

Characterization of static feedback realizable transfer functions for nonlinear control systems

Citation for published version (APA):

Huijberts, H. J. C. (1997). *Characterization of static feedback realizable transfer functions for nonlinear control systems*. (RANA : reports on applied and numerical analysis; Vol. 9705). Technische Universiteit Eindhoven.

Document status and date:

Published: 01/01/1997

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.tue.nl/taverne

Take down policy

If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computing Science

RANA 97-05
May 1997

Characterization of static feedback
realizable transfer functions for
nonlinear control systems

by

H.J.C. Huijberts



Reports on Applied and Numerical Analysis
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven
The Netherlands
ISSN: 0926-4507

Characterization of static feedback realizable transfer functions for nonlinear control systems*

H.J.C. Huijberts
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven
The Netherlands
Email: hjch@win.tue.nl

Abstract

This paper gives a characterization of all transfer functions that may be realized via regular static state feedback for a SISO nonlinear control system. It is shown that the question whether there exists a feedback realizable transfer function with a given number of zeros may be reduced to a well known problem from real algebraic geometry. As a byproduct, solvability conditions for the linear model matching problem via static state feedback, as well as conditions for the existence of linear subsystems of not necessarily maximal dimension are obtained.

1 Introduction

In this paper we consider an analytic SISO control system Σ of the form

$$\Sigma \begin{cases} \dot{x} &= f(x) + g(x)u & , x \in \mathbb{R}^n, u \in \mathbb{R} \\ y &= h(x) & , y \in \mathbb{R} \end{cases} \quad (1)$$

around a point $x_0 \in \mathbb{R}^n$. Let a strictly proper transfer function $g(s) = \frac{v(s)}{w(s)} \in \mathbb{R}(s)$ be given, with $v(s) = \sum_{k=1}^{d+1} c_k s^{k-1}$, $w(s) = s^{\bar{n}} + \sum_{k=1}^{\bar{n}} b_k s^{k-1}$. Then $g(s)$ is said to be *static feedback realizable* for Σ around x_0 , if for Σ around x_0 there exists a regular static state feedback $Q_s : u = \alpha(x) + \beta(x)v$ such that the input-output behavior of $\Sigma \circ Q_s$ around x_0 is described by $g(s)$, i.e., given v , the output y of $\Sigma \circ Q_s$ satisfies the linear differential equation $y^{(\bar{n})} + \sum_{k=1}^{\bar{n}} b_k y^{(k-1)} = \sum_{k=1}^{d+1} c_k v^{(k-1)}$. It is the purpose of this paper to give a characterization of *all* static feedback realizable transfer functions for Σ .

The main motivation for studying this problem is in the areas of input-output linearization and linear model matching of nonlinear control systems. Input-output linearization methods are among the most commonly used methods in practical nonlinear control systems design. Conditions for the existence of input-output linearizing feedbacks are known for a relatively long time (see [13],[11]). A drawback of the static input-output linearizing feedback proposed

*Completely revised version of RANA-report 96-09. This paper is to appear in the International Journal of Robust and Nonlinear Control.

in [13],[11] is that it always results in a closed loop system with transfer function $g(s) = \frac{1}{s^r}$, where r denotes the relative degree of Σ . When using input-output linearization techniques in practical nonlinear control systems design, however, the input-output linearization *per se* is not the only thing that counts. Indeed, it is equally important, or maybe even more important, to know whether linear input-output behaviors with desirable characteristics (e.g. stable poles, stable zeros, \dots) may be realized. This automatically leads to the question whether, for a given nonlinear control system, a characterization of all feedback realizable transfer functions can be given. To the best of our knowledge, the results presented in this paper give a first complete answer to this question for the case of static state feedback.

The problem of linear model matching has received quite some attention in the literature (see [11],[12],[15] and the references therein). Roughly, this problem may be stated as follows: given a transfer function $g(s)$, does there exist a static or dynamic state feedback for Σ such that the input-output behavior of Σ after feedback is described by $g(s)$. In [11] necessary and sufficient conditions for the solvability of the linear model matching problem via *dynamic* state feedback were given. However, the dimension of the dynamic feedbacks proposed in [11] to solve the linear model matching problem may be unnecessarily high. This is due to the fact that typically these dynamic state feedbacks contain, amongst others, a realization of $g(s)$. This raises the question whether the minimal dimension of a dynamic state feedback solving the linear model matching problem may be characterized. The results in this paper may be used to give a partial answer to this question, in that an answer is given to the question whether there exists a *static* state feedback (or, alternatively, a dynamic state feedback of dimension zero) solving the linear model matching problem. An alternative approach to the linear model matching problem via static state feedback may be found in [16].

The organization of the paper is as follows. In the next section we will introduce some notations, concepts and results that will be used in the rest of the paper. In Section 3 necessary and sufficient conditions for the existence of a static feedback realizable transfer function with a given number of zeros will be derived. In Section 4 it is shown that these conditions may be checked by reducing them to a well known problem from real algebraic geometry. In Section 5, some conclusions are drawn. Throughout, the different steps in the exposition will be illustrated by means of one example.

2 Preliminaries

2.1 Relative degree of one-forms

In this subsection we give a differential-geometric treatment of the relative degree of one-forms. The concept of relative degree of a one-form was introduced in [2] in an algebraic framework. Define the manifold $M_0 := \mathbb{R}^n$ with local coordinates x , and the manifolds $M_k := M_{k-1} \times \mathbb{R}$ with local coordinates $(x, u, \dots, u^{(k-1)})$ ($k = 1, \dots, 2n+1$). Clearly, M_k is an embedded submanifold of M_ℓ ($k = 0, \dots, 2n$; $\ell = k+1, \dots, 2n+1$), with the natural embedding $i_{k\ell} : M_k \rightarrow M_\ell$ defined by $i_{k\ell}(x, u, \dots, u^{(k-1)}) = (x, u, \dots, u^{(k-1)}, 0, \dots, 0)$. Let Ξ_k denote the codistribution $\text{span}\{dx\}$ on M_k ($k = 0, \dots, 2n+1$). On M_{2n+1} , we define the extended vector field

$$f^e := (f + gu) \frac{\partial}{\partial x} + \sum_{i=0}^{2n} u^{(i+1)} \frac{\partial}{\partial u^{(i)}} \quad (2)$$

For a one-form ω on M_k ($k = 0, \dots, n+1$), we define $\omega^{(\ell)}$ on M_{2n+1} by

$$\omega^{(\ell)} := \mathcal{L}_{f_e}^\ell((i_{k2n+1})_*\omega) \quad (\omega \in M_k; k = 0, \dots, n+1; \ell = 0, \dots, 2n+1-k) \quad (3)$$

Then $\omega^{(\ell)}$ may be interpreted as a one-form on $M_{k+\ell}$, in the sense that

$$(i_{k+\ell2n+1})_*(i_{k+\ell2n+1})^*\omega^{(\ell)} = \omega^{(\ell)}$$

$$(\omega \in M_k; k = 0, \dots, n+1; \ell = 0, \dots, 2n+1-k)$$

Let $\omega \in \Xi_k$ ($k = 0, \dots, n$), and assume that there exists an $\ell \in \{1, \dots, n\}$ such that $\omega^{(\ell)} \notin \Xi_{2n+1}$. Then the smallest such ℓ is called the *relative degree* of ω , to be denoted by r_ω . If for all $\ell \in \{1, \dots, n\}$ we have that $\omega^{(\ell)} \in \Xi_{2n+1}$, we define $r_\omega := +\infty$. For a function ϕ satisfying $d\phi \in \Xi_k$, we define its relative degree by $r_\phi := r_{d\phi}$. Define the codistributions \mathcal{H}_k^ℓ ($k = 1, \dots, n; \ell = k-1, \dots, 2n+1-k$) by

$$\mathcal{H}_k^\ell := \{\omega \in \Xi_\ell \mid r_\omega \geq k\} \quad (4)$$

It may then be shown that \mathcal{H}_k^ℓ may be identified with \mathcal{H}_k^{k-1} , in the sense that

$$(i_{k-1\ell})_*(i_{k-1\ell})^*\mathcal{H}_k^\ell = (i_{k-1\ell})_*\mathcal{H}_k^{k-1} \quad (k = 1, \dots, n; \ell = k-1, \dots, 2n+1-k) \quad (5)$$

We further define the codistribution \mathcal{H}_∞^n on M_n by

$$\mathcal{H}_\infty^n := \{\omega \in \Xi_n \mid r_\omega = +\infty\} \quad (6)$$

Next, define

$$\mathcal{H}_k := (i_{k-12n+1})_*\mathcal{H}_k^{k-1} \quad (k = 1, \dots, n) \quad (7)$$

$$\mathcal{H}_\infty := (i_{n2n+1})_*\mathcal{H}_\infty^n \quad (8)$$

We then have the following properties (for a proof, see (mutatis mutandis) [2]).

