Multibody dynamics notation (version 2)

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1 Introduction

This document provides a revision of the notation originally introduced in [20] for describing kinematics and dynamics quantities of mechanical systems composed by several rigid bodies. Relative to the first edition, this new version includes an expanded section on frame acceleration (Section 5.4), the correction of a few typos, and the change of the fonts used in the notation from single face to bold face.

The notation detailed in this document is inspired by the well-known Featherstone notation introduced in [7], also used, with small adaptations, in the Handbook of Robotics [16]. Featherstone's notation, while being extremely compact and pleasant for the eye, is not fully in accordance with Lie group formalism, with the potential of creating a misunderstanding between the robotics and geometric mechanics communities.

The Lie group formalism is well established in the robotics literature [13, 14, 10]. However, it is less compact than Featherstone's notation [7], leading to long expressions when several rigid bodies are present as in the case of a complete dynamic model of humanoid or quadruped robots.

This report aims, therefore, at getting the best from these two worlds. The notation strives to be compact, precise, and in harmony with Lie Group formalism. The document furthermore introduces a flexible and unambiguous notation to describe the Jacobians mapping generalized velocities of an arbitrary frame to Cartesian linear and angular velocities, expressed with respect to a reference frame of choice.

2 A quick overview on the developed notation

Quick reference list for the symbols used in this document. Precise definition is given in the text below.

A,B,C, ...	coordinate frames
$\mathbf{p}$	an arbitrary point
$\mathbf{o}_B$	origin of B
$[A]$	orientation frame associated to A
$B[A]$	frame with origin o_B and orientation $[A]$
${}^A\mathbf{p}$	coordinates of point $\mathbf{p}$ w.r.t. to A
${}^A\mathbf{o}_B$	coordinates of the origin $\mathbf{o}_B$ w.r.t. to A
${}^A\mathbf{H}_B$	homogeneous transformation from B to A
${}^A\mathbf{X}_B$	velocity transformation from B to A

${}^{C}\mathbf{v}_{A,B}$	twist expressing the velocity of B w.r.t. to A written in C
${}^{C}\mathbf{v}_{A,B}^{\wedge}$	4×4 matrix representation of ${}^{C}\mathbf{v}_{A,B}$
${}^{C}\mathbf{v}_{A,B}\times$	6×6 matrix representation of the twist cross product
${}^{C}\mathbf{v}_{A,B}\bar{\times}^{*}$	6×6 matrix representation of the dual cross product
${}^{C}\mathbf{a}_{A,B}$	acceleration of a frame B w.r.t. frame A, written in C
${}_{B}\mathbf{f}$	coordinates of the wrench $\mathbf{f}$ w.r.t. B
${}_{A}\mathbf{X}^{B}$	wrench transformation from B to A
$\langle {}_{B}\mathbf{f}, {}^{B}\mathbf{v}_{A,B}\rangle$	duality pairing between a wrench and a twist
${}^{C}\mathbf{J}_{A,B}$	Jacobian relating the velocity of B w.r.t. A expressed in C
${}^{C}\mathbf{J}_{A,B/F}$	Jacobian relating the velocity of B w.r.t. A expressed in C, where the moving-base velocity is expressed in F
${}_{B}\mathbb{M}_{B}^{L}$	6×6 inertia tensor of link L expressed w.r.t. frame B
${}_{B}\mathbb{I}_{B}^{L}$	3×3 inertia tensor of link L expressed w.r.t. frame B

3 Math preliminaries

3.1 Notation

The following notation is used throughout the document.

- The set of real numbers is denoted by $\mathbb{R}$. Let $\mathbf{u}$ and $\mathbf{v}$ be two n-dimensional column vectors of real numbers, i.e. $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$, then their inner product is denoted as $\mathbf{u}^T\mathbf{v}$, with "T" the transpose operator.

- The identity matrix of dimension n is denoted $\mathbf{I}_n\in\mathbb{R}^{n\times n}$; the zero column vector of dimension n is denoted $\mathbf{0}_n\in\mathbb{R}^n$; the zero matrix of dimension $n\times m$ is denoted $\mathbf{0}_{n\times m}\in\mathbb{R}^{n\times m}$.

- The set SO(3) is the set of $\mathbb{R}^{3\times3}$ orthogonal matrices with determinant equal to one, namely

$$\mathrm{SO}(3) := \{\,\mathbf{R}\in\mathbb{R}^{3\times3} \mid \mathbf{R}^T\mathbf{R}=\mathbf{I}_3,\ \det(\mathbf{R})=1\,\}. \tag{1}$$

  When endowed with matrix multiplication, SO(3) becomes a Lie group, the *Special Orthogonal* group of dimension three.

- The set $\mathfrak{so}(3)$, read *little* $\mathfrak{so}(3)$, is the set of 3×3 skew-symmetric matrices,

$$\mathfrak{so}(3) := \{\,\mathbf{S}\in\mathbb{R}^{3\times3} \mid \mathbf{S}^T=-\mathbf{S}\,\}. \tag{2}$$

  When endowed with the matrix commutator as operation, the set becomes a Lie algebra.

- The set SE(3) is defined as

$$\mathrm{SE}(3) := \left\{\begin{bmatrix}\mathbf{R} & \mathbf{p}\\ \mathbf{0}_{1\times3} & 1\end{bmatrix}\in\mathbb{R}^{4\times4} \mid \mathbf{R}\in\mathrm{SO}(3),\mathbf{p}\in\mathbb{R}^3\right\}. \tag{3}$$

  When endowed with matrix multiplication, it becames the *Special Euclidean* group of dimension three, a Lie group that can be used to represent rigid transformations and their composition in the 3D space.

- The set $\mathfrak{se}(3)$ is defined as

$$\mathfrak{se}(3) := \left\{ \begin{bmatrix} \mathbf{\Omega} & \boldsymbol{v} \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \mid \mathbf{\Omega} \in \mathfrak{so}(3), \boldsymbol{v} \in \mathbb{R}^3 \right\}. \tag{4}$$

  When endowed with the matrix commutator as operation, $\mathfrak{se}(3)$ becomes the Lie algebra of the Lie group SE(3).

- Given the vector $w = (x; y; z) \in \mathbb{R}^3$, we define $w^\wedge$ (read *w hat*) as the 3×3 *skew-symmetric* matrix

$$\mathbf{w}^\wedge = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^\wedge := \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \in \mathfrak{so}(3). \tag{5}$$

  Given the *skew-symmetric matrix* $W = w^\wedge$, we define $W^\vee \in \mathbb{R}^3$ (read *W vee*) as

$$\mathbf{W}^\vee = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}^\vee := \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3. \tag{6}$$

  Clearly, the vee operator is the inverse of the hat operator.

- Given a vector $\mathbf{v} = (\boldsymbol{v}; \boldsymbol{\omega}) \in \mathbb{R}^6$, $\boldsymbol{v}$ and $\boldsymbol{\omega} \in \mathbb{R}^3$, we define

$$\mathbf{v}^\wedge = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix}^\wedge := \begin{bmatrix} \boldsymbol{\omega}^\wedge & \boldsymbol{v} \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix} \in \mathfrak{se}(3). \tag{7}$$

- Similarly to what done for vectors in $\mathbb{R}^3$ few lines above, we define the *vee* operator as the inverse of the hat operator such that

$$\begin{bmatrix} \boldsymbol{\omega}^\wedge & \boldsymbol{v} \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix}^\vee := \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix} = \mathbf{v} \in \mathbb{R}^6. \tag{8}$$

- Given two normed vector spaces, E and F, and a function $f : E \mapsto F$, we define (where it exists) the differential of f at $\bar{\mathbf{x}} \in E$ as the linear function $D\mathbf{f}(\bar{\mathbf{x}}) : E \mapsto F$ such that

$$\lim_{\mathbf{x} \to \bar{\mathbf{x}}} \frac{||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\bar{\mathbf{x}}) - D\mathbf{f}(\bar{\mathbf{x}}) \cdot (\mathbf{x} - \bar{\mathbf{x}})||}{||\mathbf{x} - \bar{\mathbf{x}}||} = 0. \tag{9}$$

  When $E = F = \mathbb{R}$, the differential $D\mathbf{f}$ evaluated at $\mathbf{x} = \bar{\mathbf{x}}$ in the direction $\mathbf{z}$ is simply the classical derivative of a function multiplied by the perturbation $z \in \mathbb{R}$, i.e.,

$$Df(\bar{x}) \cdot z = \left. \frac{df}{dx} \right|_{x=\bar{x}} z.$$

  When $E = \mathbb{R}^n$ and $F = \mathbb{R}^m$, then $D\mathbf{f}(\bar{\mathbf{x}})$ has the following matrix representation

$$[\![D\mathbf{f}(\bar{\mathbf{x}})]\!] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{\mathbf{x}=\bar{\mathbf{x}}}$$

and $D\mathbf{f}(\bar{\mathbf{x}}) \cdot \mathbf{z}$ should be interpreted as the vector $[\![D\mathbf{f}(\bar{\mathbf{x}})]\!][\![\mathbf{z}]\!]$ obtained by multiplying the matrix $[\![D\mathbf{f}(\bar{\mathbf{x}})]\!]$ with the vector $[\![\mathbf{z}]\!]$.

The power of the notation $D\mathbf{f}(\mathbf{x}) \cdot \mathbf{z}$ lies on the fact that it can deal with even more general maps such as those where the input and output spaces are (normed) matrix vector spaces, such as $E = \mathbb{R}^{n\times m}$ and $F = \mathbb{R}^{l\times p}$ equipped with the Frobenius norm. This is particularly useful when dealing with maps such as a robot's forward kinematics, where one deals with maps of the form $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^{4\times 4}$ representing the pose (position and orientation) of each rigid link.

