Consistent and accurate frequency oracles under local differential privacy

Citation for published version (APA):

DOI:
10.48550/arXiv.1905.08320

Document status and date:
Published: 20/05/2019

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the “Taverne” license above, please follow below link for the End User Agreement:
www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:
openaccess@tue.nl
providing details and we will investigate your claim.

Download date: 06. Apr. 2024
Consistent and Accurate Frequency Oracles under Local Differential Privacy

Tianhao Wang\textsuperscript{1}, Milan Lopuhaä-Zwakenberg\textsuperscript{2}, Zitao Li\textsuperscript{1}, Boris Skoric\textsuperscript{2}, Ninghui Li\textsuperscript{1}
\textsuperscript{1}Purdue University, \textsuperscript{2}Eindhoven University of Technology
\{wang2842,li2490,ninghui\}@purdue.edu, \{m.a.lupuhaa,b.skoric\}@tue.nl

ABSTRACT
Local Differential Privacy (LDP) protects user privacy from the data collector. LDP protocols have been increasingly deployed in the industry. A basic building block is frequency oracle (FO) protocols, which estimate frequencies of values. While several FO protocols have been proposed, their resulting estimations, however, do not satisfy the natural consistency requirement that estimations are non-negative and sum up to 1. In this paper, we study how to add post-processing steps to FO protocols to make them consistent while achieving high accuracy for a wide range of tasks, including frequencies of individual values, frequencies of the most frequent values, and frequencies of subsets of values. We consider 10 different methods, some of them explicitly proposed before in the literature, and others introduced in this paper. We establish theoretical relationships between some of them and conducted extensive experimental evaluations. Among them, we propose to use Norm-Hyb, which keeps estimations above a certain threshold unchanged, and then converts negative estimations to 0 while adding a value (typically negative) to all other estimations to ensure a sum of 1.

CCS CONCEPTS
- Security and privacy → Privacy-preserving protocols.

KEYWORDS
local differential privacy

1 INTRODUCTION
Differential privacy [14] has been accepted as the \textit{de facto} standard for data privacy. Most early works on DP are in the centralized setting, where a trusted data curator obtains data from all individuals, and processes the data in a way that protects privacy of individual users. For example, the data curator can publish a private synopsis of the data, enabling analysis on the data, while hiding individual information.

Recently, techniques for satisfying differential privacy (DP) in the local setting, which we call LDP, have been studied and deployed. In the local setting for DP, there are many users and one aggregator. Unlike the centralized setting, the aggregator does not see the actual private data of each individual. Instead, each user sends randomized information to the aggregator, who attempts to infer the data distribution based on that. LDP techniques enable the gathering of statistics while preserving privacy of every user, without relying on trust in a single trusted third party. LDP techniques have been deployed by companies like Apple [4], Google [16], and Microsoft [10]. Exemplary use cases include collecting users’ default browser homepage and search engine, in order to understand the unwanted or malicious hijacking of user settings; or frequently typed emoji’s and words, to help with keyboard typing predictions.

The fundamental tools in LDP are mechanisms to estimate frequencies of values. Existing research [3, 6, 16, 34, 38] has developed frequency oracle (FO) protocols, where the aggregator can estimate the frequency of any chosen value in the specified domain (fraction of users reporting that value).

For utility, existing works focus only on accuracy of individual values. For example, Wang et al. [34] showed how to choose parameters to minimize estimation variances in existing protocols. While these protocols work fine for the purpose of identifying heavy hitters (i.e., frequent values) and estimating the frequencies for these heavy hitters, some applications naturally require the total frequencies of subset of values. For example, with the estimation of each emoji’s frequency, one may be interested in understanding what categories of emoji’s are more popular and need to issue subset frequency queries. For another example, in [40], a few attributes are encoded together and reported using LDP, and recovering the distribution for any attribute requires computing the frequencies of sets of encoded values. In our experiment, as a case study, we use FO to estimate the population of cities in a Census dataset, and query the population of different areas by summing up estimations for cities.

For frequencies of a subset of values, simply summing up the estimations of all values is far from optimal, especially when the input domain is large. Due to the significant amount of noise needed to satisfy LDP, the estimations for many values may be negative. By exploiting the knowledge that the frequencies are all non-negative and sum up to 1 and imposing this as a consistency requirement, the accuracy for subset frequency queries can be greatly improved.

In this paper, we study how to add post-processing steps to existing frequency oracles to make them consistent while achieving high accuracy for a wide range of tasks, including frequencies of individual values, frequencies of the most frequent values, and frequencies of subsets of values. We consider 10 different methods, some of them explicitly proposed before in the literature, and other introduced in this paper. One promising method is Norm-Sub, which converts negative estimations to 0 while adding a value δ (typically negative) to all other estimations to ensure a sum of 1. Norm-Sub is the solution to a Constraint Least Square formulation of the posterior estimation problem, which is a (highly accurate) approximation of the Maximum Likelihood Estimation (MLE) formulation of the problem. The only weakness of Norm-Sub is that it has lower accuracy on estimating the most frequent values. Before applying Norm-Sub, these estimations are unbiased. Norm-sub adds δ to them. We thus introduce Norm-Hyb, which keeps estimations...
We consider the setting where there are many users. We study 10 different post-processing methods for FO protocols, including introducing Norm-Sub, MLE-Apx (approximate MLE solution), and Norm-Hyb.

We established theoretical relationships between Norm-Sub, MLE-Apx, and those based on constraint least squares. We conducted extensive experimental evaluations that show: The frequency oracle is unbiased, and the variance of the estimation is minimized. Existing approaches, however, do not produce frequency oracles that satisfy the following natural consistency property.

\section{2 Problem Setting}

We consider the setting where there are many users and one aggregator. Each user possesses a value $v$ from a finite domain $D$, and the aggregator wants to learn the distribution of values among all users, in a way that protects the privacy of individual users. More specifically, the aggregator wants to estimate, for each value $v \in D$, the fraction of users having $v$ (the number of users having $v$ divided by the population size). Such protocols are called frequency oracle (FO) protocols under Local Differential Privacy (LDP), and they are the key building blocks of other LDP tasks.

\subsection{2.1 Frequency Oracles under LDP}

An FO protocol is specified by a pair of algorithms: $\Psi$ is used by each user to perturb her input value, and $\Phi$ is used by the aggregator. Each user sends $\Psi(v)$ to the aggregator. The formal privacy requirement is that the algorithm $\Psi(\cdot)$ satisfies the following property:

\begin{definition}[$(\epsilon,\delta)$-Local Differential Privacy] An algorithm $\Psi(\cdot)$ satisfies $(\epsilon,\delta)$-local differential privacy (LDP), where $\epsilon \geq 0$, if and only if for any input $v, v' \in D$, we have
\[
\forall y \in \Psi(D) : \Pr [\Psi(v) = y] \leq e^\epsilon \Pr [\Psi(v') = y],
\]
where $\Psi(D)$ denotes the set of all possible outputs of $\Psi$.
\end{definition}

Since a user never reveals $v$ to the aggregator and reports only $\Psi(v)$, the user’s privacy is still protected even if the aggregator is malicious.

The aggregator uses $\Phi$, which takes the vector of all reports from users as the input, and produces an oracle function that estimates the frequencies of the $v \in D$ (i.e., the fraction of users who have input value $v$). As $\Psi$ is a randomized function, the result of $\Phi$ becomes inaccurate.

For the utility goal, existing works focus on accuracy of individual values. That is, the design goal for $\Psi$ and $\Phi$ is that the estimated frequency for each $v$ is unbiased, and the variance of the estimation is minimized. Existing approaches, however, do not produce frequency oracles that satisfy the following natural consistency property.

\begin{definition}[Consistency] A frequency oracle is consistent if and only if the following two conditions are satisfied:
\begin{enumerate}
\item The estimated frequency of each value is in the range of $[0,1]$. 
\item The sum of the estimated frequencies is 1.
\end{enumerate}
\end{definition}

In this paper, we study how to add post-processing steps to $\Phi$ so that the resulting estimations are consistent and accurate for a wide range of tasks, including frequency of individual values, frequency of the most frequent values, and frequency of subsets of values. In order to do so, we first present several existing FO and analyze them.