Lemma 2.1 *Let $x_0 \in \mathbb{R}^n$ be given, and assume that the codistributions \mathcal{H}_k ($k \in \{1, \dots, n, \infty\}$) have constant dimension around $(x_0, 0, \dots, 0)$. Then around x_0 these codistributions have the following properties.*

- (i) $\mathcal{H}_1 \supset \mathcal{H}_2 \supset \dots \supset \mathcal{H}_n \supset \mathcal{H}_\infty$.
- (ii) \mathcal{H}_∞ is integrable.
- (iii) Σ is strongly accessible if and only if $\mathcal{H}_\infty = \{0\}$.
- (iv) $\mathcal{H}_k = \{\omega \in \mathcal{H}_{k-1} \mid ((i_{k-22n+1})^*\omega)^{(1)} \in \mathcal{H}_k\}$ ($k = 1, \dots, n$).
- (v) $\mathcal{H}_\infty = \{\omega \in \mathcal{H}_n \mid ((i_{n-12n+1})^*\omega)^{(1)} \in \mathcal{H}_n\}$.
- (vi) Define

$$\sigma := n+1 - \dim(\mathcal{H}_\infty) \quad (9)$$

Then

$$\dim(\mathcal{H}_k) = n+1-k \quad (k = 1, \dots, \sigma) \quad (10)$$

and

$$\mathcal{H}_k = \mathcal{H}_\infty \quad (k = \sigma, \dots, n) \quad (11)$$

(vii) Let $\lambda \in \mathcal{H}_{\sigma-1} \setminus \mathcal{H}_\infty$. Then we have for $k \in \{1, \dots, \sigma-1\}$:

$$\mathcal{H}_k = \mathcal{H}_\infty \oplus \text{span}\{((i_{n-2n+1})^* \lambda)^{(\ell)} \mid \ell = 0, \dots, \sigma-1-k\} \quad (12)$$

(viii) \mathcal{H}_k ($k \in \{1, \dots, n, \infty\}$) is invariant under regular static state feedback. \blacksquare

Example Consider on $\{x \in \mathbb{R}^4 \mid x_1 > 0\}$ the SISO system Σ given by

$$\Sigma \begin{cases} \dot{x}_1 &= x_1 x_2 - x_1 \\ \dot{x}_2 &= 2x_2 - x_2^2 - 1 + \frac{1}{x_1} u \\ \dot{x}_3 &= 3x_1 + x_3 - 3x_1^2 - 2x_1 x_2 + 2x_1^2 x_2 \\ \dot{x}_4 &= -x_1^4 + x_1^3 + 2x_1^2 - x_3^2 - x_1 x_3 + 2x_1^2 x_3 \\ y &= x_1 x_2 \end{cases} \quad (13)$$

Using Lemma 2.1.(iv),(v), we find for Σ that $\mathcal{H}_1 = \text{span}\{dx\}$, $\mathcal{H}_2 = \text{span}\{dx_1, dx_3, dx_4\}$, $\mathcal{H}_3 = \text{span}\{2(1-x_1)dx_1 + dx_3, dx_4\}$, $\mathcal{H}_4 = \text{span}\{\omega\}$, $\mathcal{H}_\infty = \{0\}$, where ω is given by

$$\omega := 6(x_1^3 - 3x_1^2 + 2x_1 - x_1 x_3 + x_3)dx_1 + 3(2x_1 - x_1^2 + x_3)dx_3 + dx_4 \quad (14)$$

Thus, it follows from Lemma 2.1.(iii) that Σ is strongly accessible. Further, note that the codistribution \mathcal{H}_4 is integrable. By [2],[14], this implies that Σ is feedback linearizable. However, we will see in the sequel that Σ is not static feedback equivalent to a linear system (i.e. a system that is linear both from a state space and an input-output point of view).

2.2 Parametrized post compensated system

In the sequel, the notion of a parametrized post compensated system will be of key importance. In this subsection we introduce this notion, and give some properties. Consider an analytic SISO system Σ of the form (1), and let $d \in \mathbb{N}$ be given. Let s_1, \dots, s_d be parameters that take their values in \mathbb{R} . We then define a *parametrized post compensated system* $\Sigma^p\langle s_1, \dots, s_d \rangle$ by

$$\Sigma^p\langle s_1, \dots, s_d \rangle \begin{cases} \dot{x} &= f(x) + g(x)u \\ \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_{d-1} &= z_d \\ \dot{z}_d &= h(x) - \sum_{k=1}^d s_k z_k \end{cases} \quad (15)$$

Similarly to what has been done in the previous subsection, one may define a sequence of parametrized codistributions $\mathcal{H}_k^p\langle s_1, \dots, s_d \rangle$ for $\Sigma^p\langle s_1, \dots, s_d \rangle$. Define $M := M_{2n+1}$, where M_{2n+1} has been defined in the previous subsection, and define $M^p := \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^{2(n+d)+1}$ with local coordinates $(x, z, u, \dots, u^{(2(n+d))})$. Clearly, M is an embedded submanifold of M^p with the natural embedding $i: M \rightarrow M^p$ defined by $i(x, u, \dots, u^{(2n)}) := (x, 0, u, \dots, u^{(2n)}, 0, \dots, 0)$. Further, let Ξ, Ξ^p denote the codistribution $\text{span}\{dx\}$ on M and M^p respectively. For $\Sigma^p\langle s_1, \dots, s_d \rangle$, we define the codistributions

$$\mathcal{H}_k^e := i_* \mathcal{H}_k \quad (k = 1, \dots, n) \quad (16)$$

$$\mathcal{H}_\infty^e := i_* \mathcal{H}_\infty \quad (17)$$

It then follows from the form of $\Sigma^p\langle s_1, \dots, s_d \rangle$ that

$$\forall_{s_1, \dots, s_d \in \mathcal{R}} \forall_{k \in \{1, \dots, n\}} \mathcal{H}_k^e \subset \mathcal{H}_k^p\langle s_1, \dots, s_d \rangle \quad (18)$$

$$\forall_{s_1, \dots, s_d \in \mathcal{R}} \forall_{k \in \{n+1, \dots, n+d, \infty\}} \mathcal{H}_\infty^e \subset \mathcal{H}_k^p\langle s_1, \dots, s_d \rangle \quad (19)$$

$$\forall_{s_1, \dots, s_d \in \mathcal{R}} \forall_{k \in \{1, \dots, n\}} \mathcal{H}_k^p\langle s_1, \dots, s_d \rangle \cap \Xi^p = \mathcal{H}_k^e \quad (20)$$

$$\forall_{s_1, \dots, s_d \in \mathcal{R}} \forall_{k \in \{n+1, \dots, n+d, \infty\}} \mathcal{H}_k^p\langle s_1, \dots, s_d \rangle \cap \Xi^p = \mathcal{H}_\infty^e \quad (21)$$

We now show that the codistributions $\mathcal{H}_k^p\langle s_1, \dots, s_d \rangle$ ($k = 1, \dots, \sigma$) may be parametrized in a polynomial way. Let \mathcal{A} denote the ring of analytic functions of $(x, u, \dots, u^{(2n)})$, and define the polynomial ring $\mathcal{R} := \mathcal{A}[s_1, \dots, s_d]$.

