Evolution equations for polynomials and rational functions which are conformal on the unit disk

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J. de Graaf

Abstract

On the unit disk D in the complex plane $\mathbf{C}$ two evolution equations for conformal mappings $\Omega(z,t)$, $z \in D$, $t \geq 0$ are studied: The quasi-linear Löwner-Kufarev equation and the quasi-linear Hopper equation. The first one has 'Hamiltonian', say, f_Ω, the second one $\mathcal{F}_\Omega$. The L-K-equation has the property that for any initial condition $\Omega_0(z)$ which is conformal on D, the solution $\Omega(z,t)$ remains conformal as long as it exists (Section 1). The H-equation has the property that for any initial condition $\Omega_0(z)$ which is polynomial/rational on D, the solution $\Omega(z,t)$ remains polynomial/rational as long as it exists (Section 2). We find conditions on the pair of Hamiltonians $\{f_\Omega, \mathcal{F}_\Omega\}$, such that both the L-K- and the H-equations describe one and the same evolution phenomenon. This implies that both the properties of being conformal and of being polynomial/rational persist (Section 3). We show that 'compatible pairs' $\{f_\Omega, \mathcal{F}_\Omega\}$ are not rare. They can both be found in physics and be 'artificially' constructed. In this paper the emphasis is on algebraic properties, although analysis cannot be avoided altogether.

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Home institution:
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven
The Netherlands
j.d.graaf@tue.nl

1 The (quasi-linear) Löwner-Kufarev equation

On the unit disk $D \subset \mathbf{C}$ we first consider the linear Löwner-Kufarev initial value problem

$$\begin{cases} \frac{\partial}{\partial t}\Omega(z,t) &= f(z,t)\, z\frac{\partial}{\partial z}\Omega(z,t), \qquad t \in I \subset \mathbf{R}, \qquad z \in D \subset \mathbf{C} \\ \Omega(z,0) &= \Omega_0(z), \end{cases} \tag{1}$$

with analytic initial condition $\Omega_0(z)$. The apriori given coefficient function f is supposed to be analytic in z and continous in t. This linear- LK-equation plays an important role in 'pure' complex analysis. See, for example, Pomerenke [5].
By applying the method of characteristics the solution of the initial value problem (1) can locally be written

$$\Omega(z,t) = \Omega_0(\varphi^{\leftarrow}(z,t)).$$

Here $\varphi^{\leftarrow}$ denotes the inverse of $z \mapsto \varphi(z,t)$, which is the solution of the initial value problem

$$\begin{cases} \dot{\varphi}(z,t) &= -f(\varphi(z,t),t)\varphi(z,t), \\ \varphi(z,0) &= z. \end{cases}$$

(The dot $\dot{}$ denotes differentiation with respect to t).
To prove this we calculate (write $'$ instead of $\frac{\partial}{\partial z}$)

$$\begin{aligned} 0 &= \tfrac{d}{dt}\varphi^{\leftarrow}(\varphi(\zeta,t),t) = \dot{\varphi}^{\leftarrow}(\varphi(\zeta,t),t) + \varphi^{\leftarrow\prime}(\varphi(\zeta,t),t)\dot{\varphi}(\zeta,t) \\ &= \dot{\varphi}^{\leftarrow}(\varphi(\zeta,t),t) - f(\varphi(\zeta,t))\varphi(\zeta,t)\varphi^{\leftarrow\prime}(\varphi(\zeta,t),t). \end{aligned}$$

With $\zeta = \varphi^{\leftarrow}(z,t)$ this reads

$$\frac{\partial}{\partial t}\varphi^{\leftarrow}(z,t) = f(z,t)\, z\, \frac{\partial}{\partial z}\varphi^{\leftarrow}(z,t).$$

Therefore $\Omega(z,t) = \Omega_0(\varphi^{\leftarrow}(z,t))$ locally solves the initial value problem (1).

From the theory of ordinary differential equations we gather the following

Properties 1.1 *Suppose that* $\forall t \geq 0$ *the mapping* $z \mapsto f(z,t)$ *is analytic on an open set* $U_t \supset D$.

I. $\forall t \geq 0$ *the mapping* $z \mapsto \varphi(z,t)$ *is an analytic bijection from an open set* $V_{0,t} \subset U_0$ *to an open set* $V_t \subset U_t$. *The sets* $V_{0,t}$, V_t *are a maximal pair and both contain* 0. *Also* $\varphi(0,t) = 0$.

