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AN ADAPTIVE TIME-STEPPING ALGORITHM FOR QUASISTATIC PROCESSES

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SUMMARY

A scheme is presented that automatically determines the magnitude of the time step in quasistatic boundary value problems discretized via some customary method (finite elements, finite differences). The basic idea stems from the path-following techniques widely employed in purely static analyses in the sense that the incremental displacements, measured in some suitable norm, are constrained to a prefixed value. Examples of trusses composed of a viscoelastic material are included to demonstrate the effectiveness of the method.

1. INTRODUCTION

In finite element analyses of quasistatic problems a time-marching procedure is normally employed to trace the evolution of stresses and strains and to monitor damage propagation. Especially during rapid damage extension, e.g. at progressive failure, a small time step is needed to obtain a converged solution in the iterative equilibrium-finding process.

In purely static analyses a similar problem exists, that is, smaller load steps should be selected with progressive damage evolution. More generally, the magnitude of the load step should be adapted to the degree of non-linearity of the discretized system. An elegant methodology to achieve this goal is the path-following procedure, also called the arc-length technique.¹⁻⁴ This method selects the magnitude of the load step such that the incremental displacements, measured in some suitable norm,^{5,6} are constant during successive equilibrium iterations within a load increment. In the limit this procedure can lead to negative load increments being selected, thus allowing the analyst to overcome limit points without manual interference. The arc-length technique is usually utilized in problems in which the external actions can be represented through an external load vector. For cases that the external loading has a component that is orthogonal to the displacement degrees-of-freedom a compatible technique was presented by Schellekens and de Borst.⁷

In this contribution we shall demonstrate how this methodology can be extended to quasistatic processes where the time plays a role. Now, the constraint acts on the time step,

which is adapted automatically when the degree of non-linearity of the discretized system changes. When the standard arc-length technique is applied in the load-displacement space for purely static loading conditions, negative load increments can be computed. This signals the onset of dynamic processes where inertia effects play a role. Similarly, negative time increments can be computed when applying an arc-length constraint to slow transient processes. Physically, this means that from this point on inertia effects can no longer be disregarded. Computationally, there is no objection to continuing a calculation under arc-length control when negative load or time increments are obtained. Then, the equilibrium path under quasistatic conditions is tracked. It is emphasized that in the case of negative time increments this may be physically not meaningful.

2. AN AUTOMATIC TIME-STEPPING PROCEDURE

When an arc-length-type method is applied in the time domain, the time increment Δt does not remain constant from iteration i to iteration $i+1$, but changes from Δt_i to Δt_{i+1} , such that $\Delta t_{i+1} = t_{i+1}^j - t_n^{j-1}$. The subscripts denote the iteration counter and the superscript the time step. t_n^{j-1} is then the last determined value of time t^{j-1} . The iterative change is denoted by the δ -symbol

$$\delta t_{i+1} = \Delta t_{i+1} - \Delta t_i \quad (1)$$

similar to the iterative change in the displacement vector:

$$\delta \mathbf{u}_{i+1} = \Delta \mathbf{u}_{i+1} - \Delta \mathbf{u}_i \quad (2)$$

We now denote by \mathbf{B} the matrix that connects the nodal velocities $\dot{\mathbf{u}}$ to the strain rates in the integration points $\dot{\boldsymbol{\epsilon}}$ of the finite elements $\dot{\boldsymbol{\epsilon}} = \mathbf{B}\dot{\mathbf{u}}$. Assembling all external actions at time t_{i+1} in a vector \mathbf{r}_{i+1} , equilibrium of the discretized system reads

$$\int_V \mathbf{B}^T \boldsymbol{\sigma}_{i+1} dV = \mathbf{r}_{i+1} \quad (3)$$

with $\boldsymbol{\sigma}_{i+1}$ the value of the stress tensor in iteration $i+1$ and the superscript T the transpose symbol. Introduction of the normalized load vector \mathbf{r} and the current load factor λ_{i+1} at iteration $i+1$, such that $\mathbf{r}_{i+1} = \lambda_{i+1}\mathbf{r}$, permits rewriting of (3) as

$$\int_V \mathbf{B}^T \boldsymbol{\sigma}_{i+1} dV = \lambda_{i+1}\mathbf{r} \quad (4)$$