Lemma 2.2 *Consider the parametrized post compensated system $\Sigma^p\langle s_1, \dots, s_d \rangle$ and the sequence of parametrized codistributions $\mathcal{H}_k^p\langle s_1, \dots, s_d \rangle$ ($k = 1, \dots, \sigma$), where σ is defined in (9). Let $x_0 \in \mathbb{R}^n$ be given, and assume that the codistributions \mathcal{H}_k ($k = 1, \dots, n$) have constant dimension around $(x_0, 0, \dots, 0)$. Let $\lambda \in \mathcal{H}_n \setminus \mathcal{H}_\infty$ satisfy*

$$(i_{n-12n+1})_*(i_{n-12n+1})^*\lambda = \lambda \quad (22)$$

Define $r := r_h$. Then around $(x_0, 0, \dots, 0)$ we have that

$$\dim(\mathcal{H}_k^p\langle s_1, \dots, s_d \rangle) = \dim(\mathcal{H}_k^e) + d \quad (k = 1, \dots, \sigma) \quad (23)$$

and there exist $\phi_{k\ell} \in \mathcal{R}$ ($k = 1, \dots, d$; $\ell = 0, \dots, \sigma - r - d - 2 + k$) such that

$$\begin{aligned} \mathcal{H}_k^p\langle s_1, \dots, s_d \rangle &= \mathcal{H}_\infty^e \oplus \text{span}\{i_*\omega_k\langle s_1, \dots, s_d \rangle - dz_k \mid k = 1, \dots, d\} \\ &\quad (k = 1, \dots, \sigma) \end{aligned} \quad (24)$$

where

$$\omega_k := \sum_{\ell=0}^{\sigma-r-d-2+k} \phi_{k\ell} \lambda^{(\ell)} \quad (25)$$

Proof Equality (23) follows straightforwardly from Lemma 2.1 and (18), ..., (21). It then follows from (19), (21), (23) that there exist parametrized one-forms $\tilde{\omega}_k\langle s_1, \dots, s_d \rangle \in \Xi^p$ ($k = 1, \dots, d$) such that

$$\mathcal{H}_\sigma^p\langle s_1, \dots, s_d \rangle = \mathcal{H}_\infty^e \oplus \text{span}\{\tilde{\omega}_k\langle s_1, \dots, s_d \rangle - dz_k \mid k = 1, \dots, d\} \quad (26)$$

From Lemma 2.1.(i) and (18), (20), (26) it then follows that

$$\begin{aligned} \mathcal{H}_\ell^p\langle s_1, \dots, s_d \rangle &= \mathcal{H}_\ell^e \oplus \text{span}\{\tilde{\omega}_k\langle s_1, \dots, s_d \rangle - dz_k \mid k = 1, \dots, d\} \\ &\quad (\ell = 1, \dots, \sigma) \end{aligned} \quad (27)$$

What remains to be shown is that $\tilde{\omega}_k = i_*\omega_k$ ($k = 1, \dots, d$), where the ω_k are of the form (25). We give the proof for $d = 2$. The proof for $d > 2$ is analogous. Since $r_h = r$, there exist $\alpha_0, \dots, \alpha_{\sigma-1-r} \in \mathcal{A}$ such that $\alpha_{\sigma-1-r} \neq 0$, and

$$dh = \sum_{\ell=0}^{\sigma-1-r} \alpha_\ell \lambda^{(\ell)} \quad (28)$$

From Lemma 2.1.(iv) and (27) it follows that

$$\dot{\tilde{\omega}}_1 - dz_1 = \dot{\tilde{\omega}}_1 - \tilde{\omega}_2 + (\tilde{\omega}_2 - dz_2) \in \mathcal{H}_{\sigma-1}^p \langle s_1, s_2 \rangle \quad (29)$$

and

$$\begin{aligned} \dot{\tilde{\omega}}_2 - dz_2 &= \dot{\tilde{\omega}}_2 + s_1 \tilde{\omega}_1 + s_2 \tilde{\omega}_2 - dh - \\ & s_1(\tilde{\omega}_1 - dz_1) - s_2(\tilde{\omega}_2 - dz_2) \in \mathcal{H}_{\sigma-1}^p \langle s_1, s_2 \rangle \end{aligned} \quad (30)$$

Let \mathcal{A}^p denote the ring of analytic functions of $(x, z, u, \dots, u^{(2(n+d))})$. With Lemma 2.1.(vii) it follows from (29),(30) that there exist parametrized functions $\beta_1 \langle s_1, s_2 \rangle, \beta_2 \langle s_1, s_2 \rangle$ satisfying $\beta_1 \langle s_1, s_2 \rangle, \beta_2 \langle s_1, s_2 \rangle \in \mathcal{A}^p, (\forall s_1, s_2 \in \mathbb{R})$ and parametrized one-forms $\pi_1 \langle s_1, s_2 \rangle, \pi_2 \langle s_1, s_2 \rangle$ satisfying $\pi_1 \langle s_1, s_2 \rangle, \pi_2 \langle s_1, s_2 \rangle \in \mathcal{H}_{\infty}^e, (\forall s_1, s_2 \in \mathbb{R})$ such that

$$\dot{\tilde{\omega}}_1 = \tilde{\omega}_2 + \beta_1(i_*\lambda) + \pi_1 \quad (31)$$

$$\dot{\tilde{\omega}}_2 = dh - s_1 \tilde{\omega}_1 - s_2 \tilde{\omega}_2 + \beta_2(i_*\lambda) + \pi_2 \quad (32)$$

From (31),(32) it follows in particular that $r_{\tilde{\omega}_1} = r + 2, r_{\tilde{\omega}_2} = r + 1$, and hence there exist parametrized functions $\tilde{\phi}_{k\ell} \langle s_1, s_2 \rangle$ ($k = 1, 2; \ell = 0, \dots, \sigma - 4 - r + k$) and parametrized one-forms $\eta_1 \langle s_1, s_2 \rangle, \eta_2 \langle s_1, s_2 \rangle$ such that

$$\forall_{s_1, s_2 \in \mathbb{R}} \eta_1 \langle s_1, s_2 \rangle, \eta_2 \langle s_1, s_2 \rangle \in \mathcal{H}_{\infty}^e \quad (33)$$

$$\forall_{s_1, s_2 \in \mathbb{R}} \forall_{k \in \{1, 2\}} \forall_{\ell \in \{0, \dots, \sigma - 4 - r + k\}} \tilde{\phi}_{k\ell} \in \mathcal{A}^p \quad (34)$$

$$\tilde{\omega}_k = \sum_{\ell=0}^{\sigma-4-r+k} \tilde{\phi}_{k\ell}(i_*\lambda)^{(\ell)} + \eta_k \quad (k = 1, 2) \quad (35)$$

Comparing (28),(31),(32),(35) we then obtain:

$$\dot{\tilde{\phi}}_{10} - \tilde{\phi}_{20} = \beta_1 \quad (36)$$

$$\dot{\tilde{\phi}}_{1\ell} + \tilde{\phi}_{1\ell-1} - \tilde{\phi}_{2\ell} = 0 \quad (\ell = 1, \dots, \sigma - 3 - r) \quad (37)$$

$$\tilde{\phi}_{1\sigma-3-r} - \tilde{\phi}_{2\sigma-2-r} = 0 \quad (38)$$

$$\dot{\tilde{\phi}}_{20} - s_1 \tilde{\phi}_{10} - s_2 \tilde{\phi}_{20} = \alpha_0 + \beta_2 \quad (39)$$

$$\dot{\tilde{\phi}}_{2\ell} + \tilde{\phi}_{2\ell-1} - s_1 \tilde{\phi}_{1\ell} - s_2 \tilde{\phi}_{2\ell} = \alpha_{\ell} \quad (\ell = 1, \dots, \sigma - 3 - r) \quad (40)$$

$$\dot{\tilde{\phi}}_{2\sigma-2-r} + \tilde{\phi}_{2\sigma-3-r} - s_2 \tilde{\phi}_{2\sigma-2-r} = \alpha_{\sigma-2-r} \quad (41)$$

$$\tilde{\phi}_{2\sigma-2-r} = \alpha_{\sigma-1-r} \quad (42)$$

From (38),(42) it follows that

$$\tilde{\phi}_{1\sigma-3-r} = \tilde{\phi}_{2\sigma-2-r} = \alpha_{\sigma-1-r} \in \mathcal{A} \subset \mathcal{R} \quad (43)$$

