System equivalence for AR-systems over rings with an application to delay-differential systems

System equivalence for AR-systems over rings
—with an application to delay-differential systems—

L.C.G.J.M. Habets
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513
NL - 5600 MB Eindhoven
The Netherlands
Telephone: +31 40 2474232
Telefax: +31 40 2442489
Email: luch@win.tue.nl

Abstract

In this paper the notion of autoregressive systems over an integral domain $\mathcal{R}$ is introduced, as a generalization of AR-systems over the rings $\mathbb{R}[s]$ and $\mathbb{R}[s, s^{-1}]$. Unlike the behavioral approach, the signal space is considered as a module $\mathcal{M}$ over the ring $\mathcal{R}$. In this setup the problem of system equivalence is studied: when do two different AR-representations characterize the same behavior? This problem is solved using a ring extension of $\mathcal{R}$, that explicitly depends on the choice of the signal space $\mathcal{M}$. In this way the usual divisibility conditions on the system-defining matrices can be recovered. The results apply to the class of delay-differential systems with (in)commensurable delays. In this particular application, the ring extension of $\mathcal{R}$ is characterized explicitly.

Key Words: Autoregressive (AR)-systems, systems over rings, delay-differential systems, behavioral approach, system equivalence, exponential polynomials.

1 Introduction

In the behavioral approach to dynamical systems, introduced by J.C. Willems (see e.g. [W1], [W2]), a system is described by a triple $(T, W, \mathcal{B})$. Here T is the time-axis, W the space in which the signals take their values, and $\mathcal{B}$—the behavior—is a subspace of the signal space W^T. The behavior $\mathcal{B}$ can be seen as the set of all time-trajectories, satisfying the laws governing the system.

As an example we consider a dynamical system, described by a set of ordinary differential equations (for continuous-time systems) or difference equations (for discrete-time systems), together with some non-dynamic linear relations among the variables. In this situation, the behavior is the set of all time-trajectories satisfying the system-defining equations. Collecting the variables in a vector w, and the equations in a polynomial matrix $P(s) \in \mathbb{R}[s]^{p \times q}$, the behavior is described by the set $\{w \mid P(\frac{d}{dt})w = 0\}$ for continuous-time systems, or $\{w \mid P(\sigma)w = 0\}$ in the discrete-time case. Here σ denotes the shift operator $\sigma(x(t)) = x(t-1)$. In both situations we see that the behavior is the kernel of an operator, described by a polynomial matrix in the differential operator $\frac{d}{dt}$ or the shift operator σ. A system described by such a kernel representation is called an *autoregressive* (AR)-system.

In this paper we study AR-systems over a more general type of ring. Instead of behaviors described by matrices over the polynomial rings $\mathbb{R}[s]$ or $\mathbb{R}[s, s^{-1}]$, we assume that the system-defining equations are described by matrices over an arbitrary *integral domain* $\mathcal{R}$. In this way we are able to study a much larger class of systems. For example, delay-differential systems with incommensurable time-delays fit into this framework.

This generalization from the rings $\mathbb{R}[s]$ and $\mathbb{R}[s, s^{-1}]$ to arbitrary integral domains resembles in a way the extension of the theory of state space systems over fields to the ring case (see e.g. [BBV], [S3]). However, there is an important difference. In the theory of state-space systems over rings, the system defining equations are studied in a rather formal way, without fixing explicitly the context in which these equations should be interpreted. For example, the same quadruple of matrices, representing a discrete-time system over a polynomial ring in state-space form, can be used to describe a continuous delay system with point delays. In this approach formal manipulations of the system-defining equations are emphasized, and these transformations are applicable independent of the way the equations are interpreted.

For the problem studied in this paper—system equivalence—the context in which the defining equations should be interpreted is important. This context is fixed by defining a *module* $\mathcal{M}$, describing the class of all time-trajectories under consideration. So $\mathcal{M}$ takes the place of the signal space W^T. $\mathcal{M}$ is a module over the ring $\mathcal{R}$. Each ring element corresponds to an operator acting on the elements of the module $\mathcal{M}$. In this framework a large class of dynamic equations can be described.

In comparison with the behavioral approach, the setup presented in this paper is slightly different. Instead of explicitly describing a time-axis T, and a space W in which the signals take their values, we take a module $\mathcal{M}$ over the ring $\mathcal{R}$ as the signal space. In this way it is possible to endow the signal space with a richer structure. Furthermore, the action of the operator corresponding to a ring element of $\mathcal{R}$, is seen as an action on a time-trajectory in $\mathcal{M}$ as a whole.

In this paper, we study the problem of *system equivalence*: when are the behaviors described by two different AR-representations the same? For AR-systems over the polynomial ring $\mathbb{R}[s]$, representing continuous-time systems described by sets of linear ordinary differential equations, it is known that system equivalence can be translated into division relations among the polynomial matrices describing the system. In this paper it becomes apparent that in general the solution to the problem of system equivalence for AR-systems over rings explicitly depends on the module $\mathcal{M}$ of all time-trajectories under consideration. Using the module $\mathcal{M}$, the ring $\mathcal{R}$ is extended to a ring $\mathcal{R}_{\mathcal{M}}$, and system equivalence is characterized by division properties of the system defining matrices over this extended ring. In this way, the well known result for AR-systems over $\mathbb{R}[s]$ is generalized to arbitrary integral domains $\mathcal{R}$.

This paper is organized as follows. First we give a formal definition of an AR-system over a ring $\mathcal{R}$. In this definition the module $\mathcal{M}$ of all time-trajectories under consideration plays an important role. It is explained how this set $\mathcal{M}$ can be seen as a module over the ring $\mathcal{R}$. In Section 3 we describe how the ring $\mathcal{R}$ can be extended to a ring $\mathcal{R}_{\mathcal{M}}$, explicitly depending on the module $\mathcal{M}$. Furthermore we show that $\mathcal{M}$ can be considered as a module over the ring extension $\mathcal{R}_{\mathcal{M}}$. Using the tools developed in Section 3, it is possible to tackle the problem of system equivalence. This solution is presented in Section 4, and consists of two parts: first we consider the square case, then the general case. In Section 5 we present an application of the results obtained in this paper. We show how the problem of system equivalence for delay-differential systems with (in)commensurable point delays can be solved within our framework. For this particular case we derive an alternative characterization of the ring $\mathcal{R}_{\mathcal{M}}$, that enables us to verify the equivalence of different AR-representations.

The results of this paper can be seen as a generalization of the case of differential-difference systems with commensurable delays described in [G-L]. In fact, [G-L] contains already some of the main ideas used here, but in a rather hidden way. Moreover, the proofs are quite different. Whereas in [G-L] most results are based on the fact that for differential-difference systems with commensurable delays the ring $\mathcal{R}_{\mathcal{M}}$ is a Bezout ring, we here use a more direct approach. In this way we solve the problem of system equivalence for a larger class of systems. Moreover, it is clarified why in the behavioral framework systems with incommensurable time-delays are more difficult to treat than systems with commensurable delays. Nevertheless, the approach in [G-L] remains interesting in its own right, because the Bezout ring property helps to solve several other system theoretic problems, for example the question of controllability of differential-difference systems with commensurable delays in a behavioral setting. Recently this question was also solved in [RW], using completely different techniques.

2 AR-systems over rings

Let $\mathcal{R}$ be an *integral domain*, i.e. $\mathcal{R}$ is a commutative ring with identity and without zero divisors. Let $\mathcal{M}$ be a module over $\mathcal{R}$. We think of $\mathcal{M}$ as the space of all time-trajectories under consideration. Each element of $\mathcal{R}$ has an action on the time-trajectories in the module $\mathcal{M}$. To distinguish between an element $r \in \mathcal{R}$ and its action on the module $\mathcal{M}$, we associate to every element $r \in \mathcal{R}$ an operator $\overline{r}$:

$$\overline{r} : \mathcal{M} \longrightarrow \mathcal{M} : \qquad \overline{r}(m) = r \cdot m. \tag{1}$$

Since $\mathcal{M}$ is a module over $\mathcal{R}$, the pair $(\mathcal{R}, \mathcal{M})$ has the following properties:

(i) $\mathcal{M}$ is a commutative group with respect to addition,

(ii) for all $r, r_1, r_2 \in \mathcal{R}$ and for all $m, m_1, m_2 \in \mathcal{M}$ we have

$$\begin{aligned} \overline{r}(m_1 + m_2) &= \overline{r}(m_1) + \overline{r}(m_2), & (2)\\ \overline{(r_1 + r_2)}(m) &= \overline{r_1}(m) + \overline{r_2}(m), & (3)\\ \overline{(r_1 r_2)}(m) &= \overline{r_1}(\overline{r_2}(m)), & (4)\\ \overline{1}(m) &= m, & (5) \end{aligned}$$

where 1 denotes the identity element of $\mathcal{R}$.

The set of all operators that can be obtained using formula (1) is again a ring, which we denote by $\overline{\mathcal{R}}$:

$$\overline{\mathcal{R}} := \{\overline{r} : \mathcal{M} \longrightarrow \mathcal{M} \mid r \in \mathcal{R}\}. \tag{6}$$

It is natural to postulate that different ring elements correspond to different operators. Throughout the paper we therefore make the following assumption.

Assumption 2.1 If $r \in \mathcal{R}$ is such that $\forall m \in \mathcal{M} : \overline{r}(m) = 0$, then $r = 0$.

Formally, Assumption 2.1 means that the surjective ring homomorphism T between $\mathcal{R}$ and $\overline{\mathcal{R}}$, described by $T(r) = \overline{r}$, is also injective. So $\mathcal{R}$ and $\overline{\mathcal{R}}$ are assumed to be *isomorphic*. Note that Assumption 2.1 puts a condition on the module $\mathcal{M}$. This module should contain enough elements in order to distinguish between operators corresponding to different elements of $\mathcal{R}$.

Next we extend our framework to the multivariable case. Let P be a $p \times q$ matrix over the integral domain $\mathcal{R}$. By replacing each entry p_{ij} of P by the corresponding operator $\overline{p_{ij}} : \mathcal{M} \longrightarrow \mathcal{M}$, the matrix P is turned into an operator $\overline{P} : \mathcal{M}^q \longrightarrow \mathcal{M}^p$, where $\mathcal{M}^q$ and $\mathcal{M}^p$ are spaces of *multivariable* time-trajectories.

Definition 2.2 Let $\mathcal{R}$ be an integral domain, and $\mathcal{M}$ a module over $\mathcal{R}$. Let $P \in \mathcal{R}^{p\times q}$. Then the pair $(P, \mathcal{M}^q)$ describes the *AR-system in $\mathcal{M}^q$, belonging to P*. The *behavior* of this system is given by

$$\mathcal{B}(P, \mathcal{M}^q) = \{m \in \mathcal{M}^q \mid \overline{P}(m) = 0\}. \tag{7}$$

In Definition 2.2 the matrix P determines the algebraic structure of the system: it contains the equations describing the laws governing the system. The module $\mathcal{M}$ determines how the equations described by P should be interpreted. The behavior of a system $(P, \mathcal{M}^q)$ is the kernel of the operator $\overline{P}$ in $\mathcal{M}^q$. If the module $\mathcal{M}$ has been fixed, or if there is no confusion on the interpretation at hand, the behavior $\mathcal{B}(P, \mathcal{M}^q)$ is also denoted by $\ker(\overline{P})$.

Example 2.3 Let $\mathcal{R} = \mathbb{R}[s]$ and $\mathcal{M}_1 = \mathcal{C}^\infty(\mathbb{R})$, the space of all arbitrarily often continuously differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. To each polynomial $r \in \mathcal{R}$ corresponds an operator $\overline{r} : \mathcal{M}_1 \longrightarrow \mathcal{M}_1$, obtained after substitution of the differentiation operator $\frac{d}{dt}$ for s. For $P \in \mathcal{R}^{p\times q}$ the behavior $\mathcal{B}(P, \mathcal{M}_1^q)$ is the solution set of a p-tuple of linear algebraic- and ordinary differential equations in q variables.

Next consider the same ring, but now in combination with the module $\mathcal{M}_2 = \mathbb{R}^{\mathbb{Z}}$ of all functions from $\mathbb{Z}$ to $\mathbb{R}$. After substitution of the unit shift operator σ defined by

$$\sigma : \mathcal{M}_2 \longrightarrow \mathcal{M}_2 : \qquad (\sigma x)(t) = x(t-1),$$

a polynomial $r \in \mathcal{R}$ is turned into a difference operator $\overline{r}$. The behavior $\mathcal{B}(P, \mathcal{M}_2^q)$, determined by a matrix $P \in \mathcal{R}^{p\times q}$, represents a discrete-time system with q variables, governed by a set of p difference equations.

If P and Q are matrices over the ring $\mathcal{R}$ with q columns, and a module $\mathcal{M}$ over $\mathcal{R}$ has been fixed, the behaviors $\mathcal{B}(P, \mathcal{M}^q)$ and $\mathcal{B}(Q, \mathcal{M}^q)$ are uniquely determined. Although the matrices P and Q may be different, it is still possible that their behaviors are the same.

Definition 2.4 Let $P \in \mathcal{R}^{p_1 \times q}$, and $Q \in \mathcal{R}^{p_2 \times q}$. Let $\mathcal{M}$ be a module over $\mathcal{R}$. Then the AR-systems $(P, \mathcal{M}^q)$ and $(Q, \mathcal{M}^q)$ are called *equivalent* if

$$\mathcal{B}(P, \mathcal{M}^q) = \mathcal{B}(Q, \mathcal{M}^q). \tag{8}$$

In this paper we study the problem of system equivalence: given a module $\mathcal{M}$ over the ring $\mathcal{R}$, we want to characterize all matrices over $\mathcal{R}$ describing the same behavior. This question is solved using division properties over a ring extension of $\mathcal{R}$, determined by the module $\mathcal{M}$. So the interpretation of the system defining equations plays a role in the problem of system equivalence.