II. *If* $U_0 \supset \overline{D}$, *then for* $t \geq 0$, *sufficiently small, we have* $V_{0,t} \supset \overline{D}$ *and* $V_t \supset \overline{D}$.

III. *If* $\Omega_0 \colon V_{0,t} \to \mathbf{C}$ *is conformal then* $\Omega(z,t) = \Omega_0(\varphi^{\leftarrow}(z,t))$ *is a conformal map on* V_t.

In [4] a quasi-linear version of the LK-equation has been introduced for modelling the behaviour of drops of viscous fluids with surface tension. The function f may now depend on Ω in a 'functional' way and we obtain the *Quasi-linear Löwner-Kufarev equation*

$$\frac{\partial}{\partial t}\Omega(z,t) = f_{\Omega(\cdot,t)}(z)\, z\frac{\partial}{\partial z}\Omega(z,t), \qquad t \in I \subset \mathbf{R}, \qquad z \in D \subset \mathbf{C}. \tag{2}$$

It is assumed that for every conformal $\Omega \colon D \to \mathbf{C}$ the mapping $\Omega \mapsto f_\Omega$ is well defined and $f_\Omega \colon D \to \mathbf{C}$ is analytic. Notation $f_{\Omega(\cdot,t)}(z) = \sum_{j=0}^{\infty} f_{\Omega,j}(t)z^j$. With the Ansatz $\Omega(z,t) =$

$\sum_{k=0}^{\infty} \omega_k(t) z^k$ the Ql-L-K-equation (2) becomes an infinite system of ordinary differential equations

$$\frac{d}{dt}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \vdots \\ \omega_n \\ \vdots \end{bmatrix} = \begin{bmatrix} f_{\Omega,0}(t) & 0 & 0 & 0 & \cdots & 0 & \cdots \\ f_{\Omega,1}(t) & f_{\Omega,0}(t) & 0 & 0 & \cdots & 0 & \cdots \\ f_{\Omega,2}(t) & f_{\Omega,1}(t) & f_{\Omega,0}(t) & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & & & \\ f_{\Omega,n-1}(t) & f_{\Omega,n-2}(t) & \cdots & \cdots & \cdots & f_{\Omega,0}(t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 1\,\omega_1 \\ 2\,\omega_2 \\ 3\,\omega_3 \\ \vdots \\ n\,\omega_n \\ \vdots \end{bmatrix}. \tag{3}$$

For some local and global existence results of this problem see [4]. Note that in the classical problem (1) with f NOT a rational function of z, a polynomial/rational initial condition will NEVER give a solution of that type. Even if f is rational this will rarely happen. In our quasi-linear case however, we have straightforwardly

Theorem 1.2 *Consider the initial value problem*

$$\begin{cases} \frac{\partial}{\partial t}\Omega(z,t) &= f_{\Omega(\cdot,t)}(z)\, z \frac{\partial}{\partial z}\Omega(z,t), \qquad t \in I \subset \mathbf{R}, \quad z \in D \subset \mathbf{C} \\ \\ \Omega(z,0) &= \Omega_0(z) = \sum_{n=1}^{N} a_n z^n, n \in \mathbf{N}. \end{cases} \tag{4}$$

Suppose that the solution $\Omega(z,t)$ exists for $z \in D$ and $t \in [0,T)$.

I. $\forall t \in [0,T) \forall N \in \mathbf{N}$ *the solution $z \mapsto \Omega(z,t)$ is a polynomial of degree N iff*

$$\Omega \mapsto f_\Omega(z) = \frac{\Phi_1(\underline{\omega})z + \omega_2 \Phi_2(\underline{\omega})z^2 + \cdots + \omega_N \Phi_N(\underline{\omega})z^N + \cdots}{\omega_1 z + 2\omega_2 z^2 + \cdots + N\omega_N z^N + \cdots}. \tag{5}$$

Here $\Omega(z) = \sum_{j=1}^{\infty} \omega_j z^j$ and $\underline{\omega} = (\omega_1, \omega_2, \cdots)$.

II. $\Omega(z,t) = \sum_{k=1}^{N} a_k(t) z^k$ *solves (4) iff*

$$\begin{cases} \frac{d\,a_1(t)}{dt} &= \Phi_1(a_1(t), \cdots, a_N(t), 0, \cdots, 0, \cdots) \\ \frac{d\,a_2(t)}{dt} &= \Phi_2(a_1(t), \cdots, a_N(t), 0, \cdots, 0, \cdots) \\ \vdots & \vdots \qquad \vdots \\ \frac{d\,a_N(t)}{dt} &= \Phi_N(a_1(t), \cdots, a_N(t), 0, \cdots, 0, \cdots). \end{cases} \tag{6}$$

Condition (5) is not easy to verify. The dynamics of a Stokes drop as studied in [3] however, has to be of this form. A similar but more complicated condition could be given in order that *rational* solutions persist.

