BACHELOR

Cost-distances in penalized scale-free random graphs

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Cost-distances in penalized scale-free random graphs

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Abstract

In this report, we discuss path costs in scale free random graphs. The degree distribution of these graphs follows a power law with $\tau \in (2, 3)$. We consider first passage percolation, i.e. information spreads from a vertex to an adjacent vertex after a certain amount of time, which we refer to as cost. Path costs consist of an i.i.d. random component L, with $\mathbb{P}\big(L < t\big) = \min(t^{\alpha}, 1)$ and a component $(\deg u \cdot \deg v)^{\mu}, \mu > 0$ which penalizes high degree vertices. Path costs between uniformly chosen vertices are shown to be a tight sequence as $n \to \infty$ under the assumption that $2\mu\alpha < 3-\tau$.

Contents

1 Introduction

Scale-free random graphs, graphs where the degree distribution follows a power law, have been used to model a variety of phenomena. Popular examples are collaboration networks, the World Wide Web [8] and social networks[9], but it can also be used for transportation networks such as metabolic networks [6].
In this report, we will discuss first-passage percolation in a scale-free random graph. [5] [7] This is a model for the spread of for example information or viruses. The information (or other phenomenon) starts in a certain vertex. From there, it spreads to neighbouring vertices after a certain amount of time, or with a certain cost. These vertices then also start transmitting the information to their neighbours. Examples include the spread of a video on the internet, or a disease across the world.

In previous work, to model the random aspect of transmission, random variables have been assigned to edges to model the path cost. [2] In certain cases, this allows for a more realistic model of first-passage percolation than counting the number of edges between vertices. One such example would be mouth-to-mouth spread of information under the assumption that people visit each other sporadically rather than on set times.
We will further expand upon this work, by introducing a penalty component on the path costs, which increases the cost of paths from and to high degree vertices. From an application standpoint, this represents high degree vertices taking longer to visit an arbitrary neighbour. An example would be that a person who knows a lot of other people may not visit them individually as frequently as someone with fewer contacts. Since the quickest paths in scale-free networks often pass through so called hubs, the vertices with the highest degrees, this could potentially impact the behaviour immensely.

The main theorem of this report states that, as long as high degree vertices are not penalized too heavily, cost distances between uniformly chosen vertices in the graph do not diverge as the graph size tends to infinity. This will be quantified later in the report.
In Section 2 we will discuss the graph model in more detail, as well as introduce the path cost function. In Section 3 an overview of the proof of our main theorem is given, which is primarily composed of three steps. These steps are elaborated on in Section 4, which will complete the proof.

2 Model Definition

2.1 Configuration and Path Costs

In this report, we will study large scale-free networks, generated by the configuration model [4], henceforth referred to as $\mathbf{CM_n(D)}$.

We will now define this model. Consider a set of n vertices, n large, with degrees D_i distributed i.i.d. The empirical distribution function satisfies the following equation:

$$\frac{c}{x^{\tau-1}} \leq 1 - F_D(x) = \mathbb{P}\big(D_i > x\big) \leq \frac{C}{x^{\tau-1}}. \tag{1}$$

We assume $\tau \in (2,3)$. Furthermore, we impose the following restrictions on the distribution function

$$\mathbb{P}\big(D \geq 2\big) = 1$$
$$\exists \epsilon, 0 < \epsilon < 1 : \max_{i \leq n} D_i \leq n^{\frac{1+\epsilon}{\tau-1}}.$$

Additionally, if the total number of half-edges generated this way is odd, a single half-edge is added to a vertex. The total number of half-edges is denoted as h_n. Now, the network will be connected by pairing up the half-edges uniformly. In more detail, the following happens

- Order the n vertices,
- For $i = 1, .., n$, use $F_D(x)$ to generate the number of half-edges D_i,
- If the total number of half-edges h_n is odd, add a half-edge to vertex n. Order the half-edges,
- Choose the first unpaired half-edge, then pair it with another half-edge, chosen uniformly at random from the still unpaired half-edges. Repeat until all half-edges are paired.

Note that this pairing procedure is exchangeable, i.e. independent of order. Each half-edge has equal probability of ending up paired with any other half-edge. [10] Note that this need not result in a connected network of size n, but w.h.p. results in a connected component of size $n(1 - o(1))$.

2.1.1 Path costs

After the graph has been created, a cost will be assigned to each edge. These costs will be composed of a penalty component and a random component.

In reality, this random cost becomes fixed only when the information spreads, but for first-passage percolation determining these costs beforehand is equivalent. For vertices u, v, the cost $C_{u,v} = C_{v,u}$ of the edge connecting them is given by

$$C_{u,v} = (\deg u \cdot \deg v)^{\mu} \cdot L_{u,v}. \tag{2}$$

Since we will take $\mu > 0$, the costs of paths connecting high degree vertices will be significantly higher. In a real-world example, this could represent a person with many contacts visiting each respective contact less frequently. The random components are i.i.d. distributed following random variable L with cumulative distribution function

$$F_L(t) = \mathcal{P}\big(L_{i,j} < t\big) = \min(t^{\alpha}, 1). \tag{3}$$

The total cost $d_C(\pi)$ of a path π is simply the sum of the costs C of all the edges the path consists of. The cost-distance $d_C(u, v)$ between two vertices is defined as

$$d_C(u, v) = \min_{\pi \in \{\text{Paths}(u,v)\}} d_C(\pi). \tag{4}$$

An important assumption on these path costs that will be required is the following.

