pm}\nabla h^{\pm}_{z_n} + \vec{b}(\nabla\sigma,\nabla\phi^{\pm})\nabla h^{\pm}_{z_n} \\
&+ \frac{h^{\pm}_{z_n}}{1+\sigma_{z_n}+\phi^{\pm}_{z_n}}(A(\nabla\sigma)\nabla\phi^{\pm}\nabla\phi^{\pm}_{z_n} + \vec{b}(\nabla\sigma,\nabla\phi^{\pm})(\nabla\sigma_{z_n}+\nabla\phi^{\pm}_{z_n})) \\
&+ \frac{\phi^{\pm}_{z_n}\hat{L}^{\pm}_{\sigma}\sigma}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^{\pm}_{z_n})}h^{\pm}_{z_n}, \\
&+ A(\nabla\sigma)\nabla\delta^{\pm}\nabla u^{\pm}_{z_n} + \vec{b}'(\nabla\sigma,\nabla\phi^{\pm})[\delta^{\pm}](\nabla U^{\pm}_{z_n}+\nabla u^{\pm}_{z_n}) \\
&+ \frac{U^{\pm}_{z_n}+u^{\pm}_{z_n}\delta^{\pm}_{z_n}}{(1+\sigma_{z_n}+\phi^{\pm}_{z_n})^2}(A(\nabla\sigma)\nabla\phi^{\pm}\nabla\phi^{\pm}_{z_n} + \vec{b}(\nabla\sigma,\nabla\phi^{\pm})(\nabla\sigma_{z_n}+\nabla\phi^{\pm}_{z_n})) \\
&+ \frac{U^{\pm}_{z_n}+u^{\pm}_{z_n}}{1+\sigma_{z_n}+\phi^{\pm}_{z_n}}(A(\nabla\sigma)\nabla\delta^{\pm}\nabla\phi^{\pm}_{z_n} + A(\nabla\sigma)\nabla\phi^{\pm}\nabla\delta^{\pm}_{z_n} \\
&\qquad\qquad + \vec{b}'(\nabla\sigma,\nabla\phi^{\pm})[\delta^{\pm}](\nabla\sigma_{z_n}+\nabla\phi^{\pm}_{z_n}) + \vec{b}(\nabla\sigma,\nabla\phi^{\pm})\nabla\delta^{\pm}_{z_n}) \\
&+ \left(\frac{\delta^{\pm}_{z_n}\hat{L}^{\pm}_{\sigma}\sigma}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^{\pm}_{z_n})} + \frac{\phi^{\pm}_{z_n}\hat{L}^{\pm}_{\sigma}\sigma\delta^{\pm}_{z_n}}{(1+\sigma_{z_n})(1+\sigma_{z_n}+\phi^{\pm}_{z_n})^2}\right)\left(u^{\pm}_{z_n} - \frac{U^{\pm}_{z_n}\phi^{\pm}_{z_n}}{1+\sigma_{z_n}}\right) \\
&+ \frac{\phi^{\pm}_{z_n}\hat{L}^{\pm}_{\sigma}\sigma U^{\pm}_{z_n}\delta^{\pm}_{z_n}}{(1+\sigma_{z_n})^2(1+\sigma_{z_n}+\phi^{\pm}_{z_n})},
\end{aligned}
$$

where

$$
\begin{aligned}
&\vec{b}'(\nabla\sigma,\nabla\phi^{\pm})[\delta^{\pm}] = \\
&\sum_{i,j=1}^{n}\int_0^1 (1-s)\partial_{ij}A(\nabla\sigma+s\nabla\phi^{\pm})\partial_i\phi^{\pm}\partial_j\delta^{\pm}\cdot\nabla\phi^{\pm}ds \\
&+\sum_{i=1}^{n}\int_0^1 (1-s)\partial_i A(\nabla\sigma+s\nabla\phi^{\pm})\partial_i\delta^{\pm}\cdot\nabla\phi^{\pm}ds \\
&+\sum_{i=1}^{n}\int_0^1 (1-s)\partial_i A(\nabla\sigma+s\nabla\phi^{\pm})\partial_i\phi^{\pm}\cdot\nabla\delta^{\pm}ds.
\end{aligned}
$$

B.2 Implicit Function Theorem

For completeness the exact version of the Implicit Function Theorem that is used in the report is stated here. It can be found in [23] Theorem 4B together with a proof.