The stress $\boldsymbol{\sigma}_{i+1}$ can be calculated from the stress at the start of the increment $\boldsymbol{\sigma}_n^{j-1}$ and the stress increment $\Delta \boldsymbol{\sigma}_{i+1}$. In general the stress increment not only depends on the strain increment but also on the time increment, e.g. in the case of viscoelastic or viscoplastic material models or in the case of thermal loading. Given a specific material behaviour, a chosen integration procedure relates the strain and time increment to the stress increment. Depending on the material behaviour and the integration procedure, there may be a difference between the derivatives $\partial \boldsymbol{\sigma} / \partial \boldsymbol{\epsilon}$ and $\partial \Delta \boldsymbol{\sigma} / \partial \Delta \boldsymbol{\epsilon}$ as well as between $\partial \boldsymbol{\sigma} / \partial t$ and $\partial \Delta \boldsymbol{\sigma} / \partial \Delta t$. Consistent linearization requires that the derivatives $\partial \Delta \boldsymbol{\sigma} / \partial \Delta \boldsymbol{\epsilon}$ and $\partial \Delta \boldsymbol{\sigma} / \partial \Delta t$ must be used.

We now expand the stress and the load factor at the end of iteration $i+1$ in a truncated Taylor series (disregarding higher-order terms):

$$\boldsymbol{\sigma}_{i+1} = \boldsymbol{\sigma}_i + \left(\frac{\partial \Delta \boldsymbol{\sigma}}{\partial \Delta \boldsymbol{\epsilon}} \right)_i \delta \boldsymbol{\epsilon}_{i+1} + \left(\frac{\partial \Delta \boldsymbol{\sigma}}{\partial \Delta t} \right)_i \delta t_{i+1} \quad (5)$$

and

$$\lambda_{i+1} = \lambda_i + \dot{\lambda}_i \delta t_{i+1} \quad (6)$$

where a superimposed dot implies time differentiation. Assuming geometric linearity, which assumption is not essential for the method, we have $\delta \boldsymbol{\varepsilon}_{i+1} = \mathbf{B} \delta \mathbf{u}_{i+1}$. Defining

$$\mathbf{D}_i = \left(\frac{\partial \Delta \boldsymbol{\sigma}}{\partial \Delta \boldsymbol{\varepsilon}} \right)_i \quad (7)$$

as the algorithmic stiffness at the end of iteration i , equation (5) can be transformed into

$$\boldsymbol{\sigma}_{i+1} = \boldsymbol{\sigma}_i + \mathbf{D}_i \mathbf{B} \delta \mathbf{u}_{i+1} + \left(\frac{\partial \Delta \boldsymbol{\sigma}}{\partial \Delta t} \right)_i \delta t_{i+1} \quad (8)$$

The next step is to substitute equations (6) and (8) into equation (4). The result is

$$\mathbf{K}_i \delta \mathbf{u}_{i+1} = \lambda_i \mathbf{r} - \int_V \mathbf{B}^T \boldsymbol{\sigma}_i \, dV + \delta t_{i+1} \left[\dot{\lambda}_i \mathbf{r} - \int_V \mathbf{B}^T \left(\frac{\partial \Delta \boldsymbol{\sigma}}{\partial \Delta t} \right)_i \, dV \right] \quad (9)$$