\subsection{2.2 Generalized Random Response (GRR)}

This FO protocol generalizes the randomized response technique [37]. Here each user with private value $v \in D$ sends the true value $v$ with probability $p$, and with probability $1-p$ sends a randomly chosen $v' \in D \setminus \{v\}$. Suppose the domain $D$ contains $d = |D|$ values, the perturbation function is formally defined as

\[
\forall y \in D \Pr [\Psi_{\text{GRR}}(\epsilon,\delta)(v) = y] = \begin{cases} 
p = \frac{e^\epsilon}{e^\epsilon + d - 1}, & \text{if } y = v \\
q = \frac{1}{e^\epsilon + d - 1}, & \text{if } y \neq v \end{cases}
\]

This satisfies $\epsilon$-LDP since $\frac{ep}{q} = e^\epsilon$.

From a population of $n$ users, the aggregator receives a length-$n$ vector $y = (y_1, y_2, \cdots, y_n)$, where $y_i \in D$ is the reported value of the $i$-th user. The aggregator counts the number of times each value $v$ appears in $y$ and produces a length-$d$ vector $c$ of natural numbers. Observe that the components of $c$ sum up to $n$, i.e., $\sum_{v \in D} c_v = n$. The aggregator then obtains the estimated frequency vector $\hat{f}$ by scaling each component of $c$ as follows:

\[
\hat{f}_v = \frac{c_v}{n} - \frac{q}{p - q} = \frac{c_v}{n} - \frac{1}{e^\epsilon + d - 1}
\]

$\hat{f}$ is the output of $\Phi$ on input $y$, i.e., $\Phi(y) = \hat{f}$. It is shown in [34] that $\hat{f}_v$ is an unbiased estimate of the true frequency of $v$, and the variance for this estimation is

\[
\sigma_v^2 = \frac{q(1-q) + f_v(p-q)(1-p-q)}{n(p-q)^2}
\]

where $f_v$ is the true frequency of value $v$. Since $\hat{f}_v$ is the result of aggregating random choices made by many users, by the central limit theorem, $\hat{f}_v$ is well-approximated by the Gaussian distribution. More precisely,

\[
\hat{f}_v \approx f_v + \mathcal{N}(0, \sigma_v).
\]
The accuracy of this protocol deteriorates fast when the domain size \( d \) increases. This is reflected in that the variance given in (3) is linear in \( d \). This motivated the development of other FO protocols.

### 2.3 Optimized Unary Encoding (OUE)

The Optimized Unary Encoding (OUE) [36] is an optimization of the Basic RAPPOR protocol in [16]. It avoids having a variance that depends on \(|D|\) by encoding the value into the unary representation. Wlog, let \( D = \{1, 2, \ldots, d\} \); each value \( v \in D \) is transformed into a binary string of length \( d \) with one-hot encoding, i.e., the \( v \)-th bit is 1 and all other bits are 0. This bit vector is then randomly perturbed. In Basic RAPPOR, each bit is flipped with probability \( \frac{1}{e^{\epsilon/2} + 1} \). This satisfies \( \epsilon \)-LDP because the unary encodings of different values differ in exactly two bits.

In OUE, the bit 1 is flipped (to 0) with probability \( p = 1/2 \), and each bit 0 is flipped (to 1) with probability \( q = \frac{1}{e^{\epsilon/2} + 1} \). While it may be a bit counter-intuitive that in OUE the bit 1 becomes a random bit after randomization, doing so enables us to transmit each of the many \((d-1)\) bits with the maximum allowed privacy budget \( \epsilon \), so that the expected number of 1’s is minimal.

The perturbed bit vector encodes a subset \( Y \) of \( D \) that includes each element whose corresponding bit is 1. The aggregator counts the number of times each value \( v \) appears in the reported sets, and produces a length-\( d \) vector \( e \), which is then scaled to obtain unbiased estimation

\[
\tilde{f}_v = \frac{c_v - q}{p - q} = \frac{c_v - \frac{1}{2}}{\frac{1}{2} - \frac{1}{e^{\epsilon/2} + 1}}
\]

It has been proved [34] that the OUE satisfies LDP, and estimated frequency \( \tilde{f}_v \) is unbiased and has variance

\[
\sigma^2 = \frac{q(1-q) + f_v(p-q)(1-p-q)}{n(p-q)^2}
\]

(5)

\[
\tilde{f}_v = \frac{1}{n} \left( \frac{4e\epsilon}{(e^{\epsilon/2} - 1)^2} + f_v \right)
\]

(6)

Ignoring the \( f_v \) factor, the factor \( d - 2 + e^g \) in Equation (3) is replaced by \( 4e\epsilon \). This suggests that for smaller \( d \) (such that \( d - 2 < 3e^g \)), one is better off with GRR; but for large \( d \), OUE is better and has a variance that does not depend on \( d \).

#### Approximate the variance.

When \( d \) is large and \( e \) is not too large, \( f_v(p-q)(1-p-q) \) is dominated by \( q(1-q) \). Thus, one can approximate Equation (6) or (3) by ignoring the \( f_v \) component to have a uniform variance for all the values. Specifically, define

\[
\sigma^2 = \frac{q(1-q)}{n(p-q)^2}
\]

(7)

One can then further approximate Equation (4) as

\[
\tilde{f}_v = f_v + N(0, \sigma).
\]

(8)

### 2.4 Other FO Protocols

Several other FO protocols have been proposed. While they take different forms when originally proposed, in essence, each has the user report some encoding of a subset \( Y \subseteq D \), so that the user’s true value has a probability \( p \) to be included in \( Y \) and another value has a probability \( q < p \) to be included in \( Y \). The estimation method used in GRR and OUE (namely, \( \tilde{f}_v = \frac{c_v/n-q}{p-q} \)) equally applies.

**Optimized Local Hashing (OLH) [36]** deals with a large domain size \( d \) by first using a random hash function to map an input value into a smaller domain of size \( g \), and then applying randomized response to the hash value in the smaller domain. In OLH, the reporting protocol is

\[
\Psi_{\text{OLH}}(v) = \langle H, \Psi_{\text{GRR}}(\epsilon, g)(H(v)) \rangle,
\]

where \( H \) is randomly chosen from a family of hash functions that hash each value in \( D \) to \( \{1 \ldots g\} \), and \( \Psi_{\text{GRR}}(\epsilon, g) \) is given in (1), while operating on the domain \( \{1 \ldots g\} \). The hash family should have the property that the distribution of each input value \( v \)’s hash value is uniform over \( \{1 \ldots g\} \) and independent from the distributions of other input values in \( D \). Since \( H \) is chosen independently of the user’s value \( v \), \( H \) by itself carries no meaningful information. Such a report \( (H, r) \) can be equivalently represented by the set \( Y = \{y \in D \mid H(y) = r\} \). The use of a hash function can be viewed as a compression technique, which results in constant size encoding of a set. For a user with value \( v \), the probability that \( v \) is in the set \( Y \) represented by the randomized report \( (H, r) \) is \( p = \frac{e^g - 1}{e^g} \) and the probability that a user with value \( \neq v \) is in \( Y \) is \( q = \frac{1}{g} \).

In OLH, both the hashing step and the randomization step result in information loss. The choice of the parameter \( g \) is a tradeoff between losing information during the hashing step and losing information during the randomization step. It is found that the estimation variance when viewed as a continuous function of \( g \) is minimized when \( g = e^g + 1 \) (or \( g = \lceil e^g + 1 \rceil \) in practice), in which case the variance is the same as in OUE [34].

**Random Matrix Projection [6]** motivated OLH, and is equivalent to using local hash with \( g \) fixed at 2. That is, each reported set consists of approximately half of elements in \( D \). When \( \epsilon < \ln 2 = 0.69 \), OLH is exactly the same as this method. With larger \( \epsilon \), OLH performs better.

**Hadamard Response [3, 5]** is similar to Random Matrix Projection, but uses Hadamard transform instead of hash functions (or random matrix as presented in [6]). The aggregation part is faster because evaluating a Hadamard entry is practically faster. Furthermore, here the partition is exact. Each user is reporting a subset that is exactly half of the domain \( D \) (assuming that the domain size is even).