Equalities (41),(43) then give

$$\tilde{\phi}_{2\sigma-3-r} = \alpha_{\sigma-2-r} - \dot{\tilde{\phi}}_{2\sigma-2-r} + s_2 \tilde{\phi}_{2\sigma-2-r} \in \mathcal{R} \quad (44)$$

Using an induction argument, it then follows from (37),(40),(43),(44) that

$$\tilde{\phi}_{k\ell} \in \mathcal{R} \quad (k = 1, 2; \ell = 1, \dots, \sigma - 4 - r + k) \quad (45)$$

It further follows from (36),(39) that $\tilde{\phi}_{10}, \tilde{\phi}_{20}$ are arbitrary. Together with (45), this establishes our claim. \blacksquare

Example (continued) Consider Σ given by (13). For $\Sigma^p\langle s \rangle$ we find

$$\begin{aligned} \mathcal{H}_5^p\langle s \rangle = \text{span}\left\{ -\frac{(s^2-1)}{4(2x_1+x_3-x_1^2)}\omega + (2(1-x_1)s - 1 + 2x_1)dx_1 + \right. \\ \left. (s-1)dx_3 - dz \right\} \end{aligned} \quad (46)$$

where ω is defined in (14). Considering the post compensated system $\Sigma^p\langle s_1, s_2 \rangle$, we obtain

$$\begin{aligned} \mathcal{H}_5^p\langle s_1, s_2 \rangle = \text{span}\left\{ \frac{s_2}{4(2x_1+x_3-x_1^2)}\omega - 2(1-x_1)dx_1 - dx_3 - dz_1, \right. \\ \left. \frac{-s_2^2+s_1+1}{4(2x_1+x_3-x_1^2)}\omega + (2(1-x_1)s_2 - 1 + 2x_1)dx_1 + (s_2-1)dx_3 - dz_2 \right\} \end{aligned} \quad (47)$$

For the post compensated system $\Sigma^p\langle s_1, s_2, s_3 \rangle$, we have

$$\begin{aligned} \mathcal{H}_5^p\langle s_1, s_2, s_3 \rangle = \text{span}\left\{ -\frac{1}{4(2x_1+x_3-x_1^2)}\omega - dz_1, \right. \\ \left. \frac{s_3}{4(2x_1+x_3-x_1^2)}\omega - 2(1-x_1)dx_1 - dx_3 - dz_2, \right. \\ \left. \frac{-s_3^2+s_2+1}{4(2x_1+x_3-x_1^2)}\omega + (2(1-x_1)s_3 - 1 + 2x_1)dx_1 + (s_3-1)dx_3 - dz_3 \right\} \end{aligned} \quad (48)$$

3 Necessary and sufficient conditions

In this section we derive necessary and sufficient conditions for the existence of a feedback realizable transfer function with a given number of zeros for a *strongly accessible* SISO system. We consider an analytic SISO system Σ of the form (1) around a point $x_0 \in \mathbb{R}^n$. We assume throughout that the relative degree $r := r_h$ of h is well-defined around x_0 , and that the codistributions \mathcal{H}_k ($k \in \{1, \dots, n, \infty\}$) have constant dimension around x_0 .

We start with some (rather trivial) observations. First note that it follows from [13],[11] that $g(s) = \frac{1}{s^r}$ is feedback realizable around x_0 . Further, it follows from the fact that the relative degree is invariant under regular static state feedback (cf. [11]), that any feedback realizable transfer function $g(s) = \frac{v(s)}{w(s)}$ satisfies $\deg(w) - \deg(v) = r$. Moreover, using arguments from linear control theory, it is easily shown that if $g(s) = \frac{v(s)}{w(s)}$ is feedback realizable, then every $\bar{g}(s) = \frac{\gamma v(s)}{\bar{w}(s)}$ satisfying $\gamma \in \mathbb{R} - \{0\}$ and $\deg(\bar{w}) = r + \deg(v)$ is feedback realizable.

We next state and prove our main result.

Theorem 3.1 Consider a strongly accessible SISO system of the form (1) around x_0 . Let $d \in \{1, \dots, n-r\}$ be given. Consider the parametrized post compensated system $\Sigma^p \langle s_1, \dots, s_d \rangle$ and the sequence of parametrized codistributions $\mathcal{H}_k^p \langle s_1, \dots, s_d \rangle$. Then there exists a feedback realizable transfer function with d zeros for Σ around x_0 if and only if there exist $a_1, \dots, a_d \in \mathbb{R}$ such that around x_0 we have

$$\mathcal{H}_\infty^p \langle a_1, \dots, a_d \rangle = \mathcal{H}_{n+1}^p \langle a_1, \dots, a_d \rangle \quad (49)$$

Proof (necessity) Assume that there exists a feedback realizable transfer function $g(s) = \frac{v(s)}{w(s)}$ with d zeros around x_0 , and assume that the static state feedback Q_s realizes $g(s)$. From the observations above, it follows that we may assume without loss of generality that $w(s) = s^{r+d}$. Write $v(s) = \sum_{k=1}^{d+1} c_k s^{k-1}$, where $c_{d+1} \neq 0$. Consider the realization (A, B, C) of $g(s)$, with (A, B) in Brunovsky canonical form, and $C = (c_1 \ \dots \ c_{d+1} \ 0 \ \dots \ 0)$. We then have that there exist new coordinates $\bar{x} = (\bar{x}_1(x), \bar{x}_2(x))$ for $\Sigma \circ Q_s$ around x_0 , such that in these new coordinates $\Sigma \circ Q_s$ takes the form

$$\Sigma \circ Q_s \begin{cases} \dot{\bar{x}}_1 &= A\bar{x}_1 + Bv, \quad \dot{\bar{x}}_2 = a(x) + b(x)v \\ y &= C\bar{x}_1 \end{cases} \quad (50)$$

Consider the post compensated system $(\Sigma \circ Q_s)^p \langle \frac{c_1}{c_{d+1}}, \dots, \frac{c_d}{c_{d+1}} \rangle$, and define new coordinates (\bar{x}, ξ) for this system, with $\xi_i := z_i - c_{d+1} \bar{x}_{1i}$ ($i = 1, \dots, d$). It is then straightforwardly checked that in these new coordinates we have $\dot{\xi}_i = \xi_{i+1}$ ($i = 1, \dots, d-1$), and $\dot{\xi}_d = -\sum_{k=1}^d \frac{c_k}{c_{d+1}} \xi_k$. This implies that

$$\mathcal{H}_\infty^p \langle \frac{c_1}{c_{d+1}}, \dots, \frac{c_d}{c_{d+1}} \rangle = \text{span}\{d\xi_1, \dots, d\xi_d\} \quad (51)$$

From Lemmas 2.1.(viii), 2.2 and the fact that $\mathcal{H}_\infty^p \langle \frac{c_1}{c_{d+1}}, \dots, \frac{c_d}{c_{d+1}} \rangle \subset \mathcal{H}_{n+1}^p \langle \frac{c_1}{c_{d+1}}, \dots, \frac{c_d}{c_{d+1}} \rangle$ it then follows that there exist $a_1, \dots, a_d \in \mathbb{R}$ such that (49) holds.