Assumption 2.1 (Constraint on μ, α) *Consider a network following the configuration model* $\mathbf{CM_n(D)}$ *with* $\tau \in (2, 3)$*. Let the path costs be defined as in equations* (2) *and* (3)*. Assume the following equation holds*

$$2\mu\alpha < 3 - \tau. \tag{5}$$

We are now ready to state the main theorem of this report.

Theorem 2.2 *Consider a connected network of size n, following the configuration model* $\mathbf{CM_n(D)}$ *and satisfying Assumption 2.1.*
Then for a pair of vertices u_n, v_n chosen uniformly at random the cost-distance $d_C(u_n, v_n)$ is a tight sequence of random variables, i.e. $d_C(u_n, v_n)$ does not tend to infinity as $n \to \infty$.

2.2 Definitions and previous work

An important concept for the proof is the so-called core, defined as

$$\text{Core}_n := \Big\{ \text{vertices } v \,\Big|\, \deg v > n^{\frac{1}{2}+\epsilon}, 0 < \epsilon \ll 1 \Big\}. \tag{6}$$

This will be a very useful tool for the process of building paths between vertices, since it will be shown later that this is a complete sub-graph.

2.2.1 Previous work

In Section 4 we are going to discuss a branching process. More details can be found there, for now we just state the following theorem, slightly rephrased [3, Proposition 4.7].

Proposition 2.3 (Coupling the forward degrees to an independent sequence) *Consider* $\mathbf{CM_n(D)}$ *with* $\tau > 2$. *Consider a branching process starting from a uniformly chosen vector. There exists a* $\rho > 0$ *such that the random vector of the first* n^ρ *encountered degrees can be coupled to an independent sequence of random variables in such a way that w.h.p. these are identical.*

The original proposition also specifies this independent sequence, which we will derive as well.

An important theorem for the random component of the path costs is taken from "Weak disorder asymptotics in the stochastic mean-field model of distance" by Bhamidi et al. We restate and simplify [2, Theorem 1.1] to our notation.

Theorem 2.4 (Cost-distance without penalty component) *Consider a complete graph of size* m*, with edge costs defined as in equation* (3)*. Let* u, v *be two uniformly chosen vertices in this graph and* $C = C(\alpha)$ *denote the cost of the optimal path between them. Then, as* $m \to \infty$*, for some a.s. finite random variable* G*,*

$$m^{\frac{1}{\alpha}} \cdot C \xrightarrow{d} c_1 \log(m) + G. \tag{7}$$

The takeaway here is that for a large complete graph of size m the random component of the cost-distance scales with $\log m \cdot m^{-\frac{1}{\alpha}}$.

Tying into this, we want to use that the core of the graph is a complete sub-graph. For this, we require the following lemma, [1, Lemma 5]. Here, let $\mathbf{H_S}$ be defined as the number half-edges connected to a set of vertices S.

Lemma 2.5 (Direct connectivity lemma) *Consider two sets of vertices* A *and* B*. If the number of half-edges* $\mathbf{H_A} = o(n)$ *and* $\mathbf{H_B}$ *satisfying*

$$\frac{\mathbf{H_A H_B}}{n} > C(n), \tag{8}$$

then, conditioning on the event $\{h_n < 2\,\mathbb{E}[D]n\}$*, with* $N(B)$ *denoting the neighbours of* B*,*

$$\mathbb{P}\big(A \cap N(B) = \emptyset\big) < \exp\bigg\{ -\frac{C(n)}{4\,\mathbb{E}[D]} \bigg\}. \tag{9}$$

If one takes a single vertex from the core for both of the sets A and B, Equation (8) is satisfied for $C(n) = n^{2\epsilon}$. Then, as n tends to infinity, the event $\{h_n \mid< 2\,\mathbb{E}[D]n\}$ becomes satisfied with high probability and the error probability in the right-hand side of Equation (9) tends to zero.
Furthermore, taking the union over all error probabilities shows that the core is a complete graph w.h.p. as n tends to infinity.

3 Overview of the proof

Recall the main theorem of this report.

Theorem 2.2 *Consider a connected network of size n, following the configuration model $\mathbf{CM_n(D)}$ and satisfying Assumption 2.1.*
Then for a pair of vertices u_n, v_n chosen uniformly at random the cost-distance $d_C(u_n, v_n)$ is a tight sequence of random variables, i.e. $d_C(u_n, v_n)$ does not tend to infinity as $n \to \infty$.

The proof of this theorem will be completed in three steps, outlined in Propositions 3.1, 3.2 and 3.3. Each of these propositions has a dedicated subsection covering its respective proof in Section 4. A path between two arbitrary vertices will be built through the core, with each proposition showing that the cost of a different component of this path remains low.

The first step is outlined in the following proposition:

Proposition 3.1 (Path from random vertex to layer 0) *Assume Assumption 2.1 is satisfied. Let u be a vertex, chosen uniformly from all vertices in the network. Let k be large. Then for all $\delta > 0$ there exists a $K = K(k)$ such that*

$$\mathbb{P}\big(\exists \textit{path } \pi_{u,\tilde{u}} : C_\pi < K\big) > 1 - \delta \tag{10}$$

with $\tilde{u}$ any vertex with $\deg \tilde{u} \geq k$.