3 An important ring extension

Given a module $\mathcal{M}$ over the integral domain $\mathcal{R}$, the ring $\overline{\mathcal{R}}$ may be considered as a set of operators, acting on the elements of $\mathcal{M}$. However, if we restrict the choice of operators to this ring $\overline{\mathcal{R}}$, it is often impossible to describe the transformation of an AR-representation into an other equivalent one. For this purpose the ring $\overline{\mathcal{R}}$ is mostly too small. To overcome this difficulty we introduce a ring extension of $\mathcal{R}$, explicitly depending on the module $\mathcal{M}$, with the property that $\mathcal{M}$ remains a module over this extended ring.

Definition 3.1 Let $\mathcal{R}$ be an integral domain, and let $\mathcal{M}$ be a module over $\mathcal{R}$. Let $r \in \mathcal{Q}(\mathcal{R})$, where $\mathcal{Q}(\mathcal{R})$ denotes the quotient field of $\mathcal{R}$. Then a pair $p, q \in \mathcal{R}$ is called an *admissible fractional representation* of r if

(i) $r = \frac{p}{q}$,

(ii) $\overline{q} : \mathcal{M} \longrightarrow \mathcal{M}$ is surjective,

(iii) $\ker(\overline{q}) \subset \ker(\overline{p})$.

The set $\mathcal{R}_{\mathcal{M}}$ is defined as the ring extension of $\mathcal{R}$, consisting of all elements in $\mathcal{Q}(\mathcal{R})$, for which there exists an admissible fractional representation.

Note that in general not every element in $\mathcal{Q}(\mathcal{R})$ has an admissible fractional representation, i.e. $\mathcal{R}_{\mathcal{M}} \subsetneq \mathcal{Q}(\mathcal{R})$.

Lemma 3.2 *$\mathcal{R}_{\mathcal{M}}$ is a subring of $\mathcal{Q}(\mathcal{R})$. $\mathcal{R}_{\mathcal{M}}$ is an integral domain and $\mathcal{R} \subset \mathcal{R}_{\mathcal{M}}$.*

Proof: Let $r_1, r_2 \in \mathcal{R}_{\mathcal{M}}$, with admissible fractional representations $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$, respectively. To show that $\mathcal{R}_{\mathcal{M}}$ is a subring of $\mathcal{Q}(\mathcal{R})$ we have to verify that both $r_1 - r_2$ and $r_1 \cdot r_2$ are elements of $\mathcal{R}_{\mathcal{M}}$.

Clearly we have $r_1 - r_2 = \frac{p_1 q_2 - p_2 q_1}{q_1 q_2}$. Moreover, $\overline{q_1 q_2}$ is surjective since both $\overline{q_1}$ and $\overline{q_2}$ are surjective. Let $x \in \ker(\overline{q_1 q_2})$, then $\overline{q_2}(x) \in \ker(\overline{q_1}) \subset \ker(\overline{p_1})$, and $\overline{q_1}(x) \in \ker(\overline{q_2}) \subset \ker(\overline{p_2})$. Hence $\overline{p_1 q_2}(x) = 0$ and $\overline{p_2 q_1}(x)) = 0$, which implies that $x \in \ker(\overline{p_1 q_2 - p_2 q_1})$, and thus $p_1 q_2 - p_2 q_1, q_1 q_2$ is an admissible fractional representation of $r_1 - r_2$.

Next, consider $r_1 \cdot r_2 = \frac{p_1 p_2}{q_1 q_2}$. Obviously $\overline{q_1 q_2}$ is surjective, and it remains to show that $\ker(\overline{q_1 q_2}) \subset \ker(\overline{p_1 p_2})$. If $\overline{q_1 q_2}(x) = 0$, then also $\overline{p_1 q_2}(x) = 0$. Since $\overline{p_1}$ and $\overline{q_2}$ commute, we have $\overline{q_2 p_1}(x) = 0$, so $\overline{p_1}(x) \in \ker(\overline{q_2}) \subset \ker(\overline{p_2})$, and thus $\overline{p_1 p_2}(x) = \overline{p_2 p_1}(x) = 0$. We conclude that $\frac{p_1 p_2}{q_1 q_2}$ is an admissible fractional representation of $r_1 r_2$.

Since every element $p \in \mathcal{R}$ has an admissible fractional representation $\frac{p}{1}$, it is obvious that $\mathcal{R} \subset \mathcal{R}_{\mathcal{M}}$. Moreover, since $\mathcal{R}$ is an integral domain, we have $1 \in \mathcal{R}_{\mathcal{M}}$, and $\mathcal{R}_{\mathcal{M}}$ is also an integral domain. ∎

Definition 3.3 Let $r \in \mathcal{R}_\mathcal{M}$, with admissible fractional representation $\frac{p}{q}$. Then the operator $\overline{r} : \mathcal{M} \longrightarrow \mathcal{M}$ is defined in the following way. Let $m \in \mathcal{M}$. Choose $w \in \mathcal{M}$ such that $\overline{q}(w) = m$. Then $\overline{r}(m) := \overline{p}(w)$.

It is not immediately clear that the action of $\overline{r}$ (with $r \in \mathcal{R}_\mathcal{M}$) on elements of $\mathcal{M}$ is well defined. If $\frac{p}{q}$ is an admissible fractional representation of r, we know that $\overline{q}$ is surjective. So there exists a $w \in \mathcal{M}$ such that $\overline{q}(w) = m$, but w is not necessarily unique. Therefore we have to show that the value of $\overline{r}(m)$ is independent of the solution w of the equation $\overline{q}(w) = m$. Furthermore, since an admissible fractional representation of $r \in \mathcal{R}_\mathcal{M}$ is not necessarily unique, we also have to prove that the operator $\overline{r}$ is independent of the chosen fractional representation $\frac{p}{q}$.

Let $m \in \mathcal{M}$. Let $w_1, w_2 \in \mathcal{M}$ be such that $\overline{q}(w_1) = \overline{q}(w_2) = m$. By property (2) this implies that $\overline{q}(w_1 - w_2) = 0$, and thus also $\overline{p}(w_1 - w_2) = 0$. Therefore $\overline{p}(w_1) = \overline{p}(w_2)$, and $\overline{r}(m)$ is independent of the solution w of $\overline{q}(w) = m$

Next, let $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ be two admissible fractional representations of $r \in \mathcal{R}_\mathcal{M}$. Then $p_1 q_2 = p_2 q_1$. Let $m \in \mathcal{M}$, and let $w_1, w_2 \in \mathcal{M}$ be such that $\overline{q_1}(w_1) = \overline{q_2}(w_2) = m$. To show that $\overline{r}$ is independent of a particular fractional representation, we have to verify that $\overline{p_1}(w_1) = \overline{p_2}(w_2)$. Since $\overline{q_1}$ is surjective, there exists a $y \in \mathcal{M}$ such that $\overline{q_1}(y) = w_2$. Because q_1 and q_2 commute, we have

$$\overline{q_1}(\overline{q_2}(y) - w_1) = \overline{q_2}(\overline{q_1}(y)) - \overline{q_1}(w_1) = \overline{q_2}(w_2) - \overline{q_1}(w_1) = m - m = 0.$$

Hence $\overline{q_2}(y) - w_1 \in \ker(\overline{q_1}) \subset \ker(\overline{p_1})$, and therefore also $\overline{p_1}(\overline{q_2}(y) - w_1) = 0$. Recalling that $p_1 q_2 = p_2 q_1$, we find

$$\overline{p_1}(w_1) = \overline{p_1 q_2}(y) = \overline{p_2 q_1}(y) = \overline{p_2}(w_2).$$

Let $\overline{\mathcal{R}_\mathcal{M}}$ denote the set of all operators from $\mathcal{M}$ to $\mathcal{M}$ that are obtained by applying Definition 3.3 to the elements of $\mathcal{R}_\mathcal{M}$. Then Assumption 2.1 guarantees that different ring elements in $\mathcal{R}_\mathcal{M}$ correspond to different operators in $\overline{\mathcal{R}_\mathcal{M}}$:

Lemma 3.4 *Let $r \in \mathcal{R}_\mathcal{M}$, and assume that*

$$\forall m \in \mathcal{M} : \overline{r}(m) = 0.$$

Then $r = 0$.

Proof: Let $r \in \mathcal{R}_\mathcal{M}$, and let $\frac{p}{q}$ be an admissible fractional representation of r. Assume that $\overline{r}(m) = 0$ for all $m \in \mathcal{M}$. Let $x \in \mathcal{M}$. Choose $m := \overline{q}(x)$. Then $\overline{p}(x) = \overline{r}(m) = 0$. So, for all $x \in \mathcal{M} : \overline{p}(x) = 0$, and Assumption 2.1 implies that $p = 0$, hence $r = 0$. ∎

In the next theorem it is shown that the module structure of $\mathcal{M}$ over $\mathcal{R}$ carries over to the ring extension $\mathcal{R}_\mathcal{M}$. This observation plays a crucial role in our solution to the problem of system equivalence.

Theorem 3.5 *Let $\mathcal{R}$ be an integral domain and $\mathcal{M}$ be a module over $\mathcal{R}$. Then $\mathcal{M}$ is also a module over $\mathcal{R}_\mathcal{M}$.*

Proof: Since $\mathcal{M}$ is a module over $\mathcal{R}$, $\mathcal{M}$ is—by definition—a commutative group with respect to addition. So we only have to prove that for all $r, r_1, r_2 \in \mathcal{R}_\mathcal{M}$, and for all $m, m_1, m_2 \in \mathcal{M}$, formulae (2) — (5) hold.

Let $r \in \mathcal{R}_{\mathcal{M}}$ with admissible fractional representation $r = \frac{p}{q}$. Let $m_1, m_2 \in \mathcal{M}$. Choose $w_1, w_2 \in \mathcal{M}$ such that $\overline{q}(w_1) = m_1$ and $\overline{q}(w_2) = m_2$. Then $\overline{q}(w_1 + w_2) = m_1 + m_2$, and thus

$$\overline{r}(m_1 + m_2) = \overline{p}(w_1 + w_2) = \overline{p}(w_1) + \overline{p}(w_2) = \overline{r}(m_1) + \overline{r}(m_2).$$

To prove (3), let $r_1, r_2 \in \mathcal{R}_{\mathcal{M}}$, with admissible fractional representations $r_1 = \frac{p_1}{q_1}$, and $r_2 = \frac{p_2}{q_2}$, respectively. Using the same arguments as in the proof of Lemma 3.2, one verifies that $\frac{p_1 q_2 + p_2 q_1}{q_1 q_2}$ is an admissible fractional representation of $r_1 + r_2$. Let $m \in \mathcal{M}$, and choose $w \in \mathcal{M}$ such that $\overline{q_1 q_2}(w) = m$. Define $w_1 := \overline{q_2}(w)$ and $w_2 := \overline{q_1}(w)$. Then $\overline{q_1}(w_1) = m$, and $\overline{q_2}(w_2) = m$, and thus

$$\overline{(r_1 + r_2)}(m) = \overline{(p_1 q_2 + p_2 q_1)}(w) = \overline{p_1}(w_1) + \overline{p_2}(w_2) = \overline{r_1}(m) + \overline{r_2}(m). \tag{9}$$

Next we prove (4). Let $r_1, r_2 \in \mathcal{R}_{\mathcal{M}}$ with admissible fractional representations $r_1 = \frac{p_1}{q_1}$ and $r_2 = \frac{p_2}{q_2}$. According to the proof of Lemma 3.2, $r_1 r_2 = \frac{p_1 p_2}{q_1 q_2}$ is an admissible fractional representation of $r_1 r_2$. Let $m \in \mathcal{M}$, and choose $w \in \mathcal{M}$ such that $\overline{q_1 q_2}(w) = m$. By definition of $\overline{r_1}$ it is obvious that $\overline{r_1}(\overline{q_1}(v)) = \overline{p_1}(v)$ for any $v \in \mathcal{M}$. Therefore we have

$$\begin{aligned}(\overline{r_1 r_2})(m) &= (\overline{p_1 p_2})(w) = \overline{p_1}(\overline{p_2}(w)) = \overline{r_1}(\overline{q_1 p_2}(w)) = \\ &= \overline{r_1}(\overline{p_2}(\overline{q_1}(w))) = \overline{r_1}(\overline{r_2}(\overline{q_2 q_1}(w))) = \overline{r_1}(\overline{r_2}(m)).\end{aligned} \tag{10}$$

Since the correctness of (5) is trivial, this completes the proof. ∎

Theorem 3.5 implies that the rings $\mathcal{R}_{\mathcal{M}}$ and $\overline{\mathcal{R}_{\mathcal{M}}}$ are isomorphic. From (9) and (10) it follows that the mapping $T : \mathcal{R}_{\mathcal{M}} \longrightarrow \overline{\mathcal{R}_{\mathcal{M}}} : T(r) = \overline{r}$ is a surjective ring homomorphism. According to Lemma 3.4, T is also injective.

Let $P \in \mathcal{R}^{p \times q}$, and consider the system $(P, \mathcal{M}^q)$ over $\mathcal{R}$. Since $\mathcal{R} \subset \mathcal{R}_{\mathcal{M}}$ and $\mathcal{M}$ is also a module over $\mathcal{R}_{\mathcal{M}}$, $(P, \mathcal{M}^q)$ can also be considered as a system over the ring $\mathcal{R}_{\mathcal{M}}$. This change in point of view does not change the behavior of the system, because both the set of laws governing the system, and the space of all time-trajectories under consideration remain the same. In fact, the class of AR-systems over $\mathcal{R}$ is embedded in the class of AR-systems over $\mathcal{R}_{\mathcal{M}}$. In this way, our freedom to manipulate the system defining equations determined by the matrix P, without actually changing the behavior, is enlarged, provided that the ring $\mathcal{R}_{\mathcal{M}}$ is indeed larger than $\mathcal{R}$. This is often the case, for example for differential-difference systems with commensurable delays.

Example 3.6 Let $\mathcal{R} = \mathbb{R}[s, z]$ and $\mathcal{M} = \mathcal{C}^{\infty}(\mathbb{R})$. A polynomial $p \in \mathbb{R}[s, z]$ is turned into a differential-difference operator $\overline{p}$ from $\mathcal{M}$ to $\mathcal{M}$ by substituting the differentiation operator $\frac{d}{dt}$ for s, and the unit shift operator σ ($\sigma : \mathcal{M} \longrightarrow \mathcal{M} : (\sigma x)(t) = x(t-1)$) for z. Since the operators $\frac{d}{dt}$ and σ commute, and Assumption 2.1 is satisfied, the polynomial ring $\mathcal{R}$ and the ring of operators $\overline{\mathcal{R}}$ are isomorphic.