with the tangent stiffness matrix conventionally defined as

$$\mathbf{K}_i = \int_V \mathbf{B}^T \mathbf{D}_i \mathbf{B} \, dV \quad (10)$$

Defining \mathbf{p}_i and \mathbf{q}_i as

$$\mathbf{p}_i = \lambda_i \mathbf{r} - \int_V \mathbf{B}^T \boldsymbol{\sigma}_i \, dV - \Delta t_i \left[\dot{\lambda}_i \mathbf{r} - \int_V \mathbf{B}^T \left(\frac{\partial \Delta \boldsymbol{\sigma}}{\partial \Delta t} \right)_i \, dV \right] \quad (11)$$

and

$$\mathbf{q}_i = \dot{\lambda}_i \mathbf{r} - \int_V \mathbf{B}^T \left(\frac{\partial \Delta \boldsymbol{\sigma}}{\partial \Delta t} \right)_i \, dV \quad (12)$$

respectively, we can rewrite equation (9) as

$$\mathbf{K}_i \delta \mathbf{u}_{i+1} = \mathbf{p}_i + \Delta t_{i+1} \mathbf{q}_i \quad (13)$$

Next we define

$$\delta \mathbf{u}_{i+1}^I = \mathbf{K}_i^{-1} \mathbf{p}_i \quad (14)$$

and

$$\delta \mathbf{u}_{i+1}^{II} = \mathbf{K}_i^{-1} \mathbf{q}_i \quad (15)$$

which can be computed without knowledge of the yet unknown time step Δt_{i+1} . The iterative correction to the displacement increment $\delta \mathbf{u}_{i+1}$ is then determined from (see equations (13)–(15))

$$\delta \mathbf{u}_{i+1} = \delta \mathbf{u}_{i+1}^I + \Delta t_{i+1} \delta \mathbf{u}_{i+1}^{II} \quad (16)$$

After computing $\delta \mathbf{u}_{i+1}^I$ and $\delta \mathbf{u}_{i+1}^{II}$, the variable time step Δt_{i+1} can be calculated from a constraint condition that the incremental displacements remain constant over the time measured in some norm. For instance, the L_2 -norm in the n -dimensional displacement space has gained much popularity.³ In its linearized version it reads^{4,5}

$$\Delta \mathbf{u}_i^T \delta \mathbf{u}_{i+1} = 0 \quad (17)$$

Substitution of (16) in (17) then yields

$$\Delta t_{i+1} = - \frac{\Delta \mathbf{u}_i^T \delta \mathbf{u}_{i+1}^I}{\Delta \mathbf{u}_i^T \delta \mathbf{u}_{i+1}^{II}} \quad (18)$$

3. ELABORATION FOR LINEAR VISCOELASTICITY

As an example of a typically time-dependent constitutive model we shall for a simple linear viscoelastic constitutive relation demonstrate how the algorithmic derivatives of the stress-strain relation can be computed with respect to the strain and the time.

A possible formulation of non-ageing linear viscoelasticity is

$$\boldsymbol{\sigma}(t) = \int_0^t R(t-\tau) \mathbf{D} \dot{\boldsymbol{\epsilon}}(\tau) d\tau \quad (19)$$

with the kernel $R(t-\tau)$ the so-called relaxation function and \mathbf{D} a dimensionless matrix, whose components are given by (in tensor format)

$$D_{ijkl} = \frac{1}{1+\nu} \left[\frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \frac{\nu}{1-2\nu} \delta_{ij} \delta_{kl} \right] \quad (20)$$

and $\dot{\boldsymbol{\epsilon}}$ the strain rate. To avoid memorizing the entire strain history as implied by the hereditary integral of equation (19) we expand R in a series of negative exponential powers,

$$R(t-\tau) = E_0 + \sum_{\alpha=1}^N E_{\alpha} \exp(-(t-\tau)/\lambda_{\alpha}) \quad (21)$$

with E_{α} and λ_{α} stiffness and the relaxation times, respectively, of the individual elements. Mechanically, the model can be interpreted as a Maxwell chain model.⁸ The resulting algorithm for computing the stress increment as a function of the strain increment and the strain history reads⁸