Note that this K does not depend on n. In other words, the path size does not tend to infinity as $n \to \infty$.
This proposition will be proven through a branching process.

For the next proposition we require the concept of layers. Choose k large, $\beta = 1+\epsilon'$, $0 < \epsilon' \ll 1$, M large. Then layer j is defined as follows.

$$\mathcal{L}_j := \left\{ \text{vertices } v \,\middle|\, \deg v \in \left(k^{\beta^j}, M \cdot k^{\beta^j}\right) \right\}. \tag{11}$$

The proposition itself is as follows:

Proposition 3.2 (Path from layer 0 to the core) *Assume Assumption 2.1 is satisfied. Consider a vertex $\tilde{u}$ in layer 0, i.e. $\deg \tilde{u} \in \big(k, M \cdot k\big)$. Then, with an error probability that can be made arbitrarily small when k is large, there is a path of small cost (arbitrarily small when k is large) connecting $\tilde{u}$ to a vertex $\bar{u}$ in the core.*

This will be proven by building a path that connects consecutive layers, finally ending up in the first layer that lies inside the core, $\mathcal{L}_\sigma$ and then showing that with high probability this path has a low cost. The following lemmas are required for this proof.

Lemma 4.3 (High probability lower bound on number of neighbours) *Consider the configuration model* $\mathbf{CM_n}(\mathbf{D})$*, layers defined as in Equation* (11)*. Then, with an error probability that can be made arbitrarily small if* k *is large, the number of connections from a vertex in* $\mathcal{L}_j$ *to* $\mathcal{L}_{j+1}$ *is greater than* $\frac{c}{n} \cdot k^{\beta^j\left(1+\beta(2-\tau)\right)}$.

And, with $\gamma = 1 + \beta(2-\tau)$,

Lemma 4.4 (Path cost) *Consider configuration model* $\mathbf{CM_n}(\mathbf{D})$*. Let* $\frac{c}{n} \cdot k^{\beta^j \gamma}$ *be a lower bound on the number of connections from a vertex in* $\mathcal{L}_j$ *to* $\mathcal{L}_{j+1}$*. Then, with an error probability that can be made arbitrarily small if* k *is large, there exists a path from layer* $\mathcal{L}_j$ *to layer* $\mathcal{L}_{j+1}$ *such that its cost is smaller than* $c \cdot k^{\beta^j\left(\mu(1+\beta)-\gamma\frac{1-\epsilon}{\alpha}\right)}$.

Finally, we show that under Assumption 2.1 this cost can be made arbitrarily small for k large, even after summing over all jumps. From these two propositions it then follows that under the assumptions of Theorem 2.2 one can build a low-cost path from a uniformly chosen vertex to the core. Since our goal was to connect two uniformly chosen vertices, to complete the proof of Theorem 2.2, all that remains is to show that if one builds a path from each of those two vertices to the core, these two paths can be connected. This proposition constitutes step 3.

Proposition 3.3 (Connecting the paths through the core) *Assume Assumption 2.1 is satisfied. Let* $\bar{u}$ *and* $\bar{v}$ *be two vertices in the core. Then, for all* $\epsilon, \delta > 0$*, if* n *is sufficiently large,*

$$\mathbb{P}\big(C_{\bar{u},\bar{v}} \leq \epsilon \mid Core_n \textit{ is a complete graph}\big) \geq 1 - \delta. \tag{12}$$

Note that the assumption of the core being a complete graph is satisfied for n large enough, as shown by Lemma 2.5.

The proof of this proposition will use that due to the large amount of possible paths through the core between these two vertices, a path exists where the random component of the cost-distance is low enough to outweigh the penalty component.

4 The proof

In this section we will first go over each of the individual propositions, than bring them together at the end to prove Theorem 2.2.
We start with Proposition 3.1.

Proposition 3.1 (Path from random vertex to layer 0) *Assume Assumption 2.1 is satisfied. Let u be a vertex, chosen uniformly from all vertices in the network. Let k be large. Then for all $\delta > 0$ there exists a $K = K(k)$ such that*

$$\mathbb{P}\big(\exists path\ \pi_{u,\tilde{u}} : C_\pi < K\big) > 1 - \delta \tag{10}$$

with $\tilde{u}$ any vertex with $\deg \tilde{u} \geq k$.

Proof: To prove this proposition, we will use a branching process. Start in vertex u, then consider its neighbours. The degree distribution of these neighbours is given by the size biased version of D, D^*, with

$$\mathbb{P}\big(D^* = d\big) = \frac{d}{\mathbb{E}[D]}\mathbb{P}\big(D = d\big), \tag{13}$$

because the probability of encountering a vertex through a branching process scales with the degree of that vertex. Continuing from these neighbours means the effective degree of these vertices is one lower than their actual degree, because one edge is connected to u. Therefore, from the point of view of vertex u, effective degrees are distributed as random variable B, described by

$$\mathbb{P}\big(B = d\big) = \mathbb{P}\big(D^* = d + 1\big). \tag{14}$$

As discussed in Section 2.2, we know from earlier work [3, Proposition 4.7] that there exists a ρ, $0 < \rho < 1$, such that this distribution is valid w.h.p. for the first n^ρ encountered vertices.