Let $r = \frac{1-z}{s} \in \mathbb{R}(s, z)$, and define $p := 1 - z$ and $q := s$. On $\mathcal{C}^{\infty}(\mathbb{R})$ the operator $\overline{q}$ (pure differentiation) is clearly surjective. Moreover

$$\ker(\overline{q}) = \{\text{constant functions}\} \subset \{\text{periodic functions in } \mathcal{C}^{\infty}(\mathbb{R}) \text{ with period } 1\} = \ker(\overline{p}),$$

so $\frac{p}{q}$ is an admissible fractional representation of r, hence $r \in \mathcal{R}_{\mathcal{M}}$. It is obvious that $r \notin \mathcal{R}$, and therefore the ring $\mathcal{R}_{\mathcal{M}}$ is strictly larger than $\mathcal{R}$. The operator $\overline{r} \in \overline{\mathcal{R}_{\mathcal{M}}}$, corresponding to r is given by

$$\overline{r} : \mathcal{C}^{\infty}(\mathbb{R}) \longrightarrow \mathcal{C}^{\infty}(\mathbb{R}) : \qquad (\overline{r}(f))(t) = \int_{t-1}^{t} f(\tau) d\tau,$$

so $\overline{r}$ may be regarded as a delay operator with *distributed* time-delay.

Given an integral domain $\mathcal{R}$ and a module $\mathcal{M}$ over $\mathcal{R}$, the ring $\mathcal{R}$ can be extended to the integral domain $\mathcal{R}_{\mathcal{M}}$, and $\mathcal{M}$ is a module over $\mathcal{R}_{\mathcal{M}}$. At this point we are in the same situation as before, so one might repeat the same extension procedure in order to obtain a ring extension of $\mathcal{R}_{\mathcal{M}}$. The next result indicates that such a repetition is useless since the ring extension of $\mathcal{R}_{\mathcal{M}}$ does not contain any new elements.

Proposition 3.7 *Let $\mathcal{R}$ be an integral domain, and let $\mathcal{M}$ be a module over $\mathcal{R}$. Let $\mathcal{R}_{\mathcal{M}}$ be the ring extension of $\mathcal{R}$ as described in Definition 3.1. Let $(\mathcal{R}_{\mathcal{M}})_{\mathcal{M}}$ denote the ring obtained after applying the same ring extension procedure to $\mathcal{R}_{\mathcal{M}}$. Then*

$$(\mathcal{R}_{\mathcal{M}})_{\mathcal{M}} = \mathcal{R}_{\mathcal{M}}. \tag{11}$$

Proof: By definition, $\mathcal{R}_{\mathcal{M}} \subset (\mathcal{R}_{\mathcal{M}})_{\mathcal{M}}$, so we only show that all elements of $(\mathcal{R}_{\mathcal{M}})_{\mathcal{M}}$ are contained in $\mathcal{R}_{\mathcal{M}}$. Let $u \in (\mathcal{R}_{\mathcal{M}})_{\mathcal{M}}$ with admissible fractional representation $u = \frac{r_1}{r_2}$ with respect to $\mathcal{R}_{\mathcal{M}}$, i.e. r_1 and r_2 satisfy conditions (i), (ii), and (iii) of Definition 3.1, with $\mathcal{R}$ replaced by $\mathcal{R}_{\mathcal{M}}$. Both r_1 and r_2 have an admissible fractional representation over $\mathcal{R}$, say $r_1 = \frac{p_1}{q_1}$ and $r_2 = \frac{p_2}{q_2}$, respectively. So u is an element of $\mathcal{Q}(\mathcal{R})$, satisfying

$$u = \frac{r_1}{r_2} = \frac{\frac{p_1}{q_1}}{\frac{p_2}{q_2}} = \frac{p_1 q_2}{p_2 q_1}.$$

We show that $\frac{(p_1 q_2)}{(p_2 q_1)}$ is an admissible fractional representation of u over $\mathcal{R}$.

By assumption, the operator $\overline{r_2}$ is surjective. For the admissible fractional representation $r_2 = \frac{p_2}{q_2}$, this implies that $\overline{p_2}$ is surjective. Since $\frac{p_1}{q_1}$ is an admissible fractional representation of r_1, also $\overline{q_1}$ is surjective, and therefore $\overline{p_2 q_1}$ is surjective.

Let $w \in \ker(\overline{p_2 q_1})$, and define $y := \overline{q_2}(\overline{q_1}(w))$. Using Definition 3.3 it follows that $\overline{r_2}(y) = \overline{p_2}(\overline{q_1}(w)) = 0$. Since $\ker(\overline{r_2}) \subset \ker(\overline{r_1})$, we have $\overline{r_1}(y) = 0$, and thus

$$(\overline{p_1 q_2})(w) = \overline{p_1}(\overline{q_2}(w)) = \overline{r_1}(\overline{q_1}(\overline{q_2}(w))) = \overline{r_1}(\overline{q_2}(\overline{q_1}(w))) = \overline{r_1}(y) = 0.$$

Hence $w \in \ker(\overline{p_1 q_2})$, and we conclude that $u = \frac{p_1 q_2}{p_2 q_1} \in \mathcal{R}_{\mathcal{M}}$. ■

Proposition 3.7 indicates that the transition from the integral domain $\mathcal{R}$ to $\mathcal{R}_{\mathcal{M}}$ is not always a ring extension. For the problem of system equivalence it is interesting to know under which conditions $\mathcal{R} = \mathcal{R}_{\mathcal{M}}$. In this paper, this question cannot be answered completely; we only mention the following partial result.

Proposition 3.8 *Let $\mathcal{R}$ be an integral domain and let $\mathcal{M}$ be a module over $\mathcal{R}$. Assume that the following two conditions are satisfied:*

(i) $\mathcal{R}$ is a Bezout domain.

(ii) If $p \in \mathcal{R}$ is such that $\ker(\overline{p}) = \{0\}$, then p is a unit in $\mathcal{R}$.

Then $\mathcal{R} = \mathcal{R}_{\mathcal{M}}$.

Proof: Since the other inclusion is trivial, we only show that $\mathcal{R}_{\mathcal{M}} \subset \mathcal{R}$. Let $r \in \mathcal{R}_{\mathcal{M}}$, with admissible fractional representation $r = \frac{p}{q}$. Because $\mathcal{R}$ is a Bezout domain, the ideal $\langle p, q\rangle$ generated by p and q is a principal ideal. So there exist a $d \in \mathcal{R}$ such that $\langle p, q\rangle = \langle d\rangle$, and $a, b \in \mathcal{R}$ such that $p = ad$ and $q = bd$. Since $\overline{q}$ is surjective, both $\overline{b}$ and $\overline{d}$ are surjective.

Next we prove that $\ker(\overline{b}) \subset \ker(\overline{a})$. Let $w \in \mathcal{M}$ be such that $\overline{b}(w) = 0$. Since $\overline{d}$ is surjective, there exists a $y \in \mathcal{M}$ such that $w = \overline{d}(y)$. Then $\overline{bd}(y) = \overline{b}(w) = 0$, hence

$y \in \ker(\overline{q}) \subset \ker(\overline{p})$, and thus we have $\overline{a}(w) = \overline{ad}(y) = \overline{p}(y) = 0$. So $w \in \ker(\overline{a})$, and we conclude that also $\frac{a}{b}$ is an admissible fractional representation of r.

By construction, the elements $a, b \in \mathcal{R}$ are coprime in the Bezout ring $\mathcal{R}$, so there exist $g, h \in \mathcal{R}$ such that

$$a \cdot g + b \cdot h = 1.$$

This implies that $\ker(\overline{b}) = \{0\}$, because if $x \in \mathcal{M}$ is such that $\overline{b}(x) = 0$, also $\overline{a}(x) = 0$, and thus

$$x = \overline{1}(x) = \overline{ag + bh}(x) = \overline{g}(\overline{a}(x)) + \overline{h}(\overline{b}(x)) = 0.$$

Using Assumption *(ii)*, we conclude that b is a unit in $\mathcal{R}$, i.e. b^{-1} exists and is an element of $\mathcal{R}$. Therefore $r = \frac{a}{b} = a \cdot b^{-1} \in \mathcal{R}$. ∎

Example 3.9 Consider the situation described in Example 2.3, with $\mathcal{R} = \mathbb{R}[s]$, i.e. $\mathcal{R}$ is a principal ideal domain, and $\mathcal{M}_1 = \mathcal{C}^\infty(\mathbb{R})$, and $\mathcal{M}_2 = \mathbb{R}^{\mathbb{Z}}$.

In the continuous-time interpretation, an operator $\overline{p}$ corresponding to a polynomial $p \in \mathcal{R}$ has kernel $\{0\}$ if and only if p is a nonzero constant. Since every linear ordinary differential equation has a nontrivial solution, only elements of $\mathbb{R}\backslash\{0\}$ correspond to injective operators from $\mathcal{M}_1$ to $\mathcal{M}_1$. Since $\mathbb{R}\backslash\{0\}$ is exactly the set of units of $\mathcal{R}$, condition *(ii)* of Proposition 3.8 is satisfied, and $\mathcal{R}_{\mathcal{M}_1} = \mathcal{R}$.

In the discrete-time interpretation the polynomial $p_1(s) = s$ corresponds to the unit shift operator from $\mathcal{M}_2$ to $\mathcal{M}_2$. Although this operator is injective, the polynomial p_1 is not a unit in $\mathcal{R}$, and condition *(ii)* of Proposition 3.8 is violated. All rational functions of the form $\frac{1}{s^k} \cdot p(s)$, with $p \in \mathcal{R}$ and $k \in \mathbb{N} \cup \{0\}$ belong to $\mathcal{R}_{\mathcal{M}_2}$. Moreover, all elements of $\mathcal{R}_{\mathcal{M}_2}$ can be written in this form. Hence $\mathcal{R}_{\mathcal{M}_2} = \mathbb{R}[s, s^{-1}]$.

4 System equivalence

For the study of the relationship between system equivalence and division properties (over the ring $\mathcal{R}_{\mathcal{M}}$) of the matrices characterizing an AR-system, we need an additional assumption.

Assumption 4.1 $\forall q \in \mathcal{R}\backslash\{0\} : \overline{q} : \mathcal{M} \longrightarrow \mathcal{M}$ is surjective.

Assumption 4.1 implies that condition (ii) in Definition 3.1 for admissible fractional representations is always satisfied. For continuous-time systems the assumption is often satisfied, e.g. for differential-difference operators with (in)commensurable time-delays (see Section 5). However, unlike Assumption 2.1, Assumption 4.1 is restrictive. For several discrete-time systems the condition is not satisfied. Some of these situations may be treated using different techniques, not included in this paper.

Remark 4.2 Most results in this section can be adapted to the situation in which Assumption 4.1 does not hold. However, using this assumption (valid in a lot of interesting applications) the theory becomes more elegant.

The results on system equivalence are divided into two groups. First we consider square matrices; subsequently these results are used in the solution of the general case.

Proposition 4.3 *Let $P \in \mathcal{R}^{p\times p}$ and $Q \in \mathcal{R}^{q\times p}$, and assume that* $\det(P) \neq 0$. *Then*

$$\mathcal{B}(P, \mathcal{M}^p) \subset \mathcal{B}(Q, \mathcal{M}^p)$$

$$\Longleftrightarrow$$

$$\exists U \in \mathcal{R_M}^{q\times p} : Q = U \cdot P.$$

In particular, if $\mathcal{B}(P, \mathcal{M}^p) \subset \mathcal{B}(Q, \mathcal{M}^p)$, then $U := Q \cdot P^{-1}$ is a matrix over $\mathcal{R}_{\mathcal{M}}^{q\times p}$, satisfying $Q = U \cdot P$.

Proof: "$\Longleftarrow$" Let $U \in \mathcal{R_M}^{q\times p}$ be such that $Q = UP$, and let $w \in \mathcal{B}(P, \mathcal{M}^p)$, i.e. $\overline{P}(w) = 0$. Then $\overline{Q}(w) = \overline{UP}(w) = \overline{U}(\overline{P}(w)) = \overline{U}(0) = 0$. So $w \in \mathcal{B}(Q, \mathcal{M}^p)$.

"$\Longrightarrow$" Assume that $\mathcal{B}(P, \mathcal{M}^p) \subset \mathcal{B}(Q, \mathcal{M}^p)$, and define

$$U := Q \cdot P^{-1} = \frac{1}{\det(P)} \cdot Q \cdot \operatorname{adj}(P). \tag{12}$$

We prove that $U \in \mathcal{R}_{\mathcal{M}}^{q\times p}$.

Since $\det(P) \neq 0$, $\overline{\det(P)}$ is surjective. For $i = 1, \ldots, q$ and for $j = 1, \ldots, p$, we denote by e_i and e_j the i-th and j-th unit vector, and by $(Q \cdot \operatorname{adj}(P))_{ij}$ the (i, j)-th entry of $Q \cdot \operatorname{adj}(P)$. We show that for all these entries the inclusion $\ker(\overline{\det(P)}) \subset \ker(\overline{(Q \cdot \operatorname{adj}(P))_{ij}})$ holds.