When $E = E_1 \times E_2 \times \cdots \times E_p$ and consequently having $\mathbf{f}$ as a map $E \ni (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p) \mapsto \mathbf{f}(\mathbf{x}_1, \ldots, \mathbf{x}_p) \in F$, we will use $D_1\mathbf{f}$, $D_2\mathbf{f}$, ..., $D_p\mathbf{f}$ to refer to the differential of $\mathbf{f}$ with respect to its first, second, ..., p-th argument. For further details on this derivative notation, we refer the reader to [1, Section 2.3] and [18, Chapter 2].

4 Points and coordinate frames

A *frame* is defined as the combination of a point (called *origin*) and an *orientation frame* in the 3D space [4, 19]. We typically employ a capital letter to indicate a frame. Given a frame A, we will indicate with $\mathbf{o}_A$ its *origin* and with $[A]$ its *orientation frame*. Formally, we write $A = (\mathbf{o}_A, [A])$.

Frames can be time moving with respect to a given reference frame and can be used, e.g., to describe the position and orientation in space of a rigid body as time evolves. They are also used to express a coordinate system for a wrench exchanged by two bodies or used to define a coordinate system to describe a robot task, such as a frame attached to the center of mass and oriented as the inertial frame.

Newton's mechanics requires the existence of an *inertial* frame. In this document, we usually indicate this inertial frame with the letter A (where A stands for *A*bsolute). As common practice, for robots operating near the Earth surface, we will assume the frame A to be fixed to the world's surface, disregarding non-inertial effects due to the Earth's motion.

4.1 Coordinate vector of a point

Given a point $\mathbf{p}$, its coordinates with respect to a frame $A = (\mathbf{o}_A, [A])$ are collected in the *coordinate vector* ${}^A\mathbf{p}$. The coordinate vector ${}^A\mathbf{p}$ represents the coordinates of the 3D geometric vector $\vec{r}_{\mathbf{o}_A,\mathbf{p}}$ connecting the origin of frame A with the point $\mathbf{p}$, pointing towards $\mathbf{p}$, expressed in the orientation frame $[A]$. Mathematically, we write this as that is

$$ {}^A\mathbf{p} := \begin{bmatrix} \vec{r}_{\mathbf{o}_A,\mathbf{p}} \cdot \vec{x}_A \\ \vec{r}_{\mathbf{o}_A,\mathbf{p}} \cdot \vec{y}_A \\ \vec{r}_{\mathbf{o}_A,\mathbf{p}} \cdot \vec{z}_A \end{bmatrix} \in \mathbb{R}^3, \tag{10} $$

where $\cdot$ denotes the *scalar product* between two vectors and $\vec{x}_A$, $\vec{y}_A$, $\vec{z}_A$, are the unit vectors defining the orientation frame $[A]$.

4.2 Change of orientation frame

Given two frames A and B, we will employ the notation

$$ {}^{A}\mathbf{R}_{B} \in \mathrm{SO}(3) \tag{11} $$

to denote the coordinate transformation from frame B to frame A. The coordinate transformation ${}^{A}\mathbf{R}_{B}$ only depends on the relative orientation between the orientation frames $[A]$ and $[B]$, irrespectively of the position of the origins $\mathbf{o}_A$ and $\mathbf{o}_B$.

4.3 Homogeneous transformation

To describe the position and orientation of a frame B with respect to another frame A, we employ the 4×4 homogeneous matrix

$$ {}^{A}\mathbf{H}_{B} := \begin{bmatrix} {}^{A}\mathbf{R}_{B} & {}^{A}\mathbf{o}_{B} \\ 0_{1\times3} & 1 \end{bmatrix}. \tag{12} $$

Given a point $\mathbf{p}$, the homogeneous transformation matrix ${}^{A}\mathbf{H}_{B}$ can be also used to map the coordinate vector ${}^{A}\mathbf{p}$ to ${}^{B}\mathbf{p}$ as follows. Let ${}^{A}\bar{\mathbf{p}}$ and ${}^{B}\bar{\mathbf{p}}$ denote the *homogenous representation* of ${}^{A}\mathbf{p}$ and ${}^{B}\mathbf{p}$, respectively. That is, let ${}^{A}\bar{\mathbf{p}} := ({}^{A}\mathbf{p}; 1) \in \mathbb{R}^4$ and likewise for ${}^{B}\bar{\mathbf{p}}$ (the symbol ; indicates row concatenation). Then

$$ {}^{A}\bar{\mathbf{p}} = {}^{A}\mathbf{H}_{B}\,{}^{B}\bar{\mathbf{p}}, \tag{13} $$

which is the matrix form of ${}^{A}\mathbf{p} = {}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{p} + {}^{A}\mathbf{o}_{B}$. We refer to [13, Chapter 2] for further details on homogeneous representation of rigid transformations.

5 Velocity vectors (twists)

In the following, given a point $\mathbf{p}$ and a frame A, we define

$$ {}^{A}\dot{\mathbf{p}} := \frac{d}{dt}\left({}^{A}\mathbf{p}\right). \tag{14} $$

In particular, when $\mathbf{p}$ is the origin of a frame, e.g., $\mathbf{p} = \mathbf{o}_B$, we have

$$ {}^{A}\dot{\mathbf{o}}_{B} = \frac{d}{dt}\left({}^{A}\mathbf{o}_{B}\right). $$

It is important to note that, by itself, expressions like $\dot{\mathbf{o}}_B$ or $\dot{\mathbf{p}}$ have *no* meaning. Similarly to (14), we also define

$$ {}^{A}\dot{\mathbf{R}}_{B} := \frac{d}{dt}\left({}^{A}\mathbf{R}_{B}\right) \tag{15} $$

and

$$ {}^{A}\dot{\mathbf{H}}_{B} := \frac{d}{dt}\left({}^{A}\mathbf{H}_{B}\right) = \begin{bmatrix} {}^{A}\dot{\mathbf{R}}_{B} & {}^{A}\dot{\mathbf{o}}_{B} \\ \mathbf{0}_{1\times3} & 0 \end{bmatrix}. \tag{16} $$

The relative velocity between a frame B with respect to a frame A can be represented by the time derivative of the homogenous transformation matrix ${}^A\mathbf{H}_B \in \mathrm{SE}(3)$. A more compact representation of ${}^A\dot{\mathbf{H}}_B$ can be obtained by multiplying it by the inverse of ${}^A\mathbf{H}_B$ on the left or on the right. In both cases, the result is an element of $\mathfrak{se}(3)$ that will be called a *twist.* Multiplying on the left, one obtains

$$
\begin{aligned}
{}^A\mathbf{H}_B^{-1}\,{}^A\dot{\mathbf{H}}_B &= \begin{bmatrix} {}^A\mathbf{R}_B^T & -{}^A\mathbf{R}_B^T\,{}^A\mathbf{o}_B \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \begin{bmatrix} {}^A\dot{\mathbf{R}}_B & {}^A\dot{\mathbf{o}}_B \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix} \\
&= \begin{bmatrix} {}^A\mathbf{R}_B^T\,{}^A\dot{\mathbf{R}}_B & {}^A\mathbf{R}_B^T\,{}^A\dot{\mathbf{o}}_B \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix}. \qquad (17)
\end{aligned}
$$

Note that ${}^A\mathbf{R}_B^T\,{}^A\dot{\mathbf{R}}_B$ appearing on the right hand side of (17) is skew symmetric. Define ${}^B\boldsymbol{v}_{A,B}$ and ${}^B\boldsymbol{\omega}_{A,B} \in \mathbb{R}^3$ so that

$$
\begin{aligned}
{}^B\boldsymbol{v}_{A,B} &:= {}^A\mathbf{R}_B^T\;{}^A\dot{\mathbf{o}}_B, \qquad (18)\\
{}^B\boldsymbol{\omega}_{A,B}^\wedge &:= {}^A\mathbf{R}_B^T\;{}^A\dot{\mathbf{R}}_B. \qquad (19)
\end{aligned}
$$

The *left trivialized* velocity of frame B with respect to frame A is

$$
{}^B\mathbf{v}_{A,B} := \begin{bmatrix} {}^B\boldsymbol{v}_{A,B} \\ {}^B\boldsymbol{\omega}_{A,B} \end{bmatrix} \in \mathbb{R}^6. \qquad (20)
$$

By construction,

$$
{}^B\mathbf{v}_{A,B}^\wedge = {}^A\mathbf{H}_B^{-1}\;{}^A\dot{\mathbf{H}}_B. \qquad (21)
$$

Note the slight abuse of notation in using the hat operator $\wedge$ in (19) and (21) that maps a vector into its corresponding matrix representation (respectively, from $\mathbb{R}^3$ to $\mathbb{R}^{3\times 3}$ using (5) in (19) and from $\mathbb{R}^6$ to $\mathbb{R}^{4\times 4}$ using (7) in (21)).
Similarly to what is done in (17), right multiplying ${}^A\dot{\mathbf{H}}_B$ by the inverse of ${}^A\mathbf{H}_B$ leads to

$$
\begin{aligned}
{}^A\dot{\mathbf{H}}_B\,{}^A\mathbf{H}_B^{-1} &= \begin{bmatrix} {}^A\dot{\mathbf{R}}_B & {}^A\dot{\mathbf{o}}_B \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix} \begin{bmatrix} {}^A\mathbf{R}_B^T & -{}^A\mathbf{R}_B^T\,{}^A\mathbf{o}_B \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \\
&= \begin{bmatrix} {}^A\dot{\mathbf{R}}_B\,{}^A\mathbf{R}_B^T & {}^A\dot{\mathbf{o}}_B - {}^A\dot{\mathbf{R}}_B\,{}^A\mathbf{R}_B^T\,{}^A\mathbf{o}_B \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix}. \qquad (22)
\end{aligned}
$$