Theorem B.6. *Suppose that:*

(i) the mapping $F : U(x_0, y_0) \subseteq X \times Y \to Z$ is defined on an open neighbourhood $u(x_0, y_0)$ and $F(x_0, y_0) = 0$, where X, Y, and Z are Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$;

(ii) F_y exists as a partial Fréchet derivative on $U(x_0, y_0)$ and the partial Fréchet derivative $F_y(x_0, y_0) : Y \to Z$ is bijective;

(iii) F and F_y are continuous at (x_0, y_0).

Then the following are true:

(a) Existence and uniqueness. There exist positive numbers r_0 and r such that for every $x \in X$ satisfying $\|x - x_0\| \leq r_0$, there is exactly one $y(x) \in Y$ for which $\|y(x) - y_0\| \leq r$ and $F(x, y(x)) = 0$.

(b) Construction of the solution. The sequence $(y_n(x))$ of successive approximations, defined by $y_0(x) \equiv y_0$, and

$$y_{n+1} = y_n(x) - F_y(x_0, y_0)^{-1}F(x, y_n(x)),$$

converges to the solution $y(x)$, as $n \to \infty$, for all points $x \in X$ satisfying $\|x - x_0\| \leq r_0$.

(c) Continuity. If F is continuous in a neighbourhood of (x_0, y_0), then $y(\cdot)$ is continuous in a neighbourhood of x_0.

(d) Continuous differentiability. If F has continuous Fréchet derivatives up to order m, $1 \le m \le \infty$, on a neighbourhood of (x_0, y_0), then so does $y(\cdot)$ on a neighbourhood of x_0.

B.3 Embedding theorems

Lemma B.7. *Let Ω be any bounded space, $J \subset \mathbb{R}$ an interval, $p > 2 + n$ and $\theta > \frac{1}{p}S$. Define*

$$X^+ = L_p(J, W_p^2(\Omega)) \cap H_p^1(J, L_p(\Omega)),$$
$$X^- = L_p(J, W_p^2(\Omega)) \cap W_p^\theta(J, W_p^{2-\theta}(\Omega)),$$
$$X^B = L_p(J, W_p^{2-1/p}(\Omega) \cap H_p^1(J, W_p^{1-1/p}(\Omega)) \cap W_p^{1+\theta}(J, L_p(\Omega)).$$

For the given value of p we have

$$X^+ \hookrightarrow C(J, C^1(\Omega)),$$
$$X^- \hookrightarrow C(J, C^1(\Omega)),$$
$$X^B \hookrightarrow C(J, C^1(\Omega)).$$

Proof. In general we have [1]

$$W_p^k(\Omega) \hookrightarrow C^\ell(\Omega) \quad \text{if} \quad k - \ell > \frac{n}{p},$$

which means that

$$W_p^t(J, W_p^s(\Omega)) \hookrightarrow C(J, C^1(\Omega)) \quad \text{if} \quad t > \frac{1}{p} \text{ and } s > 1 + \frac{n}{p}. \tag{B.4}$$

Now we consider each of the cases separately.

- Firstly,
$$X^+ \hookrightarrow W_p^t(J, W_p^s(\Omega)) \quad \text{for} \quad s + 2t \le 2.$$
Combining this with condition (B.4) gives that $X^+ \hookrightarrow C(J, C^1(\Omega))$ if $\left(1 + \frac{n}{p}\right) + 2\left(\frac{1}{p}\right) < 2$ and thus if $p > 2 + n$.
- Secondly,
$$X^- \hookrightarrow W_p^t(J, W_p^s(\Omega)) \quad \text{for} \quad s + t \le 2 \text{ and } t \le \theta.$$
Combining this with condition (B.4) gives that $X^- \hookrightarrow C(J, C^1(\Omega))$ if $\left(1 + \frac{n}{p}\right) + \left(\frac{1}{p}\right) < 2$ or equivalently $p > 1 + n$ and that $p > \frac{1}{\theta}$.
- Finally,
$$X^B \hookrightarrow W_p^t(J, W_p^s(\Omega)) \quad \text{for} \quad s + t \le 2 - \frac{1}{p} \text{ and } \theta s + \left(1 - \frac{1}{p}\right) t \le (1 + \theta)\left(1 - \frac{1}{p}\right).$$
Combining this with condition (B.4) gives that $X^B \hookrightarrow C(J, C^1(\Omega))$ if $\left(1 + \frac{n}{p}\right) + \left(\frac{1}{p}\right) \le 2 - \frac{1}{p}$ and $\theta\left(1 + \frac{n}{p}\right) + \left(1 - \frac{1}{p}\right)\frac{1}{p} < (1 + \theta)\left(1 - \frac{1}{p}\right)$. Note that both are satisfied for $p > 2 + n$.