Let $w_1 \in \ker(\overline{\det(P)})$, and define $w := w_1 \cdot e_j \in \mathcal{M}^p$. Then, according to Cramer's rule

$$\overline{P \cdot \operatorname{adj}(P)}(w) = \overline{\det(P) \cdot I}(w) = \overline{\det(P)}(w_1) \cdot e_j = 0.$$

Hence—by assumption—$\overline{\operatorname{adj}(P)}(w) \in \ker(\overline{P}) \subset \ker(\overline{Q})$, so in particular $\overline{Q \cdot \operatorname{adj}(P)}(w) = 0$. This implies that

$$0 = e_i^T \overline{Q \cdot \operatorname{adj}(P)}(w) = e_i^T \overline{Q \cdot \operatorname{adj}(P)}(e_j w_1) = \overline{(Q \cdot \operatorname{adj}(P))_{ij}}(w_1),$$

and thus $w_1 \in \ker(\overline{(Q \cdot \operatorname{adj}(P))_{ij}})$. Since $w_1 \in \ker(\overline{\det(P)})$ and the entry (i, j) were chosen arbitrarily, we conclude that $U \in \mathcal{R}_{\mathcal{M}}^{q\times p}$. Using (12) it is obvious that $UP = QP^{-1}P = Q$. ∎

Lemma 4.4 *Let $P \in \mathcal{R}^{p\times p}$ and $Q \in \mathcal{R}^{q\times p}$. Assume that* $\det(P) \neq 0$ *and $\mathcal{B}(Q, \mathcal{M}^p) \subset \mathcal{B}(P, \mathcal{M}^p)$. Then* $\operatorname{rank}(Q) = p$, *so in particular $q \geq p$.*

Proof: Assume that $\operatorname{rank}(Q) < p$. Then there exists a $0 \neq v \in \mathcal{R}^p$ such that $Qv = 0$. This implies that for all $w \in \mathcal{M}$, $\overline{v}(w) \in \mathcal{M}^p$ is an element of $\ker(\overline{Q})$. Since $\ker(\overline{Q}) \subset \ker(\overline{P})$, it follows that for all $w \in \mathcal{M}$, $(\overline{Pv})(w) = \overline{P}(\overline{v}(w)) = 0$. So, according to Assumption 2.1, $Pv = 0$, and because $\det(P) \neq 0$, we conclude $v = 0$. This contradicts our assumption, and therefore $p = \operatorname{rank}(Q) \leq \min(p, q) \leq q$. ∎

Theorem 4.5 *Let $P, Q \in \mathcal{R}^{p\times p}$, and assume that* $\det(P) \neq 0$. *Then*

$$\mathcal{B}(P, \mathcal{M}^p) = \mathcal{B}(Q, \mathcal{M}^p),$$

$$\Longleftrightarrow$$

$$\exists U \in \mathcal{R_M}^{p\times p}, U \textit{ invertible over } \mathcal{R_M}, \textit{ such that } Q = U \cdot P.$$

In particular, if $\mathcal{B}(P, \mathcal{M}^p) = \mathcal{B}(Q, \mathcal{M}^p)$, then $U := Q \cdot P^{-1}$ is a matrix over $\mathcal{R_M}$, that is invertible over $\mathcal{R_M}$, and satisfies $Q = U \cdot P$.

Proof: "$\Longleftarrow$" This direction is obvious: since both U and U^{-1} are matrices over $\mathcal{R}_{\mathcal{M}}$, one may use the same argument as in the proof of Proposition 4.3.

"$\Longrightarrow$" Since $\mathcal{B}(P, \mathcal{M}^p) \subset \mathcal{B}(Q, \mathcal{M}^p)$, Proposition 4.3 implies that $U := Q \cdot P^{-1}$ is a matrix over $\mathcal{R}_{\mathcal{M}}$, satisfying $U \cdot P = Q$. We show that U is invertible over $\mathcal{R}_{\mathcal{M}}$.

Because $\mathcal{B}(Q, \mathcal{M}^p) \subset \mathcal{B}(P, \mathcal{M}^p)$, Lemma 4.4 indicates that $\det(Q) \neq 0$. Define $V := PQ^{-1} = \frac{1}{\det(Q)} P \cdot \mathrm{adj}(Q)$. According to Proposition 4.3, $V \in \mathcal{R}_{\mathcal{M}}^{p\times p}$, and $VQ = P$. Furthermore

$$\begin{aligned} UV &= \frac{1}{\det(P)} Q \cdot \mathrm{adj}(P) \frac{1}{\det(Q)} P \cdot \mathrm{adj}(Q) = \frac{1}{\det(Q)} Q \left(\frac{1}{\det(P)} \mathrm{adj}(P) P \right) \mathrm{adj}(Q) = \\ &= \frac{1}{\det(Q)} Q \cdot \mathrm{adj}(Q) = I, \end{aligned}$$

so U is invertible over $\mathcal{R}_{\mathcal{M}}$ with inverse V. ∎

Theorem 4.5 solves the question of system equivalence for AR-systems described by a square matrix P (with $\det(P) \neq 0$, but this condition is not very restrictive). According to Proposition 4.3, the behavior $\ker(\overline{P})$ is contained in the behavior $\ker(\overline{Q})$, if and only if P is a right divisor of Q over $\mathcal{R}_{\mathcal{M}}$. If additionally Q is a right divisor of P over $\mathcal{R}_{\mathcal{M}}$, the behaviors $\ker(\overline{P})$ and $\ker(\overline{Q})$ are the same. Conversely, Theorem 4.5 also characterizes the transformations on P that do not change its behavior: premultiplication with invertible matrices over $\mathcal{R}_{\mathcal{M}}$ is allowed.

Remark 4.6 To test system equivalence it suffices to compute the matrix $U = QP^{-1}$. Then $\ker(\overline{P}) \subset \ker(\overline{Q})$ if and only if U is a matrix over $\mathcal{R}_{\mathcal{M}}$. If additionally U is invertible over $\mathcal{R}_{\mathcal{M}}$, the behaviors $\ker(\overline{P})$ and $\ker(\overline{Q})$ are equal.

Next we consider the general (non-square) case.

Proposition 4.7 *Let $P = (P_1 \mid P_2)$, with $P_1 \in \mathcal{R}^{p\times p}$ and $P_2 \in \mathcal{R}^{p\times m}$, and assume that $\det(P_1) \neq 0$. Let $Q = (Q_1 \mid Q_2)$, with $Q_1 \in \mathcal{R}^{q\times p}$ and $Q_2 \in \mathcal{R}^{q\times m}$. Then*

$$\begin{aligned} & \mathcal{B}(P, \mathcal{M}^{p+m}) \subset \mathcal{B}(Q, \mathcal{M}^{p+m}) \\ \Longleftrightarrow & \\ & \exists U \in \mathcal{R}_{\mathcal{M}}{}^{q\times p} : Q = U \cdot P. \end{aligned}$$

In particular, if $\mathcal{B}(P, \mathcal{M}^{p+m}) \subset \mathcal{B}(Q, \mathcal{M}^{p+m})$, then $U := Q_1 P_1^{-1}$ is a $q \times p$ matrix over $\mathcal{R}_{\mathcal{M}}$, satisfying $Q = U \cdot P$.

In the proof of this result we need the following lemma:

Lemma 4.8 *Let $P \in \mathcal{R}^{p\times p}$ with $\det(P) \neq 0$. Then $\overline{P} : \mathcal{M}^p \longrightarrow \mathcal{M}^p$ and $\overline{\mathrm{adj}(P)} : \mathcal{M}^p \longrightarrow \mathcal{M}^p$ are surjective.*

Proof: According to Cramer's rule we have

$$\overline{\mathrm{adj}(P)} \cdot \overline{P} = \overline{P} \cdot \overline{\mathrm{adj}(P)} = \overline{\det(P)} \cdot I.$$

Since $\overline{\det(P)} : \mathcal{M} \longrightarrow \mathcal{M}$ is surjective, also $\overline{\det(P)} \cdot I : \mathcal{M}^p \longrightarrow \mathcal{M}^p$ is surjective. This immediately implies that both $\overline{P}$ and $\overline{\mathrm{adj}(P)}$ are surjective. ∎

Proof of Proposition 4.7: Since "$\Longleftarrow$" is obvious, we only prove "$\Longrightarrow$".

Assume that $\mathcal{B}(P, \mathcal{M}^{p+m}) \subset \mathcal{B}(Q, \mathcal{M}^{p+m})$. Then also $\mathcal{B}(P_1, \mathcal{M}^p) \subset \mathcal{B}(Q_1, \mathcal{M}^p)$, which can be seen as follows. Let $w_1 \in \mathcal{B}(P_1, \mathcal{M}^p)$, and extend w_1 to a vector $w = \begin{pmatrix} w_1 \\ 0 \end{pmatrix} \in \mathcal{M}^{p+m}$, by defining the last m entries of w equal to 0. Then $w \in \ker(\overline{P}) \subset \ker(\overline{Q})$, and thus $0 = \overline{Q}(w) = \overline{Q_1}(w_1) + \overline{Q_2}(0) = \overline{Q_1}(w_1)$, i.e. $w_1 \in \ker(\overline{Q_1})$.

Next apply Proposition 4.3 to the matrices P_1 and Q_1. Defining $U := Q_1 P_1^{-1}$, we know that $U \in \mathcal{R}_{\mathcal{M}}^{q \times p}$ and $Q_1 = UP_1$. So it suffices to show that $Q_2 = UP_2$.

Let $w_2 \in \mathcal{M}^m$. Since $\det(P_1) \neq 0$, we know, according to Lemma 4.8, that there exists a $w_1 \in \mathcal{M}^p$ such that $\overline{P_1}(w_1) = -\overline{P_2}(w_2)$. Then

$$\overline{(P_1 \mid P_2)} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \overline{P_1}(w_1) + \overline{P_2}(w_2) = 0,$$

and thus also $\overline{(Q_1 \mid Q_2)} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0$. Therefore

$$\overline{(Q_2 - UP_2)}(w_2) = \overline{Q_2}(w_2) - \overline{UP_2}(w_2) = \overline{Q_2}(w_2) + \overline{UP_1}(w_1) = \overline{Q_2}(w_2) + \overline{Q_1}(w_1) = 0.$$

Since $w_2 \in \mathcal{M}^m$ was arbitrary, Lemma 3.4 implies that $Q_2 - UP_2 = 0$. ∎

At this point we may generalize Theorem 4.5 to the non-square case. Two matrices P and Q of full row rank characterize the same behavior if and only if P and Q are right divisors of each other over the ring $\mathcal{R}_{\mathcal{M}}$.

Theorem 4.9 *Let $P \in \mathcal{R}^{p \times m}$, with $\operatorname{rank}(P) = p$, and $Q \in \mathcal{R}^{q \times m}$, with $\operatorname{rank}(Q) = q$. Then*

$$\mathcal{B}(P, \mathcal{M}^m) = \mathcal{B}(Q, \mathcal{M}^m)$$

$$\Longleftrightarrow$$

$$p = q \text{ and } \exists U \in \mathcal{R}_{\mathcal{M}}^{p \times p}, U \text{ invertible over } \mathcal{R}_{\mathcal{M}}, \text{ such that } Q = U \cdot P.$$

In particular, if $\mathcal{B}(P, \mathcal{M}^m) = \mathcal{B}(Q, \mathcal{M}^m)$, and if P_1 is a $p \times p$ block of P satisfying $\det(P_1) \neq 0$, and Q_1 is the corresponding $p \times p$ block of Q, then $U := Q_1 P_1^{-1}$ is an element of $\mathcal{R}_{\mathcal{M}}^{p \times p}$, that is invertible over $\mathcal{R}_{\mathcal{M}}$, and satisfies $Q = U \cdot P$.

Proof: "$\Longleftarrow$" is trivial, so we only prove "$\Longrightarrow$".

Since $\mathcal{B}(P, \mathcal{M}^m) \subset \mathcal{B}(Q, \mathcal{M}^m)$, and $m \geq p = \operatorname{rank}(P)$, we may apply Proposition 4.7: there exists a $U \in \mathcal{R}_{\mathcal{M}}^{q \times p}$ such that $Q = UP$.

However, also the inclusion in the opposite direction holds: $\mathcal{B}(Q, \mathcal{M}^m) \subset \mathcal{B}(P, \mathcal{M}^m)$. Furthermore, $m \geq q = \operatorname{rank}(Q)$, and again using Proposition 4.7, we find a matrix $V \in \mathcal{R}_{\mathcal{M}}^{p \times q}$ such that $P = VQ$.

Combining both equalities, we conclude that $P = VUP$, and $Q = UVQ$. Since both P and Q have full row rank, both matrices are right-invertible over $\mathcal{Q}(\mathcal{R})$, and thus $VU = I_p$, and similarly $UV = I_q$. This implies that $p = q$, hence $U \in \mathcal{R}_{\mathcal{M}}^{p \times p}$ with inverse V satisfies the claim. ∎

Remark 4.10 If $P \in \mathcal{R}^{p \times m}$ with $\operatorname{rank}(P) = p$ and $Q \in \mathcal{R}^{q \times m}$ with $\operatorname{rank}(Q) = q$, it is easily verified whether $\ker(\overline{P}) = \ker(\overline{Q})$. Let P_1 be a $p \times p$ block of P such that $\det(P_1) \neq 0$, and collect the remaining columns of P in the matrix P_2. Partition the matrix Q correspondingly into matrices $Q_1 \in \mathcal{R}^{q \times p}$ and $Q_2 \in \mathcal{R}^{q \times (m-p)}$. Then $\ker(\overline{P}) \subset \ker(\overline{Q})$ if and only if $U := Q_1 P_1^{-1}$ is a $q \times p$ matrix over $\mathcal{R}_{\mathcal{M}}$ such that $U \cdot P_2 = Q_2$. If additionally $p = q$ and U is invertible over $\mathcal{R}_{\mathcal{M}}$, then the behaviors $\ker(\overline{P})$ and $\ker(\overline{Q})$ are equivalent.

In all results on system equivalence given in this section, the matrices P and Q describing an AR-system are assumed to be matrices over the ring $\mathcal{R}$. The transformation matrix U however, is a matrix over $\mathcal{R}_{\mathcal{M}}$. Note that in this way also the case of AR-systems over $\mathcal{R}_{\mathcal{M}}$ (i.e. with matrices P and Q over $\mathcal{R}_{\mathcal{M}}$) is included. According to Proposition 3.7, the ring $\mathcal{R}_{\mathcal{M}}$ is not extended after application of Definition 3.1: $(\mathcal{R}_{\mathcal{M}})_{\mathcal{M}} = \mathcal{R}_{\mathcal{M}}$. Therefore, the results of this section remain valid if $\mathcal{R}$ is replaced by $\mathcal{R}_{\mathcal{M}}$. This means that system equivalence for AR-systems over the ring $\mathcal{R}_{\mathcal{M}}$ has been characterized by division properties of the system-defining matrices over the same ring $\mathcal{R}_{\mathcal{M}}$. In particular, if a matrix P over $\mathcal{R}_{\mathcal{M}}$ is premultiplied by an invertible matrix over $\mathcal{R}_{\mathcal{M}}$, the behavior does not change.