(sufficiency) Assume that there exist $a_1, \dots, a_d \in \mathbb{R}$ such that (49) holds. It then follows from Lemma 2.2 that there exist one-forms $\omega_1, \dots, \omega_d \in \text{span}\{dx\}$ such that

$$\mathcal{H}_\infty^p \langle a_1, \dots, a_d \rangle = \text{span}\{\omega_1 - dz_1, \dots, \omega_d - dz_d\} \quad (52)$$

and

$$d\omega_i \in \text{span}\{\pi \wedge \rho \mid \pi, \rho \in \text{span}\{dx, du, \dots, du^{(2n)}\}\} \quad (i = 1, \dots, d) \quad (53)$$

From (52), Lemma 2.1.(v) and the form of $\Sigma^p \langle a_1, \dots, a_d \rangle$ it follows that $\dot{\omega}_i = \omega_{i+1}$ ($i = 1, \dots, d-1$) and $dh = \dot{\omega}_d + \sum_{k=1}^d a_k \omega_k$. Combining these equalities, we obtain

$$dh = \omega_1^{(d)} + \sum_{k=1}^d a_k \omega_1^{(k-1)} \quad (54)$$

We next show that ω_1 is exact. From Lemma 2.1.(ii) we know that $\mathcal{H}_\infty^p \langle a_1, \dots, a_d \rangle$ is integrable. By the Frobenius Theorem, this implies in particular that

$$\begin{aligned} 0 &= d(\omega_1 - dz_1) \wedge (\omega_1 - dz_1) \wedge \dots \wedge (\omega_d - dz_d) = \\ &= d\omega_1 \wedge (\omega_1 - dz_1) \wedge \dots \wedge (\omega_d - dz_d) \end{aligned} \quad (55)$$

From (53),(55) it then follows that $d\omega_1 = 0$, and hence, by Poincaré's Lemma, ω_1 is (locally) exact. Let \bar{x}_{11} be such that $\omega_1 = d\bar{x}_{11}$. It follows from Lemma 2.2 that $r_{\bar{x}_{11}} = r + d$. Defining $\bar{x}_{1k} := \mathcal{L}_f^{k-1}\bar{x}_{11}$ ($k = 2, \dots, r + d$), this then gives that the differentials $d\bar{x}_{11}, \dots, d\bar{x}_{1r+d}$ are linearly independent, and that $\dot{\bar{x}}_{1r+d} = a(x) + b(x)u$, where $b(x) \neq 0$. Further, it follows from (54) that $y = \sum_{k=1}^d a_k \bar{x}_{1k} + \bar{x}_{1d+1}$. Defining the static state feedback $Q_s : u = b(x)^{-1}(v - a(x))$, it is then established that the input-output behavior of $\Sigma \circ Q_d$ is described by $g(s) = \frac{v(s)}{s^{r+d}}$, where $v(s) = s^d + \sum_{k=1}^d a_k s^{k-1}$. Hence there exists a feedback realizable transfer function with d zeros. ■

Corollary 3.2 *Consider a strongly accessible SISO system of the form (1) around x_0 . Consider for $d \in \{1, \dots, n - r\}$ the parametrized post compensated system $\Sigma^p(s_1, \dots, s_d)$ and the sequence of parametrized codistributions $\mathcal{H}_k^p(s_1, \dots, s_d)$. Then the feedback realizable transfer functions for Σ are given by $g(s) = \frac{\gamma_d v_d(s)}{w_d(s)}$, where $d \in \{0, \dots, n - r\}$, $\gamma_d \in \mathbb{R} - \{0\}$, $w_d \in \mathbb{R}[s]$, $\deg(w_d) = r + d$, and $v_d(s) = s^d + \sum_{k=1}^d a_k s^{k-1}$, where $a_1, \dots, a_d \in \mathbb{R}$ satisfy (49). ■*

From Theorem 3.1 and Corollary 3.2 we obtain the following result concerning the linear model matching problem.

Corollary 3.3 *Consider a strongly accessible SISO system of the form (1) around x_0 . Let a strictly proper transfer function $g(s) = \frac{v(s)}{w(s)}$, be given, where $v(s) = \sum_{k=1}^{d+1} c_k s^{k-1}$, $c_{d+1} \neq 0$, and $\deg(w) = r + d$. Then the linear model matching problem via static state feedback is solvable for Σ and $g(s)$ if and only if*

$$\mathcal{H}_\infty^p \left\langle \frac{c_1}{c_{d+1}}, \dots, \frac{c_d}{c_{d+1}} \right\rangle = \mathcal{H}_{n+1}^p \left\langle \frac{c_1}{c_{d+1}}, \dots, \frac{c_d}{c_{d+1}} \right\rangle \quad (56)$$

Remark 3.4 (i) Let $d \in \{1, \dots, n - r\}$ be given. Checking the proof of Theorem 3.1, one sees that for a strongly accessible system Σ there exists a feedback realizable transfer function with d zeros if and only if there exist a function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ and $a_1, \dots, a_d \in \mathbb{R}$ such that $r_\phi = r + d$ and $h = \mathcal{L}_f^d \phi + \sum_{k=1}^d a_k \mathcal{L}_f^{k-1} \phi$. Rewriting the first equality as $\mathcal{L}_\tau \phi = 0$ ($\forall \tau \in \mathcal{H}_{r+d}^1$) one obtains a set of nonlinear PDE's for ϕ . The integrability conditions for this set of PDE's are given by (49).

(ii) Note that in the necessity-part of the proof of Theorem 3.1, we did not use the assumption that Σ is strongly accessible. Thus, the existence of $a_1, \dots, a_d \in \mathbb{R}$ such that (49) is satisfied is also a necessary condition for the existence of a feedback realizable transfer function with d zeros when Σ is not strongly accessible. However, it is not a sufficient condition. This raises the question what extra integrability conditions are needed in the case of not necessarily strongly accessible systems. This remains a topic for future research.

(iii) In the literature, a system Σ is said to have a linear subsystem if there exist a regular static state feedback and new coordinates $\bar{x}(x) = (\bar{x}_1(x), \bar{x}_2(x))$, such that Σ after feedback and coordinate transformation takes the form (50). Some results on the existence of linear subsystems have appeared, for SISO-systems as well as for MIMO-systems. In [3], the question whether a MIMO system has a linear subsystem of dimension n was

addressed. In [9], SISO systems were considered, and sufficient conditions were given for the existence of a linear subsystem of dimension larger than the relative degree. In [20],[21] a characterization of the maximal dimension of a linear subsystem was given. The result of Theorem 3.1 may be used to check whether there exist linear subsystems of not necessarily maximal dimension.

Example (continued) We continue our study of the system Σ defined in (13). First note that the relative degree of h equals 1. From the observations at the beginning of this section it then follows that all transfer functions $g(s) = \frac{1}{w_1(s)}$, with $\deg(w_1) = 1$, are feedback realizable. We now check whether there are feedback realizable transfer functions with one zero. It may be checked that from (46) it follows that $a \in \mathbb{R}$ satisfies $\mathcal{H}_\infty^p\langle a \rangle = \mathcal{H}_5^p\langle a \rangle$ if and only if it satisfies the equalities

$$\begin{aligned} (a^2 - 1)((2x_1 - x_1^2 + x_3)a + (x_1^2 - x_1 - x_3)) &= 0 \\ (a^2 - 1)((2x_1 - x_1^2 + x_3)a + (-x_1^2 + x_1 + x_3)) &= 0 \end{aligned} \quad (57)$$

It is easily seen that this implies that either $a = 1$ or $a = -1$. Thus all transfer functions $g(s) = \frac{s \pm 1}{w_2(s)}$, with $\deg(w_2) = 2$ are feedback realizable.

We next check whether there exist feedback realizable transfer functions with two zeros. From (47) it follows that $a_1, a_2 \in \mathbb{R}$ satisfy $\mathcal{H}_5^p\langle a_1, a_2 \rangle = \mathcal{H}_\infty^p\langle a_1, a_2 \rangle$ if and only if they satisfy the equalities

$$\begin{aligned} (2x_1 + x_3 - x_1^2)(-6a_2^2 + 5a_2 + 6a_1 + 6) - (4x_1 - x_3 + x_1^2)a_2 &= 0 \\ (2x_1 + x_3 - x_1^2)(-3a_2^2 + a_2 + 3a_1 + 3) - (x_1 + 2x_3 - 2x_1^2)a_2 &= 0 \\ (2x_1 + x_3 - x_1^2)(-a_2^2 + a_1 + 1) - (x_1 + x_3 - x_1^2)a_2 &= 0 \end{aligned} \quad (58)$$

Substituting $x_1 = 1, x_3 = 0$ in (58), this gives

$$\begin{aligned} -3a_2^2 + 2a_2 + 3a_1 + 3 &= 0 \\ -a_2^2 + a_1 + 1 &= 0 \end{aligned} \quad (59)$$

These equations have the unique solution $a_1 = -1, a_2 = 0$. It is readily seen that with these values also the equalities (58) are satisfied. Thus, all transfer functions $g(s) = \frac{s^2 - 1}{w_3(s)}$ with $\deg(w_3) = 3$ are feedback realizable.