■

Lemma B.8. *Let $p' < p$, $\theta'' < \theta' < \theta$ such that $\frac{1}{p'} + \theta' \leq \frac{1}{p} + \theta$, $\frac{1}{p'} - \theta' \leq \frac{1}{p} - \theta''$ and $I \subset J = [0,T]$ an interval of length $\ell < 1$. Then there exist $c > 0$, $\beta > 0$ such that*

$$\|u\|_{L_{p'}(I)} \leq \ell^\beta \|u\|_{L_p(J)}, \tag{B.5}$$

$$\|u(t)\|_{L_{p'}(\Omega)} \leq c\|u(t)\|_{L_p(\Omega)}, \tag{B.6}$$

$$\|u\|_{L_{p'}(\Omega\times I)} \leq c\ell^\beta \|u\|_{L_p(\Omega\times J)}, \tag{B.7}$$

$$\|u\|_{W^{\theta'}_{p'}(I,L_{p'}(\Omega))} \leq c\ell^\beta \|u\|_{W^\theta_p(J,L_p(\Omega))}, \tag{B.8}$$

$$\|u\|_{W^{1-1/p'}_{p'}(\Omega)} \leq c\|u\|_{W^{1-1/p}_p(\Omega)}, \tag{B.9}$$

$$\|u\|_{L_{p'}(I,W^{1-1/p'}_{p'}(\Omega))} \leq c\ell^\beta \|u\|_{L_p(J,W^{1-1/p}_p(\Omega))}, \tag{B.10}$$

$$\|u\|_{W^{-\theta'}_{p'}(\Omega)} \leq c\|u\|_{W^{-\theta''}_p(\Omega)}, \tag{B.11}$$

$$\|u\|_{W^{\theta'}_{p'}(I,W^{-\theta'}_{p'}(\Omega))} \leq c\ell^\beta \|u\|_{W^\theta_p(J,W^{-\theta''}_p(\Omega))}. \tag{B.12}$$

Proof. Let $\alpha > 1$ be such that $p = \alpha p'$, define α' by $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$ and let $\beta = \frac{1}{\alpha' p'}$. We apply the Hölder inequality to prove each of the inequalities.

(B.5) Firstly

$$\begin{aligned}\|u\|_{L_{p'}(I)} &= \left(\int_I |u|^{p'}\right)^{\frac{1}{p'}} = \left(\int_I 1\cdot|u|^{p'}\right)^{\frac{1}{p'}} \leq \left(\|1\|_{L_{\alpha'}(I)}\cdot \||u|^{p'}\|_{L_\alpha(I)}\right)^{\frac{1}{p'}} \\ &= \|1\|^{\frac{1}{p'}}_{L_{\alpha'}(I)} \cdot \left(\int_I |u|^{\alpha p'}\right)^{\frac{1}{\alpha p'}} = \ell^{\frac{1}{\alpha' p'}} \cdot \|u\|_{L_p(I)} \leq \ell^\beta \|u\|_{L_p(J)}.\end{aligned}$$

(B.6) Secondly, there exists a $c > 0$ such that

$$\begin{aligned}\|u\|_{L_{p'}(\Omega)} &= \left(\int_\Omega |u|^{p'}\right)^{\frac{1}{p'}} = \left(\int_\Omega 1\cdot|u|^{p'}\right)^{\frac{1}{p'}} \leq \left(\|1\|_{L_{\alpha'}(\Omega)}\cdot \||u|^{p'}\|_{L_\alpha(\Omega)}\right)^{\frac{1}{p'}} \\ &= \|1\|^{\frac{1}{p'}}_{L_{\alpha'}(\Omega)} \cdot \left(\int_\Omega |u|^{\alpha p'}\right)^{\frac{1}{\alpha p'}} = c\|u\|_{L_p(\Omega)},\end{aligned}$$

which proves the second inequality.