**Subset Selection [33, 38]** method reports a randomly selected subset of a fixed size \( k \). The sensitive value \( v \) is included in the set with probability \( p = 1/2 \). For any other value, it is included with probability \( q = p - \frac{k-1}{d-1} + (1-p) \cdot \frac{k}{d-1} \). To minimize estimation variance, \( k \) should be an integer equal or close to \( d/(e^{\epsilon/2} + 1) \). Ignoring the integer constraint, we have \( q = \frac{1}{2} \cdot \frac{2k-1}{d-1} = \frac{1}{2} \cdot \frac{2d e^\epsilon - 1}{d-1} = \frac{e^\epsilon - 1}{e^\epsilon} < \frac{1}{e^\epsilon} \). Putting \( p \) and \( q \) in Equation (6), its variance is smaller than that of OUE and OLH. However, as \( d \) increases, the term \( \frac{d-e^{\epsilon/2}+1}{d-1} \) gets closer and closer to 1. For a larger domain, this offers essentially the same accuracy as OLH, with higher communication cost.
3 TOWARDS CONSISTENT FREQUENCY ORACLES

Given the frequency oracles, our goal is to develop post-processing methods that produce consistent frequency oracles, are efficient to compute, and give accurate estimations for a wide variety of queries. Specifically, we want to derive the post-processed estimation \( \tilde{f}' \) from \( f \) given by the frequency oracle. The main challenge occurs when the domain of possible values is large so that many values have true frequencies that are either 0 or very low.

3.1 Baseline Methods

For the purpose of comparison, we first consider three baseline methods, which are straightforward extensions of the estimation procedures in existing FO protocols. These methods, however, do not result in consistent frequency oracles.

**Base.** We use the standard FO as presented in Section 2 to obtain estimations of each value.

Under GRR, the sum of all estimations is 1. Under OUE, the sum of all estimations is a random variable with expected value 1. When the domain is large, there will be many values in the domain that have a zero or very low true frequency; the estimation of them may be negative.

**Base-Po’s.** After applying the standard FO, we convert all negative estimations to 0.

This results in non-negative individual estimations, but the sum of all estimations is likely to be above 1. For each individual value, this method results in an estimation that is always as accurate as (and sometimes more accurate than) the Base method, because the true frequency of each value cannot be negative. However, this step introduces biases into aggregates, because some negative noises are removed or reduced by the process, but the positive noises are never removed. As a result, the expected value for the sum of all estimations is greater than 1. This effect will be reflected when answering subset queries, for which Base-Po results in biased estimations. For larger-range queries, the bias can be significant.

**Post-Po’s.** For each query result, if it is negative, we convert it to 0.

This method does not post-process the distribution. Rather, it post-process each query result individually. For subset queries, as the results are typically positive, Post-Po is similar to Base. On the other hand, when the query is on a single item, Post-Po is equivalent to Base-Po.

**Base-Cu’t.** After standard FO, convert everything below some sensitivity threshold to 0.

The original design goal for frequency oracles is to recover frequencies for frequent values, and oftentimes there is a sensitivity threshold so that only estimations above the threshold are considered. Specifically, for each value, we compare its estimation with a threshold

\[
T = F^{-1} \left(1 - \frac{\alpha}{d}\right) \sigma, \tag{9}
\]

where \( d \) is the domain size, \( F^{-1} \) is the inverse of cumulative distribution function of the standard normal distribution, and \( \sigma \) is the standard deviation of the LDP mechanism (i.e., as in Equation (7)). The intuition is that estimations below the threshold are considered to be noises. When using such a threshold, for any value \( v \in D \) whose original count is 0, the probability that it will have an estimated frequency above \( T \) (or the probability a zero-mean Gaussian variable with standard deviation \( \sigma \) is above \( T \)) is at most \( \frac{\alpha}{d} \). Thus when we observe an estimated frequency above \( T \), the probability that the true frequency of the value is 0 is (by union bound) at most \( d \times \frac{\alpha}{d} = \alpha \). In [16], it is recommended to set \( \alpha = 5\% \), following conventions in the statistical community.

Empirically we observe that such a threshold can be too high when the population size is not very large and/or the \( \epsilon \) is not large. A large threshold results in all except a few estimations below the threshold and set to 0. We note that the choice of \( \alpha \) is trading off false positives with false negatives, and setting \( \alpha = 0.05 \) is not necessary. Given a large domain, there are likely between several and a few dozen values that have quite high frequencies, with most of the remaining values having low true counts. We want to keep an estimation if it is a lot more likely to be from a frequent value than from a very low frequency one. In this paper, we choose to set \( \alpha = 2 \), which ensures that the expected number of false positives, i.e., values with very low true frequencies but estimated frequencies above \( T \), to be around 2. If there are around 20 values that are truly frequent and have estimated frequencies above \( T \), then the ratio of true positives to false positives when using this threshold is 10:1.

This method ensures that all estimations are non-negative. It does not ensure that the sum of estimations is 1. Typically, this results in under-estimations of many values whose true frequencies are non-zero, but not very high, but estimations for high frequency values are unbiased.

3.2 Methods from the Literature

Some methods for post-processing have been proposed in the literature. Under these methods, the estimated frequency of one value is affected by other estimations.

**Norm.** After standard FO, add \( \delta \) to each estimation so that the overall sum is 1.

The method is formally proposed for the centralized setting [18] of DP (and is used in the local setting, e.g., [23, 32]). Note that the method does not enforce non-negativity. For GRR, this method actually does nothing, since each user reports a single value. For the other FO’s, however, each user reports a randomly selected subset whose size is a random variable, and Norm would change the estimations.

By exploiting that the sum of all estimations should be 1, Norm can slightly reduce the variance. Specifically, for each \( v \in D \), its estimate \( f'_{v} \) can be viewed as its true frequency \( f_{v} \) plus a Gaussian noise drawn from \( \mathcal{N}(0, \sigma) \) (as in Equation (8)). The sum \( \sum_{v \in D} f'_{v} \) can be viewed as the original sum \( \sum_{v \in D} f_{v} \) plus \( d \) samples from \( \mathcal{N}(0, \sigma) \). Norm finds

\[
\delta = \frac{1 - \sum_{v \in D} f'_{v}}{d},
\]

which can be viewed as \( 1/d \) plus (or minus, since Gaussian noise is symmetric) the average of \( d \) Gaussian noises. Thus, \( f'_{v} \) can be viewed as \( f_{v} \) plus one sample of Gaussian noise plus an average of \( d \) Gaussian noises. Since the noises are zero-mean, \( f'_{v} \) is unbiased.
Moreover, as
\[
f'_v = \tilde{f}_v + \delta = \frac{1}{d} + \frac{(d - 1)\tilde{f}_v - \sum_{x \in D_v(v)} \tilde{f}_x}{d}
\]
it can be proved that the variance of \(f'_v\) is actually \(\frac{d-1}{d}\) times that of \(\tilde{f}_v\). When \(d\) is in hundreds, Norms improves over the baseline by less than one percent.

**Power.** Fit a Power-Law distribution, and then minimize the expected squared error.

Jia et al. [19] proposed a method in which one first assumes that the data follows some type of distribution (but the parameters are unknown), then uses the estimates to fit the parameters of the distribution, and finally updates the estimates.

Formally, for each value \(v\), the estimate \(\tilde{f}_v\) given by FO is regarded as the addition of two parts: the true frequency \(f_v\) and noise following the normal distribution (as shown in Equation (8)). The method then finds \(f'_v\) that minimizes \(E[(\tilde{f}_v - f'_v)^2]\). To solve this problem, the authors estimate the true distribution \(f_v\) from the estimates \(\tilde{f}\) (where \(\tilde{f}\) is the vector of the \(\tilde{f}_v\)’s).