Checking whether there exist $a_1, a_2, a_3 \in \mathbb{R}$ satisfying $\mathcal{H}_5^p\langle a_1, a_2, a_3 \rangle = \mathcal{H}_\infty^p\langle a_1, a_2, a_3 \rangle$, one obtains amongst others that a_3 has to satisfy

$$(2x_1 + x_3 - x_1^2)(3a_3 - 4) + (x_1 + x_3 - x_1^2) = 0 \quad (60)$$

Clearly, there does not exist an $a_3 \in \mathbb{R}$ satisfying this equality. Hence, in spite of the fact that Σ is feedback linearizable, there do not exist feedback realizable transfer functions with three zeros.

4 Reduction to an algebro-geometric problem

In the example we treated in the foregoing sections, we could check condition (49) in a relatively easy way. Amongst others, this was due to the fact that n and d were small. In this section we present a method to check the conditions of Theorem 3.1 that may be used for all values of n and d . For reasons of clarity of exposition, we first restrict to the case $d = 1$. At the end of the section we make some remarks about the case $d > 1$. Let $x_0 \in \mathbb{R}^n$

be given, and assume that Σ is strongly accessible around x_0 . Further, assume that the codistributions \mathcal{H}_k ($k = 1, \dots, n$) have constant dimension around $(x_0, 0, \dots, 0)$, and that the relative degree $r := r_h$ of h is well-defined around x_0 . Let $\lambda \in \mathcal{H}_n - \{0\}$ be such that (12),(22) hold. Then there exist $\alpha_0, \dots, \alpha_{n-r} \in \mathcal{A}$ such that $\alpha_{n-r} \neq 0$ and $dh = \sum_{\ell=0}^{n-r} \alpha_\ell \lambda^{(\ell)}$. Consider the parametrized post compensated system $\Sigma^p\langle s \rangle$. It then follows from Lemma 2.2 that there exist $\phi_\ell \in \mathcal{R}$ ($\ell = 0, \dots, n-r-1$) such that $\mathcal{H}_{n+1}^p\langle s \rangle = \text{span}\{\sum_{\ell=0}^{n-r-1} \phi_\ell\langle s \rangle \lambda^{(\ell)} - dz\}$. Define $\psi_0 := \dot{\phi}_0 + s\phi_0 - \alpha_0$, $\psi_\ell := \dot{\phi}_\ell + \phi_{\ell-1} + s\phi_\ell - \alpha_\ell$ ($\ell = 1, \dots, n-r-1$), and $\psi_{n-r} := \phi_{n-r-1} - \alpha_{n-r}$. Further, let $0_{\mathcal{A}}$ denote the zero-function. We now have the following result.

Theorem 4.1 *Consider a strongly accessible SISO system Σ of the form (1) around x_0 . Let $\psi_0, \dots, \psi_{n-r}$ be defined as above. Then there exists a feedback realizable transfer function with one zero for Σ around x_0 if and only if $\psi_0, \dots, \psi_{n-r}$ have a common real zero, i.e.,*

$$\exists a \in \mathbb{R} \quad \forall \ell \in \{0, \dots, n-r\} \quad \psi_\ell\langle a \rangle = 0_{\mathcal{A}} \quad (61)$$

Proof From Theorem 3.1 it follows that there exists a feedback realizable transfer function with one zero for Σ if and only if there exists an $a \in \mathbb{R}$ such that $\mathcal{H}_{n+1}\langle a \rangle = \mathcal{H}_\infty\langle a \rangle$. It is straightforwardly shown that this is equivalent to the existence of an $a \in \mathbb{R}$ such that $\frac{d}{dt}(\sum_{\ell=0}^{n-r-1} \phi_\ell\langle a \rangle \lambda^{(\ell)}) + a(\sum_{\ell=0}^{n-r-1} \phi_\ell\langle a \rangle \lambda^{(\ell)}) = dh$. It then easily follows that this is equivalent to (61). \blacksquare

We next show how (61) may be checked by reducing it to the question whether a set of polynomials in $\mathbb{R}[s]$ has a common real zero. Define $\xi := \text{col}(x, u, \dots, u^{(2n)}) \in \mathbb{R}^{3n+1}$, and let ν denote the maximal degree in s of the polynomials $\psi_0, \dots, \psi_{n-r}$. Then there exist functions $\psi_\ell^k \in \mathcal{A}$ such that

$$\psi_\ell\langle s \rangle(\xi) = \sum_{k=0}^{\nu} \psi_\ell^k(\xi) s^k \quad (\ell = 0, \dots, n-r) \quad (62)$$

Define the $(n-r+1, \nu+1)$ -matrix $P(\xi)$ with entries $P_{ij}(\xi) := \psi_i^j(\xi)$ ($i = 0, \dots, n-r$; $j = 0, \dots, \nu$). Further, define for $s \in \mathbb{R}$ the vector $v_s := \text{col}(1, s, \dots, s^\nu)$. Then the question to be considered is whether there exists a *real* solution to the equation $P(\xi)v_s \equiv 0$. Obviously, there exists a real solution to this equation *only if* there exists a $v \in \mathbb{R}^{\nu+1} - \{0\}$ satisfying the equation $P(\xi)v \equiv 0$. Note that this equation may be extended by the equations $(\partial/\partial \xi_i(P(\xi)))v \equiv 0$ ($i = 1, \dots, 3n+1$) and equations obtained by taking higher-order partial derivatives. Consider the following algorithm that performs this extension in a controlled way. The algorithm was suggested by [19], and is reminiscent of the Structure Algorithm ([12],[15]).

Algorithm 4.2

Step 0 Define $p^1 := n-r+1$, $q^1 := \nu+1$, $P^1(\xi) := P(\xi)$.

Step k Define $\rho^k := \text{rank}P^k(\xi)$. There exist an invertible (p^k, p^k) -matrix $Q^k(\xi)$ and a (q^k, q^k) -permutation matrix R^k such that

$$Q^k(\xi)P^k(\xi)R^k = \begin{pmatrix} I_{\rho^k} & \bar{P}^k(\xi) \\ 0 & 0 \end{pmatrix} \quad (63)$$

where \bar{P}^k is a $(\rho_k, q^k - \rho_k)$ -matrix. If either $\rho_k = q^k$, or $\bar{P}^k(\xi)$ is a constant matrix, we **STOP**. Otherwise, define $p^{k+1} := (3n + 1)\rho_k$, $q^{k+1} := q^k - \rho_k$, and

$$P^{k+1} := \begin{pmatrix} \frac{\partial \bar{P}^k}{\partial \xi_1} \\ \vdots \\ \frac{\partial \bar{P}^k}{\partial \xi_{3n+1}} \end{pmatrix} \quad (64)$$

and go to Step $k + 1$. ■

It may be shown that Algorithm 4.2 terminates in a finite number, say k^* of steps. We have the following results.

Lemma 4.3 *Assume that $q^{k^*} - \rho_{k^*} > 0$. Let for $k = 1, \dots, k^*$ the (q^k, ρ_k) -matrix \hat{R}^k and the $(q^k, q^k - \rho_k)$ -matrix \tilde{R}^k be such that*

$$R^k = \begin{pmatrix} \hat{R}^k & \tilde{R}^k \end{pmatrix} \quad (k = 1, \dots, k^*) \quad (65)$$

and define the matrices

$$S^k(\xi) := \tilde{R}^k - \hat{R}^k \bar{P}^k(\xi) \quad (k = 1, \dots, k^*) \quad (66)$$

Then the matrix $S(\xi)$ defined by

$$S(\xi) := S^1(\xi)S^2(\xi) \cdots S^{k^*}(\xi) \quad (67)$$

is constant and left-invertible.