5 An application: differential-difference systems

Consider a differential-difference system with k incommensurable time-delays $\tau_1, \ldots, \tau_k$. Incommensurable means that the numbers $\tau_1, \ldots, \tau_k \in \mathbb{R}^+$ are linearly independent over $\mathbb{Q}$. In this section we show that this class of systems fits into the framework proposed in this paper, and derive a simple characterization for the ring $\mathcal{R}_{\mathcal{M}}$ in this particular case.

Define $\mathcal{R} := \mathbb{R}[s, z_1, \ldots, z_k]$, and let $\mathcal{M} = \mathcal{E}(\mathbb{R})$ be the space of all infinitely differentiable complex functions on $\mathbb{R}$, under the topology of compact convergence in all derivatives. This means that a sequence of functions $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{E}(\mathbb{R})$ converges to $f \in \mathcal{E}(\mathbb{R})$ if f_n converges uniformly to f on every compact set $\Omega \subset \mathbb{R}$, and the same holds for all derivatives of f_n and f, respectively. Equipped with this topology, the complex vector space $\mathcal{C}^\infty(\mathbb{R})$ is turned into the Fréchet space $\mathcal{E}(\mathbb{R})$ (see e.g. [S1, p. 107]).

For $i = 1, \ldots, k$ we define σ_i as the delay operator on $\mathcal{M}$ corresponding to time-delay τ_i:

$$\sigma_i : \mathcal{M} \longrightarrow \mathcal{M} : \quad \sigma_i(x(t)) = x(t - \tau_i).$$

Note that the differentiation operator $\frac{d}{dt}$ and the delay operators $\sigma_1, \ldots, \sigma_k$ mutually commute. Each polynomial $p \in \mathcal{R}$ corresponds to an operator $\overline{p} : \mathcal{M} \longrightarrow \mathcal{M}$, obtained after substitution in p of $\frac{d}{dt}$ for s and σ_i for z_i ($i = 1, \ldots, k$). Now the Fréchet space $\mathcal{M}$ is regarded as a module over the ring $\mathcal{R}$ by defining the product of elements of $\mathcal{R}$ and $\mathcal{M}$ in the canonical way: for every $p = p(s, z_1, \ldots, z_k) \in \mathcal{R}$ and $m \in \mathcal{M}$:

$$p \cdot m := \overline{p}(m) = p(\frac{d}{dt}, \sigma_1, \ldots, \sigma_k)(m).$$

Let $\overline{\mathcal{R}} = \{\overline{p} : \mathcal{M} \longrightarrow \mathcal{M} \mid p \in \mathcal{R}\}$. Since the time-delays $\tau_1, \ldots, \tau_k$ are incommensurable, different ring elements $p \in \mathcal{R}$ correspond to different operators $\overline{p} \in \overline{\mathcal{R}}$, i.e. Assumption 2.1 is satisfied, and $\mathcal{R}$ and $\overline{\mathcal{R}}$ are isomorphic rings. By choosing a suitable $P \in \mathcal{R}^{n\times m}$, a set of n differential-difference equations in m variables can be described as an AR-system $(P, \mathcal{M}^m)$ over the ring $\mathcal{R}$ with behavior $\mathcal{B}(P, \mathcal{M}^m)$.

Next we have to check whether Assumption 4.1 is satisfied: for all $q \in \mathcal{R}\backslash\{0\}$, $\overline{q} : \mathcal{M} \longrightarrow \mathcal{M}$ is surjective. In the particular application at hand, this nontrivial question was studied by Ehrenpreis in [E2]. The results in [E2, Section 3] guarantee that indeed Assumption 4.1 is satisfied. Alternative (more direct) approaches to prove the surjectivity of non-zero elements in $\overline{\mathcal{R}}$ are given in [EH] and [G-L]. So the class of differential-difference systems fits into the framework of AR-systems over rings, and the solution to the problem of system equivalence using the ring extension $\mathcal{R}_{\mathcal{M}}$ applies. Moreover, for this particular application, the elements of the ring $\mathcal{R}_{\mathcal{M}}$ can be characterized explicitly. This alternative description is of great practical interest for the investigation of system equivalence of delay-differential systems.

Let $p \in \mathcal{R}$. After substitution of the exponential functions $e^{-\tau_i s}$ for the indeterminates z_i $(i = 1, \ldots, k)$, the polynomial p is turned into an *exponential* (or *quasi*) *polynomial* $\tilde{p}$:

$$\tilde{p}(s) = p(s, e^{-\tau_1 s}, \ldots, e^{-\tau_k s}).$$

$\tilde{p}$ can be considered as the frequency description of the operator $\overline{p}$ corresponding to the polynomial $p \in \mathcal{R}$, obtained by formally applying the Laplace transformation. Define $\tilde{\mathcal{R}} := \{\tilde{p} \mid p \in \mathcal{R}\}$. Since the time-delays $\tau_1, \ldots, \tau_k$ are incommensurable, the rings $\mathcal{R}$, $\overline{\mathcal{R}}$, and $\tilde{\mathcal{R}}$ are isomorphic.

Let $H(\mathbb{C})$ denote the ring of all entire functions, i.e. $H(\mathbb{C})$ consists of all functions that are holomorphic on $\mathbb{C}$. It is clear that $\tilde{\mathcal{R}}$ is a subring of $H(\mathbb{C})$. Additionally the elements of $\tilde{\mathcal{R}}$ satisfy several growth conditions; they belong to a (non-classical) Paley-Wiener algebra of holomorphic functions (see e.g. [S2, pp. 862—863], [E1], [K2, p. 16]):

Definition 5.1 The *Paley-Wiener algebra* $PW(\mathbb{C})$ consists of all functions $f \in H(\mathbb{C})$ for which there exist $C, a > 0$ and $N \in \mathbb{N} \cup \{0\}$, depending on f, such that for all $s \in \mathbb{C}$:

$$|f(s)| \leq C \cdot (1 + |s|)^N e^{a|\operatorname{Re} s|}. \tag{13}$$

Or equivalently, f is of exponential type and polynomially bounded on the imaginary axis.

Obviously, $\tilde{\mathcal{R}}$ is a subring of the Paley-Wiener algebra $PW(\mathbb{C})$.

Remark 5.2 In the literature (see e.g. [S2], [E1], [K2]) the Paley-Wiener algebra is usually defined by taking Fourier instead of Laplace transforms. Hence the variable s in (13) is replaced by $i\omega$, with $\omega \in \mathbb{C}$. Definition 5.1 is transformed to this classical setup by rotation of the complex plane over $\frac{\pi}{2}$ radians.

The Laplace transform $\tilde{p} \in \tilde{\mathcal{R}}$ is the spectral description of the corresponding operator $\overline{p} : \mathcal{E}(\mathbb{R}) \longrightarrow \mathcal{E}(\mathbb{R})$, and describes the action of $\overline{p}$ on Bohl functions. For $\lambda \in \mathbb{C}$ and $j \in \mathbb{N}$, let $e_{\lambda,j}$ denote the Bohl function $e_{\lambda,j}(t) = t^{j-1}e^{\lambda t}$. Obviously $e_{\lambda,j} \in \mathcal{E}(\mathbb{R})$. Since $\frac{d}{dt}(e_{\lambda,1}) = \lambda e_{\lambda,1}$, and $\sigma_i(e_{\lambda,1}) = e^{-\lambda\tau_i}e_{\lambda,1}$ for all $i = 1, \ldots, k$, we obtain for every polynomial $p \in \mathcal{R}$:

$$\overline{p}(e_{\lambda,1}) = \tilde{p}(\lambda)e_{\lambda,1}. \tag{14}$$

Hence every exponential function $e_{\lambda,1}$ is an eigenvector of $\overline{p}$ with spectral value $\tilde{p}(\lambda)$.

Next we extend formula (14) to Bohl functions $e_{\lambda,j}$ with $j > 1$. Let $j \in \mathbb{N}$, and consider for a moment $e_{\lambda,j}$ as a function of two variables, t and λ. Then

$$\begin{aligned} \overline{p}(e_{\lambda,j}) &= \overline{p}(\left(\frac{d}{d\lambda}\right)^{j-1} e_{\lambda,1}) = \left(\frac{d}{d\lambda}\right)^{j-1} (\overline{p}(e_{\lambda,1})) = \\ &= \left(\frac{d}{d\lambda}\right)^{j-1} (\tilde{p}(\lambda)e_{\lambda,1}) = \sum_{\ell=0}^{j-1} \binom{j-1}{\ell} \tilde{p}^{(\ell)}(\lambda)e_{\lambda,j-\ell}, \end{aligned} \tag{15}$$

where we used formula (14) and the fact that in this particular situation the operators $(\frac{d}{d\lambda})^{j-1}$ and $\overline{p}$ commute. Combining formula (14) and (15) it is easily verified whether a Bohl function belongs to the kernel of an operator $\overline{p} \in \overline{\mathcal{R}}$ (see also [BC, p. 55] and [HVL, p. 18] for similar results):

Proposition 5.3 *Let $p \in \mathcal{R}$, and let for $\lambda \in \mathbb{C}$ and $\ell \in \mathbb{N}$, $e_{\lambda,\ell}$ denote the Bohl function $e_{\lambda,\ell}(t) = t^{\ell-1}e^{\lambda t}$. Then for every $j \in \mathbb{N}$:*

$$
\begin{aligned}
& e_{\lambda,j} \in \ker(\overline{p}), \\
\Longleftrightarrow & \\
& e_{\lambda,\ell} \in \ker(\overline{p}) \textit{ for } \ell = 1, \ldots, j, \\
\Longleftrightarrow & \\
& \lambda \textit{ is a zero of } \tilde{p} \textit{ of multiplicity at least } j.
\end{aligned}
$$

■

The relationship between the differential-difference operator $\overline{p}$ and the exponential polynomial $\tilde{p}$ is now used to obtain an alternative characterization of the ring $\mathcal{R}_{\mathcal{M}}$. For this purpose we introduce the ring

$$
\mathcal{Q}(\mathcal{R})_{\text{holo}} := \left\{ \frac{p}{q} \in \mathcal{Q}(\mathcal{R}) \mid p, q \in \mathcal{R} \text{ and } \frac{\tilde{p}}{\tilde{q}} \in H(\mathbb{C}) \right\}. \tag{16}
$$

Theorem 5.4 *For differential-difference systems with incommensurable time-delays, modeled as AR-systems over the ring $\mathcal{R} = \mathbb{R}[s, z_1, \ldots, z_k]$, with the module $\mathcal{M} = \mathcal{E}(\mathbb{R})$ as signal space, we have*

$$
\mathcal{R}_{\mathcal{M}} = \mathcal{Q}(\mathcal{R})_{\text{holo}}. \tag{17}
$$

In the proof of formula (17) we need some results on kernels of delay-differential operators, known in the literature. For every $p \in \mathcal{R}$, it is obvious that if $f \in \ker(\overline{p})$ and $\tau \in \mathbb{R}$, then the translation g of f, defined by $g(t) := f(t - \tau)$ $(t \in \mathbb{R})$, is also an element of $\ker(\overline{p})$. Furthermore, since the operator $\overline{p} : \mathcal{E}(\mathbb{R}) \longrightarrow \mathcal{E}(\mathbb{R})$ is continuous, its kernel is closed. Therefore $\ker(\overline{p})$ is a closed translation invariant subspace of $\mathcal{E}(\mathbb{R})$. However, the general structure of closed translation invariant subspaces of the Fréchet space $\mathcal{E}(\mathbb{R})$ has been investigated in the literature. In the next proposition we state a result of De Rijcke, given in [R, Theorem 3.33]. The proof of this result is based on earlier works of Schwartz and Kahane (see [S2], [K1], [K2]).

Proposition 5.5 *A closed linear subspace V of $\mathcal{E}(\mathbb{R})$ is translation invariant if and only if there exists a countable set $\Sigma \subset \mathbb{C}$ and a mapping $N : \Sigma \longrightarrow \mathbb{N}$ such that*

$$
V = \overline{\operatorname{span}\{e_{\lambda,j} \mid \lambda \in \Sigma, j = 1, \ldots, N(\lambda)\}}.
$$

■

Corollary 5.6 *Let V and W be closed linear translation-invariant subspaces of $\mathcal{E}(\mathbb{R})$. Then $V \subset W$ if and only if*

$$
\forall \lambda \in \mathbb{C}\, \forall j \in \mathbb{N} : [e_{\lambda,j} \in V \Longrightarrow e_{\lambda,j} \in W].
$$

■

Proof of Theorem 5.4: "$\mathcal{R}_{\mathcal{M}} \subset \mathcal{Q}(\mathcal{R})_{\text{holo}}$" Let $r \in \mathcal{R}_{\mathcal{M}}$ with admissible fractional representation $\frac{p}{q}$. Then $\ker(\overline{q}) \subset \ker(\overline{p})$, and we prove that $\frac{\tilde{p}}{\tilde{q}}$ is holomorphic in $\mathbb{C}$.

By definition, $\frac{\tilde{p}}{\tilde{q}}$ is a meromorphic function, and therefore it suffices to show that every zero of $\tilde{q}$ is also a zero of $\tilde{p}$ (including multiplicities). Let λ be a zero of $\tilde{q}$ of multiplicity j.

Then Proposition 5.3 implies that $\overline{q}(e_{\lambda,j}) = 0$. Then $e_{\lambda,j} \in \ker(\overline{q}) \subset \ker(\overline{p})$, and applying Proposition 5.3 again, we conclude that λ is a zero of $\tilde{p}$ of multiplicity at least j.

"$\mathcal{R}_{\mathcal{M}} \supset \mathcal{Q}(\mathcal{R})_{\text{holo}}$" Let $\frac{p}{q} \in \mathcal{Q}(\mathcal{R})_{\text{holo}}$, i.e. $\frac{p}{q} \in \mathcal{Q}(\mathcal{R})$ and $\frac{\tilde{p}}{\tilde{q}}$ is holomorphic in $\mathbb{C}$. We show that $\frac{p}{q}$ is an admissible fractional representation.