(B.7) For the third inequality we use Equation (B.6). This gives for some c_1, $c_2 > 0$ that

$$\begin{aligned}\|u\|_{L_{p'}(\Omega\times I)} &= \left(\int_I \|u(t)\|^{p'}_{L_{p'}(\Omega)} dt\right)^{\frac{1}{p'}} \leq \left(\int_I c_1\|u(t)\|^{p'}_{L_p(\Omega)} dt\right)^{\frac{1}{p'}} \\ &\leq c_2\left(\int_I 1\cdot\|u(t)\|^{p'}_{L_p(\Omega)} dt\right)^{\frac{1}{p'}} \leq c_2\left(\|1\|_{L^{\alpha'}(I)} \cdot \left(\int_I \left(\|u(t)\|^{p'}_{L_p(\Omega)}\right)^\alpha dt\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{p'}} \\ &= c_2\|1\|^{\frac{1}{p'}}_{L_{\alpha'}(I)} \cdot \left(\int_I \|u(t)\|^{\alpha p'}_{L_p(\Omega)} dt\right)^{\frac{1}{\alpha p'}} = c_2\ell^\beta \|u\|_{L_p(\Omega\times I)} \leq c_2\ell^\beta\|u\|_{L_p(\Omega\times J)}.\end{aligned}$$

(B.8) In the fourth equation, by definition, we have

$$\|u\|_{W^{\theta'}_{p'}(I,L_{p'}(\Omega))} = \|u\|_{L_{p'}(\Omega\times I)} + \left(\int_I\int_I \frac{\|u(t)-u(s)\|^{p'}_{L_{p'}(\Omega)}}{|t-s|^{1+\theta' p'}} dtds\right)^{\frac{1}{p'}}.$$

For the first part use Equation (B.7), which states there exists a $c_1 > 0$ such that

$$\|u(t)\|_{L_{p'}(\Omega\times I)} \leq c_1\ell^\beta \|u(t)\|_{L_p(\Omega\times I)}.$$

For the second part we have to use the fact that for $t, s \in I$ we have $|t-s| \leq 1$. Note that this gives that $|t-s|^{\frac{1}{p'}+\theta'} \geq |t-s|^{\frac{1}{p}+\theta}$. Together with (B.6) we get that there exist c_3, c_4 such that

$$\left(\int_I\int_I \frac{\|u(t)-u(s)\|_{L_{p'}(\Omega)}^{p'}}{|t-s|^{1+\theta' p'}} dtds\right)^{\frac{1}{p'}} = \left(\int_I\int_I \left(\frac{\|u(t)-u(s)\|_{L_{p'}(\Omega)}}{|t-s|^{\frac{1}{p'}+\theta'}}\right)^{p'} dtds\right)^{\frac{1}{p'}}$$

$$\leq \left(\int_I\int_I \left(c_3 \frac{\|u(t)-u(s)\|_{L_p(\Omega)}}{|t-s|^{\frac{1}{p}+\theta}}\right)^{p'} dtds\right)^{\frac{1}{p'}}$$

$$= c_4 \left(\int_I\int_I 1\cdot \left(\frac{\|u(t)-u(s)\|_{L_p(\Omega)}}{|t-s|^{\frac{1}{p}+\theta}}\right)^{p'} dtds\right)^{\frac{1}{p'}}$$

$$\leq c_4 \|1\|_{L^{\alpha'}(I\times I)}^{\frac{1}{p'}} \left(\int_I\int_I \left(\frac{\|u(t)-u(s)\|_{L_p(\Omega)}}{|t-s|^{\frac{1}{p}+\theta}}\right)^{\alpha p'} dtds\right)^{\frac{1}{\alpha p'}}$$

$$= c_4 \ell^{2\beta} \left(\int_I\int_I \frac{\|u(t)-u(s)\|_{L_p(\Omega)}^{p}}{|t-s|^{1+\theta p}} dtds\right)^{\frac{1}{p}} = c_4 \ell^{\beta} \left(\int_J\int_J \frac{\|u(t)-u(s)\|_{L_p(\Omega)}^{p}}{|t-s|^{1+\theta p}} dtds\right)^{\frac{1}{p}}$$

Combining both results proves the inequality.