Define ${}^A\boldsymbol{v}_{A,B}$ and ${}^A\boldsymbol{\omega}_{A,B} \in \mathbb{R}^3$ as

$$
\begin{aligned}
{}^A\boldsymbol{v}_{A,B} &:= {}^A\dot{\mathbf{o}}_B - {}^A\dot{\mathbf{R}}_B\,{}^A\mathbf{R}_B^T\,{}^A\mathbf{o}_B \qquad (23)\\
{}^A\boldsymbol{\omega}_{A,B}^\wedge &:= {}^A\dot{\mathbf{R}}_B\,{}^A\mathbf{R}_B^T. \qquad (24)
\end{aligned}
$$

The *right trivialized* velocity of B with respect to A is then defined as

$$
{}^A\mathbf{v}_{A,B} := \begin{bmatrix} {}^A\boldsymbol{v}_{A,B} \\ {}^A\boldsymbol{\omega}_{A,B} \end{bmatrix} \in \mathbb{R}^6. \qquad (25)
$$

By construction,

$$
{}^A\mathbf{v}_{A,B}^\wedge = {}^A\dot{\mathbf{H}}_B\,{}^A\mathbf{H}_B^{-1}. \qquad (26)
$$

5.1 Expressing a twist with respect to an arbitrary frame

Straightforward algebraic calculations allow to show that the right and left trivialized velocities ${}^A\mathbf{v}_{A,B}$ and ${}^B\mathbf{v}_{A,B}$ are related via a linear transformation. Inspired by the notation introduced in [7], we denote this linear transformation with ${}^A\mathbf{X}_B$ and define it as

$$
{}^A\mathbf{X}_B := \begin{bmatrix} {}^A\mathbf{R}_B & {}^A\mathbf{o}_B^\wedge {}^A\mathbf{R}_B \\ \mathbf{0}_{3\times 3} & {}^A\mathbf{R}_B \end{bmatrix} \in \mathbb{R}^{6\times 6}. \tag{27}
$$

As mentioned, the right and left velocities then satisfy

$$
{}^A\mathbf{v}_{A,B} = {}^A\mathbf{X}_B \, {}^B\mathbf{v}_{A,B}. \tag{28}
$$

The inverse transformation of ${}^A\mathbf{X}_B$ is given by ${}^B\mathbf{X}_A$ and it is straightforward to show that ${}^B\mathbf{X}_A = {}^A\mathbf{X}_B^{-1}$ (recall that ${}^A\mathbf{o}_B = -{}^A\mathbf{R}_B \, {}^B\mathbf{o}_A$).

Lie group theory related notes. To draw a connection with Lie group theory, indicating with $g = g_{A,B} := {}^A\mathbf{H}_B \in \mathrm{SE}(3)$ an arbitrary element of the Special Euclidean group (i.e., a rigid transformation), ${}^A\mathbf{X}_B$ is nothing else than Ad_g. Given $g \in \mathrm{SE}(3)$ and $\xi \in \mathfrak{se}(3)$, then

$$
\mathrm{Ad}_g \, \xi := g \, \xi \, g^{-1} \in \mathfrak{se}(3). \tag{29}
$$

The operator $\mathrm{Ad} : \mathrm{SE}(3) \times \mathfrak{se}(3) \to \mathfrak{se}(3)$ is the *adjoint action* of the group SE(3) to its algebra $\mathfrak{se}(3)$. Taking $g = {}^A\mathbf{H}_B$ and $\xi = {}^B\mathbf{v}_{A,B}^\wedge$, one sees immediately that $g \, \xi \, g^{-1}$ appearing in the right hand side of (29) equals

$$
{}^A\mathbf{H}_B \, {}^B\mathbf{v}_{A,B}^\wedge \, {}^A\mathbf{H}_B^{-1}, \tag{30}
$$

which, recalling the definition of ${}^B\mathbf{v}_{A,B}$ given in (21), is equivalent to

$$
{}^A\dot{\mathbf{H}}_B \, {}^A\mathbf{H}_B^{-1} = {}^A\mathbf{v}_{A,B}^\wedge, \tag{31}
$$

by definition of ${}^A\mathbf{v}_{A,B}$ given in (26). The adjoint action of the group SE(3) to its algebra $\mathfrak{se}(3)$, given by (29), is linear with respect to its second argument. It is therefore possible, when representing $\mathfrak{se}(3)$ as a vector in $\mathbb{R}^6$ (as done in (7)) to define the adjoint action (with a slight abuse of notation) as a map $\mathrm{Ad} : \mathrm{SE}(3) \times \mathbb{R}^6 \to \mathbb{R}^6$. In this way, for $g = {}^A\mathbf{H}_B$, one gets with straightforward computations that ${}^A\mathbf{v}_{A,B} = \mathrm{Ad}_g \, {}^B\mathbf{v}_{A,B}$, with $\mathrm{Ad}_g = {}^A\mathbf{X}_B$ given in (27).

Given the ubiquity of the velocity transformation $\mathrm{Ad}_{g_{A,B}}$ (and its associate wrench transformation $\mathrm{Ad}^*_{g_{A,B}}$ that we will introduce in Section 6) in multibody dynamics computations, it is convenient to indicate it simply with the compact notation ${}^A\mathbf{X}_B$ (respectively, ${}_B\mathbf{X}^A$). We stress here, however, the importance to not forget its connection with Lie group theory: this can help, in particular, to be able to understand the body of literature on geometric mechanics written with the standard Ad notation. ∎

We conclude this section by introducing the notation ${}^C\mathbf{v}_{A,B}$, indicating the velocity of frame B with respect to frame A expressed in frame C. The left and right trivialized velocities ${}^B\mathbf{v}_{A,B}$ and ${}^A\mathbf{v}_{A,B}$, respectively given by (21) and (26), are special cases of this concept. Formally, we define

$$ {}^C\mathbf{v}_{A,B} = \begin{bmatrix} {}^C\boldsymbol{v}_{A,B} \\ {}^C\boldsymbol{\omega}_{A,B} \end{bmatrix} \in \mathbb{R}^6 \tag{32} $$

as

$$ {}^C\mathbf{v}_{A,B} := {}^C\mathbf{X}_A {}^A\mathbf{v}_{A,B} = {}^C\mathbf{X}_B {}^B\mathbf{v}_{A,B}. \tag{33} $$

The latter equality follows from (28) and the identity ${}^C\mathbf{X}_A {}^A\mathbf{X}_B = {}^C\mathbf{X}_B$.

5.2 On the linear and angular components of a twist

As evident from (19) and (24), the angular component of the twists ${}^B\mathbf{v}_{A,B}$ and ${}^A\mathbf{v}_{A,B}$ depends only on the relative orientation between the frames A and B (given by the rotation matrix ${}^A\mathbf{R}_B$) and its time evolution. This angular component corresponds to the classic concept of *angular velocity* found in undergraduate physics textbooks and it can be expressed with respect to a different orientation frame simply by multiplying its coordinates by a suitable rotation matrix. One gets, in this way, that

$$ {}^C\boldsymbol{\omega}_{A,B} = {}^C\mathbf{R}_B {}^B\boldsymbol{\omega}_{A,B} = {}^C\mathbf{R}_A {}^A\boldsymbol{\omega}_{A,B}. \tag{34} $$

The linear component of the twists ${}^B\mathbf{v}_{A,B}$ and ${}^A\mathbf{v}_{A,B}$ requires, instead, a bit more of attention. While ${}^B\boldsymbol{v}_{A,B}$ in (18) is the time derivative of ${}^A\mathbf{o}_B$ (the coordinate vector of the origin of B with respect to the frame A) expressed in the frame B, ${}^A\boldsymbol{v}_{A,B}$ is *not* the time derivative of ${}^A\mathbf{o}_B$, but instead the (initially) somehow counterintuitive expression given in (23). At each instant of time, the linear velocity ${}^A\boldsymbol{v}_{A,B}$ is the linear velocity of the point, thought as fixed with respect to frame B, that finds itself at the origin of frame A at the given instant of time. The right trivialized velocity ${}^A\mathbf{v}_{A,B}$ is a key ingredient in understanding the efficient numerical algorithms for multibody dynamics described, e.g., in [7, 9, 14]. It also finds application in geometric mechanics when defining concepts such as a mechanical symmetry or a momentum map [11, 2, 12].

There are situations in which, however, one would like to describe the linear and angular velocity of a frame with the somehow natural velocities ${}^A\dot{\mathbf{o}}_B$ and ${}^A\boldsymbol{\omega}_{A,B}$, respectively. With the notation introduced in this documents, this is possible by introducing a special frame obtained combining the frames A and B. Namely, one needs to express the velocity of frame B with respect frame A in the new frame $B[A] := (o_B, [A])$, that is, in the frame whose origin coincides with the origin of B and whose orientation coincides with the orientation of A. In this way, one gets

$$ {}^{B[A]}\mathbf{v}_{A,B} = {}^{B[A]}\mathbf{X}_B {}^B\mathbf{v}_{A,B} = \begin{bmatrix} {}^A\mathbf{R}_B & 0 \\ 0 & {}^A\mathbf{R}_B \end{bmatrix} \begin{bmatrix} {}^B\mathbf{R}_A {}^A\dot{\mathbf{o}}_B \\ {}^B\boldsymbol{\omega}_{A,B} \end{bmatrix} = \begin{bmatrix} {}^A\dot{\mathbf{o}}_B \\ {}^A\boldsymbol{\omega}_{A,B} \end{bmatrix}. \tag{35} $$

In [3], the velocity (35) is referred to as the *hybrid velocity* of frame B with respect to frame A. To avoid confusion with hybrid systems theory, however, in this document we will call (35) the *mixed velocity* of frame B with respect to frame A (we call it *mixed* as it has both the flavor of a left trivialized velocity due to the linear velocity part and of a right trivialized velocity due to the angular velocity part).