In particular, it is assume in [19] that the distribution follows Power-Law. That is, the count of a value \(f_v \cdot n\) is assumed to appear with probability proportional to \((f_v \cdot n)^{-s}\). Formally,
\[
\Pr[f_v \cdot n] = \frac{(f_v \cdot n)^{-s}}{\sum_{i=1}^{n} t^{-s}}
\]
Here only one parameter \(s\) is unknown; and it is fitted from the estimation \(\tilde{f}\). One can then incorporate the Power-Law distribution \(\Pr[f_v \cdot n]\) to derive better results for minimizing the objective. Specifically, for each value \(v \in D\), output
\[
f'_v = \sum_{i=1}^{n} \frac{\Pr(\tilde{f}_v \cdot n - i) \cdot \mathcal{N}(0, \sigma)}{\sum_{j=1}^{n} \Pr(\tilde{f}_v \cdot n - j) \cdot \mathcal{N}(0, \sigma)}} \cdot i^{-s}
\]
\[
= \sum_{i=1}^{n} \frac{\Pr(\tilde{f}_v \cdot n - i) \cdot \mathcal{N}(0, \sigma))}{\sum_{j=1}^{n} \Pr(\tilde{f}_v \cdot n - j) \cdot \mathcal{N}(0, \sigma))} \cdot j^{-s}
\]
where \(\Pr[x \sim \mathcal{N}(0, \sigma)]\) denotes the pdf of the normal distribution at \(x\), and the normal distribution has standard deviation \(\sigma\) (as in Equation (8)). Based on Equation (10), we make three observations. First, it preserves the ordering of the estimation, i.e., \(f_{v1} < f_{v2}\) iff \(\tilde{f}_{v1} < \tilde{f}_{v2}\). Second, all the processed estimates are positive, as all components of the summation are positive. Third, when \(\tilde{f}_v\) is large, \(f'_v - f'_v\) is small; otherwise \(f'_v - f'_v\) is large, because the exponential factor is dominating the polynomial.

Using this method requires knowledge and/or assumption of the distribution to be estimated. If there are too much noise, or the underlying distribution is not Power-Law, forcing the observations to fit a distribution could lead to poor accuracy. Moreover, this method does not ensure the frequencies sum up to 1, as Equation (10) only considers the frequency of each value \(v\) independently.

Finally, as Power evaluates the summation of \(n\) products for each estimation, it has a \(O(n \cdot d)\) time complexity. One can reduce this by evaluating only values within \(10 \times \epsilon\), reducing time complexity to \(O(\sqrt{n} \cdot d)\), but this is still slow. In the implementation, following

the description in [19], the method is about 100\times slower than other methods.

### 3.3 Normalization Methods

We now consider other methods that can be used to construct consistent frequency oracles without making assumptions about the data distribution.

We point out that an unavoidable cost for consistency is giving up on unbiased estimations. Without post-processing, the expected estimation sum is 1. After negative estimations are turned into 0, the expected sum is greater than 1. To adjust the estimations so that they sum up to 1, some estimations must be adjusted lower, making them no longer unbiased.

**Norm-Mul.** After standard FO, convert negative value to 0. Then multiply each value by a multiplicative factor so that the sum is 1.

More precisely, given estimation vector \(\tilde{f}\), we find \(y\) such that
\[
\sum_{v \in D} \max(y \times \tilde{f}_v, 0) = 1,
\]
and assign \(f'_v = \max(y \times \tilde{f}_v, 0)\) as the estimations. This results in a consistent FO. However, multiplying by a factor may result in the estimation of high frequency values to be significantly lower than their true values.

**Norm-Sub.** After standard FO, convert negative values to 0, while maintaining overall sum of 1 by adding \(\delta\) (which is typically negative) to each remaining value.

More precisely, given estimation vector \(\tilde{f}\), we want to find \(\delta\) such that
\[
\sum_{v \in D} \max(\tilde{f}_v + \delta, 0) = 1
\]
Then the estimation for each value \(v\) is \(f'_v = \max(\tilde{f}_v + \delta, 0)\). This extends the method Norm and results in a consistent FO.

**Norm-Cut.** After standard FO, convert negative and small positive values to 0 so that the total sums up to 1.

We note that Norm-Sub also results in biased estimations for higher frequency items. For a value with a high estimation, the original estimation is unbiased, and any change to it adds some biases. Since low estimations are likely to be caused by values with zero or very low frequencies, one natural idea is to turn the low estimations to 0 to ensure consistency, without changing the estimations of high-frequency values. This is the idea of Norm-Cut. More precisely, given the estimation vector \(\tilde{f}\), there are two cases. When \(\sum_{v \in D} \max(\tilde{f}_v, 0) \leq 1\), we simply change each negative estimations to 0. When \(\sum_{v \in D} \max(\tilde{f}_v, 0) > 1\), we want to find the smallest \(\theta\) such that
\[
\sum_{v \in D: f_v > \theta} \tilde{f}_v \leq 1
\]
Then the estimation for each value \(v\) is \(\tilde{f}_v < \theta\) and \(\tilde{f}_v \geq \theta\) if \(\tilde{f}_v \geq \theta\). This is similar to Base-cut in that both methods change all estimated values below some thresholds to 0. The differences lie in how the threshold is chosen. This results in non-negative estimations, and
typically results in estimations that sum up to 1, but might result in a sum < 1.

**Norm-Hyb.** After standard FO, convert negative value to 0, keep all estimated values above a certain threshold unchanged, and add δ to each remaining value (and changing all negative estimations to 0) to ensure a sum of 1.

More precisely, we use the same threshold as in Base-Cut, \( T = F^{-1}(1 - \frac{2}{\delta}) \sigma \). If the sum of all estimations above \( T \) is \( \leq 1 \), we keep all these estimations unchanged, and apply Norm-Sub to the other estimations (with the consistency changes to they sub up to 1 minus sum of estimations above \( T \)). If the sum of all estimations above \( T \) is greater than 1, we find the largest \( k \) such that the \( k \) highest estimations sum up to a value less than 1, and apply Norm-Sub to the other estimations.

This method is motivated by the observation that the main problem of methods such as Norm-Sub is that high-frequency estimations are adjusted away from unbiased estimations. In this method, a threshold is selected so that estimations above the threshold are not adjusted. This is a hybrid between Norm-Sub and Base-Cut. Also note that if one uses \( \theta \) as in Norm-Cut instead of \( T \), Norm-Hyb becomes equivalent to Norm-Cut.

### 3.4 Maximum Likelihood Estimation

We have presented several methods for achieving consistent FO that are based on different intuitions. A natural question is can they be justified in a more principled way. Conceptually, the problem of recovering distributions given the reports is an estimation problem. And the most natural way is to start from Bayesian inference. In particular, we want to find the \( f' \) such that

\[
\Pr[f'] \Pr[f] = \frac{\Pr[f'] \Pr[f']} {\Pr[f]} \quad (11)
\]

is maximized. Note that here what we observe is the user reports \( c \), but since the result by FO \( f \) is a linear combination of \( c \), we can just use \( f \). In (11), \( \Pr[f'] \) is the prior, and the prior distribution influences the result. In our setting, as we assume there is no such prior, \( \Pr[f'] \) is uniform. That is, \( \Pr[f'] \) is a constant that does not influence the result. As a result, we are seeking for \( f' \) which is the maximal likelihood estimator (MLE), i.e., \( \Pr[f'] \) is maximized.