(B.9) This inequality follows easily from

$$\|u\|_{W_{p'}^{1-1/p'}(\Omega)} = \|u\|_{L_{p'}(\Omega)} + \left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p'}}{|x-y|^{1+(1-1/p')p'}} dxdy\right)^{\frac{1}{p'}}$$

$$= \|u\|_{L_{p'}(\Omega)} + \left(\int_\Omega\int_\Omega \left(\frac{|u(x)-u(y)|}{|x-y|}\right)^{p'} dxdy\right)^{\frac{1}{p'}}$$

$$\leq c_1\|u\|_{L_p(\Omega)} + \left(\|1\|_{L^{\alpha'}(\Omega\times\Omega)} \int_\Omega\int_\Omega \left(\frac{|u(x)-u(y)|}{|x-y|}\right)^{\alpha p'} dxdy\right)^{\frac{1}{\alpha p'}}$$

$$= c_1\|u\|_{L_p(\Omega)} + \left(\|1\|_{L^{\alpha'}(\Omega\times\Omega)} \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{1+(1-1/p)p}} dxdy\right)^{\frac{1}{p}}$$

$$\leq c_2\|u\|_{W_p^{1-1/p}(\Omega)}.$$

(B.10) This inequality can be proven analogously to Equation (B.7), where now the inequality (B.9) is used instead of (B.6).

(B.11) By definition we have

$$\|u\|_{W_{p'}^{-\theta'}(\Omega)} = \|u\|_{L_{p'}(\Omega)} + \left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p'}}{|x-y|^{1-\theta' p'}} dxdy\right)^{\frac{1}{p'}}.$$

From (B.6) it already follows that

$$\|u\|_{L_{p'}(\Omega)} \leq c_1\|u\|_{L_p(\Omega)}.$$

For the second part we note that Ω is bounded and thus that there exists a constant c such that for all $x, y \in \Omega$ and $b > a$ we have $|x-y|^b \leq c|x-y|^a$. Since $\frac{1}{p'} - \theta' \leq \frac{1}{p} - \theta''$, this gives

$$\left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p'}}{|x-y|^{1-\theta' p'}} dxdy\right)^{\frac{1}{p'}} = \left(\int_\Omega \int_\Omega 1 \cdot \left(\frac{|u(x)-u(y)|}{|x-y|^{\frac{1}{p'}-\theta'}}\right)^{p'} dxdy\right)^{\frac{1}{p'}}$$

$$\leq \left(\int_\Omega \int_\Omega 1 \cdot \left(\frac{|u(x)-u(y)|}{|x-y|^{\frac{1}{p}-\theta''}}\right)^{p'} dxdy\right)^{\frac{1}{p'}}$$

$$\leq \left(\|1\|_{L_{\alpha'}(\Omega\times\Omega)} \int_\Omega \int_\Omega \left(\frac{|u(x)-u(y)|}{|x-y|^{\frac{1}{p}-\theta''}}\right)^{\alpha p'} dxdy\right)^{\frac{1}{\alpha p'}} \leq c \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{1-\theta'' p}} dxdy\right)^{\frac{1}{p}}.$$

Combining both inequalities gives that indeed

$$\|u\|_{W_{p'}^{-\theta'}(\Omega)} \leq c\|u\|_{W_p^{-\theta''}(\Omega)}$$

(B.12) Finally we have

$$\|u\|_{W_{p'}^{\theta'}(I, W_p^{-\theta}(\Omega))} = \|u\|_{L_{p'}(I, W_{p'}^{-\theta'}(\Omega))} + \left(\int_I \int_I \frac{\|u(t)-u(s)\|^{p'}_{W_{p'}^{-\theta'}(\Omega)}}{|t-s|^{1+\theta' p'}} dtds\right)^{\frac{1}{p'}}$$

For the first part we get, using (B.11), that there exist c_1, $c_2 > 0$ such that

$$\|u\|_{L_{p'}(\Omega\times I)} = \left(\int_I \|u(t)\|^{p'}_{W_{p'}^{-\theta'}} dt\right)^{\frac{1}{p'}} \leq \left(\int_I c_1 \|u(t)\|^{p'}_{W_p^{-\theta''}} dt\right)^{\frac{1}{p'}}$$

$$\leq c_2 \left(\int_I 1 \cdot \|u(t)\|^{p'}_{W_p^{-\theta''}} dt\right)^{\frac{1}{p'}} \leq c_2 \left(\|1\|_{L_{\alpha'}(I)} \cdot \left(\int_I \left(\|u\|^{p'}_{W_p^{-\theta''}}\right)^\alpha dt\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{p'}}$$

$$= c_2 \|1\|^{\frac{1}{p'}}_{L_{\alpha'}(I)} \cdot \left(\int_I \|u\|^{\alpha p'}_{W_p^{-\theta''}} dt\right)^{\frac{1}{\alpha p'}} = c_2 \ell^\beta \|u\|_{L_p(I, W_p^{-\theta''})} \leq c_2 \ell^\beta \|u\|_{L_p(J, W_p^{-\theta''})}.$$