Since $\overline{q}$ is surjective, we only have to verify that $\ker(\overline{q}) \subset \ker(\overline{p})$. According to Corollary 5.6, it suffices to check that every Bohl function $e_{\lambda,j} \in \ker(\overline{q})$ is also an element of $\ker(\overline{p})$. However, this follows immediately from Proposition 5.3. Let $\lambda \in \mathbb{C}$ and $j \in \mathbb{N}$, and assume that $\overline{q}(e_{\lambda,j}) = 0$. Then λ is a zero of $\tilde{q}$ of multiplicity at least j, and because $\frac{\tilde{p}}{\tilde{q}} \in H(\mathbb{C})$, λ is also a zero of $\tilde{p}$ of multiplicity at least j. Hence $\overline{p}(e_{\lambda,j}) = 0$, and $\ker(\overline{q}) \subset \ker(\overline{p})$. ∎

Theorem 5.4 gives an alternative characterization of the ring $\mathcal{R}_{\mathcal{M}}$. To verify whether a rational function $r = \frac{p}{q} \in \mathcal{Q}(\mathcal{R})$ belongs to $\mathcal{R}_{\mathcal{M}}$, it suffices to check whether the corresponding meromorphic function $\frac{\tilde{p}}{\tilde{q}}$ is an entire function. In principle this is still difficult to test because the denominator $\tilde{q}$ may have infinitely many zeros.

The following result on the quotient of two exponential polynomials gives a more precise characterization of the elements of $\mathcal{Q}(\mathcal{R})_{\text{holo}}$; the corresponding holomorphic functions satisfy the growth conditions of Definition 5.1:

Theorem 5.7 *Let $p, q \in \mathcal{R}$ be such that $\frac{p}{q} \in \mathcal{Q}(\mathcal{R})_{\text{holo}}$, i.e. $\frac{\tilde{p}}{\tilde{q}}$ is holomorphic in $\mathbb{C}$. Then $\frac{\tilde{p}}{\tilde{q}} \in PW(\mathbb{C})$.* ∎

Theorem 5.7 may be considered as a corollary of two results in [M, pp. 310—312]. An explicit proof, based on standard results on Fréchet spaces, is given in [EH]. The fact that differential-difference operators are surjective, establishes the main argument that in the definition of $\mathcal{Q}(\mathcal{R})_{\text{holo}}$ the ring of holomorphic functions $H(\mathbb{C})$ may be replaced by the Paley-Wiener algebra $PW(\mathbb{C})$.

The next step in the explicit characterization of the ring $\mathcal{Q}(\mathcal{R})_{\text{holo}}$ is based on the following result:

Theorem 5.8 ([BD]) *Consider the exponential polynomials F, G, given by*

$$F(s) = \sum_{i=1}^{n} P_i(s) e^{\rho_i s}, \qquad G(s) = \sum_{j=1}^{m} Q_j(s) e^{\mu_j s},$$

with for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$: $\rho_i, \mu_j \in \mathbb{R}$ and $P_i, Q_j \in \mathbb{R}[s]$. If the quotient $\frac{F(s)}{G(s)}$ is an entire function, it can be written in the form

$$\frac{F}{G} = \frac{H}{Q}, \tag{18}$$

with H an exponential polynomial of the same form as F and G, and $Q \in \mathbb{R}[s]$, i.e. Q is an ordinary polynomial in the variable s. ∎

In [BD] a generalized version of Theorem 5.8 is proved, which is also valid for exponential polynomials in several variables, and for complex exponents ρ_i and μ_j. A simplified version of this proof, based on the same ideas used in [BD], but adapted to the case considered here, is given in Appendix A.

Next we combine Theorem 5.4 and Theorem 5.8. Let $\frac{f}{g} \in \mathcal{Q}(\mathcal{R})_{\text{holo}}$, then $\frac{\tilde{f}}{\tilde{g}} \in H(\mathbb{C})$, and thus, according to Theorem 5.8, $\frac{\tilde{f}}{\tilde{g}}$ can be written as the quotient of an exponential polynomial H and a polynomial $q \in \mathbb{R}[s]$. Since the exponents occurring in $\tilde{f}$ and $\tilde{g}$ are linear combinations of the fixed delays $\tau_1, \ldots, \tau_k$ with integer coefficients, the same is true for the exponential polynomial H. Collecting the negative integer coefficients in one term, we conclude that there exists a polynomial $p \in \mathcal{R}$ and a monomial $m \in \mathbb{R}[z_1, \ldots, z_k]$ such that

$$H = \frac{\tilde{p}}{\tilde{m}}.$$

Since $q \in \mathbb{R}[s]$ (hence $\tilde{q} = q$), and using the fact that $\tilde{\mathcal{R}}$ and $\mathcal{R}$ are isomorphic, we obtain the following representation of the ring $\mathcal{Q}(\mathcal{R})_{\text{holo}}$:

Theorem 5.9 *Let*

$$MON(z_1, \ldots, z_k) := \{z_1^{n_1} z_2^{n_2} \cdots z_k^{n_k} \mid n_1, \ldots, n_k \in \mathbb{N} \cup \{0\}\}$$

denote the set of all monomials in the indeterminates $z_1, \ldots, z_k$. *Define the ring* $\mathcal{H}$ *by*

$$\mathcal{H} := \left\{ \frac{p}{m \cdot q} \quad \middle| \quad p \in \mathcal{R}, m \in MON(z_1, \ldots, z_k), q \in \mathbb{R}[s], \right. \\ \left. \text{such that } \frac{\tilde{p}}{\tilde{m} \cdot \tilde{q}} \in H(\mathbb{C}) \right\} \tag{19}$$

Then $\mathcal{R}_{\mathcal{M}} = \mathcal{Q}(\mathcal{R})_{\text{holo}} = \mathcal{H}$. ∎

Remark 5.10 Note that $\{C \cdot m \mid C \in \mathbb{R}\backslash\{0\}, m \in MON(z_1, \ldots, z_k)\}$ is the set of all elements in $\mathcal{R}$, that are units of the ring $\mathcal{R}_{\mathcal{M}}$. So the elements of $MON(z_1, \ldots, z_k)$, multiplied by nonzero constants, correspond to the operators in $\overline{\mathcal{R}}$, that have an inverse in $\overline{\mathcal{R}_{\mathcal{M}}}$.

The importance of Theorem 5.9 is based on the fact that characterization (19) enables us to verify whether a rational function $\frac{f}{g} \in \mathcal{Q}(\mathcal{R})$ belongs to $\mathcal{R}_{\mathcal{M}}$.

Proposition 5.11 *Let* $r = \frac{f}{g} \in \mathcal{Q}(\mathcal{R})$, *and write the denominator* g *as*

$$g = \sum_{i=1}^{N} m_i q_i,$$

with for all $i = 1, \ldots, N$: $q_i \in \mathbb{R}[s]$ *and* $m_i \in MON(z_1, \ldots, z_k)$, *such that* $m_i \neq m_j$ *for* $i \neq j$. *Let* m *denote the greatest common divisor of* $m_1, \ldots, m_N$ *in the ring* $\mathbb{R}[z_1, \ldots, z_k]$, *and* q *the greatest common divisor of* $q_1, \ldots, q_N$ *in* $\mathbb{R}[s]$. *Define*

$$d := \sum_{i=1}^{N} \frac{m_i}{m} \cdot \frac{q_i}{q} \in \mathcal{R}.$$

Then $r \in \mathcal{H}$ *if and only if*

(i) d *is a divisor of* f *in* $\mathcal{R}$, *i.e.* $p := \frac{f}{d} \in \mathcal{R}$,

(ii) all zeros of $\tilde{q}$ *are zeros of* $\tilde{p}$ *(also counting multiplicities).*

In particular, if (i) and (ii) are satisfied, then $r = \frac{p}{m \cdot q}$.

Proof: (Sufficiency) Since $g = mqd$, condition *(i)* implies that $r = \frac{f}{g} = \frac{p}{mq}$. Now $m \in MON(z_1, \ldots, z_k)$ and therefore the exponential function $\tilde{m}$ has no zeros. So, if all zeros of the polynomial $\tilde{q}$ are zeros of $\tilde{p}$ (including multiplicities), then $\frac{\tilde{f}}{\tilde{g}} = \frac{\tilde{p}}{\tilde{m}\tilde{q}}$ is an entire function. Hence $r \in \mathcal{H}$.

(Necessity) Assume that $r = \frac{f}{g} \in \mathcal{H}$. According to Theorem 5.9, there exist $p_1 \in \mathcal{R}$, $n_1 \in MON(z_1, \ldots, z_k)$, and $g_1 \in \mathbb{R}[s]$ such that

$$\frac{f}{g} = \frac{p_1}{n_1 \cdot g_1}.$$

Without loss of generality we assume that neither n_1 and p_1, nor g_1 and p_1 have a nontrivial common factor. Since $p_1 g = f n_1 g_1$, and $\mathcal{R} = \mathbb{R}[s, z_1, \ldots, z_k]$ is a unique factorization domain, there exists a $d_1 \in \mathcal{R}$ such that $f = p_1 d_1$ and $g = d_1 n_1 g_1$. Using the definitions of m, g, and d, we conclude that $n_1 | m$, $g_1 | q$ and $d | d_1$. Hence, d is a divisor of f over the ring $\mathcal{R}$, i.e. $p = \frac{f}{d} \in \mathcal{R}$, which proves *(i)*. At this point we know that $\frac{f}{g} = \frac{p}{mq}$. By assumption $\frac{\tilde{f}}{\tilde{g}}$ is an entire function, and thus the same is true for $\frac{\tilde{p}}{\tilde{m}\tilde{q}}$. This implies that all zeros of $\tilde{q}$ are zeros of $\tilde{p}$, also counting multiplicities. ■

Remark 5.12 Proposition 5.11 describes a constructive method to test whether an element $\frac{f}{g} \in \mathcal{Q}(\mathcal{R})$ belongs to $\mathcal{Q}(\mathcal{R})_{\text{holo}} = \mathcal{H}$. First of all, in Proposition 5.11 the monomial $m \in MON(z_1, \ldots, z_k)$, and the polynomials $q \in \mathbb{R}[s]$ and $p \in \mathcal{R}$ are obtained from f and g using standard techniques. Therefore condition *(i)* is easily verified. Furthermore, $\tilde{q} \in \mathbb{R}[s]$ has only finitely many zeros in $\mathbb{C}$ (including multiplicities). To check condition *(ii)*, one only has to test whether in this *finite* number of points, the function $\tilde{p}$ is annihilated, also taking the multiplicity of the zeros into account. This is the main advantage of characterization (19) of $\mathcal{Q}(\mathcal{R})_{\text{holo}} = \mathcal{H}$ in comparison with (16): the number of points in which a pole-zero cancellation has to be tested is reduced from an (in principle) infinite number of points to a finite number of points.

For differential-difference systems with commensurable delays, i.e. the case $k = 1$, the ring $\mathcal{H}$ has recently been introduced in [G-L]. Using this ring, the problem of system equivalence was solved in a different way. In [G-L] it is shown that for $k = 1$ (commensurable delays) the ring $\mathcal{H}$ is a Bezout ring, i.e. every finitely generated ideal of $\mathcal{H}$ is principal. This implies that every matrix over $\mathcal{H}$ admits a Smith form, and thus the usual approach to the problem of system equivalence (compare [W1], [W2]) applies. For $k > 1$ however, the ring $\mathcal{H}$ is not a Bezout ring, as is shown in the next counterexample.

Example 5.13 Let $\mathcal{R} = \mathbb{R}[s, z_1, z_2]$, and consider $f, g \in \mathcal{R}$, given by $f(s, z_1, z_2) = s + z_1$ and $g(s, z_1, z_2) = z_1 - z_2$. Let $\tau_1, \tau_2 \in \mathbb{R}^+$ be the incommensurable time-delays corresponding to the indeterminates z_1 and z_2 respectively. It is obvious that all zeros of $\tilde{g}$ lie on the imaginary axis. Using the fact that τ_1 and τ_2 are incommensurate, one may show that none of these zeros is also a zero of $\tilde{f}$. Hence $\tilde{f}$ and $\tilde{g}$ have no common zeros in $\mathbb{C}$.

Consider the ideal $\langle f, g \rangle_{\mathcal{H}}$ in the ring $\mathcal{H}$. If $\mathcal{H}$ is a Bezout ring, this ideal is a principal ideal, generated by a single element $h = \frac{h_1}{h_2} \in \mathcal{H}$: $\langle f, g \rangle_{\mathcal{H}} = \langle h \rangle_{\mathcal{H}}$. Every zero of $\tilde{h} = \frac{\tilde{h_1}}{\tilde{h_2}}$ is a common zero of $\tilde{f}$ and $\tilde{g}$. Therefore $\tilde{h}$ has no zeros in $\mathbb{C}$, and thus (see e.g. [H2, p. 6]) $\tilde{h}$ is an exponential function. So h is the quotient of two monomials in z_1 and z_2, and we conclude that h is a unit of the ring $\mathcal{H}$. Hence $\langle h \rangle_{\mathcal{H}} = \mathcal{H}$, and there exist $c_1, c_2 \in \mathcal{H}$ such that

$$c_1 \cdot f + c_2 \cdot g = 1. \tag{20}$$

According to (19), c_1 and c_2 may be written in the form $c_i = \frac{d_i}{m_i n_i}$, with $d_i \in \mathcal{R}$, $m_i \in MON(z_1, z_2)$, and $n_i \in \mathbb{R}[s]$ $(i = 1, 2)$. Multiplying (20) by $n_1 \cdot n_2$ we obtain

$$\frac{d_1 n_2}{m_1} \cdot f + \frac{d_2 n_1}{m_2} \cdot g = n_1 \cdot n_2. \tag{21}$$

Since $\frac{d_1 n_2}{m_1}, \frac{d_2 n_1}{m_2} \in \mathbb{R}[s, z_1, z_2, z_1^{-1}, z_2^{-1}]$ and $n_1 n_2 \in \mathbb{R}[s]$ we conclude that the ideal $\mathcal{I}$ in $\mathbb{R}[s, z_1, z_2, z_1^{-1}, z_2^{-1}]$ generated by f and g contains a univariate polynomial in s.