Let $F_B(k)$ denote the probability of a random vertex encountered through this branching process having degree lower than k. The probability of not encountering a vertex with degree k or higher in the first n^ρ vertices is given by

$$\big(F_B(k)\big)^{n^\rho}. \tag{15}$$

Since $0 < F_B(k) < 1$ and $\rho > 0$, this converges to zero as $n \to \infty$.
So with an error probability that converges to zero as n tends to infinity this branching process finds a vertex with degree k or higher.

Furthermore, the cost of the path from vertex u to the first such vertex $\tilde{u}$ only contains vertices with degree lower than k (by definition). The cost is therefore bounded by

$$\text{Cost}(u, \tilde{u}) \leq k^{2\mu} \cdot (L_1 + ... + L_R) \tag{16}$$

with R the number of steps until $\tilde{u}$ is reached.

Let us return to the stated Proposition 3.1. Let $\delta > 0$ be given. Choose $n = n_0$ high enough such that $\big(F_B(k)\big)^{n^\rho} < \frac{1}{2}\delta$, which we have just shown to be possible. Choose K high enough such that the error probability of $k^{2\mu} \cdot (L_1 + ... + L_R)$ exceeding K is lower than $\frac{1}{2}\delta$.
Then, note that for a network with size $n > n_0$, this same K remains valid, since it only depends on R and by extent n_0.
This completes the proof of Proposition 3.1. □

Next, we will take a look at Proposition 3.2,

Proposition 3.2 (Path from layer 0 **to the core)** *Assume Assumption 2.1 is satisfied. Consider a vertex* $\tilde{u}$ *in layer* 0*, i.e.* $\deg \tilde{u} \in \big(k, M \cdot k\big)$*. Then, with an error probability that can be made arbitrarily small when k is large, there is a path of small cost (arbitrarily small when k is large) connecting* $\tilde{u}$ *to a vertex* $\bar{u}$ *in the core.*

First, recall our definition of layers

$$\mathcal{L}_j := \Big\{\text{vertices } v \,\Big|\, \deg v \in \Big(k^{\beta^j}, M \cdot k^{\beta^j}\Big)\Big\}. \tag{11 restated}$$

To illustrate the approach we will take, consider Figure 1 on the following page.

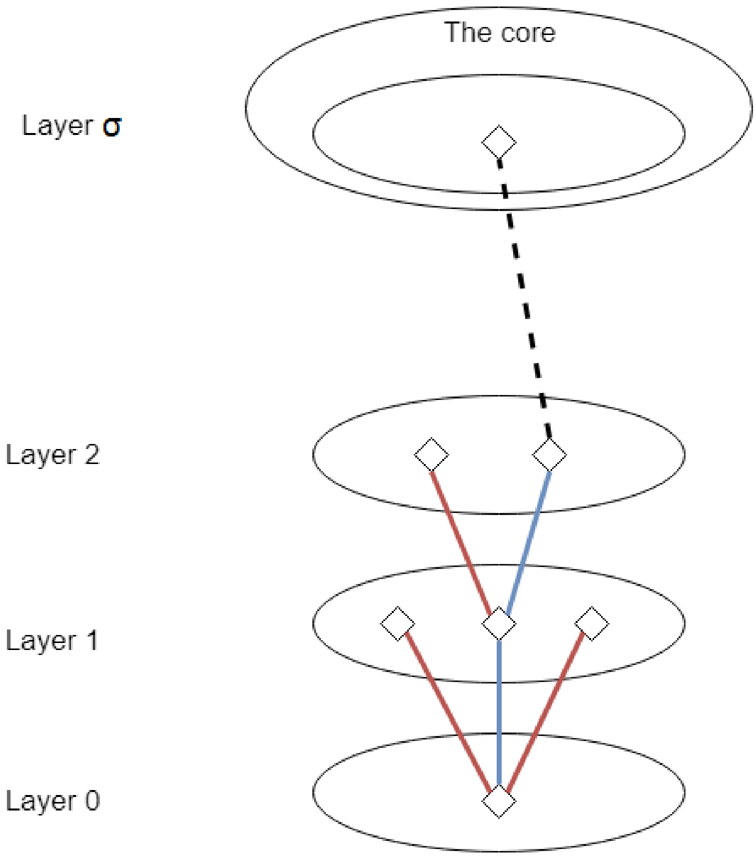

Figure 1: A diagram illustrating the greedy path through the layers.

We start with our vertex $\tilde{u}$ in the bottom layer, layer 0. Then, using a greedy path algorithm, we construct a path to the core. Choose the cheapest connection from vertex $\tilde{u}$ to layer 1. Then continue from that vertex to layer 2 by choosing the cheapest connection and repeat until you end up in layer ρ, which is defined as the first layer that lies in the core.

We will investigate the path at step-level. Consider a vertex v in $\mathcal{L}_j$ and its neighbours in $\mathcal{L}_{j+1}$. We will need a high probability lower bound on the number of these neighbours.
To this end, an upper and lower bound on the expected value of the number of these neighbours and a lower bound on the variance will be derived.