$$\Delta \boldsymbol{\sigma} = E_0 \mathbf{D} \Delta \boldsymbol{\epsilon} + \sum_{\alpha=1}^N [1 - \exp(-\Delta t/\lambda_{\alpha})] \left[\frac{E_{\alpha}}{\Delta t/\lambda_{\alpha}} \mathbf{D} \Delta \boldsymbol{\epsilon} - \boldsymbol{\sigma}_{\alpha}(t^{j-1}) \right] \quad (22)$$

with $\boldsymbol{\sigma}_{\alpha}$ a set of internal variables. \mathbf{D}_i and $(\partial \Delta \boldsymbol{\sigma} / \partial \Delta t)_i$ are then found by differentiating relation (22) with respect to the strain increment and the time increment, respectively, as follows:

$$\mathbf{D}_i = E_0 \mathbf{D} + \sum_{\alpha=1}^N [1 - \exp(-\Delta t_i/\lambda_{\alpha})] \frac{E_{\alpha}}{\Delta t_i/\lambda_{\alpha}} \mathbf{D} \quad (23)$$

and

$$\begin{aligned} \left(\frac{\partial \Delta \boldsymbol{\sigma}}{\partial \Delta t} \right)_i &= \sum_{\alpha=1}^N \frac{1}{\lambda_{\alpha}} \exp(-\Delta t_i/\lambda_{\alpha}) \left[\frac{E_{\alpha}}{\Delta t_i/\lambda_{\alpha}} \mathbf{D} \Delta \boldsymbol{\epsilon}_i - \boldsymbol{\sigma}_{\alpha}(t^{j-1}) \right] \\ &\quad - \sum_{\alpha=1}^N [1 - \exp(-\Delta t_i/\lambda_{\alpha})] \frac{E_{\alpha}}{\Delta t_i^2/\lambda_{\alpha}} \mathbf{D} \Delta \boldsymbol{\epsilon}_i \end{aligned} \quad (24)$$

For the first iteration, $\Delta t_0 = 0$. However, no difficulties arise when dividing by Δt_0 , since for $i = 0$, $\partial \Delta \boldsymbol{\sigma} / \partial \Delta t$ reduces to

$$\left(\frac{\partial \Delta \boldsymbol{\sigma}}{\partial \Delta t} \right)_0 = - \sum_{\alpha=1}^N \frac{\boldsymbol{\sigma}_{\alpha}(t^{j-1})}{\lambda_{\alpha}} \quad (25)$$

This result can be derived easily considering that $\lim_{\Delta t_0 \rightarrow 0} [1 - \exp(-\Delta t/\lambda_{\alpha})]/(\Delta t/\lambda_{\alpha}) = 1$.

4. EXAMPLE CALCULATIONS

The performance of the method is shown in Figures 1–4. In Figure 1 a simple shallow truss structure is shown composed of a viscoelastic material. The model consists of a single Maxwell element with a stiffness $E_1 = 30 \times 10^3 \text{ Nmm}^{-2}$ and a relaxation time $\lambda_1 = 100 \text{ s}$. The cross-section of the trusses is 1.0 mm^2 and the centre node is 268 mm higher than the outer nodes, leading to an angle of 15 degrees between trusses and x -axis. Firstly, a constant load of 100 N is placed on top of the structure. Next, time increments are imposed according to the method outlined in the preceding Sections. An initial time increment of 40 s is used. In subsequent time increments the displacement increment is required to be equal at every iteration. The convergence is checked by relating the norm of the out-of-balance force vector to the out-of-balance force norm of the first iteration. The usual check against the norm of the load increment cannot be applied since no load is added during the time increments. The analysis is terminated if a negative pivot is found in the system of equations, indicating that a snap-through occurs. Typically four to six equilibrium iterations were needed to achieve an out-of-balance force ratio of 10^{-4} . Figure 2 gives the time–displacement diagram. We observe that, when the structure softens, the time step is made smaller while the displacement increments remain constant.