This property of the ideal $\mathcal{I}$ may be verified, using a constructive elimination technique, like the Gröbner basis algorithm. However, this method cannot be applied directly, because the Gröbner basis algorithm cannot handle the fractions z_1^{-1}, z_2^{-1}. This problem is circumvented by introducing two new variables z_3 and z_4 and the polynomials $p_1 := z_1 z_3 - 1$ and $p_2 := z_2 z_4 - 1$. So z_3 and z_4 play the role of z_1^{-1} and z_2^{-1}, respectively. Now one computes the Gröbner basis G of the ideal $\mathcal{I}'$ in $\mathbb{R}[s, z_1, z_2, z_3, z_4]$ generated by f, g, p_1, and p_2, with respect to the pure lexicographic term ordering $s \prec z_1 \prec z_2 \prec z_3 \prec z_4$. Then the Elimination Theorem (see [CLS, p. 114]) implies that $\mathcal{I}'$ contains a polynomial in $\mathbb{R}[s]$ if and only if G does. In our example $G \cap \mathbb{R}[s] = \emptyset$, and thus $\mathcal{I}' \cap \mathbb{R}[s] = \emptyset$. However, if the ideal $\mathcal{I}'$ does not contain a univariate polynomial in s, one may verify that also $\mathcal{I} \cap \mathbb{R}[s] = \emptyset$. This contradicts formula (21) and we conclude that $\mathcal{H}$ is not a Bezout ring.

In comparison with [G-L] the results of the present paper are more general, because they also include the situation of delay-differential systems with incommensurable delays. For these systems the ring $\mathcal{R}_{\mathcal{M}} = \mathcal{H}$ is not a Bezout ring, and the method to solve the problem of system equivalence developed in [G-L] fails. Note however that in the results on system equivalence in Section 4, the system defining matrix P is assumed to be of full row rank. Although this is not a severe restriction, the results in [G-L] are valid without this assumption, and therefore somewhat stronger. The approach developed in this paper is adapted to the situation where $\mathcal{H}$ is not a Bezout ring; in our method the Smith form (only available if $\mathcal{H}$ is a Bezout ring) is not involved. Also the construction of a unimodular matrix, transforming two equivalent system representations into each other, is explicitly described.

Example 5.14 Consider the behaviors in $\mathcal{E}(\mathbb{R})^3$ of the delay-differential systems with incommensurable time-delays $\tau_1, \tau_2 \in \mathbb{R}^+$ described by

$$P = \begin{pmatrix} s & s^2 z_1 + s z_2 & 0 \\ 0 & z_2 & 1 \end{pmatrix}, \text{ and}$$

$$Q = \begin{pmatrix} z_1 - z_2 & s z_1(z_1 - z_2) + z_2(z_1 - z_2 + 1) & 1 \\ s(z_1^2 - z_1 z_2 - 1) & s^2 z_1(z_1^2 - z_1 z_2 - 1) + s z_2(z_1^2 - z_1 z_2 + z_1 - 1) & s z_1 \end{pmatrix}.$$

Let P_1 and Q_1 be the matrices consisting of the first and third columns of P and Q, respectively. Then $\det(P_1) = s$ and

$$U := Q_1 P_1^{-1} = \begin{pmatrix} \frac{z_1 - z_2}{s} & 1 \\ z_1^2 - z_1 z_2 - 1 & s z_1 \end{pmatrix}$$

is the candidate matrix for transforming P into Q. First we show that U is a matrix over $\mathcal{H}$. Let $u_{11} = \frac{p}{q}$ with $p = z_1 - z_2$ and $q = s$. Then $\frac{\tilde{p}}{\tilde{q}} = \frac{e^{-\tau_1 s} - e^{-\tau_2 s}}{s}$ is holomorphic in $\mathbb{C}$ because 0 is a zero of $\tilde{p} = e^{-\tau_1 s} - e^{-\tau_2 s}$, and thus $u_{11} \in \mathcal{H}$. By direct computation it is verified that $U \cdot P = Q$, and thus $\ker(\overline{P}) \subset \ker(\overline{Q})$. Furthermore $\det(U) = 1$, so U is invertible over $\mathcal{H}$, and the equivalence of the behaviors $\ker(\overline{P})$ and $\ker(\overline{Q})$ follows with Theorem 4.9. Note that in this example there does not exist a matrix over $\mathcal{R}$, that transforms P into Q, i.e. P is not a right divisor of Q over $\mathcal{R}$.

6 Conclusions

In this paper we solved the problem of system equivalence for AR-systems over an arbitrary integral domain $\mathcal{R}$. For this type of systems, the signal space is assumed to be a module $\mathcal{M}$ over $\mathcal{R}$. After introduction of a ring extension $\mathcal{R}_{\mathcal{M}}$ of $\mathcal{R}$, explicitly depending on the module $\mathcal{M}$, system equivalence was characterized by division properties on the system-defining matrices over the ring $\mathcal{R}_{\mathcal{M}}$.

The theory is applicable to differential-difference systems. In this case the ring $\mathcal{R}_{\mathcal{M}}$ takes a special form: every element of $\mathcal{R}_{\mathcal{M}}$ is described by the pole-zero cancellation properties of the quotient of an exponential polynomial (the formal Laplace transform of the corresponding delay-differential operator), and an ordinary polynomial. Using this alternative characterization, a constructive method for the verification of system equivalence is obtained. Further research is necessary to study the algebraic structure of the ring $\mathcal{R}_{\mathcal{M}}$ in the case of differential-difference systems. Also the characterization of the ring $\mathcal{R}_{\mathcal{M}}$ in other applications remains an open problem.

Appendix A

This appendix is devoted to the proof of Theorem 5.8. The statement of this result is valid in a far more general context, and was proved in full generality in [BD]. We here confine ourselves to the restricted version, needed in Section 5. Our proof is based on the same ideas as in [BD], but for this simplified version the technicalities are less involved. It consists of a constructive iteration, in combination with a stopping criterion, that guarantees that the number of iterations necessary to obtain the final result is finite. We start with the elaboration of the stopping criterion.

Theorem A.1 *Consider the exponential polynomials F, G, given by*

$$F(s) = \sum_{i=1}^{n} P_i(s) e^{\rho_i s}, \qquad G(s) = \sum_{j=1}^{m} Q_j(s) e^{\mu_j s},$$

with $\rho_1 > \rho_2 > \cdots > \rho_n$ and $\mu_1 > \mu_2 > \cdots > \mu_m$, and $P_i, Q_j \in \mathbb{R}[s]$ $(i = 1, \ldots, n; j = 1, \ldots, m)$. Assume that $P_1 \neq 0$, $P_n \neq 0$, $Q_1 \neq 0$, and $Q_m \neq 0$. If the quotient $K = \frac{F}{G}$ is an entire function, then $\rho_1 - \rho_n \geq \mu_1 - \mu_m$.

Theorem A.1 gives a necessary condition for the quotient of two exponential polynomials to be entire: the length of the delay interval of the numerator (i.e. the difference of the largest and smallest exponent occurring in the numerator) has to be larger or equal to the length of the delay interval of the denominator.

The proof of Theorem A.1 is based on the following fact on subharmonic functions (see e.g. [BG, Corollary 4.4.33]):

Lemma A.2 *There is no subharmonic function which is bounded above in $\mathbb{C}$, except for the constants.* ∎

Proof of Theorem A.1: Assume that $K = \frac{F}{G}$ is entire, and define the function $u : \mathbb{C} \longrightarrow \mathbb{R}$ by

$$u(s) := \limsup_{t \to \infty} \frac{1}{t} \log |K(ts)|. \tag{22}$$

First we derive an explicit function description for u. Since

$$K(s)=\frac{\sum_{i=1}^{n}P_i(s)e^{\rho_i s}}{\sum_{j=1}^{m}Q_j(s)e^{\mu_j s}}=\frac{e^{(\rho_1-\mu_1)s}(P_1(s)+\sum_{i=2}^{n}P_i(s)e^{(\rho_i-\rho_1)s})}{Q_1(s)+\sum_{j=2}^{m}Q_j(s)e^{(\mu_j-\mu_1)s}},$$

and $\rho_i-\rho_1<0$ for $i=2,\dots,n$, and $\mu_j-\mu_1<0$ for $j=2,\dots,m$, we find for $\operatorname{Re}s>0$:

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log|K(ts)| &= \limsup_{t\to\infty}\frac{1}{t}\Bigg((\rho_1-\mu_1)t\operatorname{Re}s+\log\left|P_1(ts)+\sum_{i=2}^{n}P_i(ts)e^{(\rho_i-\rho_1)ts}\right| \\ &\qquad -\log\left|Q_1(ts)+\sum_{j=2}^{m}Q_j(ts)e^{(\mu_j-\mu_1)ts}\right|\Bigg)=(\rho_1-\mu_1)\cdot\operatorname{Re}s.\end{aligned}$$

Completely analogously, by factoring out the exponential function $e^{(\rho_n-\mu_m)s}$, one verifies that for $\operatorname{Re}s<0$:

$$\limsup_{t\to\infty}\frac{1}{t}\log|K(ts)|=(\rho_n-\mu_m)\cdot\operatorname{Re}s.$$

To determine the values of u on the imaginary axis, we use Theorem 5.7. Since K belongs to the Paley-Wiener class $PW(\mathbb{C})$, K is polynomially bounded on the imaginary axis: there exist $C>0$ and $N\in\mathbb{N}$ such that $|K(s)|\leq C(1+|s|)^N$ for all $s\in\mathbb{C}$ with $\operatorname{Re}s=0$. This immediately implies that if $\operatorname{Re}s=0$, then

$$\limsup_{t\to\infty}\frac{1}{t}\log|K(ts)|=0.$$

Summarizing we have

$$u(s)=\begin{cases}(\rho_1-\mu_1)\cdot\operatorname{Re}s & \text{if } \operatorname{Re}s>0\\ 0 & \text{if } \operatorname{Re}s=0\\ (\rho_n-\mu_m)\cdot\operatorname{Re}s & \text{if } \operatorname{Re}s<0\end{cases}\tag{23}$$

Since K is an entire function, we know that for every $t>0$ the function $u_t(s)=\frac{1}{t}\log|K(ts)|$ is a subharmonic function in the variable s. Furthermore, the limes superior u of u_t $(t\to\infty)$ is continuous on $\mathbb{C}$, and therefore Hartogs's Lemma (see e.g. [BG, Proposition 4.4.40]) implies that also u is a subharmonic function.

To prove the theorem, assume that $\rho_1-\rho_n<\mu_1-\mu_m$. Then $\rho_1-\mu_1<\rho_n-\mu_m$, and there exists an $\alpha\in\mathbb{R}$ such that $\rho_1+\alpha-\mu_1<0<\rho_n+\alpha-\mu_m$. Define $\hat{K}(s)=e^{\alpha s}K(s)$. $\hat{K}$ is an entire function, and quotient of the exponential polynomials $e^{\alpha s}F(s)$ and $G(s)$, so according to formula (23) the corresponding function $\hat{u}(s)=\limsup_{t\to\infty}\frac{1}{t}\log|\hat{K}(ts)|$ satisfies:

$$\hat{u}(s)=\begin{cases}(\rho_1+\alpha-\mu_1)\cdot\operatorname{Re}s & \text{if } \operatorname{Re}s>0\\ 0 & \text{if } \operatorname{Re}s=0\\ (\rho_n+\alpha-\mu_m)\cdot\operatorname{Re}s & \text{if } \operatorname{Re}s<0\end{cases}$$

Now $\rho_1+\alpha-\mu_1<0$ and $\rho_n+\alpha-\mu_m>0$, thus the subharmonic function $\hat{u}$ is bounded above by 0. Lemma A.2 implies that $\hat{u}=0$, and thus $\rho_1+\alpha-\mu_1=0=\rho_n+\alpha-\mu_m$. This contradicts our assumption and we conclude that $\rho_1-\rho_n\geq\mu_1-\mu_m$. ∎

Corollary A.3 *Consider the exponential polynomials F,G, given by*

$$F(s)=\sum_{i=1}^{n}P_i(s)e^{\rho_i s},\qquad G(s)=\sum_{j=1}^{m}Q_j(s)e^{\mu_j s},$$

with $\rho_1 > \rho_2 > \cdots > \rho_n$ *and* $\mu_1 > \mu_2 > \cdots > \mu_m$, *and* $P_i, Q_j \in \mathbb{R}[s]$ $(i = 1, \ldots, n; j = 1, \ldots, m)$. *Assume that* $Q_1 \neq 0$, *and* $Q_m \neq 0$. *If* $K = \frac{F}{G} \in H(\mathbb{C})$ *and* $\rho_1 - \rho_n < \mu_1 - \mu_m$, *then* $F = 0$ *and* $K = 0$. ■

Theorem A.1 or Corollary A.3 is applied as stopping criterion for the iteration that is used to prove Theorem 5.8. For this purpose a sequence (F_i) of exponential polynomials is constructed with the property that $K_i = \frac{F_i}{G}$ is entire. Moreover, in every step the length of the delay interval of F_i decreases with an a priori fixed number. According to Corollary A.3 this iteration will stop after finitely many steps with $F_i = 0$ and $K_i = 0$.

To start the iteration, consider two exponential polynomials F, G:

$$F(s) = \sum_{i=1}^{n} P_i(s) e^{\rho_i s}, \qquad G(s) = \sum_{j=1}^{m} Q_j(s) e^{\mu_j s},$$

with $\rho_1 > \rho_2 > \cdots > \rho_n$, and $\mu_1 > \mu_2 > \cdots > \mu_m$, and $P_i, Q_j \in \mathbb{R}[s]$ $(i = 1, \ldots, n; j = 1, \ldots, m)$. Assume that $m \geq 2$, and $Q_j \neq 0$ for $j = 1, \ldots, m$. Let the quotient $K = \frac{F}{G}$ be an entire function, and assume that $\rho_1 - \rho_n \geq \mu_1 - \mu_m$.