Claim 4.1 *Consider configuration model* $\mathbf{CM_n(D)}$*. Let* v *be a vertex in* $\mathcal{L}_j$ *and let* $Y_{j+1}(v)$ *denote the number of connections between this vertex and* $\mathcal{L}_{j+1}$*. Then*

$$\frac{c}{n} \cdot k^{\beta^j \cdot \big(1+\beta(2-\tau)\big)} \le \mathbb{E}[Y_{j+1}(v)] \le \frac{c}{n} \cdot M^2 \cdot k^{\beta^j \cdot \big(1+\beta(2-\tau)\big)}. \tag{17}$$

Proof: For the expected value, the following holds

$$\mathbb{E}[Y_{j+1}(v)] = \mathbb{E}[\sum_{i=1}^{\deg v} \sum_{w_k \in \mathcal{L}_{j+1}} \mathbb{1}\{v_i \leftrightarrow w_k\}] \tag{18}$$

Here we have v_i the i-th half-edge of vertex v, so the first sum sums over all the half-edges of vertex v. The second sum sums over all the half-edges connected to vertices in layer $\mathcal{L}_{j+1}$. From the pairing algorithm in Section 2, we obtain, with $\mathbf{H}_{j+1}$ the number of half-edges in layer $\mathcal{L}_{j+1}$:

$$\mathbb{E}[Y_{j+1}(v)] \ge k^{\beta^j} \cdot \mathbf{H}_{j+1} \cdot \frac{1}{h_n - 1} \tag{19}$$

Our empirical distribution function from Equation (1) allows us to compute $\mathbf{H}_{j+1}$. Since we are looking for a lower bound, we use the minimal degree of a layer $\mathcal{L}_{j+1}$ vertex for all vertices.

$$\begin{aligned} \mathbb{E}[Y_{j+1}(v)] &\ge k^{\beta^j} \cdot k^{\beta^{j+1}(2-\tau)} \cdot \frac{c}{n} \\ &= k^{\beta^j \cdot \big(1+\beta(2-\tau)\big)} \cdot \frac{c}{n}. \end{aligned} \tag{20}$$

Similarly, by taking the maximum instead of minimum degree for both v and vertices in layer $\mathcal{L}_j$,

$$\mathbb{E}[Y_{j+1}(v)] \le M^2 \cdot k^{\beta^j \cdot \big(1+\beta(2-\tau)\big)} \cdot \frac{c}{n}. \tag{21}$$

□

Henceforth, we shall denote $\gamma = 1+\beta(2-\tau)$ and $\mathbb{1}_{i,k} = \mathbb{1}\{v_i \leftrightarrow \text{half-edge } k \text{ in } \mathcal{L}_{j+1}\}$. Next, we compute an upper bound for the variance of $Y_{j+1}(v)$.

Claim 4.2 *Consider the configuration model* $\mathbf{CM_n(D)}$*. Let* v *be a vertex in* $\mathcal{L}_j$ *and let* $Y_{j+1}(v)$ *denote the number of connections between this vertex and* $\mathcal{L}_{j+1}$*. Then*

$$\mathrm{Var}\big(Y_{j+1}(v)\big) \le \mathbb{E}[Y_{j+1}(v)] + \frac{C}{n} \cdot \mathbb{E}[Y_{j+1}(v)]^2. \tag{22}$$

Proof:

$$
\begin{aligned}
\operatorname{Var}\big(Y_{j+1}(v)\big) &= \operatorname{Var}\Big(\sum_{\substack{i\le \deg v\\ k\le \mathbf{H}_{j+1}}} \mathbb{1}_{i,k}\Big)\\
&= \sum_{\substack{i\le \deg v\\ k\le \mathbf{H}_{j+1}}} \operatorname{Var}\big(\mathbb{1}_{i,k}\big) + \sum_{\substack{i\le \deg v\\ k\le \mathbf{H}_{j+1}}}\sum_{\substack{s\le \deg v\\ t\le \mathbf{H}_{j+1}}} \operatorname{Cov}\big(\mathbb{1}_{i,k},\mathbb{1}_{s,t}\big)\\
&= \sum_{\substack{i\le \deg v\\ k\le \mathbf{H}_{j+1}}} \Big(\mathbb{E}[\mathbb{1}_{i,k}^2]-\mathbb{E}[\mathbb{1}_{i,k}]^2\Big) + \sum_{(i,k)\neq(s,t)} \mathbb{E}[\mathbb{1}_{i,k}\mathbb{1}_{s,t}] - \mathbb{E}[\mathbb{1}_{i,k}]\,\mathbb{E}[\mathbb{1}_{s,t}]\\
&\le \mathbb{E}[Y_{j+1}(v)] + \sum_{(i,k)\neq(s,t)} \left(\frac{1}{(h_n-1)(h_n-3)} - \frac{1}{(h_n-1)^2}\right)\\
&\le \mathbb{E}[Y_{j+1}(v)] + \sum_{(i,k)\neq(s,t)} \frac{2}{h_n^3}\\
&\le \mathbb{E}[Y_{j+1}(v)] + \sum_{(i,k)\neq(s,t)} \frac{2}{h_n^3}.
\end{aligned}
\tag{23}
$$

The choice of pairs corresponds to binomial coefficients. Furthermore $h_n = c\cdot n$ for some c, so

$$
\operatorname{Var}\big(Y_{j+1}(v)\big) \le \mathbb{E}[Y_{j+1}(v)] + \binom{k^{\beta^j}}{2}\cdot\binom{\mathcal{H}_{\ge k^{\beta^{j+1}}}}{2}\cdot\frac{c}{n^3}. \tag{24}
$$