For the purpose of this paper the main attraction of the quasi-linear Löwner-Kufarev equation lies in the following

Theorem 1.3 *Consider the initial value problem*

$$\begin{cases} \frac{\partial}{\partial t}\Omega(z,t) &= f_{\Omega(\cdot,t)}(z)\, z \frac{\partial}{\partial z}\Omega(z,t), \qquad t \in I \subset \mathbf{R}, \quad z \in D \subset \mathbf{C} \\ \\ \Omega(z,0) &= \Omega_0(z) \text{ is conformal on } D. \end{cases} \tag{7}$$

Suppose $\Omega(z,t)$ for $z \in D$ and $0 \le t < T$.
Then $\forall t \in [0,T)$ the mapping $z \mapsto \Omega(z,t)$ is conformal on D.

Proof The proof is reduced to Properties 1.1 by taking there $f(z,t) = f_{\Omega(\cdot,t)}(z)$ with our special solution $\Omega(z,t)$ substituted. □

2 The (quasi-linear) Hopper equation

In [3] Hopper introduced an evolution equation which describes the behaviour of a drop of a Stokes fluid driven by surface tension. The unknown function in Hopper's equation is a conformal map from the unit disk $D \subset \mathbf{C}$ to the region occupied by the fluid. For mathematical considerations on Hopper's equation see [2] and [1]. In this section we look at a slightly modified and more general version of this equation. First some notation.

Notation 2.1 *Let g be an analytic function on an open set $W \subset \mathbf{C}$. The analytic function $g^\dagger$ is defined by $g^\dagger(z) = \overline{g(\frac{1}{\bar z})}$. Clearly $z \in W^*$, the domain of $g^\dagger$, iff $\frac{1}{\bar z} \in W$.*

Observe the following

Properties 2.2
- *If $\partial D \subset W$, then $\partial D \subset W^*$ and $g^\dagger(z) = \overline{g(z)}$ if $|z| = 1$.*
- *If g is analytic on $\overline{D}$, then $g^\dagger$ is analytic on $D^c \cup \{\infty\}$.*
- *If $g(z) = \sum_{k=-\infty}^{\infty} g_k z^k$ then $g^\dagger(z) = \sum_{k=-\infty}^{\infty} \overline{g}_k \frac{1}{z^k}$.*
- $(z\frac{d}{dz}g(z))^\dagger = -z\frac{d}{dz}g^\dagger(z)$ *and* $(g')^\dagger(z) = -z^2(g^\dagger)'(z)$.
- *[g conformal on D] $\Rightarrow$ [$\frac{1}{2\pi i}\int_{|z|=1} g'(z)g^\dagger(z)\,dz = \sum_{k=1}^{\infty} k\,|g(k)|^2 = \frac{1}{\pi}Area[g(D)]$].*

We now formulate our version of Hopper's initial value problem for the evolution of a conformal map $\Omega(\cdot,t)\colon D \to \mathbf{C}$.

$$\begin{cases} \frac{\partial}{\partial t}(z\Omega'(z,t)\Omega^\dagger(z,t)) - z\frac{\partial}{\partial z}(\mathcal{F}_{\Omega(\cdot,t)}(z)z\Omega'(z,t)\Omega^\dagger(z,t)) &= \Theta_{\Omega(\cdot,t)}(z), \\ \qquad\text{with } t \in I \subset \mathbf{R},\ z \in D \subset \mathbf{C}. & \\ \Omega(z,0) &= \Omega_0(z). \end{cases} \tag{8}$$