5.3 The cross product on $\mathbb{R}^6$ ($\times$)

The defining equation for the velocity ${}^B\mathbf{v}_{A,B}$ given by (21) can be rewritten as

$$
{}^A\dot{\mathbf{H}}_B = {}^A\mathbf{H}_B \, {}^B\mathbf{v}^\wedge_{A,B}. \tag{36}
$$

By differentiating with respect to time the velocity transformation ${}^A\mathbf{X}_B$ given in (27), a formula similar in structure to (36) can be obtained that prescribes the time evolution of ${}^A\mathbf{X}_B$ as a function of ${}^B\mathbf{v}_{A,B}$. Namely, one gets

$$
{}^A\dot{\mathbf{X}}_B = {}^A\mathbf{X}_B \, {}^B\mathbf{v}_{A,B}\times \tag{37}
$$

where the term ${}^B\mathbf{v}_{A,B}\times$ is defined as

$$
{}^B\mathbf{v}_{A,B}\times := \begin{bmatrix} {}^B\boldsymbol{\omega}^\wedge_{A,B} & {}^B\boldsymbol{v}^\wedge_{A,B} \\ 0_{3\times 3} & {}^B\boldsymbol{\omega}^\wedge_{A,B} \end{bmatrix}. \tag{38}
$$

We will refer to (38) as the matrix representation of the *cross product on* $\mathbb{R}^6$.

Basic properties of the cross product. The cross product between vectors of $\mathbb{R}^6$ that derives from (38) satisfies the classical anticommutative property

$$
{}^C\mathbf{v}_{A,B} \times {}^C\mathbf{v}_{D,E} = -\,{}^C\mathbf{v}_{D,E} \times {}^C\mathbf{v}_{A,B}. \tag{39}
$$

A direct consequence of the anticommutativity is that, for any ${}^C\mathbf{v}_{A,B}$,

$$
{}^C\mathbf{v}_{A,B} \times {}^C\mathbf{v}_{A,B} = 0_{6\times 1}. \tag{40}
$$

Velocity transformation and the cross product. The cross product of velocity vectors defined via (38) satisfies the distributive property

$$
{}^A\mathbf{X}_B \, {}^B\mathbf{v}_{A,B}\times = ({}^A\mathbf{X}_B \, {}^B\mathbf{v}_{A,B}) \times {}^A\mathbf{X}_B = {}^A\mathbf{v}_{A,B} \times {}^A\mathbf{X}_B. \tag{41}
$$

Lie group theory related notes. For someone knowledgeable with the theory of Lie groups, a deeper look at the cross product defined via (38) reveals that this operation turns $\mathbb{R}^6$ into a Lie algebra (a vector space with a anticommutative bilinear operation satisfying the Jacobi identity [11, Chapter 9]).

Indeed, (38) is nothing else then the matrix representation of the adjoint action of $\mathbb{R}^6$ on itself, indicated with ad, once we interpret $\mathbb{R}^6$ as the Lie algebra *induced* by the Lie algebra homeomorphism between $\mathbb{R}^6$ and $\mathfrak{se}(3)$ provided by the hat ($^\wedge$) operator defined in (7). Defining $g = {}^A\mathbf{H}_B \in \mathrm{SE}(3)$, then (37) can be rewritten in the usual form (cf. [11, Chapter 9, equation (9.3.4)]) as

$$
\frac{d}{dt}\,\mathrm{Ad}_g = \mathrm{Ad}_g \ \mathrm{ad}_{g^{-1}\dot{g}}, \tag{42}
$$

where $\mathrm{Ad}_g = {}^A\mathbf{X}_B$, $\mathrm{ad}_{g^{-1}\dot{g}} = {}^A\mathbf{v}_{A,B}\times$, and $g^{-1}\dot{g} = {}^B\mathbf{v}^\wedge_{A,B}$. This standard Lie group notation, employing Ad and ad is found in well-known robotic literature such as, e.g., [8] and [14].

Finally, the distributive property (41) is equivalent to the identity (cf., e.g., [11, Chapter 9])

$$
\mathrm{Ad}_g \, \mathrm{ad}_{g^{-1}\dot{g}} = \mathrm{ad}_{\mathrm{Ad}_g\, g^{-1}\dot{g}} \, \mathrm{Ad}_g = \mathrm{ad}_{\dot{g}g^{-1}} \, \mathrm{Ad}_g, \tag{43}
$$

once we pose, as in (42), $g = {}^A\mathbf{H}_B \in \mathrm{SE}(3)$ and $\dot{g}g^{-1} = {}^A\mathbf{v}^\wedge_{A,B}$. ■

5.4 Frame acceleration and acceleration vectors

Several definitions of frame accelerations are present in the robotic literature, such conventional and spatial accelerations [6]. In [7], "coordinate free" (or "absolute") frame accelerations are introduced by only considering twists with respect to an (implicitly defined) inertial frame. In our experience, this particular definition of acceleration is convenient in obtaining computational efficient algorithms for multibody dynamics, but it is not natural for robot task specification and closed-loop control, where it is common to use the classical concept of linear acceleration as the derivative of the (inertial) coordinates of a point in space.

In this section, we start defining the *apparent acceleration* of a frame B with respect to a frame A seen and expressed in a frame C simply as the time-derivative of the corresponding velocity ${}^C\mathbf{v}_{A,B}$, that is

$$ {}^C\dot{\mathbf{v}}_{A,B} := \frac{d}{dt}\left({}^C\mathbf{v}_{A,B}\right). \tag{44} $$

Writing ${}^C\mathbf{v}_{A,B}$ as the product ${}^C\mathbf{X}_B{}^B\mathbf{v}_{A,B}$ and using the time derivative formula for a change of coordinates given by (37), one gets

$$ {}^C\dot{\mathbf{v}}_{A,B} = {}^C\mathbf{X}_B\left({}^B\mathbf{v}_{C,B} \times {}^B\mathbf{v}_{A,B} + {}^B\dot{\mathbf{v}}_{A,B}\right). \tag{45} $$

The equation above shows that, in general, ${}^C\dot{\mathbf{v}}_{A,B} \neq {}^C\mathbf{X}_B{}^B\dot{\mathbf{v}}_{A,B}$. However, for the special case $C = A$, one obtains the fundamental and at first-sight surprising relationship between left and right trivialized accelerations given by

$$ {}^A\dot{\mathbf{v}}_{A,B} = {}^A\mathbf{X}_B{}^B\dot{\mathbf{v}}_{A,B}. \tag{46} $$

Due to this last equality, that does not involve any cross product, it is possible to define the (intrinsic) *acceleration* of a frame B with respect to a frame A expressed in a frame C as

$$ {}^C\mathbf{a}_{A,B} := {}^C\mathbf{X}_A{}^A\dot{\mathbf{v}}_{A,B} = {}^C\mathbf{X}_B{}^B\dot{\mathbf{v}}_{A,B}. \tag{47} $$

Component-wise, the intrinsic and apparent accelerations (47) and (44) are written as

$$ {}^C\mathbf{a}_{A,B} = \begin{bmatrix} {}^C\boldsymbol{a}_{A,B} \\ {}^C\boldsymbol{\alpha}_{A,B} \end{bmatrix} \in \mathbb{R}^6 \quad \text{and} \quad {}^C\dot{\mathbf{v}}_{A,B} = \begin{bmatrix} {}^C\dot{\boldsymbol{v}}_{A,B} \\ {}^C\dot{\boldsymbol{\omega}}_{A,B} \end{bmatrix} \in \mathbb{R}^6. \tag{48} $$

Combining the above definitions and equalities and using the equality ${}^C\mathbf{v}_{C,B} \times {}^C\mathbf{v}_{A,B} = ({}^C\mathbf{v}_{C,B} + {}^C\mathbf{v}_{B,A}) \times {}^C\mathbf{v}_{A,B} = {}^C\mathbf{v}_{C,A} \times {}^C\mathbf{v}_{A,B}$, leads to the following relationship between the intrinsic and apparent accelerations

$$ {}^C\mathbf{a}_{A,B} = {}^C\dot{\mathbf{v}}_{A,B} + {}^C\mathbf{v}_{A,C} \times {}^C\mathbf{v}_{A,B}, \tag{49} $$

which coincides with [6, equation(4)]. Component-wise, (49) equals

$$ {}^C\boldsymbol{a}_{A,B} = {}^C\dot{\boldsymbol{v}}_{A,B} + {}^C\boldsymbol{\omega}_{A,C} \times {}^C\boldsymbol{v}_{A,B} + {}^C\boldsymbol{v}_{A,C} \times {}^C\boldsymbol{\omega}_{A,B}, \tag{50} $$

$$ {}^C\boldsymbol{\alpha}_{A,B} = {}^C\dot{\boldsymbol{\omega}}_{A,B} + {}^C\boldsymbol{\omega}_{A,C} \times {}^C\boldsymbol{\omega}_{A,B}. \tag{51} $$