A more complicated example is the space truss structure of Figure 3. Now the stresses in the different truss members are not equal. Some members are in compression, while others are in tension. The material parameters are the same as in the previous example. The cross-section of each member is 317 mm^2 . The elevation of the centre node is 82.16 mm and that of the other unsupported nodes is 62.16 mm. The total span width in the y -direction is 1000 mm. The outer nodes are fixed in all directions. A constant load of 1500 N is applied followed by time

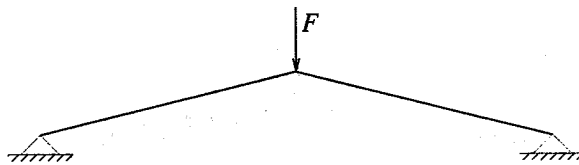


Figure 1. Simple truss model used for assessment of automatic time-stepping procedure

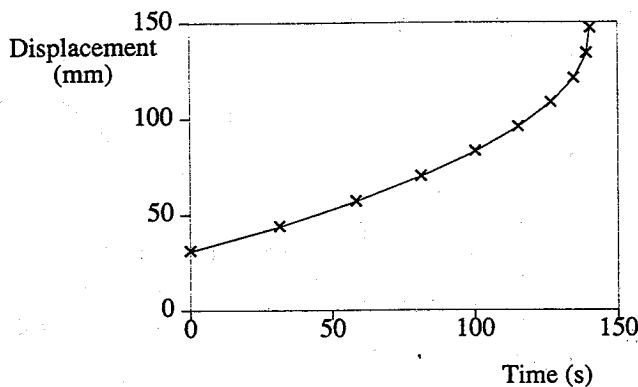


Figure 2. Time-displacement diagram for simple truss structure of Figure 1

steps. Two analyses have been performed, the first with an initial time step of 20 s and the second with 100 s. After the first iteration, the norm of the incremental displacements is kept constant by the described algorithm. Using the same convergence criterion as in the previous example, typically three iterations were needed to achieve equilibrium with the smaller time steps and six iterations with the larger time steps. Figure 4 gives the time-displacement diagrams.