Remarks 2.3
- $\mathcal{F}_\Omega$ is an analytic function on D. It 'regulates the dynamics' and may depend on Ω in a functional way. In [3] Hopper takes $\mathcal{F}_\Omega$ such that $\mathrm{Re}\,\mathcal{F}_\Omega(z) = \frac{1}{2|\Omega'(z)|}$ if $|z| = 1$ and $\mathrm{Im}\,\mathcal{F}_\Omega(0) = 0$. Hence in his case $\mathcal{F}_\Omega(z) = \frac{1}{2\pi i}\int_{|\sigma|=1}\frac{\sigma+z}{2|\Omega'(\sigma)|\sigma(\sigma-z)}\,d\sigma$.
- Θ_Ω is required to be analytic on D. Therefore the singularities on the lefthand side of (8) have to cancel $\forall t \in I$. This gives 'the dynamics'. In [3] the constant term in the Taylorseries of Θ vanishes. Division of both sides by z in this case leads to Hopper's original equation as introduced in [3].
- Put $\Theta_{\Omega(\cdot,t)}(z) = v_{\Omega(\cdot,t)}(z) + 2zX'_{\Omega(\cdot,t)}(z)$. Here v_Ω is the constant term in the Taylor expansion for Θ_Ω. For later convenience the remaining term is written with the derivative of a suitable analytic function X_Ω. Note that the functional $v_{\Omega(\cdot,t)}$ gives the $\frac{1}{\pi}\times$ the growth rate of the area of $\Omega(D,t)$. Indeed, evaluation of the contour integral $\frac{1}{2\pi i}\int_{|z|=1}\frac{\cdots}{z}\,dz$ of both sides of the equality (8) leads to $\frac{d}{dt}\mathrm{Area}[\Omega(D,t)] = \pi v_{\Omega(\cdot,t)}$, use Properties 2.2. Because of this, v_Ω has to be a $\mathbf{R}-$valued functional. In [3] $v_\Omega = 0$ because of area conservation for incompressible Stokes fluids.
- (Rotation symmetry) If for some constant $\alpha \in \mathbf{R}$ it happens that $\mathcal{F}_{e^{i\alpha}\Omega}(z) = \mathcal{F}_\Omega(z)$ and also $v_{e^{i\alpha}\Omega} = v_\Omega$ for all Ω, then $e^{i\alpha}\Omega(z,t)$ is a solution if $\Omega(z,t)$ is. In [3] there is rotation symmetry.
- (Reflection symmetry) Write $\Omega^{\sim}(z)$ for $\overline{\Omega(\bar z)}$. If it happens that $\mathcal{F}_{\Omega^{\sim}}(z) = \mathcal{F}_\Omega^{\sim}(z)$ and $v_{\Omega^{\sim}} = v_\Omega$ for all Ω, then $\Omega^{\sim}(z,t)$ is a solution if $\Omega(z,t)$ is. Reflection symmetry implies that

all Taylor coefficients of $\mathcal{F}_\Omega$ are real if all Taylor coefficients of Ω are real. In [3] there is reflection symmetry.

• The function $H(z) = z\Omega'(z)\Omega^\dagger(z)$ plays a prominent role in (8). In fact (8) can be considered as an evolution equation for the non-positive Laurent part of H. For the interesting problem 'how to recover Ω from H' see [1].

Write $\mathcal{F}_{\Omega(\cdot,t)}(z) = \sum_{j=0}^\infty \mathcal{F}_{\Omega,j}(t)z^j = \sum_{j=0}^\infty \mathcal{F}_j(\underline{\omega})z^j$, with $\underline{\omega} = (\omega_1, \omega_2, \omega_3, \cdots)$. Substitution of the Ansatz $\Omega(z,t) = \sum_{k=1}^\infty \omega_k(t)z^k$, with $\omega_1(t) > 0$ and $\omega_j(t) \in \mathbf{C}$, $j > 1$, in the left hand side of (8) and demanding that the negative part of the Laurent series should vanish, turns the H-equation in to an infinite system of ordinary differential equations which is now of upper diagonal type:

$$\frac{d}{dt}\begin{bmatrix} \omega_1 & 2\omega_2 & 3\omega_3 & 4\omega_4 & \cdots \\ 0 & \omega_1 & 2\omega_2 & 3\omega_3 & \cdots \\ 0 & 0 & \omega_1 & 2\omega_2 & \cdots \\ 0 & 0 & 0 & \omega_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}\begin{bmatrix} \overline{\omega}_1 \\ \overline{\omega}_2 \\ \overline{\omega}_3 \\ \overline{\omega}_4 \\ \vdots \end{bmatrix} = \tag{9}$$