Note. For task specification, the following (mixed) apparent acceleration

$$ {}^{B[A]}\dot{\mathbf{v}}_{A,B} = \begin{bmatrix} {}^{B[A]}\dot{\boldsymbol{v}}_{A,B} \\ {}^{B[A]}\dot{\boldsymbol{\omega}}_{A,B} \end{bmatrix} = \begin{bmatrix} {}^{A}\ddot{\mathbf{o}}_{B} \\ {}^{A}\dot{\boldsymbol{\omega}}_{A,B} \end{bmatrix}. \tag{52} $$

is of common use in robotics, because the linear acceleration corresponds to the Cartesian acceleration of the origin of B with respect to frame A. From (49), this apparent acceleration can be expressed in terms of the intrinsic acceleration as

$$ {}^{B[A]}\dot{\mathbf{v}}_{A,B} = {}^{B[A]}\mathbf{a}_{A,B} - {}^{B[A]}\mathbf{v}_{A,B[A]} \times {}^{B[A]}\mathbf{v}_{A,B}. \tag{53} $$

Component-wise, (53) reads

$$ {}^{B[A]}\dot{\boldsymbol{v}}_{A,B} = {}^{B[A]}\boldsymbol{a}_{A,B} - {}^{B[A]}\boldsymbol{v}_{A,B} \times {}^{B[A]}\boldsymbol{\omega}_{A,B}, \tag{54} $$

$$ {}^{B[A]}\dot{\boldsymbol{\omega}}_{A,B} = {}^{B[A]}\boldsymbol{\alpha}_{A,B}, \tag{55} $$

where we used ${}^{B[A]}\boldsymbol{\omega}_{A,B[A]} = 0$ and ${}^{B[A]}\boldsymbol{v}_{A,B[A]} = {}^{B[A]}\boldsymbol{v}_{A,B}$.

Lie group theory related note. The formula (45), relating the acceleration with the apparent acceleration, is written in the standard notations of Lie groups as

$$ \frac{d}{dt}\left(\mathrm{Ad}_g\, \xi\right) = \mathrm{Ad}_g\left(\mathrm{ad}_{g^{-1}\dot{g}}\, \xi + \dot{\xi}\right), \tag{56} $$

and it is a well-known result (cf. [11, Proposition 9.3.8]).

6 Force covectors (wrenches)

The coordinates of a wrench $\mathbf{f}$ with respect to a given frame B are indicated as

$$ {}_{B}\mathbf{f} := \begin{bmatrix} {}_{B}\boldsymbol{f} \\ {}_{B}\boldsymbol{\tau} \end{bmatrix} \in \mathbb{R}^6. \tag{57} $$

Note how, in contrast to twists, just the coordinate frame with respect to which the wrench $\mathbf{f}$ is expressed is indicated explicitly.

As for a twist, we can define a linear map to change the coordinates of a wrench from a frame B to another frame A. This coordinate transformation is denoted ${}_{A}\mathbf{X}^{B}$, so that we have

$$ {}_{A}\mathbf{f} = {}_{A}\mathbf{X}^{B}\, {}_{B}\mathbf{f}. \tag{58} $$

The mapping ${}_{A}\mathbf{X}^{B}$ is actually strictly connected to the transformation ${}^{B}\mathbf{X}_{A}$ given in (27), and can be defined as

$$ {}_{A}\mathbf{X}^{B} := {}^{B}\mathbf{X}_{A}^{T} = \begin{bmatrix} {}^{A}\mathbf{R}_{B} & \mathbf{0}_{3\times 3} \\ -{}^{A}\mathbf{R}_{B}\, {}^{B}\mathbf{o}_{A}^{\wedge} & {}^{A}\mathbf{R}_{B} \end{bmatrix} = \begin{bmatrix} {}^{A}\mathbf{R}_{B} & \mathbf{0}_{3\times 3} \\ {}^{A}\mathbf{o}_{B}^{\wedge}\, {}^{A}\mathbf{R}_{B} & {}^{A}\mathbf{R}_{B} \end{bmatrix} \tag{59} $$

where, for the last equality, we made use of the identity ${}^{A}\mathbf{o}_{B} = -{}^{A}\mathbf{R}_{B}\, {}^{B}\mathbf{o}_{A}$. The definition (59) leads, in particular, to the expected coordinate independency of power

$$ \left\langle {}_{B}\mathbf{f}, {}^{B}\mathbf{v}_{A,B} \right\rangle = \left\langle {}_{A}\mathbf{f}, {}^{A}\mathbf{v}_{A,B} \right\rangle. \tag{60} $$

In the above expression, $\mathbf{f}$ can be interpreted as a wrench applied to a rigid body and expressed with respect to a frame B which is fixed with respect to the body and A as the inertial frame.

6.1 The dual cross-product between a twist and a wrench

The time derivative of the wrench coordinate transformation ${}_A\mathbf{X}^B$ has a structure that resembles the velocity coordinate transformation ${}^A\mathbf{X}_B$, given in (37). Straightforward computations lead to the expression

$$ {}_A\dot{\mathbf{X}}^B = {}_A\mathbf{X}^{B\,B}\mathbf{v}_{A,B}\bar{\times}^* \tag{61}$$

where $\bar{\times}^*$ represents an operation between a twist and a wrench, that we call the dual cross-product and indicate with $\bar{\times}^*$, whose matrix representation is

$$ {}^B\mathbf{v}_{A,B}\bar{\times}^* := \begin{bmatrix} {}^B\boldsymbol{\omega}^\wedge_{A,B} & \mathbf{0}_{3\times 3} \\ {}^B\boldsymbol{v}^\wedge_{A,B} & {}^B\boldsymbol{\omega}^\wedge_{A,B} \end{bmatrix}. \tag{62}$$

Note how (62) can be obtained from (38) by simply transposing it and changing its sign. This fact is actually encoded in the symbol $\bar{\times}^*$ itself, in the sense that the overline and the the star represent, respectively, the sign change and transposition (more formally, its adjoint linear map, that is typically indicated with a star).

The dual cross product (62) takes one twist and one wrench and returns one wrench (as opposed to one twist from two twists as in the case of the cross product (38)): this is also the reason why the sub- and superscripts in (61) are also correct: when ${}_A\dot{\mathbf{X}}^B$ is applied to a wrench ${}_B\mathbf{f}$ expressed in B, the dual cross product between ${}^B\mathbf{v}_{A,B}$ and ${}_B\mathbf{f}$ will return a wrench expressed in B that can then be converted into a wrench expressed in A via ${}_A\mathbf{X}^B$. The dual cross-product also satisfies the geometrically intuitive equality

$$ {}_A\mathbf{X}^{B\,B}\mathbf{v}_{A,B}\bar{\times}^* = {}^A\mathbf{v}_{A,B}\bar{\times}^*\,{}_A\mathbf{X}^B. \tag{63}$$

The result is straightforward to prove.

Lie group theory related note. In the language of differential geometry, the dual space of $\mathfrak{se}(3)$ (namely, the space of linear applications from $\mathfrak{se}(3)$ to $\mathbb{R}$) is indicated with $\mathfrak{se}(3)^*$. Wrenches belong to this space as opposed to twists that live instead in $\mathfrak{se}(3)$. In terms of standard Lie group notation, the wrench coordinate transformation ${}_A\mathbf{X}^B$ is written

$$ {}_A\mathbf{X}^B = \mathrm{Ad}^*_{g^{-1}} \tag{64}$$

where $g := {}^A\mathbf{H}_B \in \mathrm{SE}(3)$. Recall that $\mathrm{Ad}_g = {}^A\mathbf{X}_B$ and $\mathrm{Ad}_{g^{-1}} = {}^B\mathbf{X}_A$. Let $\xi^\wedge := {}^B\mathbf{v}^\wedge_{A,B} \in \mathfrak{se}(3)$, then one gets

$$ {}^B\mathbf{v}_{A,B}\bar{\times}^* = -\,\mathrm{ad}^*_\xi\,. \tag{65}$$

The formula above makes it clear, once again, that the notation $\bar{\times}^*$ has been explicitly chosen to remind the fact that (62) is obtained from the cross product $\times$ given in (38) and indicated with ad in standard Lie group notation, by computing its adjoint ($*$) and changing its sign ($-$). Finally, the time derivate of (64) is simply

$$ \frac{d}{dt}\mathrm{Ad}^*_{g^{-1}} = -\,\mathrm{Ad}^*_{g^{-1}}\,\mathrm{ad}^*_\xi \tag{66}$$

for $\dot{g} = g\xi$, with $g = {}^A\mathbf{H}_B$ and $\xi = {}^B\mathbf{v}^\wedge_{A,B}$, which is equivalent to (61). ∎

7 Generalized inertia tensor

The 6×6 generalized inertia of a rigid body L (where L stands for *link*), when expressed with respect to a frame C whose origin coincides with the body center of mass is denoted ${}_C\mathbb{M}_C^L$ and explicitly given by

$$ {}_C\mathbb{M}_C^L = \begin{bmatrix} m_L\,\mathbf{I}_3 & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & {}_C\mathbb{I}_C^L \end{bmatrix}, \tag{67} $$

with m_L body mass and ${}_C\mathbb{I}_C^L$ the 3×3 inertia tensor of L expressed in C.