Proof See Appendix. ■

Lemma 4.4 *Assume that there exists a $v \in \mathbb{R}^{\nu+1} - \{0\}$ such that $P(\xi)v \equiv 0$. Define the matrices*

$$T^k(\xi) := S^1(\xi) \cdots S^k(\xi) \quad (k = 1, \dots, k^*) \quad (68)$$

Then there exist $\tilde{v}^k \in \mathbb{R}^{q^k - \rho_k} - \{0\}$ ($k = 1, \dots, k^*$) such that

$$v = T^k(\xi)\tilde{v}^k \quad (k = 1, \dots, k^*) \quad (69)$$

Proof See Appendix. ■

Proposition 4.5 *There exists a $v \in \mathbb{R}^{\nu+1} - \{0\}$ such that $P(\xi)v \equiv 0$ if and only if $q^{k^*} - \rho_{k^*} > 0$. Moreover, if $q^{k^*} - \rho_{k^*} > 0$, then*

$$\{v \in \mathbb{R}^{\nu+1} \mid P(\xi)v \equiv 0\} = \text{Im}S \quad (70)$$

Proof Assume that $q^{k^*} - \rho_{k^*} = 0$. Then it follows from Lemma 4.4 that $v = 0$, which gives a contradiction. Conversely, if $q^{k^*} - \rho_{k^*} > 0$, it immediately follows from Lemma 4.4 that there exists a $v \in \mathbb{R}^{\nu+1} - \{0\}$ such that $P(\xi)v \equiv 0$. We next prove (70). It follows from Lemma 4.4 that $\{v \in \mathbb{R}^{\nu+1} \mid P(\xi)v \equiv 0\} \subset \text{Im}T^{k^*}(\xi) = \text{Im}S$. Conversely, let $v \in \text{Im}S$, say $v = S\tilde{v}$, where $\tilde{v} \in \mathbb{R}^{q^{k^*} - \rho_{k^*}}$. It is straightforwardly checked that $Q^1(\xi)P(\xi)S^1(\xi) = 0$, and hence

$P(\xi)S^1(\xi) = 0$. This gives $P(\xi)v = P(\xi)S\bar{v} \equiv 0$, which yields $\text{Im}S \subset \{v \in \mathbb{R}^{\nu+1} \mid P(\xi)v \equiv 0\}$. This establishes (70). ■

We now return to our original problem. Assume that $q^{k^*} - \rho_{k^*} > 0$, and let the matrix S be defined by (67). Let \bar{P} be a right-invertible matrix such that $\text{Im}S = \text{Ker}\bar{P}$, and define the polynomials $\bar{p}_1, \dots, \bar{p}_{q^{k^*}} \in \mathbb{R}[s]$ by

$$\bar{p}_i(s) := \sum_{j=1}^{\nu+1} \bar{P}_{ij} s^{j-1} \quad (i = 1, \dots, q^{k^*})$$

It then follows from Proposition 4.5 that $a \in \mathbb{R}$ satisfies (61) if and only if $\bar{P}v_a = 0$, i.e., if and only if a is a common zero of the polynomials \bar{p}_i ($i = 1, \dots, q^{k^*}$). Let $\langle \bar{p}_1, \dots, \bar{p}_{q^{k^*}} \rangle$ denote the polynomial ideal in $\mathbb{R}[s]$ spanned by $\bar{p}_1, \dots, \bar{p}_{q^{k^*}}$. Since $\mathbb{R}[s]$ is a principal ideal domain, there exists a polynomial $\hat{p} \in \mathbb{R}[s]$ with the property that $\langle \bar{p}_1, \dots, \bar{p}_{q^{k^*}} \rangle = \langle \hat{p} \rangle$ (see e.g. [18]). Thus, we have reduced our problem to the problem whether a monovariate polynomial has a real root. This is a well-known problem from real algebraic geometry, that has received attention since the times of Newton and Descartes. Obviously, there exists a real root when the polynomial \hat{p} is of odd degree. When \hat{p} is of even degree, one can check whether \hat{p} has a real zero (in fact one can even determine the number of real zeros) using the so called Newton sums and Hankel forms associated with the polynomial. We refer to [6] for details on this topic.

In case one is trying to answer the question whether there exists a feedback realizable transfer function with $d \in \{2, \dots, n - r\}$ zeros for Σ , one can proceed roughly in the same way as above. In this case, it may be shown that there exists a feedback realizable transfer function with d zeros if and only if a set of polynomials $\psi_0, \dots, \psi_\gamma \in \mathcal{A}[s_1, \dots, s_d]$ has a common real zero. Applying the same kind of algorithm as indicated above, the problem may then reduced to the problem whether a set of polynomials $\bar{p}_1, \dots, \bar{p}_q \in \mathbb{R}[s_1, \dots, s_d]$ has a common real zero. This problem has first been solved by Tarski ([17]). Later on, the problem has been considered by Collins ([4], see also [1],[5]) by using the concept of Cylindrical Algebraic Decomposition (CAD) of \mathbb{R}^n . By now, MAPLE-implementations of the algorithm for Cylindrical Algebraic Decomposition are available. A drawback, however, is that the complexity of existing algorithms is doubly exponential. Further, with the method of CAD one can also tackle problems in which polynomial equalities as well as polynomial inequalities play a role. By using the polynomial inequalities obtained from the Routh-Hurwitz test, it follows that this also allows to check whether there exist feedback realizable transfer functions with a prescribed number of *stable* zeros.

5 Conclusions

In this paper we have characterized the feedback realizable transfer functions of a nonlinear SISO system. Further, it has been shown that the existence of a feedback realizable transfer function with a given number of zeros can be checked by reducing the problem to a well known problem from real algebraic geometry, that can be tackled by means of a so called Cylindrical Algebraic Decomposition (CAD) of \mathbb{R}^n . A drawback of using CAD is that the complexity of existing algorithms is doubly exponential. This brings up the question whether the use of CAD could be circumvented. One way to do this might be to investigate whether or not the

polynomial equations obtained have some special (preferably triangular) structure that can be employed. This remains a topic for future research.

In the paper, we have restricted ourselves on the one hand to SISO systems, and on the other hand to regular static state feedback. We expect that an extension of the results in the paper to MIMO systems (using regular static state feedback) is possible. Also an extension to the regular dynamic feedback case (at least for square systems having an invertible decoupling matrix) seems possible. These remain topics for future research.

Acknowledgments

The research was performed while the author was visiting the Laboratoire d'Automatique de Nantes, Ecole Centrale de Nantes/Université de Nantes, France, supported by a grant from the Région Pays de Loire. The grant as well as the hospitality of the laboratory are gratefully acknowledged. I would also like to thank Claude H. Moog and Xiaohua Xia for some motivating discussions and suggestions. Further, I thank Kees Praagman for some algebraic help, and Krister Forsman for stopping me from trying to invent something like Cylindrical Algebraic Decomposition myself.

References

- [1] Arnon, D.S., G.E. Collins and S. McCallum, *Cylindrical algebraic decomposition I: The basic algorithm*, SIAM J. Comput., **13**, (1984), pp. 865-877.
- [2] Aranda-Bricaire, E., C.H. Moog and J.B. Pomet, *A linear algebraic framework for dynamic feedback linearization*, IEEE Trans. Automat. Control, **40**, (1995), pp. 127-132.
- [3] Cheng, D., A. Isidori, W. Respondek and T.J. Tarn, *Exact linearization of nonlinear systems with outputs*, Math. Syst. Theory, **21**, (1988), pp. 63-83.
- [4] Collins, G.E., *Quantifier elimination for real closed fields by cylindrical algebraic decomposition*, In **Second GI Conf. Automata theory and formal languages**, LNCS 33, Springer-Verlag, Berlin, 1975, pp. 134-183.
- [5] Davenport, J.H., Y. Siret and E. Tournier, **Computer algebra. Systems and algorithms for algebraic computation**, Academic Press, 1988.
- [6] Gantmacher, F.R., **Théorie des matrices**, Editions Jacques Gabay, Sceaux, 1990.
- [7] Grizzle, J.W., H.J.C. Huijberts and C.H. Moog, *Characterizations of linear subsystems of nonlinear control systems*, in preparation.
- [8] Hunt, L.R., R. Su, and G. Meyer, *Design for multi-input nonlinear systems*, in **Differential geometric control theory**, R.S. Millman and H. Sussmann (Eds.), Birkhäuser, Boston, 1983, pp. 268-298.
- [9] Hunt, L.R., and M.S. Verma, *Linear dynamics hidden by input-output linearization*, Int. J. Control, **53**, (1991), pp. 731-740.
- [10] Jakubczyk, B., and W. Respondek, *On linearization of control systems*, Bull. Acad. Polon. Sci. Ser. Sci. Math., **28**, (1980), pp. 517-522.