$$= -\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & \mathcal{F}_0(\underline{\omega}) & \mathcal{F}_1(\underline{\omega}) & \mathcal{F}_2(\underline{\omega}) & \cdots \\ 0 & 0 & 2\mathcal{F}_0(\underline{\omega}) & 2\mathcal{F}_1(\underline{\omega}) & \cdots \\ 0 & 0 & 0 & 3\mathcal{F}_0(\underline{\omega}) & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}\begin{bmatrix} \omega_1 & 2\omega_2 & 3\omega_3 & 4\omega_4 & \cdots \\ 0 & \omega_1 & 2\omega_2 & 3\omega_3 & \cdots \\ 0 & 0 & \omega_1 & 2\omega_2 & \cdots \\ 0 & 0 & 0 & \omega_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}\begin{bmatrix} \overline{\omega}_1 \\ \overline{\omega}_2 \\ \overline{\omega}_3 \\ \overline{\omega}_4 \\ \vdots \end{bmatrix} + \begin{bmatrix} \upsilon(\underline{\omega}) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Theorem 2.4 *Consider the initial value problem (8) with initial condition Ω_0.*

I. *If $\Omega_0(z) = \sum_{j=1}^N \omega_j(0)z^j$, with $\omega_1(0) > 0$, then the solution$\Omega(z,t)$ remains a polynomial of degree N: $\Omega(z,t) = \sum_{j=1}^N \omega_j(t)z^j$. Here the coefficient functions $\omega_j(t)$ satisfy the system of ordinary differential equations obtained from (9) by taking there $\omega_{N+1}(t) = \omega_{N+2}(t) = \cdots = 0$. Note that we obtain in this way 1 real and $(N-1)$ complex ordinary differential equations. Note also that $\omega_1(t) > 0$ as long as the solution exists.*

II. *If we take $\Omega_0(z) = \omega_1(0)z + \omega_N(0)z^N$, then $\Omega(z,t) = \omega_1(t)z + \omega_N(t)z^N$. The system of ordinary differential equations for ω_1 , ω_N is obtained by equating to 0 in (9) all functions ω_j with $1 < j < N$ and $j > N$.*

III. *If $\mathcal{F}_\Omega$ has reflection symmetry, as e.g. in [3], then a polynomial initial condition with real coefficients leads to a polynomial solution with real coefficients.*

Proof All straightforward verifications. □

Next we turn to the persistence of rational initial conditions. Hopper's equation (8) can be reformulated as

$$\Omega^\dagger(z,t)\Big[\frac{\dot{\Omega}'(z,t)}{\Omega'(z,t)} - \mathcal{F}_{\Omega(\cdot,t)}(z)\Big(1 + \frac{z\Omega''(z,t)}{\Omega'(z,t)}\Big) - z\mathcal{F}'_{\Omega(\cdot,t)}(z)\Big] + \tag{10}$$

$$+\frac{1}{z}(\Omega')^\dagger(z,t)\mathcal{F}_{\Omega(\cdot,t)}(z) + \dot{\Omega}^\dagger(z,t) - \frac{\upsilon_{\Omega(\cdot,t)}}{z}\frac{1}{\Omega'(z,t)} = 2\frac{X'(z,t)}{\Omega'(z,t)},$$

or, equivalently, omitting the arguments of Ω, X, and $\mathcal{F}_\Omega$,

$$\Omega^\dagger\Big[\frac{\dot\Omega'}{\Omega'} - z\mathcal{F}_\Omega\frac{\Omega''}{\Omega'}\Big] + \dot\Omega^\dagger - \Big(\Omega^\dagger\big[z\mathcal{F}_\Omega\big]\Big)' - \frac{v_\Omega}{z}\frac{1}{\Omega'} = 2\frac{X'}{\Omega'}. \tag{11}$$

Note that the functions between [ ] are analytic on D. The requirement is again, that for given $\Omega(\cdot,t)$ the time derivative $\dot\Omega(\cdot,t)$ has to be such that the left hand side of (10) extends to an analytic function on D. Thereby the evolution of Ω is determined.

Theorem 2.5 *Consider the initial value problem (8) with the Ansatz*

$$\Omega(z,t) = A_0(t)z + z\sum_{n=0}^{N}\frac{A_n(t)}{1-\alpha_n(t)z}, \tag{12}$$

with $N\in\mathbf{N}, A_0(0)\in\mathbf{R},\ A_n(t)\in\mathbf{C},\ \alpha_n(t)\in\mathbf{C},$ *such that* $\Omega'(z,0) = A_0(0)+\sum_{n=1}^N A_n(0) > 0$ *and* $|\alpha_n(t)|<1$.