The generalized inertia expressed with a generic frame B, whose origin is not necessarily coinciding with the center of mass, is denoted and computed as

$$ \begin{aligned} {}_B\mathbb{M}_B^L &= {}_B\mathbf{X}^C \, {}_C\mathbb{M}_C^L \, {}^C\mathbf{X}_B \\ &= \begin{bmatrix} m_L\mathbf{I}_3 & -m_L\,{}^B\mathbf{o}_C^\wedge \\ m_L\,{}^B\mathbf{o}_C^\wedge & {}_B\mathbb{I}_B^L \end{bmatrix}, \end{aligned} \tag{68} $$

where

$$ {}_B\mathbb{I}_B^L = {}^C\mathbf{R}_B^T \, {}_C\mathbb{I}_C^L \, {}^C\mathbf{R}_B - m_L\,{}^B\mathbf{o}_C^\wedge \, {}^B\mathbf{o}_C^\wedge. \tag{69} $$

We recall that ${}^B\mathbf{o}_C = -{}^B\mathbf{R}_C\,{}^C\mathbf{o}_B$. The term $-m_L\,{}^B\mathbf{o}_C^\wedge\,{}^B\mathbf{o}_C^\wedge$ appearing in ${}_B\mathbb{I}_B^L$ is the classic correction term of the Huygens-Steiner (also known as parallel axis) theorem. The matrix product $\mathbf{o}^\wedge\mathbf{o}^\wedge$ is sometimes written as $\mathbf{o}\cdot\mathbf{o}\;\mathbf{I}_3 + \mathbf{o}\otimes\mathbf{o}$, with $\cdot$ and $\otimes$ denoting the scalar and outer products, respectively.

8 The geometric Jacobians

The goal of this section is to define a precise and unambiguous notation for the geometric Jacobians for fixed-base and, in particular, for moving-base multibody systems (also known as free-floating multibody systems). Geometric Jacobians are essential tools in defining contact and constraint forces in multibody dynamics as well as express position and force tasks in robot control.

In this section, A will denote an *inertial frame* and B the *moving-base* frame, i.e., a frame rigidly attached to one of the bodies composing the multibody system, selected to represent the relative pose of the system with respect to the world frame A. The configuration of a moving-base multibody system is parametrized as $\mathbf{q} = (\mathbf{H}, \mathbf{s}) \in \mathrm{SE}(3)\times\mathbb{R}^{n_J}$, with $\mathbf{H} = {}^A\mathbf{H}_B \in \mathrm{SE}(3)$ representing the pose (position and orientation) of the moving-base frame B and $\mathbf{s} \in \mathbb{R}^{n_J}$ the internal joint displacements ($\mathbf{s}$ stands for *shape*). The configuration space (more correctly, the configuration manifold) has correspondingly dimension $n = 6+n_J$.

Let E denote a frame (rigidly) attached to an arbitrary body to be used, e.g., for the specification of a task to be executed by the robot or a possible point of contact with the environment. The frame E could represent, e.g., the pose of a specific frame rigidly attached to an *end effector* of a robot manipulator or a hand or foot on a humanoid robot. Let

$$ {}^A\mathbf{H}_E = {}^A\mathbf{H}_E(\mathbf{q}) = {}^A\mathbf{H}_E(\mathbf{H}, \mathbf{s}) \tag{70} $$

denote the homogeneous transformation expressing E with respect to A as a function of the configuration $\mathbf{q} = (\mathbf{H}, \mathbf{s})$.

Let $\delta\mathbf{H}$ denote an infinitesimal perturbation of the pose of the moving base ($\delta\mathbf{H} \in T_{\mathbf{H}}\mathrm{SE}(3)$, in the language of differential geometry) and $\delta\mathbf{s}$ an infinitesimal perturbation of the joint displacements. Then, the corresponding infinitesimal perturbation of frame E can be computed as

$$^{A}\delta\mathbf{H}_{E} = {}^{A}D_1\mathbf{H}_E(\mathbf{H}, \mathbf{s}) \cdot \delta\mathbf{H} + {}^{A}D_2\mathbf{H}_E(\mathbf{H}, \mathbf{s}) \cdot \delta\mathbf{s}, \tag{71}$$

where $\mathbf{H}$ is short for ${}^{A}\mathbf{H}_B$ and $\delta\mathbf{H}$ is short for ${}^{A}\delta\mathbf{H}_B$. Let ${}^{E}\mathbf{\Delta}_{A,E}$ and ${}^{B}\mathbf{\Delta}_{A,B} \in \mathbb{R}^6$ denote the trivialized infinitesimal perturbations

$$^{E}\mathbf{\Delta}^{\wedge}_{A,E} := {}^{A}\mathbf{H}_E^{-1}\,{}^{A}\delta\mathbf{H}_E, \tag{72}$$

$$\mathbf{\Delta}^{\wedge} = {}^{B}\mathbf{\Delta}^{\wedge}_{A,B} := {}^{A}\mathbf{H}_B^{-1}\,{}^{A}\delta\mathbf{H}_B = \mathbf{H}^{-1}\,\delta\mathbf{H}. \tag{73}$$

Combining (72) and (73) together with (71), note how ${}^{E}\mathbf{\Delta}_{A,E}$ depends linearly on ${}^{B}\mathbf{\Delta}_{A,B}$ and $\delta\mathbf{s}$. Such a linear map defines the *geometric Jacobian* for the (moving-base) multibody system and will be indicated with the symbol ${}^{E}\mathbf{J}_{A,E/B} \in \mathbb{R}^{6\times(6+n_J)}$.

The subscript $A, E/B$ appearing in ${}^{E}\mathbf{J}_{A,E/B}$ indicate that the Jacobian allows to compute the infinitesimal perturbation of frame E relative to frame A, based on the infinitesimal perturbation of the internal joint configuration and that of the moving base, this latest perturbation being expressed with respect to frame B. The superscript E appearing in ${}^{E}\mathbf{J}_{A,E/B}$ specifies that the infinitesimal perturbation is expressed with respect to frame E.

The Jacobian is therefore obtained by means of two left-trivializations (one in the output ${}^{E}\mathbf{\Delta}_{A,E}$ and one in the input ${}^{B}\mathbf{\Delta}_{A,B}$) and for this reason, we will sometimes refer to ${}^{E}\mathbf{J}_{A,E/B}$ as the *doubly left-trivialized* geometric Jacobian associated to the rigid transformation ${}^{A}\mathbf{H}_E = {}^{A}\mathbf{H}_E({}^{A}\mathbf{H}_B, \mathbf{s})$. In formulas,

$$^{E}\mathbf{\Delta}_{A,E} = {}^{E}\mathbf{J}_{A,E/B}(\mathbf{H}, \mathbf{s}) \begin{bmatrix} \mathbf{\Delta} \\ \delta\mathbf{s} \end{bmatrix}, \tag{74}$$

where we recall $\mathbf{H} = {}^{A}\mathbf{H}_B$ and $\mathbf{\Delta} = {}^{B}\mathbf{\Delta}_{A,B}$.

The infinitesimal perturbation of frames E and B can be expressed with respect to other arbitrary frames, let us say C and D. In this case, we defined the geometric Jacobian ${}^{D}\mathbf{J}_{A,E/C}$ via two changes of coordinates from the doubly left-trivialized Jacobian ${}^{E}\mathbf{J}_{A,E/B}$ as

$$^{D}\mathbf{J}_{A,E/C} = {}^{D}\mathbf{X}_E\,{}^{E}\mathbf{J}_{A,E/B}\,{}^{B}\mathbf{Y}_C, \tag{75}$$

where the combined twist-joint velocity transformation

$$^{B}\mathbf{Y}_C := \begin{bmatrix} {}^{B}\mathbf{X}_C & \mathbf{0}_{6\times n_J} \\ \mathbf{0}_{n_J\times 6} & \mathbf{I}_{n_J} \end{bmatrix}, \tag{76}$$

with $\mathbf{I}$ and $\mathbf{0}$ denoting identity and zero matrix of indicated dimensions. For the control of humanoid robots, it is common to express the moving-base and end-effector frame twists in the mixed frames $B[A]$ and $E[A]$, respectively. The associated geometric Jacobian ${}^{E[A]}\mathbf{J}_{A,E/B[A]}$ obtained from (75) when $D = E[A]$ and $C = B[A]$ will be called the *doubly mixed* geometric Jacobian associated to the rigid transformation ${}^{A}\mathbf{H}_E = {}^{A}\mathbf{H}_E({}^{A}\mathbf{H}_B, \mathbf{s})$.

Geometric Jacobians for fixed-based systems. For fixed-base systems such as robot manipulators, the configuration variable is simply $\mathbf{q} = \mathbf{s}$. In this context, as there is no moving base, the geometric Jacobians is simply written ${}^{C}\mathbf{J}_{A,B}$. One speaks then of a mixed Jacobian for ${}^{B[A]}\mathbf{J}_{A,B}$ and left-trivialized Jacobian for ${}^{B}\mathbf{J}_{A,B}$.

9 Moving-base multibody dynamics

In this section, we write the equations of motion of a moving-base multibody system in a compact form. As introduced in Section 8, the configuration of such a system will be denoted $\mathbf{q} = (\mathbf{H}, \mathbf{s}) := ({}^{A}\mathbf{H}_{B}, \mathbf{s})$, with A being the inertial frame, B the (selected) moving-base frame, and $\mathbf{s} \in \mathbb{R}^{n_J}$ the joint displacements.