To compute \( \Pr[f'] \), we use Equation (4), which states that, given the original distribution \( f' \), the vector \( f \) is a set of independent random variables, where each component \( f_v \) follows Gaussian distribution with mean \( f'_v \) and variance \( \sigma'^2_v \). The likelihood of \( f \) given \( f' \) is thus

\[
\Pr[f|f'] = \prod_v \Pr[f_v|f'_v] = \prod_v \frac{1}{\sqrt{2\pi}\sigma'^2_v} \cdot e^{-\frac{(f'_v - f_v)^2}{2\sigma'^2_v}} = \prod_v \frac{1}{\sqrt{2\pi}\sigma'^2_v} \cdot e^{-\frac{(f'_v - f_v)^2}{2\sigma'^2_v}} \quad (12)
\]

From (12), we first simplify the exponent plugging in the value of \( \sigma'^2_v \) as in Equation (2) or (5):

\[
\sum_v \frac{(f'_v - f_v)^2}{2\sigma'^2_v} = \sum_v \frac{(f'_v - f_v)^2}{q(1 - q) + f'_v(p - q)(1 - p - q)} \approx \sum_v \frac{(f'_v - f_v)^2}{q(1 - q) + f'_v(p - q)(1 - p - q)} = \sum_v \frac{(c_v/n - q - (p - q)f'_v)^2}{q(1 - q) + f'_v(p - q)(1 - p - q)}
\]

The factor \( \frac{1}{\sqrt{2\pi}\sigma'^2_v} \) in the exponent ensures that for large \( n \) the exponent will vary the most with \( f' \), which dominates the coefficient \( \frac{1}{\sqrt{2\pi}\sigma'^2_v} \). Thus approximately we find \( f' \) that achieves the following optimization goal:

\[
\begin{align*}
\text{minimize:} & \quad \sum_v \frac{(c_v/n - q - (p - q)f'_v)^2}{q(1 - q) + f'_v(p - q)(1 - p - q)} f'_v \\
\text{subject to:} & \quad \sum_v f'_v = 1, \\
& \quad \forall v, 0 \leq f'_v \leq 1.
\end{align*}
\]

**Approximate Solution for MLE.** In Appendix A, we use the KKT condition [21, 22] to solve optimization problem in Eq. (13). The result is presented below:

Partition the domain \( D \) into \( D_0 \) and \( D_1 \), where \( D_0 \cap D_1 = \emptyset \) and \( D_0 \cup D_1 = D \). For \( v \in D_0 \), assign \( f'_v = 0 \). For \( v \in D_1 \),

\[
f'_v = \frac{c_v/n - q - (p - q)f'_v}{p - q + (p(1 - p) - q(1 - q))x_v} \quad (14)
\]

where

\[
x_v = \frac{\sum_{x \in D_1} c_v/n - |D_1|q - (p - q)}{(p - q) + (p(1 - p) - q(1 - q))x_v},
\]

It can be verified \( \sum_{v \in D} f'_v = 1 \). We call this solution MLE-Apx.

**Analysis.** As \( \tilde{f}'_v = \frac{c_v/n - q}{p - q} \), we can regard Equation (14) as

\[
f'_v = \tilde{f}'_v \cdot y + \delta,
\]

where

\[
y = \frac{p - q}{p - q + (p(1 - p) - q(1 - q))x_v}, \quad \delta = \frac{-q(1 - q)x_v}{p - q + (p(1 - p) - q(1 - q))x_v}.
\]

Hence MLE-Apx appears to represent some hybrid of Norm-Sub and Norm-Mult. In general, Norm-Sub and MLE-Apx give very close results, as \( y \sim 1 \).

In the actual implementation, we initialize \( D_0 = \emptyset \) and \( D_1 = D \); \( \forall v \in D_1 \), if \( f'_v < 0 \), we move \( v \) into \( D_0 \). We keep doing this until \( \forall v \in D_1, f'_v > 0 \).

### 3.5 Constrained Least Squares

The algebraic solution to the MLE formulation is quite complicated. However, if we use the approximate variance that is the same for each value (i.e., Equation (7), assuming \( g(1 - q) \) dominates \( f_0(p - q)(1 - p - q) \)), one can use least squares with constraints (sum up to
1 and non-negative) to recover input distribution when given the estimations. We call this method CLS (Constrained Least Squares).

**CLS.** First use standard FO, then use least squares with constraints (sum up to 1 and non-negative) to recover the rest of the values.

Specifically, given the estimates \( \hat{f} \) by FO, the method outputs \( f' \) that is a solution of the following problem:

\[
\begin{align*}
\text{minimize:} & \quad ||f' - \hat{f}||_2 \\
\text{subject to:} & \quad \forall v, f'_v \geq 0 \\
& \quad \sum_v f'_v = 1
\end{align*}
\]

**Solution for CLS.** Similar to MLE, in Appendix B, we use the KKT condition \([21, 22]\) to solve the problem. The solution is as follows:

Partition the domain \( D \) into \( D_0 \) and \( D_1 \), where \( D_0 \cap D_1 = \emptyset \) and \( D_0 \cup D_1 = D \). For \( v \in D_0 \), assign \( f'_v = 0 \). For \( v \in D_1 \),

\[
f'_v = \frac{c_v/n - q}{p - q} \left( \frac{1}{|D_1|} \sum_{v \in D_1} \frac{c_v/n - q}{p - q} - 1 \right)
\]

It can be verified \( \sum_{v \in D} f'_v = 1 \).

**Analysis.** As \( \hat{f}_v = \frac{c_v/n - q}{p - q} \), CLS is equivalent to Norm-Sub, and \( \delta = -\frac{1}{|D_1|} \left( \sum_{v \in D_1} \frac{c_v/n - q}{p - q} - 1 \right) \) is the \( \delta \) we want to find in Norm-Sub.

### 3.6 Discussions

In summary, Norm-Sub is the solution to the Constraint Least Square (CLS) formulation to the problem. Furthermore, when the \( f_0 \) component in variance is dominated by the other component (as in Equation (7)), the CLS formulation is equivalent to our MLE formulation. In that case, Norm-Sub is equivalent to MLE-Apx. Norm-Sub is theoretically well justified as the approximate MLE solution. Norm-Hyb additional considers the prior knowledge that a large input domain means that the vast majority of the true values are 0 or close to 0, and avoids disturbing the most frequent values.

Table 1 gives a summary of the methods. First of all, all of the methods preserve the frequency order of the value, i.e., \( f'_v \leq f''_v \) iff \( \hat{f}_v \leq \hat{f}_v \). Except Power, all methods rely on evaluating simple functions on the estimations, plus searching for an additive value \( \delta \) (as in Norm-Sub, Norm-Hyb) or a partition of \( D \) (as in MLE-Apx), which are typically efficient as one can binary search them. Thus these methods have time complexities of \( O(d) \). On the other hand, Power evaluates the summation of \( n \) products for each estimation, thus has a \( O(n \cdot d) \) (or \( O(\sqrt{n} \cdot d) \), if ignoring some terms) time complexity.

### 4 EVALUATION

#### 4.1 Experimental Setup

**Datasets.** We run experiments on six datasets (two synthetic and four real). All of the datasets has some values that are frequent. Due to space limit, we only present results on the following two in this section, and include results on the other four datasets in the appendix.

- **IPUMS [29]:** The US Census data for the year 1940. We sample 1% of users, and use the city attribute (N/A are discarded). This results in \( n = 602325 \) users and \( d = 915 \) cities.
- **Synthetic Zipf’s distribution with 1024 values and 1 million users and**

**year 1940.** We sample 1% of users, and use the city attribute (N/A are discarded). This results in \( n = 602325 \) users and \( d = 915 \) cities.

**Power.** Fit Power-Law dist., then minimize expected squared error [19] Yes No \( O(\sqrt{n} \cdot d) \)

<table>
<thead>
<tr>
<th>Method</th>
<th>Description</th>
<th>Non-neg</th>
<th>Sum to 1</th>
<th>Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base</td>
<td>Use existing estimation</td>
<td>No</td>
<td>No</td>
<td>( O(d) )</td>
</tr>
<tr>
<td>Base-Pos</td>
<td>Convert negative est. to 0</td>
<td>Yes</td>
<td>No</td>
<td>( O(d) )</td>
</tr>
<tr>
<td>Post-Pos</td>
<td>Convert negative query result to 0</td>
<td>Yes</td>
<td>No</td>
<td>( O(d) )</td>
</tr>
<tr>
<td>Base-Cut</td>
<td>Convert est. below threshold ( \theta ) to 0</td>
<td>Yes</td>
<td>No</td>
<td>( O(d) )</td>
</tr>
<tr>
<td>Norm</td>
<td>Add ( \delta ) to est. [18]</td>
<td>No</td>
<td>Yes</td>
<td>( O(d) )</td>
</tr>
<tr>
<td>Norm-Mul</td>
<td>Convert negative est. to 0, then multiply ( \gamma ) to positive est.</td>
<td>Yes</td>
<td>Yes</td>
<td>( O(d) )</td>
</tr>
<tr>
<td>Norm-Cut</td>
<td>Convert negative and small positive est. below ( \theta ) to 0</td>
<td>Yes</td>
<td>Almost</td>
<td>( O(d) )</td>
</tr>
<tr>
<td>Norm-Sub</td>
<td>Convert negative est. to 0 while adding ( \delta ) to positive est.</td>
<td>Yes</td>
<td>Yes</td>
<td>( O(d) )</td>
</tr>
<tr>
<td>Norm-Hyb</td>
<td>Keep est. above ( \theta ), then apply Norm-Sub to others</td>
<td>Yes</td>
<td>Yes</td>
<td>( O(d) )</td>
</tr>
<tr>
<td>MLE-Apx</td>
<td>Convert negative est. to 0, then add ( \delta ) to positive est.</td>
<td>Yes</td>
<td>Yes</td>
<td>( O(d) )</td>
</tr>
</tbody>
</table>

| Table 1: Summary of Methods. |
estimating the full domain, the top-$k$ frequent values, and sets of values. Specifically, for the full domain, we compute
\[
MSE = \frac{1}{d} \sum_{v \in D} (f_v - \hat{f_w})^2.
\]