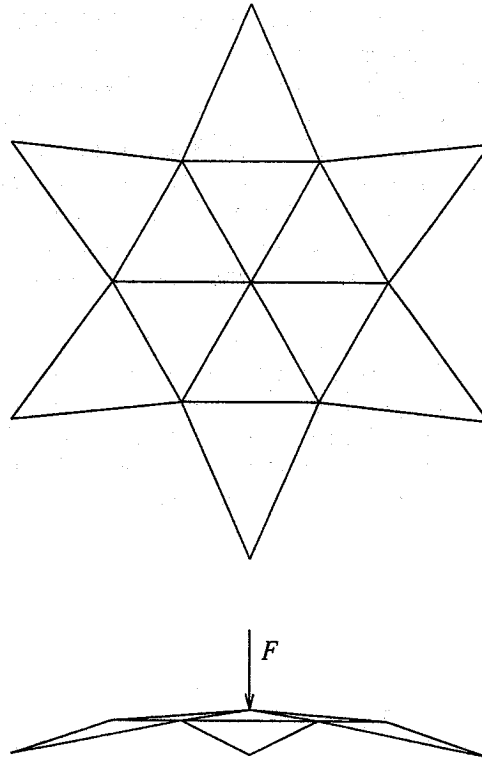


Figure 3. Space truss structure

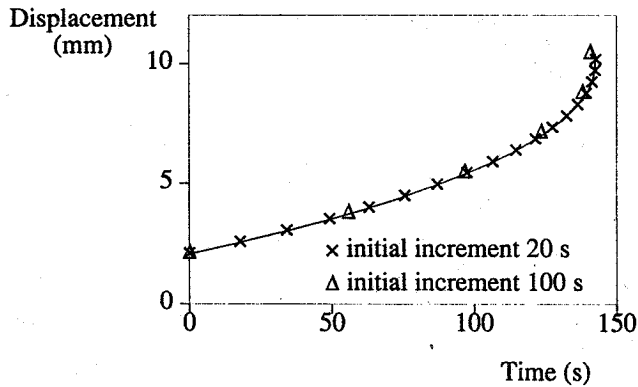


Figure 4. Time-displacement diagram for space truss structure of Figure 3

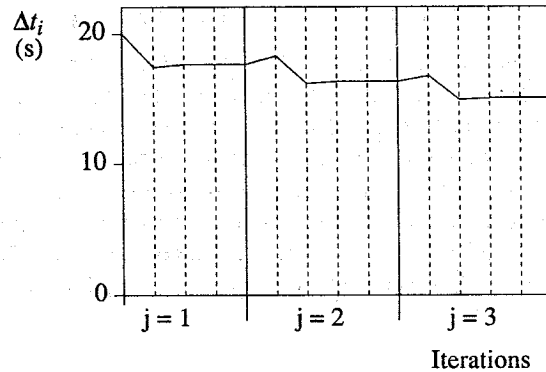


Figure 5. Calculated time increments during iterations; initial increment 20 s

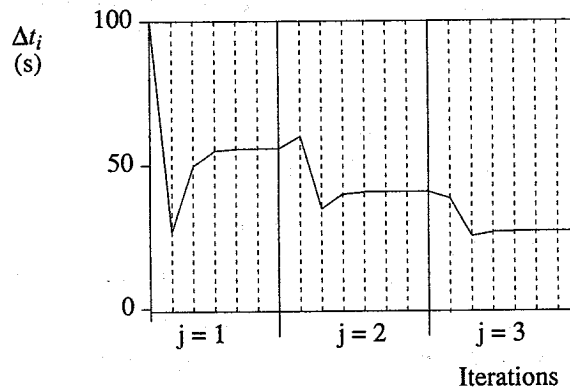


Figure 6. Calculated time increments during iterations; initial increment 100 s

In Figures 5 and 6 the calculated time increments during the iterations of the first three time increments are presented for the space truss example, for initial time increments of 20 and 100 s, respectively. We observe that, especially for the larger initial time step in Figure 6, an oscillatory behaviour is computed in the first few iterations. The too strong reduction in the time increment that is computed in the second iteration of each time step is caused by the linearization procedure used to compute the value of the time step. Specifically, the substitution of $\Delta t_0 = 0$ in the derivative $\partial \Delta \sigma / \partial \Delta t$ of equation (5) causes a too high relaxation, which in turn leads to an underestimation of the desired time increment (see also equation (25)). Under certain conditions a negative value of the time increment can be computed, e.g. in viscoelasticity when the initial value of the time step is in the order of the relaxation time. It is emphasized that computation of this negative value for the time increment can be corrected in later iterations and is different from the case discussed in the Introduction, namely that computation of a negative time increment in the final iteration marks the onset of a dynamic process. If, for the latter case, the constitutive model does not allow negative values of the time increment, a lower bound on the time step can be imposed in the calculations. This just violates the arc-length constraint (e.g. equation (17)), but has no influence on the accuracy of the computed structural response.

5. CONCLUDING REMARKS

An algorithm has been derived that allows for the adaptation of time increments during equilibrium iterations within a time step. The algorithm can be used in situations where large time steps can be used in the beginning of the analysis but where smaller time steps must be used at a later stage in the analysis. This occurs in particular when time-dependent material non-linearities play an important role in the structural response, e.g. in structural collapse of viscoelastic members due to geometric destabilization, or in crack propagation problems under thermal loadings. Then, application of the method leads to a significant reduction of user intervention in non-linear, time-dependent finite element calculations.

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