It is common to employ the *mixed* velocity ${}^{B[A]}\mathbf{v}_{A,B}$ to express the motion of the moving base and therefore we here employ the *mixed* generalized velocity $\nu := {}^{B[A]}\nu = ({}^{B[A]}\mathbf{v}_{A,B}, \dot{\mathbf{s}}) =: (\mathbf{v}, \mathbf{r})$ to parameterize the velocity of the entire system. The kinematics of the moving-base system is therefore written as

$$\dot{\mathbf{H}} = \mathbf{H}(\mathbf{X}\mathbf{v})^{\wedge} \tag{77}$$

$$\dot{\mathbf{s}} = \mathbf{r} \tag{78}$$

with $\mathbf{X} = {}^{B}\mathbf{X}_{B[A]}$. More compactly, we will write both equations above with a single equation[1] as as $\dot{\mathbf{q}} = \mathbf{q}(\mathbf{Y}\boldsymbol{\nu})^{\wedge}$ where $\mathbf{Y} := {}^{B}\mathbf{Y}_{B[A]} = \mathrm{diag}({}^{B}\mathbf{X}_{B[A]}, \mathbf{I}_{n_J})$ is the combined twist-joint velocity transformation from mixed to body-fixed velocity. For more details, cf. (76) in Section 8. The total equations of motion for the moving-base multibody system are written as

$$\dot{\mathbf{q}} = \mathbf{q}(\mathbf{Y}\boldsymbol{\nu})^{\wedge}, \tag{79}$$

$$\mathbf{M}(\mathbf{q})\dot{\boldsymbol{\nu}} + \mathbf{C}(\mathbf{q}, \boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{G}(\mathbf{q}) = \mathbf{S}\,\boldsymbol{\tau} + \sum_{k \in \mathcal{I}_C} \mathbf{J}_k^T(\mathbf{q})\ \mathbf{f}_k \tag{80}$$

where $\mathbf{M}$ the mass matrix, $\mathbf{C}$ the Coriolis matrix, $\mathbf{G}$ the potential force vector, $\mathbf{S} := [\mathbf{0}_{6\times n_J}; \mathbf{I}_{n_J}]$ the *joint selection* matrix [2], $\boldsymbol{\tau}$ the joint torques, $\mathcal{I}_C$ the set of closed contacts, $\mathbf{f}_k := {}_{C_k[A]}\mathbf{f}_k$ the $k-th$ contact wrench, and

$$\mathbf{J}_k(\mathbf{q}) := {}^{C_k[A]}\mathbf{J}_{A,L_i/B[A]}(\mathbf{q}) \tag{81}$$

the geometric Jacobian (see Section 8 for details on the notation) associated to the frame L_i *rigidly attached* to the link i that is experiencing the k-th contact and C_k denotes the contact frame[3] where the contact interaction forces and torques are expressed.

[1] This notation is derived from Lie group theory, when considering the configuration manifold $Q = \mathrm{SE}(3) \times \mathbb{R}^{n_J}$ as the Lie group defined by the direct product of the groups SE(3) and $(\mathbb{R}^{n_J}, +)$.

[2] The joint selection matrix (see, e.g., [5]) simply emphasizes the fact that moving-based systems are typically underactuated, requiring establishing physical contact with the environment in order to fully control their posture.

[3] Note that C_k is allowed to move with respect to the link frame L_i, at it happens, e.g., for a rolling contact. This is exactly the reason why one needs to use ${}^{C_k[A]}\mathbf{J}_{A,L_i/B[A]}$ and not ${}^{C_k[A]}\mathbf{J}_{A,C_k/B[A]}$ as contact Jacobian.

On velocity parametrization. Note that $\mathbf{M}$, $\mathbf{C}$, $\mathbf{G}$, $\mathbf{J}$ appearing in (80) depend on the choice of the moving-base velocity: if we would have chosen, e.g., ${}^{B}\mathbf{v}_{A,B}$ instead of ${}^{B[A]}\mathbf{v}_{A,B}$ as done above, this would have led to matrices/vectors with different entries. It is possible to pass from one representation to another by suitable left- and right-multiplication by means of a suitable generalized velocity transformation $\mathbf{Y}$.

On contact Jacobians. To better understand (81), recall that the infinitesimal power injected into the multibody system by a contact wrench ${}_{C}\mathbf{f}$ is given by

$$\begin{aligned}&\left\langle {}_{L}\mathbf{X}^{C}{}_{C}\mathbf{f}, {}^{L}\mathbf{J}_{A,L/B[A]}(\mathbf{q})\,\boldsymbol{\nu}\right\rangle = \\ &\left\langle {}_{C}\mathbf{f}, {}^{C}\mathbf{X}_{L}{}^{L}\mathbf{J}_{A,L/B[A]}(\mathbf{q})\,\boldsymbol{\nu}\right\rangle = \left\langle {}_{C}\mathbf{f}, {}^{C}\mathbf{J}_{A,L/B[A]}(\mathbf{q})\,\boldsymbol{\nu}\right\rangle.\end{aligned} \tag{82}$$

It is essential to note that, in general, ${}^{C}J_{A,L/B[A]} \neq {}^{C}J_{A,C/B[A]}$ because ${}^{C}X_{L}$ can be *time varying* because the contact frame C might move with respect to the link (and consequently with respect to L) which is experiencing the contact wrench. At each instant of time, the twist ${}^{C}\mathbf{J}_{A,L/B[A]}(\mathbf{q})\,\boldsymbol{\nu}$ represents the (combined linear and angular) velocity of a frame rigidly attached to the link that, at that moment, has the same position and orientation of frame C.

Lie group theory related note. In the robotics literature, the equations of motions (80) are often referred to, with abuse of terminology, as forced Euler-Lagrange equations. While, indeed, the variational principle and the Lagrangian play a central role in obtaining the unforced equations

$$\mathbf{M}(\mathbf{q})\dot{\boldsymbol{\nu}} + \mathbf{C}(\mathbf{q}, \boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{G}(\mathbf{q}) = 0, \tag{83}$$

it is important to realize that the Lagrangian is a mapping defined on the tangent bundle of $Q = SE(3) \times \mathbb{R}^{n_J}$, that is $L : TQ \to \mathbb{R}$, $(\mathbf{q}, \dot{\mathbf{q}}) \mapsto L(\mathbf{q}, \dot{\mathbf{q}})$. The classical Euler-Lagrange equations, typically written in coordinates as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = \{1, 2, \dots, n\}, \tag{84}$$

with n the configuration space dimension, do not apply in our context because Q is not $\mathbb{R}^n$. Indeed, to obtain (83), one needs to resolve to geometric mechanics [11]. In this context, one first defines a *trivialized* Lagrangian as the difference between kinetic and potential energy, where the velocity is parameterized via the trivialized velocity. Typically, one defines the trivialized Lagrangian as $\mathbf{l}(\mathbf{q}, \boldsymbol{\xi}) := \mathbf{L}(\mathbf{q}, \mathbf{q}\boldsymbol{\xi}) = 1/2\,\langle \mathbf{I}(\mathbf{q})\boldsymbol{\xi}, \boldsymbol{\xi}\rangle - \mathbf{V}(\mathbf{q})$, where $\boldsymbol{\xi} = {}^{B}\boldsymbol{\nu}$ is the left-trivialized generalized velocity, $\mathbf{V}$ denotes the potential energy, and $\mathbf{I}(\mathbf{q})$ the inertia matrix satisfying $\mathbf{M}(\mathbf{q}) = \mathbf{Y}^{T}\mathbf{I}(\mathbf{q})\mathbf{Y}$, if we employ the mixed mass matrix $\mathbf{M}(\mathbf{q})$ and combined twist-joint velocity transformation $\mathbf{Y}$ as in (79) and (80), respectively. One then applies a modified version of the Euler-Lagrange equations (typically referred to as the Hamel equations, cf., e.g., [11, Section 13.6]) to the trivialized Lagrangian $\mathbf{l}$, obtaining a differential equation in $(\mathbf{q}, \xi)$. The interested reader is referred to [15, Section II] and references therein for further reading.

10 Acknowledgements

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A Comparison with other existing notations

In this section, our notation is compared with equivalent notations commonly appearing in the literature. Namely, Featherstone's notation appearing in [7, 16],[16], Siciliano's notation appearing in [17], Spong's notation appearing in [19], and the Lie group notation appearing in [13, 14, 10]

A.1 Featherstone's notation

In [7] and in the second chapter of [16], based on it, the concept of *spatial* velocity and acceleration is used to explain rigid body algorithms. It is worth noting that in [7] the term *spatial* has a totally different meaning with respect to how it is used in [13]. In particular, in [7], *spatial* is used to indicate a 6D vector, being it a twist, a link acceleration, a wrench, or momentum, while in [13], the term spatial is used to indicate a 6D vector expressed with respect to an *inertial* reference frame.

In [7], 6D vectors are composed using the angular-linear serialization. In this report, we use instead the linear-angular serialization. In the remining of this section, we explicitly show the difference between this report's and Featherstone's notation (disregarding the difference in angular-linear serialization).

Homogeneous transformations. In Featherstone's notation, the homogeneous transformation is seldom used, as most of the theory is introduced using directly 6D vectors. For this reason there is not direct equivalent of the notation.