For the top domain, we consider the top $k$ values with highest $f_v$ instead of the whole domain $D$, and for the sets of values, instead of measuring errors for singletons, we measure errors for sets, that is, we first sum the frequencies for a set of values, and then measure the difference. The sets are randomly sampled from $D$ (without replacement) and have a fixed size.

**Plotting Convention.** Because there are 11 algorithms (10 post-processing methods plus Base), and for any single metric there are often multiple methods that perform very similarly, resulting their lines overlapping. To make the figures readable, we plot results on two separate figures on the same row. On the left, we plot 6 methods, Base, Base-Pos, Post-Pos, Norm, Norm-Mul, and Norm-Sub. On the right, we plot Norm-Sub with the remaining 5 methods, MLE-Apx, Base-Cut, Norm-Cut, Norm-Hyb, and Power. We mainly want to compare the methods in the right column.

### 4.2 Full-domain Evaluation

Figure 1 shows MSE when querying the frequency of every value in the domain. We vary $\epsilon$ from 0.2 to 2. Let us fist focus on the figures on the left. Base performs very close to Norm, since the adjustment of Norm can be either positive or negative as the expected value of the estimation sum is 1. As Base-Pos (which is equivalent to Post-Pos in this setting) converts negative results to 0, its MSE is around half that of Base (note the y-axis is in log-scale). Norm-Sub is able to reduce the MSE of Base by about a factor of 10. Norm-Mul behaves differently from other methods. In particular, the MSE decreases much slower than other methods. This is because Norm-Mul multiplies the original estimations by the same factor. The higher the estimate, the greater the adjustment. Since the estimations are individually unbiased, this is not the correct adjustment. When $\epsilon$ is low, all methods perform poorly due to the amount of noise. When $\epsilon$ increases, this drawback of Norm-Mul is more clearly seen.

For the right part of Figure 1, we observe that, Norm-Sub and MLE-Apx perform almost exactly the same, validating the prediction from theoretical analysis; they are better than other methods. Norm-Sub, MLE-Apx, Power, Norm-Hyb and Base-Cut perform very similarly. Base-Cut converts results below a threshold $T$ to 0. This suggests that if one considers average accuracy of all estimations, the dominating source of errors comes from the fact many values have true frequencies close or equal to 0 are randomly perturbed. Norm-Cut also convert low estimations to 0, but the threshold $\theta$ is likely to be lower than $T$, because $\theta$ is chosen to achieve a sum of 1.

### 4.3 Set-value Evaluation

Estimating set-values plays an important role in the interactive data analysis setting (e.g., estimating which category of emoji’s is more...
popular). Keeping $\varepsilon = 1$, we evaluate the performance of different methods by changing the size of the set. For the set-value queries, we uniformly sample $\rho \%$ of the domain (without replacement) and evaluate the MSE between the sum of the true frequencies of values in the set and that of the estimated frequencies. Formally, define $D_{\rho}$ as a random subset of $D$ that has $\rho \% \times |D|$ elements; and define $f_{D_{\rho}} = \sum_{v \in D_{\rho}} f_v$. We sample $D_{\rho}$ multiple times and measure MSE between $f_{D_{\rho}}$ and $f'_{D_{\rho}}$.

**Vary $\rho$ from 10 to 90.** Following the layout convention, we show results for set-value estimations in Figure 2, where we first vary $\rho$ from 10 to 90. Overall, as the results are normalized, the norm-based approaches perform well, especially when $\rho$ is large; and their MSE is symmetric with $\rho = 50$. In particular, when $\rho = 90$, the best norm-based method, Norm-Hyb, performs 1.5 to 4 orders of magnitude better than any of the non-norm based methods.

For each specific method, it is observed the MSE for Base-Pos is 1 to 2 orders of magnitude higher than other methods, because it turns negative estimates to 0, thus introduces systematic bias. Post-Pos is slightly better than Base, as most of the query results are positive. In some cases, Base-Cut performs better than Base. This happens when the threshold $T$ is high, where converting estimates below $T = 0$ is more likely to make the summation $f_{D_{\rho}}$ close to one. Finally, Power performs better than Base especially when $\rho$ is small, but in the IPUMS dataset, when $\rho$ is approaching 90, Power can be worse than Base. This is because Power converts negative values to be positive, while Base does not. From $\rho = 10$ to 90, the decay of Power is 1 to 2 orders of magnitude, while Base is less than 1 order.

**Vary $\rho$ from 1 to 10.** Having examined the performance of set-queries for larger $\rho$, we then vary $\rho$ from 1 to 10 and demonstrate the results in Figure 3. Within this $\rho$ range, the errors of all methods increase with $\rho$, which is as expected. When $\rho$ becomes small, the performance of different methods approaches to that of full-domain estimation.

Norm-Cut somehow varies the threshold so that after cutting, the remaining estimates sum up to one. Thus the performance of Norm-Cut is better than Base-Cut especially when $\rho \geq 2$. Intuitively, the norm-based methods should perform better answering set-queries. But Norm-Mul does not. This is because the multiplication operation keeps the small positive estimates almost unchanged, while they should be reduced. On the other hand, the larger estimates are reduced a lot more, making them biased. This also demonstrates that enforcing sum-to-one is not enough. Different approaches perform significantly different.

**Fixed set queries.** Besides random set queries, we include a case study of subset queries for the IPUMS dataset. The queries ask the number of residents in each:

1. State Economic Area (SEA): there are 370 such SEAs; on average, each SEA includes $\rho \% \times d = 0.2\% \times 951 = 2$ cities.
2. State: 49 States, average $\rho = 2$.
3. Region: 9 Regions, average $\rho = 11$. 

![Figure 2: Results on set-value estimation, varying set size percentage $\rho$ from 10 to 90.](image-url)
The MSE varying $\epsilon$ is reported in Figure 4. Here we see that Norm-Sub, MLE-Apx, and Norm-Hyb perform consistently better than the other methods. In the Region query, Norm-Mul performs comparable to Norm-Sub. This is because the population in each region is more balanced. Power perform poor, as the distribution is different from Power-Law.

In general, the shape of Figure 4 is similar to Figure 1, as they both evaluate MSE varying $\epsilon$. However, as the tasks are full-domain and set-value queries, respectively, we observe different relationships among the methods. In particular, now Base-Pos is generally higher as it introduces systematic bias to the results. And Post-Pos is slightly better than Base and Norm, especially when $\rho$ is smaller. Comparing Base-Cut and Norm-Cut, we observe the decay of Base-Cut becomes smaller than Norm-Cut, and their starting point (at small $\epsilon$) depends on $\rho$. This is consistent with Figure 3.

### 4.4 Frequent-value Evaluation

Finally, we evaluate different methods varying the top values to be considered. Define $D_{ik}$ as \{v $\in$ D | $f_v$ ranks top $k$\}. We measure MSE between $(f'_{v})_{v \in D_{ik}}$ and $(f_{v})_{v \in D_{ik}}$ for different values of $k$ (from 2 to 32), fixing $\epsilon = 1$. Note that neither the frequency oracle nor the subsequent post-processing operation is aware of $D_{ik}$.