For the second part we again use that from $|t-s| \leq 1$ follows that $|t-s|^{\frac{1}{p'}+\theta'} \geq |t-s|^{\frac{1}{p}+\theta}$. Together with (B.11) we get that there exist c_3, c_4 such that

$$\left(\int_I \int_I \frac{\|u(t)-u(s)\|^{p'}_{W_{p'}^{-\theta'}(\Omega))}}{|t-s|^{1+\theta' p'}} dtds\right)^{\frac{1}{p'}} = \left(\int_I \int_I \left(\frac{\|u(t)-u(s)\|_{W_{p'}^{-\theta'}(\Omega))}}{|t-s|^{\frac{1}{p'}+\theta'}}\right)^{p'} dtds\right)^{\frac{1}{p'}}$$

$$\leq \left(\int_I \int_I \left(c_3 \frac{\|u(t)-u(s)\|_{W_p^{-\theta''}(\Omega))}}{|t-s|^{\frac{1}{p}+\theta}}\right)^{p'} dtds\right)^{\frac{1}{p'}}$$

$$\leq c_4 \left(\int_I \int_I 1 \cdot \left(\frac{\|u(t)-u(s)\|_{W_p^{-\theta''}(\Omega))}}{|t-s|^{\frac{1}{p}+\theta}}\right)^{p'} dtds\right)^{\frac{1}{p'}}$$

$$\leq c_4 \left(\|1\|_{L_{\alpha'}(I\times I)} \int_I \int_I \left(\frac{\|u(t)-u(s)\|_{W_p^{-\theta''}(\Omega))}}{|t-s|^{\frac{1}{p}+\theta}}\right)^{\alpha p'} dtds\right)^{\frac{1}{\alpha p'}}$$

$$\leq c_4 \ell^{2\beta} \left(\int_I \int_I \frac{\|u(t)-u(s)\|^p_{W_p^{-\theta''}(\Omega))}}{|t-s|^{1+\theta p}} dtds\right)^{\frac{1}{p}} \leq c_4 \ell^\beta \left(\int_J \int_J \frac{\|u(t)-u(s)\|^p_{W_p^{-\theta''}(\Omega))}}{|t-s|^{1+\theta p}} dtds\right)^{\frac{1}{p}}.$$

Combining both results above proves the last inequality.

■

Lemma B.9. *Let $p' < p$, $\theta'' < \theta' < \theta$ such that $\frac{1}{p'} + \theta' \leq \frac{1}{p} + \theta$, $\frac{1}{p'} - \theta' \leq \frac{1}{p} - \theta''$ then for all $\varepsilon > 0$ there exists $\ell < 1$ such that on any interval $I \subset J = [0, T]$ with a length smaller than ℓ we have*

$$\begin{aligned}
\|u\|_{L_{p'}(I)} &\leq \varepsilon && \text{for all } u \in L_p(J),\\
\|u\|_{L_{p'}(\Omega\times I)} &\leq \varepsilon && \text{for all } u \in L_p(\Omega \times J),\\
\|u\|_{W^{\theta'}_{p'}(I, L_{p'}(\Omega))} &\leq \varepsilon && \text{for all } u \in W^{\theta}_p(J, L_p(\Omega)),\\
\|u\|_{L_{p'}(I, W^{1-1/p'}_{p'}(\Omega))} &\leq \varepsilon && \text{for all } u \in L_p(J, W^{1-1/p}_p(\Omega)),\\
\|u\|_{W^{\theta'}_{p'}(I, W^{-\theta'}_{p'}(\Omega))} &\leq \varepsilon && \text{for all } u \in W^{\theta}_p(J, W^{-\theta''}_p(\Omega)).
\end{aligned}$$

In all of these cases ℓ only depends on ε, J, Ω, p, p' and the norm of u in the corresponding space.

Proof. This lemma is a simple consequence of Lemma B.8 by choosing the maximal interval length ℓ such that for example $\ell^{\beta}\|u\|_{L_p(J)} \leq \varepsilon$ in the first case. ■

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