Velocities. In Featherstone's notation, the 6D rigid body velocity of a body-frame B expressed in a frame C is indicated as

$$ {}^{C}\boldsymbol{v}_{B}. $$

All velocities in Featherstone's are always relative to an *implicitly defined* inertial frame A. In this report's notation, we prefer to explictly indicate this dependency, and therefore the equivalent expression for this velocity is

$$ {}^{C}\mathbf{v}_{A,B}. $$

Accelerations. Featherstone [7, 16] uses the dot notation $(\dot{\cdot})$ to indicate the differentiation with respect to an implicitly defined inertial frame, and the ring notation $(\mathring{\cdot})$ to indicate the differentiation with respect to the frame in which the quantity is expressed. As we do not implicitly assume the existence of an absolute inertial frame, we just use the $(\cdot)$ to indicate the differentiation in coordinates. Using this definition, it is easy to see that the body (*spatial*) acceleration defined in Featherstone as

$$ {}^{C}\dot{\boldsymbol{v}}_{B} = {}^{C}\boldsymbol{a}_{B} $$

is equivalent, in this report's notation, to

$$ {}^{C}\mathbf{a}_{A,B} = {}^{C}\mathbf{X}_{A}\,{}^{A}\dot{\mathbf{v}}_{A,B}, \tag{85}$$

where A is the inertial frame implicitly used in Featherstone's, and ${}^{C}\mathbf{a}_{A,B}$ is the (intrinsic) acceleration defined in (47). Note that from (46), using this report's notation, we get

$$ {}^{B}\dot{\mathbf{v}}_{A,B} = {}^{B}\mathbf{X}_{A}\,{}^{A}\dot{\mathbf{v}}_{A,B}, \tag{86}$$

that in Featherstone's notation is written

$$ {}^{B}\mathring{\boldsymbol{v}}_{B} = {}^{B}\dot{\boldsymbol{v}}_{B}. \tag{87}$$

Adjoint transformations. The adjoint transform that maps a motion vector expressed in a frame B in one expressed in a frame C is indicated in this report as ${}^{C}X_{B}$. This notation is directly take from Featherstone's, where it is indicated with ${}^{C}\boldsymbol{X}_{B}$. However, the transformation matrix for a 6D force vector is indicated with ${}^{C}\boldsymbol{X}^{*}_{B}$ in Featherstone's, while in this report's we use ${}_{C}\mathbf{X}^{B}$. The main reasons behind this choice are: a) the star is typically used to indicate the adjoint (in the sense of adjoint linear transformation in linear algebra) and indeed, in this report's notation we get ${}_{C}\mathbf{X}^{B} = {}^{B}\mathbf{X}^{*}_{C}$, which is not the case in Featherstone's; b) ${}_{C}\mathbf{X}^{B}$ maps wrenches into wrenches while ${}^{B}\mathbf{X}_{C}$ maps twists into twist and we use a right superscript to indicate a twist and a right subscript to indicate a wrench.

6D Cross Product. In Featherstone's, the 6D Cross product of a 6D motion vector $\boldsymbol{v}$ and a 6D motion vector $\boldsymbol{u}$ is indicated as

$$\boldsymbol{v} \times \boldsymbol{u}.$$

A very similar notation is used in this report, namely

$$\mathbf{v} \times \mathbf{u}.$$

The 6D cross product of a 6D motion vector $\boldsymbol{v}$ and a 6D motion vector $\boldsymbol{f}$ is indicated in Featherstone's as

$$\boldsymbol{v} \times^{*} \boldsymbol{f}.$$

To indicate explicitly that $\times^{*}$ is nothing else that minus the adjoint representation of the Lie algebra of $SE(3)$ to itself, we write the same operation as

$$\mathbf{v} \bar{\times}^{*} \mathbf{f}.$$

Further details are given in the explanation of (62).

Recap on this report's and Featherstone's notation comparison. Summarizing, the main difference and similarities of the two notations are the following.

This report	Featherstone [7]
${}^{C}\mathbf{v}_{A,B}$	${}^{C}\boldsymbol{v}_{B}$
${}^{C}\mathbf{a}_{A,B}$	${}^{C}\dot{\boldsymbol{v}}_{B} = {}^{C}\boldsymbol{a}_{B}$
${}^{C}\dot{\mathbf{v}}_{A,B}$	${}^{C}\mathring{\boldsymbol{v}}_{B}$
${}^{C}\mathbf{X}_{B}$	${}^{C}\boldsymbol{X}_{B}$
${}_{C}\mathbf{X}^{B}$	${}^{C}\boldsymbol{X}^{*}_{B}$
$\mathbf{v}\times$	$\boldsymbol{v}\times$
$\mathbf{v}\bar{\times}^{*}$	$\boldsymbol{v}\times^{*}$

A.2 Siciliano's notation

In this section, we compare this report's notation the notation used in the classical book of Siciliano et al. [17].

Homogenous transformation. In [17], the homogeneous transformation that maps the coordinates of a point from a frame A to a frame B is indicated with

$$\mathbf{T}_B^A = \begin{bmatrix} \mathbf{R}_B^A & \mathbf{o}_B^A \\ \mathbf{0}_{3\times 1} & 1 \end{bmatrix}. \tag{88}$$

Comparing it with (12), we obtain the following comparison table.

This report	Siciliano et al. [17]
${}^A\mathbf{H}_B$	$\mathbf{T}_B^A$
${}^A\mathbf{R}_B$	$\mathbf{R}_B^A$
${}^A\mathbf{o}_B$	$\mathbf{o}_B^A$

Note that, in Siciliano et al.'s notation, $\mathbf{o}_B^A$ is simply denoted $\mathbf{p}_B$ whenever A is an inertial frame.

Velocity of a frame. In [17], the velocity of a frame B is denoted

$$\mathbf{v}_B = \begin{bmatrix} \dot{\mathbf{p}}_B \\ \boldsymbol{\omega}_B \end{bmatrix}. \tag{89}$$

Comparing it with (35), indicating with A the inertial frame implicitly assumed by the Siciliano notation, we have

This report	Siciliano et al. [17]
${}^{B[A]}\mathbf{v}_{A,B}$	$\mathbf{v}_B$
${}^A\dot{\mathbf{o}}_B$	$\dot{\mathbf{p}}_B$
${}^A\boldsymbol{\omega}_{A,B}$	$\boldsymbol{\omega}_B$

A.3 Spong's notation

In this section, we compare this report's notation the notation used in the classical book of Spong et al. [19]. In [19] the base frame of the fixed robot is indicated with 0, while the frame of the end-effector is indicated with n. To simplify the comparison between the two notations, we will use A to indicate the frame 0 and B to indicate the frame n.

Homogenous transformation. In [19, Section 2.6], the homogeneous transformation that maps the coordinates of a point from a frame A to a frame B is indicated with

$$H_B^A = \begin{bmatrix} R_B^A & d_B^A \\ \mathbf{0}_{3\times 1} & 1 \end{bmatrix}. \tag{90}$$

Comparing it with (12), we obtain the following comparison table.

This report	Spong et al. [19]
${}^A\mathbf{H}_B$	H_B^A
${}^A\mathbf{R}_B$	R_B^A
${}^A\mathbf{o}_B$	d_B^A

Velocity of a frame. In [19, Section 4.6], the velocity of a frame B w.r.t. to a inertial frame A is denoted

$$\xi_B^A = \begin{bmatrix} v_B^A \\ \omega_B^A \end{bmatrix} = \begin{bmatrix} \dot{o}_B^A \\ \omega_B^A \end{bmatrix}. \tag{91}$$

Comparing it with (35), we have

This report	Spong et al. [19]
${}^{B[A]}\mathbf{v}_{A,B}$	ξ_B^A
${}^{A}\dot{\mathbf{o}}_B$	$v_B^A = \dot{o}_B^A$
${}^{A}\boldsymbol{\omega}_{A,B}$	ω_B^A

A.4 Lie group theory notation

In this section, we compare this report's notation the notation used in the classical book of Murray et al. [13]. In [13] the inertial frame is indicated with A, while the body frame is indicated with B. However, when used as suffix these letters are used lower-case, so as a or b.

Homogenous transformation. In [13, Section 4.2], the homogeneous transformation that maps the coordinates of a point from a frame A to a frame B is indicated with

$$g_{ab} = \begin{bmatrix} R_{ab} & p_{ab} \\ \mathbf{0}_{3\times 1} & 1 \end{bmatrix}. \tag{92}$$

Comparing it with (12), we obtain the following comparison table.

This report	Murray et al. [13]
${}^{A}\mathbf{H}_B$	g_{ab}
${}^{A}\mathbf{R}_B$	R_{ab}
${}^{A}\mathbf{o}_B$	p_{ab}

Velocity of a frame. In [13, Equation 2.53, Section 4.2], the so-called *spatial* velocity of a frame B with respect to a frame A is defined as

$$\hat{V}_{ab}^s = \begin{bmatrix} v_{ab}^s \\ \omega_{ab}^s \end{bmatrix} = \begin{bmatrix} -\dot{R}_{ab} R_{ab}^T p_{ab} p_{ab} + \dot{p}_{ab} \\ (\dot{R}_{ab} R_{ab}^T)^\vee \end{bmatrix}. \tag{93}$$

Comparing it with (23), it is clear that this is equivalent to what in this report is referred as *right trivialized* velocity ${}^{A}\mathbf{v}_{A,B}$.

Similarly, in [13, Equation 2.55, Section 4.2], the *body* velocity of a frame B with respect to a frame A is defined as

$$\hat{V}_{ab}^b = \begin{bmatrix} v_{ab}^b \\ \omega_{ab}^b \end{bmatrix} = \begin{bmatrix} R_{ab}^T \dot{p}_{ab} \\ (R_{ab}^T \dot{R}_{ab})^\vee \end{bmatrix}. \tag{94}$$

Comparing it with (23), we see that this is equivalent to what in this report is referred as the *left trivialized* velocity ${}^{B}\mathbf{v}_{A,B}$.

The overall comparison of velocities between this report and [13] is given in the following table:

This report	Murray et al. [13]
${}^{A}\mathbf{v}_{A,B}$	$\hat{V}^{s}_{ab}$
${}^{A}\boldsymbol{v}_{A,B}$	v^{s}_{ab}
${}^{A}\boldsymbol{\omega}_{A,B}$	ω^{a}_{ab}
${}^{B}\mathbf{v}_{A,B}$	$\hat{V}^{b}_{ab}$
${}^{B}\boldsymbol{v}_{A,B}$	v^{b}_{ab}
${}^{B}\boldsymbol{\omega}_{A,B}$	ω^{b}_{ab}

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