Fix $d := \mu_1 - \mu_2$. Since $\rho_1 - \rho_n \geq \mu_1 - \mu_m \geq d$, there exists an $r \in \{1, \ldots, n-1\}$ such that

$$\left[\begin{array}{lll} \rho_i & > & \rho_1 - d \quad \text{for } i = 1, \ldots, r, \\ \rho_i & \leq & \rho_1 - d \quad \text{for } i = r+1, \ldots, n. \end{array}\right.$$

We distinguish two different cases:

Case 1 $\exists \ell \in \{1, \ldots, r-1\}$ such that $\rho_\ell - \rho_n \geq \mu_1 - \mu_m$ and $\rho_{\ell+1} - \rho_n < \mu_1 - \mu_m$. Define $\tilde{n} := \ell$.

Case 2 $\forall i \in \{1, \ldots, r\} : \rho_i - \rho_n \geq \mu_1 - \mu_m$ (or equivalently $\rho_r - \rho_n \geq \mu_1 - \mu_m$). Define $\tilde{n} := r$.

In both cases we make one step in the iteration procedure by defining the functions g, F_1 and K_1 in the following way:

$$g(s) = G(s) - Q_1(s) e^{\mu_1 s}, \tag{24}$$

$$F_1(s) = Q_1(s) \left(F(s) - \sum_{i=1}^{\tilde{n}} P_i(s) e^{\rho_i s} \right) - g(s) \sum_{i=1}^{\tilde{n}} P_i(s) e^{(\rho_i - \mu_1)s}, \tag{25}$$

$$K_1(s) = Q_1(s) K(s) - \sum_{i=1}^{\tilde{n}} P_i(s) e^{(\rho_i - \mu_1)s}. \tag{26}$$

It is obvious that g and F_1 are exponential polynomials, and because $K \in H(\mathbb{C})$ we also know that $K_1 \in H(\mathbb{C})$. Furthermore, K_1 is the quotient of the exponential polynomials F_1 and G:

$$F_1 = K_1 \cdot G, \tag{27}$$

which, by Theorem 5.7, implies that $K_1 \in PW(\mathbb{C})$. The validity of formula (27) is shown by direct computation, using the fact that $F = K \cdot G$:

$$\begin{aligned} K_1(s)G(s) &= Q_1(s)K(s)G(s) - \sum_{i=1}^{\tilde{n}} P_i(s) e^{(\rho_i - \mu_1)s} G(s) = \\ &= Q_1(s)F(s) - (g(s) + Q_1(s)e^{\mu_1 s}) \sum_{i=1}^{\tilde{n}} P_i(s) e^{(\rho_i - \mu_1)s} = \\ &= Q_1(s) \left(F(s) - \sum_{i=1}^{\tilde{n}} P_i(s) e^{\rho_i s} \right) - g(s) \sum_{i=1}^{\tilde{n}} P_i(s) e^{(\rho_i - \mu_1)s} = F_1(s). \end{aligned}$$

We conclude that the new triple (F_1, G, K_1) has the same properties as the original triple (F, G, K). Moreover, if K_1 can be written as the quotient of an exponential polynomial and an ordinary polynomial, formula (26) implies that the same is true for K.

Next we study the exponents occurring in the exponential polynomial F_1. It is obvious that the term $Q_1(s)(F(s) - \sum_{i=1}^{\tilde{n}} P_i(s)e^{\rho_i s})$ has exponents $\rho_{\tilde{n}+1}, \ldots, \rho_n$, at least if $P_i \neq 0$ for $i = \tilde{n}+1, \ldots, n$. $g(s) = G(s) - Q_1(s)e^{\mu_1 s}$ has exponents $\mu_2, \ldots, \mu_m$, and therefore the exponents of $g(s) \sum_{i=1}^{\tilde{n}} P_i(s)e^{(\rho_i - \mu_1)s}$ belong to the set

$$\{\rho_i - \mu_1 + \mu_j \mid i = 1, \ldots, \tilde{n}; j = 2, \ldots, m\}. \tag{28}$$

It is evident that the smallest element of this set is $\rho_{\tilde{n}} - \mu_1 + \mu_m$. So for the smallest exponent of F_1 we have two candidates: ρ_n and $\rho_{\tilde{n}} - \mu_1 + \mu_m$. The definition of $\tilde{n}$ implies that ρ_n is the smallest: $\rho_{\tilde{n}} - \rho_n \geq \mu_1 - \mu_m$, and thus $\rho_n \leq \rho_{\tilde{n}} - \mu_1 + \mu_m$. To obtain an upper bound for the exponents of F_1 we return to the set of all possible exponents of $g(s) \sum_{i=1}^{\tilde{n}} P_i(s)e^{(\rho_i - \mu_1)s}$, given in (28). Every exponent in this set satisfies

$$\rho_i - \mu_1 + \mu_j \leq \rho_1 - \mu_1 + \mu_2 = \rho_1 - d,$$

i.e. all these exponents are bounded above by $\rho_1 - d$. For a concise analysis, we now distinguish between **Case 1** and **Case 2**:

Case 1: In addition to the previous results on the exponents of F_1, we know that $\tilde{n} < r$. Therefore the largest possible exponent of F_1 is $\rho_{\tilde{n}+1}$, which is —by definition of r— larger than $\rho_1 - d$. We conclude that the exponents of F_1 are contained in the interval $[\rho_n, \rho_{\tilde{n}+1}]$.

Case 2: Here $\tilde{n} = r$, and thus $\rho_{\tilde{n}+1} \leq \rho_1 - d$. This implies that the exponents of F_1 are bounded above by $\rho_1 - d$. Therefore the exponents of F_1 are contained in the interval $[\rho_n, \rho_1 - d]$.

At this point the significance of Corollary A.3 becomes apparent. In **Case 1** the quotient $K_1 = \frac{F_1}{G}$ satisfies the conditions of Corollary A.3: the difference of the largest and smallest exponent in the numerator F_1 is at most $\rho_{\tilde{n}+1} - \rho_n$, which is —by definition of $\tilde{n}$— smaller than the difference $\mu_1 - \mu_m$ of the largest and smallest exponent in the denominator G. Furthermore K_1 is an entire function and thus $F_1 = 0$ and $K_1 = 0$. Using formula (26), we obtain

$$K(s) = \frac{1}{Q_1(s)} \sum_{i=1}^{\tilde{n}} P_i(s)e^{(\rho_i - \mu_1)s},$$

i.e. K is the quotient of an exponential polynomial and a polynomial in $\mathbb{R}[s]$. Although in **Case 2** we cannot draw the same conclusion yet, we know that in this situation the length of the delay interval of F_1 is at least d smaller than the length of the delay interval of F.

Theorem A.4 (Theorem 5.8) *Consider the exponential polynomials F, G, given by*

$$F(s) = \sum_{i=1}^{n} P_i(s)e^{\rho_i s}, \qquad G(s) = \sum_{j=1}^{m} Q_j(s)e^{\mu_j s},$$

with $\rho_1 > \rho_2 > \cdots > \rho_n$, and $\mu_1 > \mu_2 > \cdots > \mu_m$, and $P_i, Q_j \in \mathbb{R}[s]$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Suppose that the quotient $K := \frac{F}{G}$ is an entire function. Then there exist an exponential polynomial H,

$$H(s) = \sum_{k=1}^{\ell} H_k(s)e^{\nu_k s},$$

with $H_k \in \mathbb{R}[s]$ and $\nu_k \in \mathbb{R}$ for $k = 1, \ldots, \ell$, and a polynomial $Q \in \mathbb{R}[s]$, such that

$$K = \frac{H}{Q}, \tag{29}$$

i.e. K can be written as the quotient of an exponential polynomial and an ordinary polynomial.

Proof: Without loss of generality we assume that $Q_j \neq 0$ for $j = 1, \ldots, m$. If $m = 1$, then K is already in the form (29), and there is nothing to prove.

If $m > 1$, fix $d := \mu_1 - \mu_2$. Define $F_0 := F$ and $K_0 := K$. As long as $F_i \neq 0$ we construct the exponential polynomial F_{i+1} and the function $K_{i+1} \in PW(\mathbb{C})$, according to formulae (25) and (26), with F and F_1 replaced by F_i and F_{i+1}, respectively, and K and K_1 replaced by K_i and K_{i+1}, respectively. In this way we obtain a sequence of exponential polynomials (F_i) and a sequence of Paley-Wiener functions (K_i) with the following properties:

(i) $F_i = K_i \cdot G$ for all i.

(ii) If K_{i+1} can be written in the form (29), i.e. if K_{i+1} is the quotient of an exponential polynomial and an ordinary polynomial, then also K_i can be written in the form (29).

(iii) If $F_i \neq 0$, then the exponents occurring in F_i are contained in the interval $[\rho_n, \rho_1 - i \cdot d]$

Properties *(i)* and *(ii)* follow immediately from formulae (25) and (26). Moreover, if $F_i \neq 0$, and $i > 0$, we know that in the i-th step of the construction, **Case 2** was valid. (If in the i-th iteration **Case 1** is valid, the algorithm terminates with $F_i = 0$). So the smallest exponent occurring in F_i is not smaller than the smallest exponent of F_{i-1}, and the largest exponent of F_i is at least d units smaller than the largest exponent of F_{i-1}. Since the exponents of F_0 are contained in $[\rho_n, \rho_1]$, and $F_i \neq 0$ implies that $F_j \neq 0$ for $j = 0, 1, \ldots, i-1$, we conclude that the exponents of F_i belong to the interval $[\rho_n, \rho_1 - i \cdot d]$.

If $\rho_1 - \rho_n < \mu_1 - \mu_m$, then Corollary A.3 implies that $F = 0$ and $K = 0$, and there is nothing to prove. Otherwise define $N := \text{entier}(\frac{(\rho_1 - \rho_n) - (\mu_1 - \mu_m)}{d}) + 1$. Then there exists an $M \leq N$ such that $F_M = 0$ and $K_M = 0$. To verify this claim, suppose that the iteration does not stop before the N-th step, and that $F_N \neq 0$. Then the exponents of F_N belong to the interval $[\rho_n, \rho_1 - Nd]$. By definition of N, the length $\rho_1 - Nd - \rho_n$ of this interval is smaller than $\mu_1 - \mu_m$ (the difference of the largest and smallest exponent occurring in G). Since $K_N \in H(\mathbb{C})$ this implies, according to Corollary A.3, that $F_N = 0$ and $K_N = 0$. We conclude that the iteration stops after a finite number of M steps, with $M \leq N$.

After termination of the algorithm, we know that $F_M = 0$ and $K_M = 0$. In particular, K_M can be written in the form (29). Using property *(ii)*, and exploiting its recursive character, we find that every element of the sequence $(K_i)_{i=0}^{M}$ can be written in the form (29). Hence, also $K = K_0$ can be written as the quotient of an exponential polynomial and an ordinary polynomial. ∎

References

[BC] R. Bellman and K.L. Cooke, *Differential-Difference Equations.* Mathematics in science and engineering, Vol. 6, Academic Press, New York, 1963.

[BD] C.A. Berenstein and M.A. Dostal, The Ritt theorem in several variables. *Ark. Mat.*, **12**, pp. 267—280, 1974.

[BG] C.A. Berenstein and R.A. Gay, *Complex Variables — An Introduction.* Graduate Texts in Mathematics, Vol. 125, Springer-Verlag, New York, 1991.

[BBV] J.W. Brewer, J.W. Bunce, and F.S. Van Vleck, *Linear systems over commutative rings*. Lecture notes in pure and applied mathematics, Vol. 104, Marcel Dekker, New York, 1986.

[CLS] D. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1992.

[E1] L.Ehrenpreis, Analytic functions and the Fourier transform of distributions, I. *Ann. of Math.*, **63**, pp. 129—159, 1956.

[E2] L. Ehrenpreis, Solution of some problems of division. Part III. Division in the Spaces, $\mathcal{D}'$, $\mathcal{H}$, $\mathcal{Q}_A$, $\mathcal{O}$. *American J. of Math.*, **78**, pp. 685—715, 1956.

[EH] S.J.L. van Eijndhoven and L.C.G.J.M. Habets, On quotients of exponential polynomials in the Paley-Wiener class. In preparation.

[G-L] H. Glüsing-Lüerssen, A behavioral approach to delay-differential systems. *SIAM J. Control Optim.*, **35**, pp. 480—499, 1997.

[H1] L.C.G.J.M. Habets, System equivalence for AR-systems over rings. *Proceedings of the 4-th European Control Conference ECC '97*, Brussels, 1997.

[HVL] J.K. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*. Applied Mathematical Sciences, Vol. 99, Springer-Verlag, New York, 1993.

[H2] A.S.B. Holland, *Introduction to the theory of entire functions*. Academic Press, New York, 1973.

[K1] J.-P. Kahane, Sur quelques problèmes d'unicité et de prolongement, relatifs aux fonctions approchables par des sommes d'exponentielles. *Ann. Inst. Fourier*, **5**, pp. 39—130, 1953/1954.

[K2] J.-P. Kahane, *Lectures on mean periodic functions*. Tata Institute of Fundamental Research, Bombay, 1959.

[M] B. Malgrange, Existence et approximations des solutions des équations aux dérivées partielles et des équations de convolution. *Ann. Inst. Fourier*, **6**, pp. 271—355, 1955/1956.

[R] M. de Rijcke, *Semigroups, Invariance and Time-Invariant Linear Systems*. Ph.D. Thesis, Eindhoven University of Technology, Eindhoven, 1996.

[RW] P. Rocha and J.C. Willems, Behavioral controllability of delay-differential systems. *SIAM J. Control Optim.*, **35**, pp. 254—264, 1997.

[S1] H.H. Schaefer, *Topological Vector Spaces*. MacMillan, New York, 1966.

[S2] L. Schwartz, Théorie générale des fonctions moyenne-périodiques. *Ann. of Math.*, **48**, pp. 857—929, 1947.

[S3] E.D. Sontag, Linear systems over commutative rings: a survey. *Ricerche di Automatica*, **7**, pp. 1—34, 1976.

[W1] J.C. Willems, Models for dynamics. In: U. Kirchgraber and H.O. Walther, editors, *Dynamics Reported*, Vol. 2, pp. 171—269, B.G. Teubner, Stuttgart/Wiley, Chichester, 1989.

[W2] J.C. Willems, Paradigms and puzzles in the theory of dynamical systems. *IEEE Trans. Aut. Control*, **AC-36**, pp. 259—294, 1991.