From the left column of Figure 5, we observe that Base, Base-Pos, Post-Pos, and Norm perform consistently well for different $k$, as the first three methods do nothing to the top values, and Norm touches them in an unbiased way. Norm-Mul performs at least 10× worse than any other methods because it multiplies the estimations, which makes the higher estimations receive a lot more reduction. Norm-Sub and MLE-Apx also perform worse than Base, but better than Norm-Mul, because the same amount is subtracted from every estimate, regardless of $k$.

To give a better comparison, we plot both Base and Norm-Sub to the right. These two methods have consistent MSE for different $k$. The rest four methods, Base-Cut, Norm-Cut, Norm-Hyb, and Power, all have MSE that grows with $k$. In particular, for Base-Cut, a fixed threshold $T$ (in Equation (9)) is used and estimates below it is converted to 0. This also suggests that at $\epsilon = 1$, around 10 values can be reliably estimated. For the same reason, this phenomenon happens to Norm-Cut. As Norm-Cut is better than Base-Cut, it suggests the threshold $\theta$ used in Norm-Cut is smaller than $T$ in Base-Cut. If $T$ is reduced (can be varied by different $\alpha$), MSE of Norm-Hyb (as well as Base-Cut) can be lowered until it matches that of Norm-Cut. Thus $T$ (or $\alpha$) is actually a utility tradeoff between frequent values versus set-values. Finally, Power performs worse than Norm-Hyb in IPUMS when $k > 12$.

### 4.5 Summary of Findings

In summary, we evaluate the 10 post-processing methods on different datasets, for different tasks, and varying different parameters. We now summarize the findings and present guidelines for using the post-processing methods.
With the experiments, we verify the theoretical connections among the methods: Norm-Sub and MLE-Apx perform similarly, and Base and Norm performs similarly.

The best choice for post-processing method depends on the queries one wants to perform. If set-value estimation is needed, especially when the set size is large, one should use Norm-Hyb. If one just wants to estimate results for the most frequent values, one should use Norm. Note that the two approaches do not conflict with each other. That is, if the value of $k$ is specified in advance, one can adjust the threshold $T$ in Norm-Hyb to be the same as the $k$-th highest raw estimation, to achieve high utility while ensuring consistency. Finally, in the case when one cares about single value queries only, as demonstrated in Appendix C, Base-Cut should be used. However, we note that this is less interesting in the categorical setting of LDP. As the amount of noise is large, one cares more about most frequent values and set-values. Overall, we recommend Norm-Hyb. But if there is a specific requirement, one can follow the guideline for choosing post-processing methods.

- When single value queries are desired, use Base-Cut.
- When frequent values are desired: when $k$ is available, use Norm-Hyb; otherwise, use Norm.
- When set-value queries are more important, use Norm-Hyb.

5 RELATED WORK

LDP frequency oracle (estimating frequencies of values) is a fundamental primitive in LDP. There have been several mechanisms [5, 6, 10, 13, 16, 34] proposed for this task. Among them, [34] introduces OLH, which achieves low estimation errors and low communication costs on large domains. Hadamard Response [3, 5] is similar...
to OLH in essence, but uses the Hadamard transform instead of hash functions. The aggregation part is faster because evaluating a Hadamard entry is practically faster, but it can only be used for binary output, which gives higher error than OLH [34]. Subset selection [33, 38] method achieves better accuracy than OLH, but with a much higher communication cost.

LDP frequency oracle is also a building block for other analytical tasks, e.g., finding heavy hitters [5–7, 17, 35], frequent itemset mining [27, 36], releasing marginals under LDP [9, 28, 40], key-value pair estimation [39], evolving data monitoring [15, 20], and empirical risk minimization [30, 31]. Mean estimation is also a building block in LDP, but most of them transform the numerical value to a discrete value using stochastic round, and then apply frequency oracles [10, 12, 13, 25]. Others (e.g. [41]) apply Gaussian noise, which can be shown to give worse utility.

There exist efforts to post-process results in the setting of centralized DP. Most of them focus on utilizing the structural information in problems other than the simple histogram, e.g., estimating marginals [11, 26] and hierarchy structure [18]. The methods do not consider the non-negativity constraint. Other than that, they are similar to Norm-Sub and minimize $L_2$ distance. On the other hand, the authors of [24] started from MLE and propose a method to minimize $L_1$ instead of $L_2$ distance, as the DP noise follows Laplace distribution.

In the LDP setting, [32] and [23] also consider the hierarchy structure and apply the technique of [18]. Jia et al. [19] is the first to focus on post-processing of FO in LDP and propose to use external information about the dataset’s distribution (e.g., assume the underlying dataset follows Gaussian or Zipf’s distribution). We note that such information may not always be available. On the other hand, we exploit the basic information in each LDP setting. That is, first, the total number of users is known; second, negative values are not possible. We found that in the LDP setting, minimizing $L_2$ distance achieves MLE. The intuition is the noise is more close to the Gaussian distribution.

When many user reports are anonymized and then mixed (shuffled), one can argue a stronger privacy guarantee [8, 15]. Such a privacy amplification effect holds only when the anonymization party is trusted. This extension can be applied in our setting.

6 CONCLUSION

In this paper, we study how to post-process results from existing frequency oracles to make them consistent while achieving high accuracy for a wide range of tasks, including frequencies of individual values, frequencies of the most frequent values, and frequencies of subsets of values. We considered 10 different methods, in addition to the baseline. We identified Norm performs similar to Base, and MLE-Apx performs similar to Norm-Sub. We then recommend that for full-domain estimation, Base-Cut should be used; when estimating frequency of the most frequent values, Norm or Norm-Hyb (when one knows how many frequent values he is interested) should be used; when answering set-value queries, Norm-Hyb should be used.
REFERENCES


[10]CONSISTENT AND ACCURATE FREQUENCY ORACLES UNDER LOCAL DIFFERENTIAL PRIVACY


[16] Zhikun Zhang, Tianhao Wang, Ninghui Li, Shibo He, and Jiming Chen. 2018. A Solution for MLE-APX

A SOLUTION FOR MLE-APX

Using the KKT condition [21, 22], we augment the optimization target with the following equations:

\[
\begin{align*}
\text{minimize} & \quad \sum_{v} \frac{(c_v - n - q - (p - q)f'_v)^2}{q(1 - q) + (p - q)(1 - p - q)f'_v} + a + b \\
\text{subject to} & \quad \sum_{v} f'_v = 1, \quad \forall v, 0 \leq f'_v \leq 1, \quad a = \mu \cdot \sum_{v} f'_v, \quad b = \sum_{v} \lambda_v \cdot f'_v, \quad \forall v : \lambda_v \cdot f'_v = 0.
\end{align*}
\]

Since \( b = 0 \) and \( a = \mu \) is a constant, the condition for minimizing the target is unchanged. Given that the target is convex, we can find the maximum by taking the partial derivative with respect to each variable:

\[
\frac{\partial}{\partial f'_v} \left( \sum_{v} \frac{(c_v - n - q - (p - q)f'_v)^2}{q(1 - q) + (p - q)(1 - p - q)f'_v} + a + b \right) = -\frac{(c_v - n - q - f'_v(p - q)) \cdot (p(1 - p) - q(1 - q))}{q(1 - q) + f'_v(p - q)(1 - p - q) + \mu + \lambda_v} = 0
\]

Define a temporary notation

\[
x_v = \frac{c_v - n - q - f'_v(p - q)}{q(1 - q) + f'_v(p - q)(1 - p - q)} = \frac{c_v - n - q - q(1 - q)x_v}{p - q + (p(1 - p) - q(1 - q))x_v}
\]

so that \( f'_v = \frac{x_v}{p - q + (p(1 - p) - q(1 - q))x_v} \)
With \( x_v \), we can simplify the previous equation:

\[
(p(1-p) - q(1-q))x_v^2 + 2(p-q)x_v = \mu + \lambda_v = 0
\]