I. *If* $v_\Omega \neq 0$ *and* $A_0(0)\neq 0$, *the initial value problem (8) is locally solved by (12). The* $A_0,\ A_1,\cdots A_N,\alpha_1,\cdots,\alpha_N$ *satisfy a coupled system of 1 real and* $2N$ *complex ordinary differential equations.*

II. *If* $v_\Omega = 0$ *as in [3], then a solution (12) exists with* A_0 *identically zero.*

III. *There exists a solution of the more general form*

$$\Omega(z,t) = c_1(t)z + \cdots + c_N(t)z^N + \sum_{m=1}^{M}\sum_{n=1}^{N(m)} c_{m,n}(t)\frac{z^n}{(1-\zeta_m(t)z)^n}. \tag{13}$$

Also in this general case the t*-dependent coefficients satisfy a system of ordinary differential equations.*

Proof (sketch)

I. Calculate

$$\begin{aligned}
&\Omega'(z) = A_0 + \textstyle\sum_{n=1}^N \frac{A_n}{(1-\alpha_n z)^2}, && \Omega''(z) = \textstyle\sum_{n=1}^N \frac{2\alpha_n A_n}{(1-\alpha_n z)^3},\\
&\dot\Omega'(z) = \dot A_0 + \textstyle\sum_{n=1}^N \frac{\dot A_n}{(1-\alpha_n z)^2} + \sum_{n=1}^N \frac{2\dot\alpha_n A_n z}{(1-\alpha_n z)^3}, && \Omega^\dagger(z) = \frac{\overline{A_0}}{z} + \textstyle\sum_{n=1}^N \frac{\overline{A_n}}{z-\overline{\alpha}_n},\\
&\dot\Omega^\dagger(z) = \frac{\overline{\dot A_0}}{z} + \textstyle\sum_{n=1}^N \frac{\overline{\dot A_n}}{z-\overline{\alpha}_n} + \sum_{n=1}^N \frac{\overline{A_n}\,\overline{\dot\alpha}_n}{(z-\overline{\alpha}_n)^2}.
\end{aligned} \tag{14}$$

Substitute (14) in (11) and equate to 0, respectively, the 1rst-order pole at $z=0$, the 1rst-order poles at $z=\overline{\alpha}_k,\ 1\le k\le N$ and the 2nd-order poles at $z=\overline{\alpha}_k,\ 1\le k\le N$. This leads to the following system of ordinary differential equations where we employ the notations $\mathcal{F}_{\underline{A},\underline{\alpha}}(z), v_{\underline{A},\underline{\alpha}}$ instead of $\mathcal{F}_\Omega, v_\Omega$

$$\begin{aligned}
&\tfrac{d}{dt}\,\overline{A_0}[A_0+\textstyle\sum_{n=1}^N A_n] = v_{\underline{A},\underline{\alpha}},\\
&\overline{\dot A}_k + \overline{A_k}\Big[\frac{\dot\Omega'(\overline{\alpha}_k)}{\Omega'(\overline{\alpha}_k)} - \overline{\alpha}_k\mathcal{F}_{\underline{A},\underline{\alpha}}(\overline{\alpha}_k)\frac{\Omega''(\overline{\alpha}_k)}{\Omega'(\overline{\alpha}_k)}\Big] = 0,\ 1\le k\le N,\\
&\dot\alpha_k + \overline{\mathcal{F}_{\underline{A},\underline{\alpha}}(\overline{\alpha}_k)}\alpha_k = 0,\ 1\le k\le N.
\end{aligned} \tag{15}$$

Because of the initial conditions and $v_{\underline{A},\underline{\alpha}}$ being real it follows that the functions $A_0(t)$ and $\Omega'(z,t) = A_0(t)+\sum_{n=1}^N A_n(t)$ are real valued. The latter remains positive as long as Ω remains conformal.

II. This is presisely the case dealt with hitherto in [4] and [1].

III. For the special case $v_\Omega = 0$ and $c_1, \cdots, c_N$ taken 0 see again [4] and [1].

□

We observe that Hopper's equation has complete sets of polynomial and rational solutions. The natural question arises whether those solutions are conformal mappings on there existence interval. The answer to this question is yes if they are simultaneously solutions of a (quasi-linear) Löwner-Kafarev equation. This is the subject of the next section.