We then use $\binom{n}{2} = n\cdot(n-1)/2$ to get

$$
\operatorname{Var}\big(Y_{j+1}(v)\big) \le \mathbb{E}[Y_{j+1}(v)] + \tilde{c}\cdot\frac{(k^{\beta^j})^2(\mathcal{H}_{\ge k^{\beta^{j+1}}})^2}{n^3}, \tag{25}
$$

which, with equation (20), leads to the final upper bound for the variance, which is

$$
\operatorname{Var}\big(Y_{j+1}(v)\big) \le \mathbb{E}[Y_{j+1}(v)] + \frac{C}{n}\cdot\mathbb{E}[Y_{j+1}(v)]^2. \tag{26}
$$

□

Next is the proof of the following lemma:

Lemma 4.3 (High probability lower bound on number of neighbours) *Consider the configuration model* $\mathbf{CM_n}(\mathbf{D})$*, layers defined as in Equation* (11)*. Then, with an error probability that can be made arbitrarily small if* k *is large, the number of connections from a vertex in* $\mathcal{L}_j$ *to* $\mathcal{L}_{j+1}$ *is greater than* $\frac{c}{n}\cdot k^{\beta^j\left(1+\beta(2-\tau)\right)}$.

Proof: We need a high probability lower bound on the number of neighbours. For this bound, we will use $\frac{1}{2}\,\mathbb{E}[Y_{j+1}(v)]$ (or rather, $\frac{1}{2}$ times the lower bound for this expected value we derived).
Now, it will be shown that with high probability, the actual number of neighbours is not lower than this amount. First, we use

$$\mathbb{P}\big(Y_{j+1}(v) < \frac{1}{2}\,\mathbb{E}[Y_{j+1}(v)]\big) \leq \mathbb{P}\big(\big|Y_{j+1}(v) - \mathbb{E}[Y_{j+1}(v)]\big| \geq \frac{1}{2}\,\mathbb{E}[Y_{j+1}(v)]\big). \tag{27}$$

Then, applying Chebyshev's inequality and using equation (26) for the variance, it follows that

$$\begin{aligned}\mathbb{P}\big(Y_{j+1}(v) < \frac{1}{2}\,\mathbb{E}[Y_{j+1}(v)]\big) &\leq \frac{4 \cdot Var\langle Y_{j+1}(v)\rangle}{\mathbb{E}[Y_{j+1}(v)]^2} \\ &\leq \frac{4}{\mathbb{E}[Y_{j+1}(v)]} + \frac{C}{n}.\end{aligned} \tag{28}$$

Since n is very large, the second part is very small. The first part is summable over j and can then be made arbitrarily small by increasing k. Therefore, $\frac{1}{2}\,\mathbb{E}[Y_{j+1}(v)]$ is a high probability lower bound for the number of neighbours in $\mathcal{L}_{j+1}$ of a vertex in $\mathcal{L}_j$. □

We will now prove the following lemma:

Lemma 4.4 (Path cost) *Consider configuration model* $\mathbf{CM_n(D)}$. *Let* $\frac{c}{n} \cdot k^{\beta^j \gamma}$ *be a lower bound on the number of connections from a vertex in* $\mathcal{L}_j$ *to* $\mathcal{L}_{j+1}$. *Then, with an error probability that can be made arbitrarily small if* k *is large, there exists a path from layer* $\mathcal{L}_j$ *to layer* $\mathcal{L}_{j+1}$ *such that its cost is smaller than* $c \cdot k^{\beta^j\left(\mu(1+\beta)-\gamma\frac{1-\epsilon}{\alpha}\right)}$.

Proof: Now that we have a high probability lower bound on the the number of connections to each next layer, we can analyze the path cost. Recall that this cost for an edge between vertices u and v was described by

$$\begin{aligned}&d_C(u, v) = (\delta_u \delta_v)^{\mu} \cdot L_{u,v} \\ &\mathbb{P}\big(L < t\big) = F_L\big(t\big) = \min(t^{\alpha}, 1).\end{aligned} \tag{29}$$

Consider the jump from $\mathcal{L}_j$ to $\mathcal{L}_{j+1}$, with N connections between the vertex in the former layer to the latter. For each connection, it then holds

$$d_C \leq M^2 \cdot k^{\mu\beta^j} \cdot k^{\mu\beta^{j+1}} \cdot L_i. \tag{30}$$

We wish to choose the connection with the lowest path cost. The high probability upper bound for the minimum of the random components will be taken equal to $t = F_L^{-1}\big(\frac{1}{N^{1-\epsilon}}\big)$ with $\epsilon < 1$. The probability of exceeding this upper bound is given by

$$\mathbb{P}\big(\min_{i=1,\dots,N}(L_i) \geq t\big) = \mathbb{P}\big(L_i \geq t\big)^N, \tag{31}$$

entering $t = F_L^{-1}\big(\frac{1}{M^{1-\epsilon}}\big)$ yields

$$\begin{aligned} \mathbb{P}\Big(\min_{i=1,\dots,N}(L_i) \geq t\Big) &= \Big(1 - \frac{1}{N^{1-\epsilon}}\Big)^N \\ &\leq \exp\Big(-\frac{N}{N^{1-\epsilon}}\Big) \\ &\leq \exp\big(-N^{\epsilon}\big). \end{aligned} \tag{32}$$