Now suppose there is a subset of domain \( D_0 \subseteq D \) s.t., \( \forall v \in D_0, f'_v = 0 \) and \( \forall v \in D_1 = D \setminus D_0, f'_v > 0 \) and \( \lambda_v = 0 \). We observe that for \( v \in D_1 \), solution of \( x_v \) does not depend on \( v \). Thus we can solve \( x_v \) by summing up \( f'_v \) for all \( v \) in \( D_1 \):

\[
\sum_{v \in D_1} f'_v = \sum_{v \in D_1} \frac{c_v/n - q - q(1-q)x_v}{p - q} = \sum_{v \in D_1} \frac{c_v/n - |D_1|q - |D_1|q(1-q)x_v}{p - q} = \sum_{v \in D_1} \frac{c_v/n - |D_1|q - |D_1|q(1-q)x_v}{p - q} + (p - q)(1 - p - q) |D_1| q(1-q)
\]

\[
\Rightarrow x_v = \frac{\sum_{x \in D_1} c_v/n - |D_1|q - |D_1|q(1-q)x_v}{(p - q)(1 - p - q) + |D_1| q(1-q)}
\]

Given \( x_v \), we can compute the solution

\[
f''_v = \frac{c_v/n - q - q(1-q)x_v}{p - q} = \frac{c_v/n - q - q(1-q)x_v}{p - q} + (p - q)(1 - p - q) |D_1| q(1-q)
\]

for each value \( v \in D_1 \) efficiently; and \( f''_v = 0 \) for \( v \in D_0 \). It can be verified \( \sum_v f''_v = 1 \).

Finally, to find \( D_0 \), one initiates \( D_0 = \emptyset \) and \( D_1 = D \), and iteratively tests whether all values in \( D_1 \) are positive. In each iteration, for any negative \( \alpha_x \), \( x \) is moved from \( D_1 \) to \( D_0 \). The process terminates when no negative \( \alpha_x \) is found for all \( x \in D_1 \).

**B SOLUTION FOR CLS**

Using the KKT condition [21, 22], we augment the optimization target with the following equations:

\[
\text{minimize } \sum_v (f'_v - \tilde{f}_v)^2 + a + b
\]

where \( \sum_v f'_v = 1 \), \( \forall v, 0 \leq f'_v \leq 1 \), \( a = \mu \cdot \sum_v f'_v \), \( b = \sum_v \lambda_v \cdot f'_v \), \( \forall v : \lambda_v \cdot f'_v = 0 \).

Since \( b = 0 \), and \( a = \mu \) is a constant, the condition that minimizing the target is unchanged. Given that the target is convex, we can find the minimum by taking the partial derivative with respect to each variable:

\[
\frac{\partial}{\partial f'_v} \left[ \sum_v (f'_v - \tilde{f}_v)^2 + a + b \right] = 0
\]

\[
\Rightarrow 2(f'_v - \tilde{f}_v) + \mu + \lambda_v = 0
\]

\[
\Rightarrow f'_v = \tilde{f}_v - \frac{1}{2}(\mu + \lambda_v)
\]

Now suppose there is a subset of domain \( D_0 \subseteq D \) s.t., \( \forall v \in D_0, f'_v = 0 \) and \( \forall v \in D_1 = D \setminus D_0, f'_v > 0 \) \& \( \lambda_v = 0 \). By summing up \( f'_v \) for all \( v \in D_1 \), we have

\[
1 = \sum_{v \in D_1} c_v/n - q/p - q = \frac{|D_1| \mu}{2}
\]

Thus for all \( v \in D_1 \), we can use the formula

\[
f''_v = \frac{c_v/n - q}{p - q} - \frac{1}{|D_1|} \left( \sum_{v \in D_1} \frac{c_v/n - q}{p - q} - 1 \right)
\]

to derive the estimate \( f''_v \) for value \( v \in D_1 \), and \( f''_v = 0 \) for \( v \in D_0 \). One can also find \( D_0 \) using a similar approach when dealing with MLE. And it can also be verified \( \sum_v f''_v = 1 \).

**C SUPPLEMENTARY EVALUATION RESULTS**

In this section, we present additional results with different datasets and settings.

**C.1 Datasets**

We focus on datasets that have values with high frequencies. The smooth distribution is less interesting in LDP because of the large amount of noise.

Four new datasets are evaluated: a synthetic one from Beta distribution and three real datasets POS, Fire, and Kosarak.

- Synthetic Beta distribution of 400 values and 100K reports. The beta distribution is continuous, so we bucketize the samples (multiply each sample by 400 and then take the integer). For the parameters of the Beta distribution, we choose to use \( a = 50 \) and \( b = 2 \).
- POS [42]: A dataset containing 0.5 million merchant transactions of half a million users. There are 1657 possible values. For each transaction, one item is randomly chosen.
- Fire [2]: San Francisco’s Fire Department Calls For Service. We use the portion for the year 2016. There are 0.3 million calls. For each call, its corresponding location unit ID (there are 742 unique ID) is reported.
- Kosarak [1]: A dataset of 1 million click streams on a Hungarian website that contains around one million users with 42178 possible values. For each stream, one item is randomly chosen.

Note that as the Kosarak dataset contains \( d = 42178 \) values, the method Power is too slow to finish (more than 4 hours for one instance of post processing), thus skipped. The statistics of the datasets are given in Table 2.

<table>
<thead>
<tr>
<th>Dataset Name</th>
<th>( n )</th>
<th>( d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>IPUMS</td>
<td>602325</td>
<td>915</td>
</tr>
<tr>
<td>Kosarak</td>
<td>990002</td>
<td>42178</td>
</tr>
<tr>
<td>POS</td>
<td>515597</td>
<td>1657</td>
</tr>
<tr>
<td>Fire</td>
<td>305132</td>
<td>742</td>
</tr>
<tr>
<td>Beta(50, 2)</td>
<td>100000</td>
<td>400</td>
</tr>
<tr>
<td>Zipf(1.5)</td>
<td>1000000</td>
<td>1024</td>
</tr>
</tbody>
</table>

**Table 2: Summary of Datasets.**

The datasets do not necessarily follow Power-Law distribution (some with heavy tails), but there exist values with high counts, which is the ideal setting FO is applicable. The densities of the datasets are plotted in Figure 6.
we can see the MSE for Base-Pos is about 1 orders of magnitude greater than other methods, because it turns negative estimates to 0, thus introducing systematic errors. Base-Cut performs well when $\epsilon$ is small. This is because the threshold is high, thus converting estimates below it, the summation $f_D'$ is close to one. But when $\epsilon$ becomes large, the threshold is low, making $f_D'$ greater than one.

### C.4 Frequent Value Estimation

We consider evaluating the top $k = 10$ values, varying $\epsilon$. The results for MSE are shown in Figure 9. Comparing different methods, we observe that Base and Base-Pos work best, especially in the low-$\epsilon$ range. This is because they do not modify the top-frequent estimations (i.e., $f'_v = f_v$ for each $v \in D_{10}$), making them unbiased estimations. On the other hand, all other methods somewhat changes $(f_v)_{v \in D_{10}}$. Specifically, Base-Cut will convert estimations below a certain threshold ($T$ in Equation (9)) to zero. When $\epsilon$ is low, $T$ is high. Thus we see the MSE of Base-Cut is higher (around $2 \times$) than the baselines when $\epsilon$ is small (roughly, $\epsilon < 1$); and is similar when $\epsilon$ is high enough. Norm-Sub (together with MLE-Apx) and Norm-Mul always modify the top-frequent estimations, thus consistently output worse results than the baselines. On the other hand, Norm-Cut and Norm-Hyb both rely on a threshold (and do not change any estimates above the threshold), thus perform similarly to Base-Cut. Power subtracts a tiny amount to the top frequent values, and therefore performs a little worse than Norm-Cut and Norm-Hyb in this task.

### C.3 Set-value Estimation

For the set-value queries, we uniformly sample $\rho = 5$ percent values (without replacement) from the domain $D$ and evaluate MSE varying $\epsilon$. We show the results in Figure 8. For each specific method,
Figure 7: MAE results on full-domain estimation, varying $\varepsilon$. 
Figure 8: Results on set-value estimation varying $\epsilon$, fixing $\rho = 5$. 
Figure 9: Results on frequent value estimation varying $\epsilon$. 