- [11] Isidori, A., **Nonlinear control systems: an introduction**, LNCIS 72, Springer-Verlag, Berlin, 1985.
- [12] Isidori, A., **Nonlinear control systems** (Second Edition), Springer-Verlag, Berlin, 1989.
- [13] Isidori, A., and A. Ruberti, *On the synthesis of linear input-output responses for nonlinear systems*, Syst. Contr. Lett., **4**, (1984), pp. 17-22.
- [14] Marino, R., *On the largest feedback linearizable subsystem*, Syst. Contr. Lett., **7**, (1986), pp. 345-351.
- [15] Nijmeijer, H., and A.J. van der Schaft, **Nonlinear dynamical control systems**, Springer, New York, 1990.
- [16] Paraskevopoulos, P.N., A.S. Tsirikos and E.A. Karagianni, *Robust tracking of an inverted pendulum via a new linear exact model matching technique*, Proceedings CDC 1995, New Orleans, USA, pp. 1676-1683.
- [17] Tarski, A., **A decision method for elementary algebra and geometry** (Second revised edition), University of California Press, Berkeley, 1951.
- [18] Van der Waerden, B.L., **Algebra**, Springer-Verlag, New York, 1991.
- [19] Xia, X., Personal communication, 1995.
- [20] Xu, Z., and L.R. Hunt, *On the largest input-output linearizable subsystem*, IEEE Trans. Automat. Control, **AC-41**, (1996), pp. 128-132.
- [21] Xu, Z., and L.R. Hunt, *Dimensions of linear subsystems of nonlinear systems*, Proceedings CDC 1995, New Orleans, USA, pp. 2263-2268.

Appendix

Proof of Lemma 4.3

Note that $S^{k^*}(\xi)$ is constant. We then have for $i = 1, \dots, 3n + 1$:

$$\frac{\partial S}{\partial \xi_i} = \sum_{k=1}^{k^*-1} S^1(\xi) \cdots S^{k-1}(\xi) \frac{\partial S^k}{\partial \xi_i}(\xi) S^{k+1}(\xi) \cdots S^{k^*}(\xi) \quad (71)$$

From (66) it easily follows that $\frac{\partial S^k}{\partial \xi_i} S^{k+1} = 0$. Together with (71) and the fact that S^{k^*} is constant this gives that S is constant. Since R^k is invertible, there exists a left-inverse $(\tilde{R}^k)^-$ of \tilde{R}^k satisfying $(\tilde{R}^k)^- \tilde{R}^k = 0$. By (66) this gives $(\tilde{R}^k)^- S^k(\xi) = (\tilde{R}^k)^- \tilde{R}^k = I_{q^k - \rho_k}$, which implies that $S^k(\xi)$ is left-invertible. This immediately implies that also S is left-invertible. ■

Proof of Lemma 4.4

By induction. First consider the case $k = 1$. Since $P^1(\xi) \equiv 0$, we also have

$$Q^1(\xi)P^1(\xi)v \equiv 0 \quad (72)$$

Let $\hat{v}^1 \in \mathbb{R}^{\rho_1}$, $\tilde{v}^1 \in \mathbb{R}^{q^1 - \rho_1}$ be such that

$$v = \hat{R}^1 \hat{v}^1 + \tilde{R}^1 \tilde{v}^1 \quad (73)$$

Then

$$0 \equiv Q^1(\xi)P^1(\xi)v = Q^1(\xi)P^1(\xi)R^1 \begin{pmatrix} \hat{v}^1 \\ \tilde{v}^1 \end{pmatrix} = \begin{pmatrix} I & \bar{P}^1(\xi) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{v}^1 \\ \tilde{v}^1 \end{pmatrix} \quad (74)$$

and hence

$$\hat{v}^1 = -\bar{P}^1(\xi)\tilde{v}^1 \quad (75)$$

From (73) and (75) it then follows that

$$v = S^1(\xi)\tilde{v}^1 \quad (76)$$

and hence (69) holds for $k = 1$. Next, assume that (69) holds for $k = 1, \dots, \ell - 1$, where $\ell \in \{2, \dots, k^*\}$. We then have in particular that there exists a $\tilde{v}^{\ell-1} \in \mathbb{R}^{q^\ell}$ such that

$$v = T^{\ell-1}(\xi)\tilde{v}^{\ell-1} = T^{\ell-2}(\xi)(\hat{R}^{\ell-1} - \hat{R}^{\ell-1}\bar{P}^{\ell-1}(\xi))\tilde{v}^{\ell-1} \quad (77)$$

Analogously to the proof of Lemma 4.3 it may be shown that

$$\begin{aligned} \frac{\partial}{\partial \xi_i} \left(T^{\ell-2}(\xi)(\hat{R}^{\ell-1} - \hat{R}^{\ell-1}\bar{P}^{\ell-1}(\xi)) \right) = \\ -T^{\ell-2}(\xi)\hat{R}^{\ell-1}\frac{\partial \bar{P}^{\ell-1}}{\partial \xi_i}(\xi) \quad (i = 1, \dots, 3n+1) \end{aligned} \quad (78)$$

It then follows from (64),(77),(78) that

$$0 \equiv -T^{\ell-1}(\xi)\hat{R}^{\ell-1}P^\ell(\xi)\tilde{v}^{\ell-1} \quad (79)$$

From the fact that $T^{\ell-2}$ and $\hat{R}^{\ell-1}$ are left-invertible, it then follows that

$$P^\ell(\xi)\tilde{v}^{\ell-1} \equiv 0 \quad (80)$$

Let $\hat{v}^\ell \in \mathbb{R}^{\rho_\ell}$, $\tilde{v}^\ell \in \mathbb{R}^{q^\ell - \rho_\ell}$ be such that

$$\tilde{v}^{\ell-1} = \hat{R}^\ell \hat{v}^\ell + \tilde{R}^\ell \tilde{v}^\ell \quad (81)$$

It then follows from (80),(81) that

$$0 \equiv Q^\ell(\xi)P^\ell(\xi)R^\ell \begin{pmatrix} \hat{v}^\ell \\ \tilde{v}^\ell \end{pmatrix} = \begin{pmatrix} I & \bar{P}^\ell(\xi) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{v}^\ell \\ \tilde{v}^\ell \end{pmatrix} \quad (82)$$

Together with (81) this implies that

$$\tilde{v}^{\ell-1} = S^\ell(\xi)\tilde{v}^\ell$$

Combining this with (77), we conclude that (69) holds for $k = \ell$. This establishes (69) for all $k \in \{1, \dots, k^*\}$. ■

PREVIOUS PUBLICATIONS IN THIS SERIES:

Number	Author(s)	Title	Month
96-24	A.F.M. ter Elst D.W. Robinson	Second-order subelliptic operators on Lie groups III: Hölder continuous coefficients	December '96
96-25	J. de Graaf	Evolution equations	December '96
96-26	J. de Graaf	Spaces of harmonic functions and evolution equations in them	December '96
96-27	R.M.M. Mattheij S.J. Wright	Parallel algorithms for parameter identification in odes	December '96
97-01	J.H.P.A. Martens J.C. Reijenga J.H.M. ten Thije Boonkkamp R.M.M. Mattheij F.M. Everaerts	Transient modelling of capillary electrophoresis. Isotachophoresis	January '97
97-02	A.A. Reusken	Approximate cyclic reduction preconditioning	February '97
97-03	J.J.A.M. Brands	Asymptotics of non-Laplacian integrals	February '97
97-04	M.J. Noot A.C. Telea J.K.M. Jansen R.M.M. Mattheij	Real Time Numerical Simulation and Visualization of Electrochemical Drilling	March '97
97-05	H.J.C. Huijberts	Characterization of static feedback realizable transfer functions for non-linear control systems	May '97