3 From Löwner-Kufarev to Hopper

Our starting point is the quasi-linear Löwner-Kufarev equation (2) with a given Hamiltonian $\Omega \mapsto f_\Omega$:

$$\tfrac{\partial}{\partial t}\Omega = f_\Omega\, z \tfrac{\partial}{\partial z}\Omega. \tag{16}$$

We apply the †-operation to (16), cf. Properties 2.2, and multiply the result by Ω', which gives $\dot{\Omega}^\dagger\Omega' = -(f_\Omega^\dagger z(\Omega^\dagger)')\Omega'$. Add to this expression $\Omega^\dagger\dot{\Omega}' = \Omega^\dagger(f_\Omega z\Omega')'$. This leads to a kind of 'balance equation for the area'

$$\tfrac{\partial}{\partial t}(\Omega'\Omega^\dagger) = \tfrac{\partial}{\partial z}(f_\Omega\, z\Omega'\Omega^\dagger) - z(\Omega^\dagger)'\Omega'\{f_\Omega + f_\Omega^\dagger\}. \tag{17}$$

By integration along the unit circle

$$v_\Omega = \frac{d}{dt}\frac{\mathrm{Area}[\Omega(D,t)]}{\pi} = \frac{d}{dt}\frac{1}{2\pi i}\int_{|z|=1}(\Omega'\Omega^\dagger)\,dz = \frac{1}{2\pi i}\int_{|z|=1}\frac{\Omega'^\dagger\Omega'}{z}(f_\Omega + f_\Omega^\dagger)\,dz. \tag{18}$$

Remark 3.1 Because of Gauss' theorem, (18) should be equal to $\frac{1}{\pi}\int_{\partial\Omega(D,t)} V_n(s)\,ds = \frac{1}{\pi}\int_0^{2\pi} V_n(\Omega(e^{i\theta}))|\Omega'(\Omega(e^{i\theta}))|\,d\theta$. Here s denotes an arclength parametrization of $\partial\Omega(D,t)$ and V_n is the 'normal velocity field' at the boundary $\partial\Omega(D,t)$. A *surface motion law* attaches to any domain $\Omega(D)$ out of a given class, a normal vectorfield V_n at its boundary $\partial\Omega(D)$, thereby determining the evolution of the domain. The above formulae strongly suggest that the relation between f_Ω and V_n is given by $\mathrm{Re}\, f_\Omega(e^{i\theta}) = \frac{V_n(\Omega(e^i\theta))}{|\Omega'(e^{i\theta})|}$. Cf.[4].

In order to link (17) to Hopper's equation we introduce an Ω-dependent analytic function $\mathcal{F}_\Omega$ on D which will be specified later. With $\mathcal{F}_\Omega$ we rewrite (17) as

$$\tfrac{\partial}{\partial t}(z\Omega'\Omega^\dagger) - z\tfrac{\partial}{\partial z}(\mathcal{F}_\Omega z\Omega'\Omega^\dagger) = z\tfrac{\partial}{\partial z}[(f_\Omega - \mathcal{F}_\Omega)z\Omega'\Omega^\dagger] + \Omega'^\dagger\Omega'(f_\Omega + f_\Omega^\dagger). \tag{19}$$

We put $(f_\Omega - \mathcal{F}_\Omega)z\Omega' = -2\Phi_\Omega$ and $-2z\frac{\partial}{\partial z}[\Phi_\Omega\Omega^\dagger] + \Omega'^\dagger\Omega'(f_\Omega + f_\Omega^\dagger) - v_\Omega = 2z\frac{\partial}{\partial z}X_\Omega$. With these notations (19) becomes

$$\tfrac{\partial}{\partial t}(z\Omega'\Omega^\dagger) - z\tfrac{\partial}{\partial z}(\mathcal{F}_\Omega z\Omega'\Omega^\dagger) = z\tfrac{\partial}{\partial z}X_\Omega + v_\Omega. \tag{20}$$

This LOOKS like Hopper's equation. However it IS Hopper's equation ONLY if we have been able to find $\Omega \mapsto \mathcal{F}_\Omega$ such that X_Ω is ANALYTIC. In general this not possible because Hopper's equation always has polynomial/rational solutions which is not true for the Löwner-Kufarev equation. We don't know conditions on f_Ω such that all solutions of a Löwner-Kufarev equation with Hamiltonian f_Ω are also solutions of some Hopper equation with a suitable Hamiltonian $\mathcal{F}_\Omega$. In the next theorem we put conditions on the pair $\{\Phi_\Omega,\, X_\Omega\}$, such that it leads to a 'compatible' pair $\{f_\Omega,\, \mathcal{F}_\Omega\}$.