Therefore we now have, with error probability $\exp\big(-N^{\epsilon}\big)$, that for the minimum path cost d_{C_j} from $\mathcal{L}_j$ to $\mathcal{L}_{j+1}$,

$$\begin{aligned} d_{C_j} &\leq c \cdot k^{\mu\beta^j} \cdot k^{\mu\beta^{j+1}} \cdot F_L^{-1}\big(\frac{1}{N^{1-\epsilon}}\big) \\ &\leq c \cdot k^{\mu\beta^j(1+\beta)} \cdot F_L^{-1}\Big(\frac{1}{k^{\beta^j\gamma(\alpha-\epsilon)}}\Big) \\ &\leq c \cdot k^{\beta^j\left(\mu(1+\beta)-\gamma\frac{1-\epsilon}{\alpha}\right)} \end{aligned} \tag{33}$$

This is the upper bound we sought to obtain. Since the error probability was $\exp\big(-N^{\epsilon}\big)$ with $N = \frac{1}{2}\mathbb{E}[Y_{j+1}(v)] \geq \frac{c}{n} \cdot k^{\beta^j\gamma}$, this upper bound indeed has an error probability that can be made arbitrarily small if k is large. Summing over all the jumps retains this property. □

The total cost of the path from layer 0 to the core is simply the sum over the costs of the separate jumps, see equation (33). To prove proposition 3.2, we need this to become arbitrarily small if k is increased. This is the case if the exponent is negative.

Claim 4.5 *Let Assumption 2.1 be satisfied, $\beta > 1, \epsilon > 0$ to be determined. Then it holds that*

$$\beta^j\Big(\mu(1+\beta) - \gamma\frac{1-\epsilon}{\alpha}\Big) < 0, \tag{34}$$

such that the path cost in equation (33) *can be made arbitrarily small by increasing k.*

Proof: The upper bound on the total cost of the path is given by

$$\sum_{j \leq \left\lceil \frac{\ln n^{\frac{1}{2}+\epsilon}}{\ln k^{\beta}} \right\rceil} c \cdot k^{\beta^j \left(\mu(1+\beta) - \gamma \frac{1-\epsilon}{\alpha}\right)} \tag{35}$$

This sum can be made arbitrarily small by increasing k if and only if the exponent is negative. Since $\beta^j > 0$, it is left to prove that under assumption 2.1 it holds that

$$\begin{aligned} &\mu(1+\beta) - \gamma \frac{1-\epsilon}{\alpha} < 0 \\ &\mu(1+\beta) < \gamma \frac{1-\epsilon}{\alpha} \\ &\mu\alpha(1+\beta) < \gamma(1-\epsilon). \end{aligned} \tag{36}$$

Recall $\gamma = 1 + \beta(2-\tau)$, equation (36) therefore becomes

$$\mu\alpha(1+\beta) < (1 + \beta(2-\tau))(1-\epsilon). \tag{37}$$

For $\beta = 1, \epsilon = 0$, this reduces to Assumption 2.1. Therefore, for any μ, α satisfying Assumption 2.1, the above equation can be satisfied by taking β and ϵ close enough to 1 and 0 respectively. □

We are now ready to prove Proposition 3.2.

Proposition 3.2 (Path from layer 0 to the core) *Assume Assumption 2.1 is satisfied. Consider a vertex $\tilde{u}$ in layer 0, i.e.* $\deg \tilde{u} \in \big(k, M \cdot k\big)$. *Then, with an error probability that can be made arbitrarily small when k is large, there is a path of small cost (arbitrarily small when k is large) connecting $\tilde{u}$ to a vertex $\bar{u}$ in the core.*

Proof: Lemma 4.3 and Lemma 4.4 together provide an upper bound on the path cost that is valid with an arbitrarily small error probability (for k large) under the assumptions of Proposition 3.2.

By building a path through the greedy path algorithm outlined in Figure 1 we obtain a path that jumps from layer to layer with the computed costs, which exists with arbitrarily small error probability due to Lemma 4.3. It has furthermore been shown that the total cost becomes arbitrarily small for k large. □

We proceed by proving Proposition 3.3

Proposition 3.3 (Connecting the paths through the core) *Assume Assumption 2.1 is satisfied. Let $\bar{u}$ and $\bar{v}$ be two vertices in the core. Then, for all $\epsilon, \delta > 0$, if n is sufficiently large,*

$$\mathbb{P}\big(C_{\bar{u},\bar{v}} \leq \epsilon \mid \mathit{Core}_n \text{ is a complete graph}\big) \geq 1 - \delta. \tag{12}$$

Proof: We will attempt to build a cheap path between vertices $\bar{u}$ and $\bar{v}$ through the core. For this path, we will only go through other vertices in the core. Recall edge costs were given by

$$\begin{aligned} C_{w_1,w_2} &= (\delta_{w_1}\delta_{w_2})^{\mu} \cdot L_{w_1,w_2} \\ F_L(t) &= \mathbb{P}\big(L_{i,j} < t\big) = \min(t^{\alpha}, 1). \end{aligned} \tag{2,3 restated}$$

For the degree of vertices in this graph, we know it is smaller than $n^{\frac{1}{2}+2\epsilon}$. Therefore, with $\tilde{C}_{\bar{u},\bar{v}}$ the cost-distance in the graph if one ignores the penalty component,

$$C_{\bar{u},\bar{v}} \leq n^{2\mu(\frac{1}{2}+2\epsilon)} \cdot \tilde{C}_{\bar{u},\bar{v}}. \tag{38}$$

We then apply Theorem 2.4 to obtain

$$C_{\bar{u},\bar{v}} \leq n^{2\mu(\frac{1}{2}+2\epsilon)} \cdot c_1 \log(m) \cdot m^{\frac{-1}{\alpha}}, \tag{39}$$

with m denoting the size of the core. From Equation (1) we know

$$\begin{aligned} \mathbb{P}\big(D_i > n^{\frac{1}{2}+\epsilon}\big) &= \frac{c}{n^{(\frac{1}{2}+\epsilon)(\tau-1)}} \\ m &= n \cdot F_D\big(n^{\frac{1}{2}+\epsilon}\big) \\ m &= c \cdot n^{1-(\frac{1}{2}+\epsilon)(\tau-1)}. \end{aligned} \tag{40}$$

Combining this estimate with Equation (2,3 restated) obtain the following relation for the cost-distance

$$d_C(\bar{u},\bar{v}) \leq c\log(n) \cdot n^{\mu(1+4\epsilon)-\frac{1-(\frac{1}{2}+\epsilon)(\tau-1)}{\alpha}}. \tag{41}$$

The expression on the right hand side can be made arbitrarily small by increasing n if and only if the exponent is negative. In other words, we want to show

$$\begin{aligned} \mu(1+4\epsilon) - \frac{1-(\frac{1}{2}+\epsilon)(\tau-1)}{\alpha} &< 0 \\ 2\mu\alpha &< \frac{2-(1+2\epsilon)(\tau-1)}{1+4\epsilon} \\ 2\mu\alpha &< \frac{2}{1+2\epsilon} + \frac{1+2\epsilon}{1+4\epsilon}(1-\tau). \end{aligned} \tag{42}$$

For $\epsilon = 0$, this simplifies to Assumption 2.1. Therefore, for any μ, α satisfying Assumption 2.1, ϵ can be chosen such that the above equation is satisfied.
So indeed, the path cost can be made arbitrarily small for n sufficiently large. □

4.1 Completing the proof

We now have all the pieces needed to complete the proof of our main theorem.

Theorem 2.2 *Consider a connected network of size n, following the configuration model $\mathbf{CM_n(D)}$ and satisfying Assumption 2.1.*
Then for a pair of vertices u_n, v_n chosen uniformly at random the cost-distance $d_C(u_n, v_n)$ is a tight sequence of random variables, i.e. $d_C(u_n, v_n)$ does not tend to infinity as $n \to \infty$.

Proof: Formally, to show this is indeed a tight sequence, we need to show that for all $\delta > 0$ there exists a $T = T(\delta)$ independent of n such that for all n

$$\mathbb{P}\big(d_C(u_n, v_n) \in [-T, T]\big) > 1 - \delta \tag{43}$$

We split up the cost-distance in parts:

$$d_C(u, v) = d_C(u, \tilde{u}) + d_C(\tilde{u}, \bar{u}) + d_C(v, \tilde{v}) + d_C(\tilde{v}, \bar{v}) + d_C(\bar{u}, \bar{v}), \tag{44}$$

where we have taken $\deg \tilde{u}, \deg \tilde{v} \geq k$ with k large (and to be determined). Furthermore $\bar{u}, \bar{v} \in \mathbf{Core}_n$. Also note that we dropped the sub-script n for convenience, but are fundamentally still looking at a sequence of such distances in n.
From Proposition 3.2 we obtain that under Assumption 2.1 we can create a path to the core from a vertex with degree k or higher with arbitrarily low cost by choosing k high enough with an error probability that can be made arbitrarily small with k large enough. For our purposes we need the cost to be smaller than 1 and the error probability to be smaller than $\frac{1}{6}\delta$ for both these paths. In equation form,

$$\begin{aligned} \mathbb{P}\big(d_C(\tilde{u}, \bar{u}) < 1\big) &> 1 - \frac{1}{6}\delta \\ \mathbb{P}\big(d_C(\tilde{v}, \bar{v}) < 1\big) &> 1 - \frac{1}{6}\delta, \end{aligned} \tag{45}$$

Then, we apply Proposition 3.1 to build a path from vertices u and v to a vertex with degree at least k. We know that for a provided error probability, we choose once again $\frac{1}{6}\delta$, there exists a K independent of n such that there exists a path with cost below K from our starting vertex to a degree k or higher vertex. We obtain

$$\begin{aligned} \mathbb{P}\big(d_C(u, \tilde{u}) < K\big) &> 1 - \frac{1}{6}\delta \\ \mathbb{P}\big(d_C(v, \tilde{v}) < K\big) &> 1 - \frac{1}{6}\delta, \end{aligned} \tag{46}$$

Finally, we use Proposition 3.3 to connect the paths through the core:

$$\mathbb{P}\big(d_C(\bar{u}, \bar{v}) < 1\mathbf{Core}_n \text{ is a complete graph}\big) > 1 - \frac{1}{6}\delta, \tag{47}$$

and we use Lemma 2.5 to obtain

$$\mathbb{P}\big(\mathbf{Core}_n \text{ is a complete graph}\big) > 1 - \frac{1}{6}\delta, \tag{48}$$

for n high enough. We can combine Equations (45), (46), (47) and (48) to show that Equation (43) is satisfied for $T = 2 \cdot K + 3$. □

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