Definition 3.2 The pair $\{f_\Omega, \mathcal{F}_\Omega\}$ is called a *compatible pair* if all solutions of a Löwner-Kufarev equation with Hamiltonian f_Ω are also solutions of a Hopper equation with $\mathcal{F}_\Omega$ as Hamiltonian and v_Ω defined by (18).

Theorem 3.3 *Consider two Ω-dependent analytic functions X_Ω and Φ_Ω on D with $\Phi_\Omega(0) = 0$. If*

$$X_\Omega(z) + \Omega^\dagger(z)\Phi_\Omega(z) \in i\mathbf{R}, \qquad \textit{for } |z| = 1, \tag{21}$$

or, equivalently,

$$\frac{1}{\Omega(z)\Omega^\dagger(z)} X_\Omega(z) + \frac{\Phi_\Omega(z)}{\Omega(z)} \in i\mathbf{R}, \qquad \textit{for } |z| = 1, \tag{22}$$

then for any given $v_\Omega \in \mathbf{R}$ the functions f_Ω and $\mathcal{F}_\Omega$ are well defined by

$$\left\{ \begin{array}{rcl} (f_\Omega - \mathcal{F}_\Omega) z\Omega' & = & -2\Phi_\Omega \\ 2z\frac{\partial}{\partial z}(X_\Omega + \Omega^\dagger\Phi_\Omega)\frac{1}{\Omega'^\dagger\Omega'} + \frac{v_\Omega}{\Omega'^\dagger\Omega'} & = & f_\Omega + f_\Omega^\dagger, \end{array} \right. \tag{23}$$

and the pair $\{f_\Omega, \mathcal{F}_\Omega\}$ is a compatible pair.

Proof Because of (21) the lefthand side of the 2nd equation in (23) is real valued if $|z| = 1$. Therefore it has a Laurent series, $\sum_{n=-\infty}^{\infty} \gamma_n z^n$ say, with $\gamma_{-n} = \overline{\gamma_n}$. Now just take $f_\Omega = \frac{1}{2}\gamma_0 + \sum_{n=1}^{\infty} \gamma_n z^n$. Next $\mathcal{F}_\Omega$ can be obtained with the 1rst equation in (23). With these choices for f_Ω and $\mathcal{F}_\Omega$ all solutions of (16) are solutions of (20). □

Next we formulate a main consequence of the results of this paper

Corollary 3.4 *Consider the intial value problem for the quasi-linear Löwner-Kufarev equation (2),(16) with a Hamiltonian f_Ω which is the first member of a compatible pair $\{f_\Omega, \mathcal{F}_\Omega\}$. For the initial condition $\Omega_0(z)$ we only consider functions which are analytic and injective (=conformal) on D.*

I. *If, in addition, $\Omega_0(z)$ is a polynomial, then the solution $\Omega_0(z,t)$ will remain a polynomial which is conformal on D for all $t \in [0, T)$, the whole existence interval.*

II. *If, in addition, $\Omega_0(z)$ is a rational function, then the solution $\Omega_0(z,t)$ will remain a rational function which is conformal on D for all $t \in [0, T)$, the whole existence interval.*

Example 3.5 If $\Omega(D) = W$ is a domain in the complex $\zeta-$plane and χ, ϕ are analytic functions on W such that $\chi(\Omega(z)) = X_\Omega(z)$ and $\phi(\Omega(z)) = \Phi_\Omega(z)$, then condition (21) corresponds to

$$\chi(\zeta) + \overline{\zeta}\phi(\zeta) \in i\mathbf{R}, \qquad \text{for } \zeta \in \partial W. \tag{24}$$

This condition is, in fact, satisfied in [3]. A function of type (24) represents the general solution of Stokes' equations in 2 dimensions. Mathematical details to this are mentioned in [2],[1].

Example 3.6 From conditions (21), (22) it follows that if one of the mappings $\Omega \mapsto \Phi_\Omega$, $\Omega \mapsto X_\Omega$ is freely chosen, then the other one can, in principle, be found: One has to solve a Dirichlet problem for finding its real part and then apply harmonic conjugation for finding its imaginary part (up to a real constant).

An artificial but elegant class of examples of this type is constructed by letting the domain $\Omega(D)$ 'evolve in an external field' given by a fixed entire function $\phi(\zeta)$ with $\phi(0) = 0$. Just define Φ_Ω by $\Phi_\Omega(z) = \phi(\Omega(z))$. Then X_Ω follows in the above described way. Of course a similar game can be started with a fixed entire function $\chi(\zeta)$ and taking $X_\Omega(z) = \chi(\Omega(z))$.

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